REFLECTION FUNCTORS FOR QUIVER VARIETIES AND WEYL GROUP ACTIONS

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Abstract. We define a Weyl group action on quiver varieties using reflection functors, which resemble ones introduced by Bernstein-Gelfand-Ponomarev [1]. As an application, we define Weyl group representations of homology groups of quiver varieties. They are analogues of Slodowy's construction of Springer representations of the Weyl group.

Introduction

Consider a finite graph with the set of vertices I. The author [15, 17] associated to $\mathbf{v}$, $\mathbf{w} \in \mathbb{Z}^I_{\geq 0}$, $\zeta \in \mathbb{R}^3 \otimes \mathbb{R}^I$ a hyper-Kähler manifold (possibly with singularities) $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ and called it a *quiver variety*. It was shown that the direct sum of homology groups $\bigoplus_{\mathbf{v}} H_*(\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}))$ has a natural structure of a representation of the Kac-Moody algebra, corresponding to the graph.

The definition of quiver varieties was motivated by author's joint work with Kronheimer [8], where we identify moduli spaces of anti-self-dual connection on ALE spaces with hyper-Kähler quotients of a finite dimensional quaternion vector spaces related to the representation theory of a quiver associated with an ADE Dynkin graph. This is called the ADHM description, since it is a generalization of the description of anti-self-dual connections on $\mathbb{R}^4$ due to Atiyah-Drinfeld-Hitchin-Manin. Quiver varieties are generalization of such hyper-Kähler quotients to arbitary graphs. The parameters $\mathbf{v}$ and $\mathbf{w}$ correspond to Chern classes and the framing at the end respectively.

There is a Weyl group action on ALE spaces. Pulling back anti-self-dual connections, we have an induced action on moduli spaces. More precisely, an element w in the Weyl group sends $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ to $\mathfrak{M}_{w\zeta}(w * \mathbf{v}, \mathbf{w})$, where $w * \mathbf{v}$ is given by $\mathbf{w} - \mathbf{C}(w * \mathbf{v}) = w(\mathbf{w} - \mathbf{C}\mathbf{v})$ ($\mathbf{C}$ is the Cartan matrix). The author [15, §9] used this observation to define an analogue of Slodowy's construction of Springer representations of the Weyl group [19].

We can transform the action by the ADHM description. Maps corresponding to elements of the Weyl group have purely quiver theoretic description, and hence make sense for quiver varieties for any finite graphs. These are what we study in this paper. We call them *reflection functors*, since they resemble ones introduced by Bernstein-Gelfand-Ponomarev [1]. As an application, we can define Weyl group representations on homology groups of quiver varieties. These representations are expected to be related to representations of the Kac-Moody algebra.

In fact, the Weyl group action on ALE spaces, whose existence was originally proved via Brieskorn's construction of simultaneous resolutions [7, §4], can be also realized by reflection functors. This observation was due to Kronheimer (private communication) and was our starting point.

Most of results of this paper were mentioned in [15, §9], and the definition of reflection functors (for simple reflections) was given in [16] about ten years ago. After these announcements, there appeared several related papers. Crawley-Boevey and Hollands [2] defined reflection

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functors for simple reflections under some condition on parameters. Lusztig [11] defined Weyl group actions on quiver varieties by using his description of the coordinate rings of quiver varieties [10]. Maffei [13] also defined reflection functors for simple reflections. We include the identification of our definition with theirs and also with our previous action [15, §9] in this paper. But others use description of quiver varieties as complex (or algebraic) manifolds by forgetting hyper-Kähler structures. So it is impossible for them to prove our assertion that reflection functors are *hyper-Kähler isometry*, namely they preserve the Riemannian metric and all complex structures I, J, K.

We also identify the reflection functor for the longest element of the Weyl group with Lusztig's *new symmetry* [12] on quiver varieties when the graph is of type ADE. His definition makes sense only on lagrangian subvarieties of quiver varieties, while ours are defined on the whole varieties. Via the ADHM description the functor corresponds to the map sending A to its dual A^*.

In this paper, if $A\colon V \to W$ is a linear map between hermitian vector spaces V and W, then $A^\dagger\colon W \to V$ denotes its hermitian adjoint:

$$(Av, w) = (v, A^\dagger w) \quad \text{for } v \in V,\ w \in W.$$

And ${}^tA\colon W^* \to V^*$ denotes the transpose:

$$\langle Av, w^* \rangle = \langle v, {}^tAw^* \rangle \quad \text{for } v \in V,\ w^* \in W^*.$$

1. Hyper-Kähler structure

In order to define reflection functors in a way compatible with hyper-Kähler structures, we need to rewrite the definition of quiver varieties. We use a formulation using quaternion and spinors. This was already used in the ADHM description of instantons on ALE spaces [8]. It is well-known among differential geometers, especially those working on the Seiberg-Witten monopole equation, but we give a detailed explanation for the sake of readers.

1(i). **A hyper-Kähler moment map.**

A *hyper-Kähler structure* on a manifold X is a Riemannian metric g together with a set of three almost complex structures (I, J, K) which are parallel with respect to the Levi-Civita connection of g and satisfy the hermitian condition and the quaternion relations:

$$g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w) \quad \text{for } v, w \in TX,$$
$$IJ = -JI = K.$$

A manifold with a hyper-Kähler structure is called a *hyper-Kähler manifold*. We have the associated Kähler forms ω_I, ω_J, ω_K defined by

$$\omega_I(v, w) = g(Iv, w), \quad \omega_J(v, w) = g(Jv, w),$$
$$\omega_K(v, w) = g(Kv, w) \quad \text{for } v,\ w \in TX$$

which are closed and parallel.

Let G be a compact Lie group acting on X so as to preserve the hyper-Kähler structure (g, I, J, K). Each element $\xi \in \mathfrak{g}$ of the Lie algebra of G defines a vector field ξ^* on X which generates the action of ξ.

Definition 1.1. A *hyper-Kähler moment map* for the action of G on X is a map $\mu = (\mu_I, \mu_J, \mu_K)\colon X \to \mathbb{R}^3 \otimes \mathfrak{g}^*$ which satisfies

$$\begin{aligned} \mu_A(g \cdot x) &= \mathrm{Ad}_g^{*-1}(\mu_A(x)), && x \in X,\ g \in G,\ A = I, J, K, \\ \langle \xi, d\mu_A(v)\rangle &= -\omega_A(\xi^*, v), && v \in TX,\ \xi \in \mathfrak{g}, A = I, J, K, \end{aligned}$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$, $\mathrm{Ad}^*\colon \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint map and $\langle\ ,\ \rangle$ denotes the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

A hyper-Kähler moment map is unique up to an element of $Z \overset{\text{def.}}{=} \{\zeta \in \mathbb{R}^3 \otimes \mathfrak{g}^* \mid \mathrm{Ad}_g^*(\zeta) = \zeta\}$ if it exists.

Suppose μ exists and choose an element ζ from Z. Then $\mu^{-1}(\zeta)$ is invariant under the G-action, so we can make the quotient space $\mu^{-1}(\zeta)/G$. Let $i\colon \mu^{-1}(\zeta) \to X$ be the inclusion, and $\pi\colon \mu^{-1}(\zeta) \to \mu^{-1}(\zeta)/G$ the projection. By the general theory of the hyper-Kähler quotient [4], if the action of G on $\mu^{-1}(\zeta)$ is free, then the quotient space $\mu^{-1}(\zeta)/G$ has a natural hyper-Kähler structure such that

(1) i is a Riemannian immersion, and π is a Riemannian submersion,
(2) the Kähler form ω'_A on $\mu^{-1}(\zeta)/G$ satisfy $\pi^*(\omega'_A) = i^*(\omega_A)$ $(A = I, J, K)$.

1(ii). **Quaternion and spinor.** Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternion field. Let $\mathrm{Re}\colon \mathbb{H} \to \mathbb{R}$ and $\mathrm{Im}\colon \mathbb{H} \to \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be maps defined by taking the real and imaginary parts. Let $\overline{}\colon \mathbb{H} \to \mathbb{H}$ be the involution given by $\overline{(x_0 + ix_1 + jx_2 + kx_3)} \overset{\text{def.}}{=} x_0 - ix_1 - jx_2 - kx_3$. We define an inner product on $\mathbb{H}$ by $((x, x')) \overset{\text{def.}}{=} \mathrm{Re}(x\overline{x'})$. It satisfies $((x, x')) = ((x', x)) = ((ix, ix')) = ((jx, jx')) = ((kx, kx'))$. We also define a complex valued skew symmetric form ω by $\omega(x, x') \overset{\text{def.}}{=} ((xj, x')) + i((xk, x')) = \mathrm{Re}(xj\overline{x'}) + i\,\mathrm{Re}(xk\overline{x'})$.

The multiplication of i, j, k from left together with the inner product make $\mathbb{H}$ into a hyper-Kähler manifold. The group $\mathrm{Sp}(1)$ of quaternions of unit length acts on $\mathbb{H}$ by $x \mapsto xg^{-1}$ ($g \in \mathrm{Sp}(1)$), preserving the hyper-Kähler structure. The hyper-Kähler moment map vanishing at the origin is given by

$$\langle \xi, \mu_I(x)\rangle = \frac{1}{2}\mathrm{Re}\,(ix\xi\overline{x}), \quad \langle \xi, \mu_J(x)\rangle = \frac{1}{2}\mathrm{Re}\,(jx\xi\overline{x}), \quad \langle \xi, \mu_K(x)\rangle = \frac{1}{2}\mathrm{Re}\,(kx\xi\overline{x}),$$

where $\xi \in \mathfrak{sp}(1)$ is regarded as a pure imaginary quaternion. We also have a more compact expression:

$$i\langle \xi, \mu_I(x)\rangle + j\langle \xi, \mu_J(x)\rangle + k\langle \xi, \mu_K(x)\rangle = -\frac{1}{2}\mathrm{Im}(x\xi\overline{x}). \tag{1.2}$$

The group $\mathrm{Sp}(1)$ has another action on $\mathbb{H}$ given by $x \mapsto gx$. In order to distinguish with the previous action, we denote this $\mathrm{Sp}(1)$ by $\mathrm{Sp}(1)_L$. This action preserves the inner product $((\ ,\ ))$ and the skew symmetric form ω, but rotate the mulitplication of i, j, k from left. More precisely, the multiplication map $\mathfrak{sp}(1)_L \times \mathbb{H} \ni (\xi, x) \mapsto \xi x \in \mathbb{H}$ is equivariant if we let $\mathrm{Sp}(1)_L$ act on $\mathfrak{sp}(1)_L$, the space of imaginary quaternions, by $\xi \mapsto g\xi g^{-1}$ ($\xi \in \mathfrak{sp}(1)_L$, $g \in \mathrm{Sp}(1)_L$). The moment map $\mu\colon \mathbb{H} \to \mathbb{R}^3 \otimes \mathfrak{sp}(1)^*$ is equivariant if we identify $\mathbb{R}^3$ with $\mathfrak{sp}(1)_L$ and let $\mathrm{Sp}(1)_L$ act as above. (The action on $\mathfrak{sp}(1)^*$ is trivial.) This can be seen from the formula (1.2): $\mathrm{Im}(gx\,\xi\,\overline{gx}) = g\,\mathrm{Im}(x\xi\overline{x})g^{-1}$.

Considering the multplication of $-i$ from right as a complex structure, we regard $\mathbb{H}$ as a complex vector space and denote it by S^+. This is the space of positive spinors. The inner product $((\ ,\ ))$ satisfies $((x(-i), x'(-i))) = ((x, x'))$. We extend it to a hermitian inner product: $(x, x') \overset{\text{def.}}{=} ((x, x')) + i((x, x'(-i)))$. Regarding an element $x \in S^+$ as an element of

$\mathrm{Hom}(\mathbb{C}, S^+)$, we denote its hermitian adjoint by $x^\dagger \in \mathrm{Hom}(S^+, \mathbb{C}) = (S^+)^*$. Similarly we have $(\ )^\dagger\colon (S^+)^* \to S^+$. The multiplication of a pure imaginary quaternion ξ from left is complex linear, trace-free and skew-hermitian: $\xi(x(-i)) = (\xi x)(-i), \mathrm{tr}(\xi) = 0, ((\xi x, y)) = -((x, \xi y))$. This allows us to identify $\mathfrak{sp}(1)_L$ with $\mathfrak{su}(S^+)$, the Lie algebra of trace-free skew-hermitian endomorphisms of S^+. The form ω is complex linear: $\omega(x(-i), x') = i\omega(x, x')$. Thus ω defines a skew-symmetric form on S^+. We also regard ω as a map $S^+ \to (S^+)^*$, by mapping x to $\omega(x, \cdot)$. This map is denoted also by ω.

Let us consider the subgroup $\mathrm{U}(1)$ of $\mathrm{Sp}(1)$ which consists of complex numbers of unit length. Its action on S^+ given by $x \mapsto x\lambda^{-1}$ is complex linear: $x\lambda^{-1}(-i) = x(-i)\lambda^{-1}$. The corresponding moment map $\mu\colon S^+ \to \mathbb{R}^3 \otimes \mathfrak{u}(1)^*$ is the composition of the previous moment map $\mu\colon S^+ = \mathbb{H} \to \mathbb{R}^3 \otimes \mathfrak{sp}(1)^*$ and the projection $\mathfrak{sp}(1)^* \to \mathfrak{u}(1)^*$. If we identify $\mathbb{R}^3$ with $\mathfrak{su}(S^+)$, then it has the following expression

$$\langle \xi, \mu(x) \rangle = (\xi x \otimes x^\dagger)_0, \tag{1.3}$$

where $(\ )_0$ denotes the trace-free part. Note also that

$$(\xi x \otimes x^\dagger)_0 = -\left(\xi(\omega x)^\dagger \otimes (\omega x)\right)_0. \tag{1.4}$$

These equations will be used frequently later.

Remark 1.5. The expression (1.3) appears in the Seiberg-Witten monopole equation.

1(iii). **Holomorphic description.** Let us choose a particular complex structure, say i. Regarding the multiplication of i from left as an endomorphism of S^+, we have the eigenspace decomposition $S^+ = L \oplus L^*$ (eigenvalue i and $-i$). If we set $z_1 = x_0 + x_1 i, z_2 = x_2 + x_3 i$ for $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, then $z_1 \in L^*, z_2 j \in L$. This induces the following identification $S^+ \cong \mathbb{C}^2$:

$$S^+ = L^* \oplus L \ni x = z_1 + z_2 j \longmapsto \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \in \mathbb{C}^2.$$

We shall simply write $x = \left[\begin{smallmatrix} \overline{z_1} \\ z_2 \end{smallmatrix}\right]$ hereafter. We say the right hand side as a *holomorphic description* of x. Note that this identification respects the complex structures: the multiplication of $-i$ from right on S^+ and the map $\left[\begin{smallmatrix} \overline{z_1} \\ z_2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} i\overline{z_1} \\ iz_2 \end{smallmatrix}\right]$. The multiplication of i, j, k from the left are given by

$$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

respectively. The hermitian inner product and skew-symmetric form are give by

$$((x, x')) = \overline{z_1} z_1' + z_2 \overline{z_2'}, \quad \omega(x, x') = -\overline{z_1} z_2' + z_2 \overline{z_1'} \quad \text{for } x = \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix}, x' = \begin{bmatrix} \overline{z_1'} \\ z_2' \end{bmatrix}.$$

The corresponding map $\omega\colon S^+ \to (S^+)^*$ is written as

$$\begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} z_2 & -\overline{z_1} \end{bmatrix}.$$

Let us consider the subgroup action of $\mathrm{U}(1)$ on S^+ as §1(ii). It is given by

$$\begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} \lambda\overline{z_1} \\ \lambda z_2 \end{bmatrix} = \begin{bmatrix} \overline{\lambda^{-1} z_1} \\ \lambda z_2 \end{bmatrix} \qquad \text{for } \lambda \in \mathrm{U}(1)$$

in the holomorphic description. The hyper-Kähler moment map is expressed as

$$\begin{cases} \langle \xi, \mu_I(x) \rangle = \dfrac{i}{2} \left(|z_1|^2 - |z_2|^2 \right) \xi, \\ \langle \xi, \mu_J(x) \rangle + i \langle \xi, \mu_K(x) \rangle = z_1 z_2 \xi, \end{cases} \qquad \text{for } x = \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix}.$$

We have the following matrix expression:

$$\begin{aligned} &\langle \xi, \mu_I(x) \rangle \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \langle \xi, \mu_J(x) \rangle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \langle \xi, \mu_K(x) \rangle \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\ &= \xi \begin{bmatrix} \frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right) & \overline{z_1 z_2} \\ z_1 z_2 & -\frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right) \end{bmatrix}. \end{aligned} \tag{1.6}$$

The equations (1.3, 1.4) are expressed as

$$\begin{aligned} \begin{bmatrix} \frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right) & \overline{z_1 z_2} \\ z_1 z_2 & -\frac{1}{2} \left(|z_1|^2 - |z_2|^2 \right) \end{bmatrix} &= \begin{bmatrix} |z_1|^2 & \overline{z_1 z_2} \\ z_1 z_2 & |z_2|^2 \end{bmatrix}_0 = \left(\begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix}^\dagger \right)_0 \\ &= - \begin{bmatrix} |z_2|^2 & -\overline{z_1 z_2} \\ -z_1 z_2 & |z_1|^2 \end{bmatrix}_0 = - \left(\left(\omega \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \right)^\dagger \omega \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \right)_0 . \end{aligned}$$

Remark 1.7. The expression (1.6) appears in the Seiberg-Witten monopole equation on a Kähler surface.

1(iv). **Hidden symmetry.** In the holomorphic description above, a natural symmetry between i, j, k are broken, and the complex structure i is chosen. So it is natural to consider S^+ as a complex manifold by i. (More generally any hyper-Kähler manifold is a complex manifold by the integrable almost complex structure I.) Then z_1, z_2 are *holomorphic* coordinates. If we make a combination $\mu_{\mathbb{C}} \overset{\text{def.}}{=} \mu_J + i\mu_K$, then $\langle \xi, \mu_{\mathbb{C}}(x) \rangle = \xi z_1 z_2$ is a holomorphic function. In this context, we denote the remaining moment map μ_I by $\mu_{\mathbb{R}}$ and we call $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ the complex and real part of the moment map respectively. If we define a holomorphic symplectic form $\omega_{\mathbb{C}}$ by $\omega_J + i\omega_K$, then the complex part $\mu_{\mathbb{C}}$ of the hyper-Kähler moment map μ can be considered as a moment map for the $\mathbb{C}^*$-action on the holomorphic symplectic manifold $(S^+, \omega_{\mathbb{C}})$. (Note that we also choose the identification {pure imaginary quaternions} $\cong \mathbb{R} \oplus \mathbb{C}$.)

The action of the group $\mathrm{Sp}(1)_L$ is expressed as

$$\begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \overline{g_1} & -\overline{g_2} \\ g_2 & g_1 \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ z_2 \end{bmatrix} \qquad \text{for } g = g_1 + g_2 j \in \mathrm{Sp}(1)_L.$$

This action is *not* holomorphic, and hence cannot be seen from the point of view of a complex manifold. In this sense, $\mathrm{Sp}(1)_L$ is a *hidden symmetry* in the holomorphic description of the theory.

In this paper, we use the following technique several times: We first construct something in an $\mathrm{Sp}(1)_L$-equivariant way by using quaternion notation. Second, we choose a complex structure i and use the holomorphic description to say something with respect to i. Then we change the complex structure and deduce the assertion for any complex structure. For example, if we want to say $\mu(x) = 0$, then we need to check that (a) $\mu(x)$ is $\mathrm{Sp}(1)_L$-equivariant and (b) $\mu_{\mathbb{C}}(x) = 0$ for some complex structure.

2(i). **Definition.** Suppose that a finite graph without edge loops (i.e., no edges joining a vertex with itself) is given. Let I be the set of vertices and E the set of edges. Let $\mathbf{A}$ be the adjacency matrix of the graph, namely

$$\mathbf{A} = (\mathbf{A}_{kl})_{k,l\in I}, \qquad \text{where } \mathbf{A}_{kl} \text{ is the number of edges joining } k \text{ and } l.$$

We associate with the graph (I, E) a symmetric generalized Cartan matrix $\mathbf{C} = 2\mathbf{I} - \mathbf{A}$, where $\mathbf{I}$ is the identity matrix. This gives a bijection between the finite graphs without edge loops and symmetric Cartan matrices. We have the corresponding symmetric Kac-Moody algebra, and its Weyl group, which is a group with generators s_k ($k \in I$) and relations

$$s_k^2 = 1, \qquad s_k s_l = s_l s_k \quad \text{if } \mathbf{A}_{kl} = 0, \qquad s_k s_l s_k = s_l s_k s_l \quad \text{if } \mathbf{A}_{kl} = 1. \tag{2.1}$$

It acts on $\mathbb{R}^I$ by $s_k(\zeta) = \zeta'$, where $\zeta'_l = \zeta_l - \mathbf{C}_{kl}\zeta_k$ for $\zeta = (\zeta_l)_{l\in I}$, $\zeta' = (\zeta'_l)_{l\in I}$ $\mathbf{C} = (\mathbf{C}_{kl})_{k,l}$. The action preserves the lattice $\mathbb{Z}^I$.

Let H be the set of pairs consisting of an edge together with its orientation. For $h \in H$, we denote by $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) the incoming (resp. outgoing) vertex of h. For $h \in H$ we denote by $\overline{h}$ the same edge as h with the reverse orientation. Choose and fix an orientation Ω of the graph, i.e., a subset $\Omega \subset H$ such that $\overline{\Omega} \cup \Omega = H$, $\Omega \cap \overline{\Omega} = \emptyset$. The pair (I, Ω) is called a *quiver*.

Let $V = (V_k)_{k\in I}$ be a collection of finite-dimensional vector spaces over $\mathbb{C}$ with hermitian inner products for each vertex $k \in I$. The dimension of V is a vector

$$\dim V = (\dim V_k)_{k\in I} \in \mathbb{Z}^I_{\geq 0}.$$

If V^1 and V^2 are such collections, we define vector spaces by

$$\mathrm{L}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{k\in I} \mathrm{Hom}(V^1_k, V^2_k), \quad \mathrm{E}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{h\in H} \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}),$$

$$\mathrm{E}_\Omega(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{h\in\Omega} \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}), \quad \mathrm{E}_{\overline{\Omega}}(V^1, V^2) \overset{\text{def.}}{=} \bigoplus_{h\in\overline{\Omega}} \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}).$$

For $B = (B_h) \in \mathrm{E}(V^1, V^2)$ and $C = (C_h) \in \mathrm{E}(V^2, V^3)$, let us define a multiplication of B and C by

$$CB \overset{\text{def.}}{=} \left(\sum_{\mathrm{in}(h)=k} C_h B_{\overline{h}} \right)_k \in \mathrm{L}(V^1, V^3).$$

Multiplications ba, Ba of $a \in \mathrm{L}(V^1, V^2)$, $b \in \mathrm{L}(V^2, V^3)$, $B \in \mathrm{E}(V^2, V^3)$ is defined in obvious manner. If $a \in \mathrm{L}(V^1, V^1)$, its trace $\mathrm{tr}(a)$ is understood as $\sum_k \mathrm{tr}(a_k)$.

For two collections V, W of hermitian vector spaces with $\mathbf{v} = \dim V$, $\mathbf{w} = \dim W$, we consider the vector space given by

$$\mathbf{M} \equiv \mathbf{M}(\mathbf{v}, \mathbf{w}) \equiv \mathbf{M}(V, W) \overset{\text{def.}}{=} S^+ \otimes_{\mathbb{C}} \mathrm{E}_\Omega(V, V) \oplus S^+ \otimes_{\mathbb{C}} \mathrm{L}(W, V), \tag{2.2}$$

where we use the notation $\mathbf{M}(\mathbf{v}, \mathbf{w})$ when the isomophism classes of hermitian vector spaces V, W are concerned, and $\mathbf{M}$ when V, W are clear in the context. The above two components for an element of $\mathbf{M}$ will be denoted by $\mathcal{A} = \bigoplus \mathcal{A}_h$, $\Psi = \bigoplus \Psi_k$ respectively.

Definition 2.3. We define an affine action of the Weyl group on $\mathbb{Z}^I$ (depending on $\mathbf{w}$) by $s_k * \mathbf{v} \overset{\text{def.}}{=} \mathbf{v}'$, where $v'_k = v_k - \sum_l \mathbf{C}_{kl} v_l + w_k$, $v'_l = v_l$ if $l \neq k$ for $\mathbf{v} = (v_l)_{l\in I}$, $\mathbf{w} = (w_l)_{l\in I}$,

$\mathbf{v}' = (v'_l)_{l\in I}$. Note that we have $\mathbf{w} - \mathbf{C}(s_k * \mathbf{v}) = s_k(\mathbf{w} - \mathbf{C}\mathbf{v})$. We denote the action by $*_{\mathbf{w}}$ if we want to emphasize the $\mathbf{w}$-dependence.

As in §1(ii), we consider S^+ as a hyper-Kähler manifold by the inner product and the multiplications of i, j, k from the left. Together with the hermitian inner product on V, W, we have an induced inner product on $\mathbf{M}$. We also define the operators I, J, K by $i\otimes\mathrm{id}$, $j\otimes\mathrm{id}$, $k\otimes\mathrm{id}$. Thus $\mathbf{M}$ has a (flat) hyper-Kähler structure.

Let $G \equiv G_{\mathbf{v}} \equiv G_V$ be the compact Lie group defined by

$$G \equiv G_{\mathbf{v}} \equiv G_V \overset{\text{def.}}{=} \prod_k \mathrm{U}(V_k),$$

where we use the notation $G_{\mathbf{v}}$ (resp. G_V) when we want to emphasize the dimension (resp. the vector space). Its Lie algebra $\mathfrak{g} \equiv \mathfrak{g}_{\mathbf{v}} \equiv \mathfrak{g}_V$ is the direct sum $\bigoplus_k \mathfrak{u}(V_k)$. The group G acts on $\mathbf{M}$ by

$$(\mathcal{A},\Psi) \mapsto g\cdot(\mathcal{A},\Psi) \overset{\text{def.}}{=} ((g\otimes \mathrm{id}_{S^+})\,\mathcal{A}g^{-1}, (g\otimes\mathrm{id}_{S^+})\,\Psi) \tag{2.4}$$

preserving the hyper-Kähler structure. Let $\mu = (\mu_I,\mu_J,\mu_K)\colon \mathbf{M}\to\mathbb{R}^3\otimes\mathfrak{g}^*$ be the hyper-Kähler moment map vanishing at the origin. Explicitly it is given by

$$\mu(\mathcal{A},\Psi) = i\left(\mathcal{A}\mathcal{A}^\dagger + (\omega\mathcal{A})^\dagger\,\omega\mathcal{A} + \Psi\Psi^\dagger\right)_0.$$

We have the following convention in the above formula: (1) $\mathcal{A}\mathcal{A}^\dagger$, $(\omega\mathcal{A})^\dagger\,\omega\mathcal{A}$, $\Psi\Psi^\dagger$ are considered as elements of $\mathrm{End}(S^+)\otimes \mathrm{L}(V,V)$ by the multiplication defined above, (2) $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via the trace, and (3) $\mathbb{R}^3$ is identified with $\mathfrak{su}(S^+)$ and $(\ )_0$ denotes the trace-free part as in §1.

Let $Z_{\mathbf{v}}\subset\mathfrak{g}_{\mathbf{v}}$ denote the center. It is the direct sum of the set of scalar matrices on V_k. Thus we have a natural projection $(i\mathbb{R})^I \to Z_{\mathbf{v}}$ given by $(\zeta_k)_{k\in I}\mapsto \bigoplus_{k\in I}\zeta_k\,\mathrm{id}_{V_k}\in Z_{\mathbf{v}}$, where we delete the summand correponding to k if $V_k = 0$.

Choosing an element $\zeta = (\zeta_I,\zeta_J,\zeta_K)\in\mathbb{R}^3\otimes(i\mathbb{R})^I$, we consider the hyper-Kähler quotient $\mathfrak{M}_\zeta$ of $\mathbf{M}$ by G:

$$\mathfrak{M}_\zeta \equiv \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \{(\mathcal{A},\Psi)\in\mathbf{M}(\mathbf{v},\mathbf{w}) \mid \mu(\mathcal{A},\Psi) = -\zeta\}/G, \tag{2.5}$$

where ζ is considered as an element of $\mathbb{R}^3\otimes Z_{\mathbf{v}}$ by the above projection. This is the *quiver variety* introduced in [15].

We say a point $(\mathcal{A},\Psi)\in\mu^{-1}(-\zeta)$ is *non-degenerate* if its stabilizer is trivial. We denote by $\mathfrak{M}_\zeta^{\mathrm{reg}}$ the set of non-degenerate G-orbits. This is an open subset of $\mathfrak{M}_\zeta$, and is a hyper-Kähler manifold by [4].

Let

$$\begin{aligned}
R_+ &\overset{\text{def.}}{=} \{\theta = (\theta_k)\in\mathbb{Z}^I_{\geq 0} \mid {}^t\theta\mathbf{C}\theta \leq 2\}\setminus\{0\},\\
R_+(\mathbf{v}) &\overset{\text{def.}}{=} \{\theta\in R_+ \mid \theta_k \leq \dim_{\mathbb{C}} V_k \quad \text{for all } k\},\\
D_\theta &\overset{\text{def.}}{=} \{x = (x_k)\in(i\mathbb{R})^I \mid \sum_k x_k\theta_k = 0\} \quad \text{for } \theta\in R_+.
\end{aligned}$$

When the graph is of Dynkin type, R_+ is the set of positive roots, and D_θ is the wall defined by the root θ. In general, R_+ may be an infinite set, but $R_+(\mathbf{v})$ is always finite.

Proposition 2.6 ([15, 2.8]). *Suppose*

$$\zeta \in \mathbb{R}^3 \otimes (i\mathbb{R})^I \setminus \bigcup_{\theta \in R_+(\mathbf{v})} \mathbb{R}^3 \otimes D_\theta. \tag{2.7}$$

Then the regular locus $\mathfrak{M}_\zeta^{\mathrm{reg}}$ *coincides with* $\mathfrak{M}_\zeta$. *Thus* $\mathfrak{M}_\zeta$ *is nonsingular. Moreover the hyper-Kähler metric is complete.*

2(ii). **A holomorphic description.** As in §1(iii) we choose a particular complex structure, say I, and use the following holomorphic description:

$$\mathcal{A}_h = \begin{bmatrix} B_{\overline{h}}^\dagger \\ B_h \end{bmatrix}, \qquad B_h \colon V_{\mathrm{out}(h)} \to V_{\mathrm{in}(h)}, \quad B_{\overline{h}} \colon V_{\mathrm{in}(h)} \to V_{\mathrm{out}(h)},$$

$$\Psi_k = \begin{bmatrix} j_k^\dagger \\ i_k \end{bmatrix}, \qquad i_k \colon W_k \to V_k, \quad j_k \colon V_k \to W_k.$$

Thus $\mathbf{M}$ is isomorphic to

$$\mathrm{E}(V,V) \oplus \mathrm{L}(W,V) \oplus \mathrm{L}(V,W).$$

We can write down the hyper-Kähler moment map explicitly:

$$\mu_{\mathbb{R}}(B,i,j) = \frac{i}{2}\left(-BB^\dagger + B^\dagger B - ii^\dagger + j^\dagger j\right) \in \mathfrak{g},$$
$$\mu_{\mathbb{C}}(B,i,j) = \varepsilon BB + ij \in \mathfrak{g} \otimes \mathbb{C},$$

where the dual of the Lie algebra of G is identified with the Lie algebra via the trace, $\varepsilon \colon H \to \{\pm 1\}$ is defined by $\varepsilon(h) = 1$ if $h \in \Omega$, $\varepsilon(h) = -1$ if $h \in \overline{\Omega}$, and $\varepsilon B \in \mathrm{E}(V,V)$ is defined by $(\varepsilon B)_h = \varepsilon(h) B_h$. **Caution**: $\mu_{\mathbb{R}}$ differs by sign from one in [15]. $\mu_{\mathbb{C}}$, $\zeta_{\mathbb{C}}$ and $G^{\mathbb{C}}$ (see below) were denoted by μ, ζ and G respectively in [17].

Let $G^{\mathbb{C}}$ be the algebraic group defined by

$$G^{\mathbb{C}} \equiv G^{\mathbb{C}}_{\mathbf{v}} \equiv G^{\mathbb{C}}_V \overset{\text{def.}}{=} \prod_k \mathrm{GL}(V_k).$$

This is the complexification of G. It acts on $\mathbf{M}$ by

$$(B,i,j) \mapsto g \cdot (B,i,j) \overset{\text{def.}}{=} (gBg^{-1}, gi, jg^{-1}) \tag{2.8}$$

preserving the holomorphic symplectic form $\omega_{\mathbb{C}}$. Let $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ be an affine algebraic variety (not necessarily irreducible) defined as the zero set of $\mu_{\mathbb{C}} + \zeta_{\mathbb{C}}$.

2(iii). **Stability.** We want to identify the hyper-Kähler quotient (2.5) with a quotient of $\mu_{\mathbb{C}}^{-1}$ devided by $G^{\mathbb{C}}$. For this purpose, we introduce a notion of the 'stability', following King's work [6].

For a collection $S = (S_k)_{k\in I}$ of subspaces of V_k and $B = \bigoplus B_h$ as above, we say S is *B-invariant* if $B_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$.

For $\zeta_{\mathbb{R}} = (\zeta_{k,\mathbb{R}})_{k\in I} \in (i\mathbb{R})^I$, let $\zeta_{\mathbb{R}}(\dim V) \overset{\text{def.}}{=} i\sum_{k\in I} \zeta_{k,\mathbb{R}} \dim V_k$.

Definition 2.9. A point $(B,i,j) \in \mathbf{M}$ is $\zeta_{\mathbb{R}}$*-semistable* if the following two conditions are satisfied:

(1) If a collection $S = (S_k)_{k\in I}$ of subspaces in V_k is contained in $\operatorname{Ker} j$ and B-invariant, then $\zeta_{\mathbb{R}}(\dim S) \le 0$.
(2) If a collection $T = (T_k)_{k\in I}$ of subspaces in V_k contains $\operatorname{Im} i$ and is B-invariant, then $\zeta_{\mathbb{R}}(\dim T) \le \zeta_{\mathbb{R}}(\dim V)$.

We say (B,i,j) is $\zeta_{\mathbb{R}}$*-stable* if the strict inequalities hold in (1),(2) unless $S=0$, $T=V$ respectively.

If $i\zeta_{k,\mathbb{R}}>0$ for all k, the condition (2) is superfluous, and the condition (1) turns out to be the nonexistence of nonzero collections $S=(S_k)$ such that $S_k\subset\operatorname{Ker} j_k$ and $B_h(S_{\mathrm{out}(h)})\subset S_{\mathrm{in}(h)}$. (In this case $\zeta_{\mathbb{R}}$-stability and $\zeta_{\mathbb{R}}$-semistability are equivalent.) This is the stability condition used in [17, 3.9]. The case when $i\zeta_{k,\mathbb{R}}<0$ for all k is also important. The condition (1) is superfluous and the condition (2) turns out to be the nonexistence of proper collections $T=(T_k)$ such that $T_k\supset\operatorname{Im} i_k$ and $B_h(T_{\mathrm{out}(h)})\subset T_{\mathrm{out}(h)}$. This coincides with the natural condition for the description of Hilbert schemes of points on $\mathbb{C}^2$ ([18, §1]). It was used also in [10].

We also need the stability condition for $B\in\mathrm{E}(V,V)$.

Definition 2.10. Suppose that $\zeta_{\mathbb{R}}(\dim V)=0$. A point $B\in\mathrm{E}(V,V)$ is $\zeta_{\mathbb{R}}$*-semistable* if the following is satisfied:

- If a collection $S=(S_k)_{k\in I}$ of subspaces in V_k is B-invariant, then $\zeta_{\mathbb{R}}(\dim S)\le 0$.

A point B is $\zeta_{\mathbb{R}}$*-stable* if the strict inequality holds unless $S=0$ or $S=V$.

Let $H^{\mathrm{s}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$ (resp. $H^{\mathrm{ss}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$) be the set of $\zeta_{\mathbb{R}}$-stable (resp. $\zeta_{\mathbb{R}}$-semistable) points in $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{R}})$.

We say two $\zeta_{\mathbb{R}}$-semistable points (B,i,j), (B',i',j') are S*-equivalent* when the closures of orbits intersect in $H^{\mathrm{ss}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$.

Proposition 2.11. (1) *A point $(B,i,j)\in\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ is $\zeta_{\mathbb{R}}$-semistable if and only if the closure of its $G^{\mathbb{C}}$-orbit intersects with $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})$. The natural map*

$$\mathfrak{M}_\zeta\to H^{\mathrm{ss}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}/\!\sim$$

is a homeomorphism. Here the right hand side denotes the quotient space of $H^{\mathrm{ss}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}$ divided by S-equialence relation.

(2) *A point $(B,i,j)\in\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ is $\zeta_{\mathbb{R}}$-stable if and only if its $G^{\mathbb{C}}$-orbit contains a non-degenerate point in $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})$. The restriction of the above map gives us a homeomorphism*

$$\mathfrak{M}^{\mathrm{reg}}_\zeta\to H^{\mathrm{s}}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}/G^{\mathbb{C}}.$$

(3) *A $G^{\mathbb{C}}$-orbit intersects with $\mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})$ if and only if it is there exists a direct sum decomposition*

$$V=V^0\oplus V^1\oplus V^2\oplus\cdots,$$

such that

(a) *$\zeta_{\mathbb{R}}(\dim V^p)=0$ for $p\ge 1$,*
(b) *each summand is invariant under B,*
(c) *the image of i is contained in V^0 and j is zero on $\bigoplus_{p\ge1}V^p$,*
(d) *$(B|_{V^0},i,j)$ considered as a data in $\mathbf{M}(V^0,W)$ is $\zeta_{\mathbb{R}}$-stable,*
(e) *the restriction of B to V^p is $\zeta_{\mathbb{R}}$-stable in the sense of Definition 2.10 for $p\ge 1$.*

The statements (1),(2) can be proved by an argument in [6] (see also [15, 3.1, 3.2, 3.5], [17, 3.8]). The statement (3) was proved in [15, 6.5], [17, 3.27].

This proposition and Proposition 2.6 imply that the $\zeta_{\mathbb{R}}$-stability and $\zeta_{\mathbb{R}}$-semistability for points in $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ are equivalent when $\zeta=(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})$ satisfies the condition (2.7).

3(i). **Admissible collection.** We fix $\zeta \in \mathbb{R}^3 \otimes (i\mathbb{R})^I$. An *admissble collection* is the following data:

(1) $V^k = (V^k_l)_{l\in I}$: a collection of hermitian vector spaces for each $k \in I$,
(2) $(\mathcal{A}^k, \Psi^k) \in \mathbf{M}(V^k, W^k)$ satisfying $\mu(\mathcal{A}^k, \Psi^k) = -\zeta$ and the non-degeneracy condition for each $k \in I$, where $W^k = (W^k_l)_{l\in I}$ is given by $W^k_l = \mathbb{C}$ if $l = k$ and 0 otherwise,
(3) $\Phi^{\overline{h}} \in S^+ \otimes \mathrm{L}(V^{\mathrm{in}(h)}, V^{\mathrm{out}(h)})$ for each $h \in \Omega$

such that

$$\begin{aligned}\left(\left(\omega\mathcal{A}^{\mathrm{out}(h)}\right)^\dagger \otimes \omega\Phi^{\overline{h}}\right)_0 + \left(\Phi^{\overline{h}} \otimes \mathcal{A}^{\mathrm{in}(h)\dagger}\right)_0 &= \left(\Psi^{\mathrm{out}(h)} \otimes \Psi^{\mathrm{in}(h)\dagger}\right)_0,\\ \left(\left(\omega\mathcal{A}^{\mathrm{in}(h)}\right)^\dagger \otimes \Phi^{\overline{h}\dagger}\right)_0 &= \left(\left(\omega\Phi^{\overline{h}}\right)^\dagger \otimes \mathcal{A}^{\mathrm{out}(h)\dagger}\right)_0,\\ \left(\Phi^{\overline{h}}_{\mathrm{in}(h)} \otimes \omega\Psi^{\mathrm{in}(h)}\right)_0 &= \left(\mathcal{A}^{\mathrm{out}(h)}_h \otimes \omega\Psi^{\mathrm{out}(h)}\right)_0,\\ \left(\left(\omega\Psi^{\mathrm{out}(h)}\right)^\dagger \otimes \omega\Phi^{\overline{h}}_{\mathrm{out}(h)}\right)_0 &= \left(\left(\omega\Psi^{\mathrm{in}(h)}\right)^\dagger \otimes \omega\mathcal{A}^{\mathrm{in}(h)}_h\right)_0.\end{aligned} \tag{3.1}$$

In the first equality, we consider $\Psi^{\mathrm{out}(h)}$ (resp. $\Psi^{\mathrm{in}(h)}$) as an element of $S^+ \otimes V^{\mathrm{out}(h)}_{\mathrm{out}(h)}$ (resp. $S^+ \otimes V^{\mathrm{in}(h)}_{\mathrm{in}(h)}$), and then $\left(\Psi^{\mathrm{out}(h)} \otimes \Psi^{\mathrm{in}(h)\dagger}\right)_0$ as an element of $\mathfrak{sl}(S^+) \otimes \mathrm{E}_\Omega(V^{\mathrm{out}(h)}, V^{\mathrm{in}(h)})$ via the inclusion $\mathfrak{sl}(S^+) \otimes \mathrm{Hom}(V^{\mathrm{out}(h)}_{\mathrm{out}(h)}, V^{\mathrm{in}(h)}_{\mathrm{in}(h)}) \to \mathfrak{sl}(S^+) \otimes \mathrm{E}_\Omega(V^{\mathrm{out}(h)}, V^{\mathrm{in}(h)})$. In the third equality, the both hand sides are considered as elements of $\mathfrak{sl}(S^+) \otimes V^{\mathrm{out}(h)}_{\mathrm{in}(h)}$. A similar identification was used in the fourth equality. These equalities will be refered as the *compatibility condition*.

Let us give few examples of admissible collections. The first one is trivial:

(1) $V^k_l = 0$ for any k, l,
(2) $\mathcal{A}^k = 0$, $\Psi^k = 0$,
(3) $\Phi^{\overline{h}} = 0$.

The next one will play an important role later. Fix a vertex k_0. We set

(1) $V^k_l = 0$ unless $k = l = k_0$ and $V^{k_0}_{k_0} = \mathbb{C}$,
(2) $\mathcal{A}^k = 0$, $\Psi^k = 0$ unless $k = k_0$, and $\mathcal{A}^{k_0} = 0$, $\Psi^{k_0} \in S^+ \otimes \mathrm{L}(V^{k_0}, W^{k_0}) \cong S^+$ is such that $(\Psi^{k_0} \otimes (\Psi^{k_0})^\dagger)_0 = -\zeta_{k_0}$, where ζ_{k_0} is the k_0-component of ζ,
(3) $\Phi^{\overline{h}} = 0$.

Note that Ψ^{k_0} is unique up to a multiplication by an element of $\mathrm{U}(1)$.

3(ii). **Holomorphic descriptions.** As in §2(ii), we write the admissible data in the holomorphic description:

$$\begin{aligned}\mathcal{A}^k_h &= \begin{bmatrix} B^{k\dagger}_{\overline{h}} \\ B^k_h \end{bmatrix}, & B^k_{\overline{h}} &\in \mathrm{Hom}(V^k_{\mathrm{in}(h)}, V^k_{\mathrm{out}(h)}), & B^k_h &\in \mathrm{Hom}(V^k_{\mathrm{out}(h)}, V^k_{\mathrm{in}(h)}),\\ \Psi^k_k &= \begin{bmatrix} j^{k\dagger}_k \\ i^k_k \end{bmatrix}, & i^k_k &\in \mathrm{Hom}(W^k_k, V^k_k), & j^k_k &\in \mathrm{Hom}(V^k_k, W^k_k),\\ \Phi^{\overline{h}}_k &= \begin{bmatrix} \phi^{h\dagger}_k \\ \phi^{\overline{h}}_k \end{bmatrix}, & \phi^h_k &\in \mathrm{Hom}(V^{\mathrm{out}(h)}_k, V^{\mathrm{in}(h)}_k), & \phi^{\overline{h}}_k &\in \mathrm{Hom}(V^{\mathrm{in}(h)}_k, V^{\mathrm{out}(h)}_k).\end{aligned}$$

We use notation $\phi^h = (\phi^h_k)_k \in \mathrm{L}(V^{\mathrm{out}(h)}, V^{\mathrm{in}(h)})$, $B^k = (B^k_h)_h \in \mathrm{E}(V^k, V^k)$, $i^k \in \mathrm{L}(W^k, V^k)$, $j^k \in \mathrm{L}(V^k, W^k)$ as before. The complex part of the compatibility condition turns out to be

$$\begin{gathered} -\varepsilon(h) i^{\mathrm{in}(h)} \otimes j^{\mathrm{out}(h)} + \phi^h B^{\mathrm{out}(h)} = B^{\mathrm{in}(h)} \phi^h, \\ \phi^h_{\mathrm{out}(h)} i^{\mathrm{out}(h)}_{\mathrm{out}(h)} = B^{\mathrm{in}(h)}_{\overline{h}} i^{\mathrm{in}(h)}_{\mathrm{in}(h)}, \quad j^{\mathrm{in}(h)}_{\mathrm{in}(h)} \phi^h_{\mathrm{in}(h)} = j^{\mathrm{out}(h)}_{\mathrm{out}(h)} B^{\mathrm{out}(h)}_{\overline{h}}. \end{gathered} \tag{3.2}$$

Lemma 3.3. *The map $\phi^h \in \mathrm{L}(V^{\mathrm{out}(h)}, V^{\mathrm{in}(h)})$ satisfying* (3.2) *is uniquely determined from $(B^{\mathrm{out}(h)}, i^{\mathrm{out}(h)}, j^{\mathrm{out}(h)})$ and $(B^{\mathrm{in}(h)}, i^{\mathrm{in}(h)}, j^{\mathrm{in}(h)})$ (if it exists).*

Proof. Suppose two maps ϕ^h, ϕ'^h satisfying (3.2) are given. Consider $\phi^h - \phi'^h$. By (3.2) the kernel of $\phi^h - \phi'^h$ contains the image of $i^{\mathrm{out}(h)}$ and is $B^{\mathrm{out}(h)}$-invariant. Hence we have

$$\zeta_{\mathbb{R}}(\dim \mathrm{Ker}(\phi^h - \phi'^h)) < \zeta_{\mathbb{R}}(\dim V^{\mathrm{out}(h)})$$

unless $\phi^h = \phi'^h$ by the stability condition for $(B^{\mathrm{out}(h)}, i^{\mathrm{out}(h)}, j^{\mathrm{out}(h)})$. Moreover, the image of $\phi^h - \phi'^h$ is contained in the kernel of $j^{\mathrm{in}(h)}$ and $B^{\mathrm{in}(h)}$-invariant. Hence we have

$$\zeta_{\mathbb{R}}(\dim \mathrm{Im}(\phi^h - \phi'^h)) < 0$$

unless $\phi^h = \phi'^h$ by the stability condition for $(B^{\mathrm{in}(h)}, i^{\mathrm{in}(h)}, j^{\mathrm{in}(h)})$. Combining two inequalities, we must have $\phi^h = \phi'^h$. $\square$

Lemma 3.4. *For each $k, l \in I$ we have*

$$\sum_{\substack{h \in \overline{\Omega} \\ \mathrm{in}(h) = k}} \left(\Phi^h_l \Phi^{h\dagger}_l\right)_0 + \sum_{\substack{h \in \Omega \\ \mathrm{in}(h) = k}} \left(\left(\omega \Phi^{\overline{h}}_l\right)^\dagger \omega \Phi^{\overline{h}}_l\right)_0 + \delta_{kl} \left(\Psi^k_k \Psi^{k\dagger}_k\right)_0 = \zeta'_k \,\mathrm{id}_{V^k_l},$$

where $\zeta'_k = \zeta_k - \sum_l {}^t(\mathbf{C}_{kl} \mathbf{v}^l) \cdot \zeta$ and $\mathbf{v}^l = \dim V^l$.

Proof. By the technique explained in §1(iv), it is enough to check

$$- \sum_{\substack{h \in H \\ \mathrm{in}(h) = k}} \varepsilon(h) \phi^h_l \phi^{\overline{h}}_l + \delta_{kl} i^k_k \otimes j^k_k = \zeta'_{k,\mathbb{C}} \,\mathrm{id}_{V^k_l}. \tag{3.5}$$

By the compatibility condition (3.2) we have

$$\begin{gathered} B^k \phi^h \phi^{\overline{h}} = \phi^h \phi^{\overline{h}} B^k + \varepsilon(h) B^k_{\overline{h}} i^k \otimes j^k - \varepsilon(h) i^k \otimes j^k B^k_h, \\ \phi^h_k \phi^{\overline{h}}_k i^k_k = B^k_h B^k_{\overline{h}} i^k_k + \varepsilon(h) \langle j^{\mathrm{out}(h)}, i^{\mathrm{out}(h)} \rangle i^k_k, \\ j^k_k \phi^h_k \phi^{\overline{h}}_k = j^k_k B^k_h B^k_{\overline{h}} + \varepsilon(h) \langle j^{\mathrm{out}(h)}, i^{\mathrm{out}(h)} \rangle j^k_k \end{gathered}$$

for $h \in H$ with $\mathrm{in}(h) = k$. Here $i^{\mathrm{out}(h)}$ and $j^{\mathrm{out}(h)}$ are considered as elements of $V^{\mathrm{out}(h)}_{\mathrm{out}(h)}$ and $(V^{\mathrm{out}(h)}_{\mathrm{out}(h)})^*$ respectively, and $\langle\ ,\ \rangle$ denote the natural pairing between them. By the equation $\mu_{\mathbb{C}}(B^m, i^m, j^m) = -\zeta_{\mathbb{C}}$, we have

$$\langle j^m, i^m \rangle = \mathrm{tr}\,(i^m j^m) = -\sum_n \dim V^m_n \zeta_{n,\mathbb{C}}. \tag{3.6}$$

Let us define $\eta = (\eta_l) \in \mathrm{L}(V^k, V^k)$ by setting η_l as the left hand side minus right hand side of (3.5). Then the above equations together with $\mu_{\mathbb{C}}(B^k, i^k, j^k) = -\zeta_{\mathbb{C}}$ implies that

$$\eta B^k = B^k \eta, \quad \eta i^k = 0, \quad j^k \eta = 0.$$

By the non-degeneracy condition for (B^k, i^k, j^k), we have $\eta = 0$. $\square$

Let $\Lambda^l = (\Lambda^l_k)_{k\in I}$, where $\Lambda^l_k \overset{\text{def.}}{=} V^k_l$. Then (Φ_l, Ψ^l) defines datum for $\mathbf{M}(\Lambda^l, W)$ for the opposite orientation $\overline{\Omega}$ and satisfies $\mu(\Phi_l, \Psi^l) = \zeta'$ for each l. Moreover, $\mathcal{A}_h$, considered as an element of $\mathrm{L}(\Lambda^{\mathrm{out}(h)}, \Lambda^{\mathrm{in}(h)})$, satisfies the compatibility condition for (Φ_l, Ψ^l). This observation will not be used later.

3(iii). **Reflection functor.** Now we define a *reflection functor* for a given admissible collection. Suppose that collections of hermitian vector spaces V, W and a datum $(\mathcal{A}, \Psi) \in \mathbf{M}(V, W)$ such that $\mu(\mathcal{A}, \Psi) = -\zeta$ is given. Set

(3.7)
$$\widetilde{V}_k \overset{\text{def.}}{=} V_k \oplus \mathrm{E}(V^k, V) \oplus \mathrm{L}(V^k, W),$$

and let $\iota_{V_k}\colon V_k \to \widetilde{V}_k$, $\iota_\Omega\colon \mathrm{E}_\Omega(V^k, V) \to \widetilde{V}_k$, $\iota_{\overline{\Omega}}\colon \mathrm{E}_{\overline{\Omega}}(V^k, V) \to \widetilde{V}_k$, $\iota_W\colon \mathrm{L}(V^k, W) \to \widetilde{V}_k$ be the inclusions.

Let us consider an operator $\mathcal{D}_k\colon S^+ \otimes \mathrm{L}(V^k, V) \to \widetilde{V}_k$ given by

(3.8)
$$\mathcal{D}_k\eta \overset{\text{def.}}{=} -\iota_{V_k} \operatorname{tr}_{S^+}(\eta_k \omega \Psi^k) + \iota_\Omega \left(\omega\mathcal{A}\eta - \operatorname{tr}_{S^+}(\eta \otimes \omega\mathcal{A}^k)\right) + \iota_{\overline{\Omega}} \left(\mathcal{A}^\dagger\eta - \operatorname{tr}_{S^+}(\eta \otimes \mathcal{A}^{k\dagger})\right) + \iota_W \Psi^\dagger \eta,$$

where Ψ^k is considered as an element of $S^+ \otimes V^k_k$. This is an analogue of the Dirac operator. A similar operator was introduced in [8, §4].

Let us rewrite this operator in terms of holomorphic descriptions. Consider the following sequence of vector spaces:

(3.9)
$$\mathrm{L}(V^k, V) \xrightarrow{\alpha_k} \widetilde{V}_k = V_k \oplus \mathrm{E}(V^k, V) \oplus \mathrm{L}(V^k, W) \xrightarrow{\beta_k} \mathrm{L}(V^k, V),$$

where

$$\alpha_k(\eta) = \begin{bmatrix} -\eta_k i^k \\ B\eta - \eta B^k \\ j\eta \end{bmatrix}, \qquad \beta_k \begin{bmatrix} v_k \\ C \\ b \end{bmatrix} = \varepsilon(BC + CB^k) + v_k \otimes j^k + ib.$$

Here $v_k \otimes j^k$ is considered as an element of $\mathrm{L}(V^k, V)$ via the embedding $\operatorname{Hom}(V^k_k, V_k) \subset \mathrm{L}(V^k, V)$. The operator $\mathcal{D}_k$ is identified with $\begin{bmatrix}\alpha_k & \beta_k^\dagger\end{bmatrix}$.

We have the following analogue of the Bochner-Weitzenböck formula.

Lemma 3.10. *We have $\mathcal{D}_k^\dagger \mathcal{D}_k = \mathrm{id}_{S^+} \otimes \Delta_k$ for a positive self-adjoint operator $\Delta_k\colon \mathrm{L}(V^k, V) \to \mathrm{L}(V^k, V)$.*

Proof. Let us show $\mathcal{D}_k^\dagger \mathcal{D}_k = \mathrm{id}_{S^+} \otimes \Delta_k$ first. This means that the trace-free part of $\mathcal{D}_k^\dagger \mathcal{D}_k$ is zero. By the technique explained in §1(iv), it is enough to check $\beta_k \alpha_k = 0$. But it follows from the equations $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}} = \mu_{\mathbb{C}}(B^k, i^k, j^k)$.

The positivity of Δ_k is equivalent to $\operatorname{Ker} \mathcal{D}_k = 0$. By above it is equivalent to $\operatorname{Ker} \alpha_k = 0$, $\operatorname{Im} \beta_k = \mathrm{L}(V^k, V)$. Take $\eta \in \operatorname{Ker} \alpha_k \subset \mathrm{L}(V^k, V)$. Suppose $\eta \neq 0$. The image $\operatorname{Im} \eta$ is contained in $\operatorname{Ker} j$ and is invariant under B. Thus we have

$$\zeta_{\mathbb{R}}(\dim \operatorname{Im} \eta) \le 0$$

by the $\zeta_{\mathbb{R}}$-semistability condition for (B, i, j). On the other hand, $\operatorname{Ker} \eta$ contains $\operatorname{Im} i^k$ and is invariant under B^k. Thus we have

$$\zeta_{\mathbb{R}}(\dim \operatorname{Ker} \eta) < \zeta_{\mathbb{R}}(\dim V)$$

by the $\zeta_{\mathbb{R}}$-stability condition for (B^k, i^k, j^k). Summing two inequalities, we get $\zeta_{\mathbb{R}}(\dim V) < \zeta_{\mathbb{R}}(\dim V)$, which is a contradiction. Thus we must have $\eta = 0$. The proof for $\operatorname{Im} \beta_k = \mathrm{L}(V^k, V)$ is similar and hence omitted. $\square$

We define a new collection of hermitian vector spaces $V' = (V'_k)_{k\in I}$ by

$$V'_k \stackrel{\text{def.}}{=} \operatorname{Ker} \mathcal{D}_k^\dagger,$$

where the hermitian structure on $\operatorname{Ker}\mathcal{D}_k^\dagger$ is the restriction of that on $\widetilde{V}_k$. Since $\operatorname{Ker}\mathcal{D}_k = 0$ by Lemma 3.10, we have

$$\dim V'_k = \dim V_k + \sum_l \dim V_l^k(\dim W_l - \sum_m \mathbf{C}_{lm} \dim V_m).$$

Set $\mathbf{v}' = (\dim V'_k)_{k\in I} \in \mathbb{Z}^I$. Then

$$\mathbf{w} - \mathbf{C}\mathbf{v}' = \mathbf{w} - \mathbf{C}\mathbf{v} - \mathbf{C}\mathbf{V}(\mathbf{w} - \mathbf{C}\mathbf{v}),$$

where $\mathbf{V} = (\mathbf{V}_{kl})_{k,l\in I} = \dim V_l^k$.

Let $I_k \colon V'_k = \operatorname{Ker}\mathcal{D}_k^\dagger \to \widetilde{V}_k$ and $P_k \colon \widetilde{V}_k \to V'_k = \operatorname{Ker}\mathcal{D}_k^\dagger$ be the inclusion and the orthogonal projection. We define a new data $(\mathcal{A}', \Psi') \in \mathbf{M}(V', W)$ by

$$(\mathcal{A}'_h I_{\operatorname{out}(h)}) \begin{bmatrix} v_{\operatorname{out}(h)} \\ C \\ b \end{bmatrix} \stackrel{\text{def.}}{=} (\operatorname{id}_{S^+} \otimes P_{\operatorname{in}(h)}) \begin{bmatrix} \mathcal{A}_h v_{\operatorname{out}(h)} + C_h \Psi^{\operatorname{out}(h)} \\ v_{\operatorname{out}(h)} \otimes \left(\omega \Psi_{\operatorname{in}(h)}^{\operatorname{in}(h)}\right)^\dagger + C\Phi^{\overline{h}} \\ b\Phi^{\overline{h}} \end{bmatrix},$$

$$\Psi'_k w_k \stackrel{\text{def.}}{=} (\operatorname{id}_{S^+} \otimes P_k) \begin{bmatrix} \Psi_k w_k \\ 0 \\ \left(\omega\Psi_k^k\right)^\dagger \otimes w_k \end{bmatrix},$$

where $v_{\operatorname{out}(h)} \otimes \left(\omega\Psi_{\operatorname{in}(h)}^{\operatorname{in}(h)}\right)^\dagger$ is considered as an element of $S^+ \otimes \operatorname{E}(V^{\operatorname{in}(h)}, V)$ via the inclusion $\operatorname{Hom}(V_{\operatorname{in}(h)}^{\operatorname{in}(h)}, V_{\operatorname{out}(h)}) \subset \operatorname{E}(V^{\operatorname{in}(h)}, V)$, and $\left(\omega\Psi_k^k\right)^\dagger \otimes w_k$ is considered as an element of $S^+ \otimes \operatorname{L}(V^k, W)$ via the inclusion $\operatorname{Hom}(V_k^k, W_k) \subset \operatorname{L}(V^k, W)$.

Now we want to give a holomorphic description of the new data $(\mathcal{A}', \Psi')$. Note that we have the canonical isomorphism $\operatorname{Ker}\mathcal{D}_k^\dagger \cong \operatorname{Ker}\beta_k / \operatorname{Im}\alpha_k$.

Let us define

$$\widetilde{B}'_h \colon \widetilde{V}_{\operatorname{out}(h)} \to \widetilde{V}_{\operatorname{in}(h)}, \quad \widetilde{i}'_k \colon W_k \to \widetilde{V}_k, \quad \widetilde{j}'_k \colon \widetilde{V}_k \to W_k$$

by

$$\widetilde{B}'_h \begin{bmatrix} v_{\operatorname{out}(h)} \\ C \\ b \end{bmatrix} \stackrel{\text{def.}}{=} \begin{bmatrix} B_h v_{\operatorname{out}(h)} + C_h i^{\operatorname{out}(h)} \\ \varepsilon(\overline{h}) v_{\operatorname{out}(h)} \otimes j^{\operatorname{in}(h)} + C\phi^{\overline{h}} \\ b\phi^{\overline{h}} \end{bmatrix},$$

$$\widetilde{i}'_k(w_k) \stackrel{\text{def.}}{=} \begin{bmatrix} i_k(w_k) \\ 0 \\ -w_k \otimes j_k^k \end{bmatrix}, \qquad \widetilde{j}'_k \begin{bmatrix} v_k \\ C \\ b \end{bmatrix} \stackrel{\text{def.}}{=} j_k(v_k) + b_k i_k^k.$$

Lemma 3.11. (1) $\widetilde{B}'_h$ *maps* $\operatorname{Ker}\beta_{\operatorname{out}(h)}$ (*resp.* $\operatorname{Im}\alpha_{\operatorname{out}(h)}$) *to* $\operatorname{Ker}\beta_{\operatorname{in}(h)}$ (*resp.* $\operatorname{Im}\alpha_{\operatorname{in}(h)}$).

(2) $\beta_k \widetilde{i}'_k = 0$, $\widetilde{j}'_k \alpha_k = 0$.

Proof. (2) is clear from the definition. Let us prove (1). We have

$$\begin{aligned}
\beta_{\mathrm{in}(h)}\widetilde{B}'_h \begin{bmatrix} v_{\mathrm{out}(h)} \\ C \\ b \end{bmatrix} &= \varepsilon BC\phi^{\overline{h}} + v_{\mathrm{out}(h)} \otimes j^{\mathrm{in}(h)} B_h^{\mathrm{in}(h)} + \varepsilon C\phi^{\overline{h}} B^{\mathrm{in}(h)} + C_h i^{\mathrm{out}(h)} \otimes j^{\mathrm{in}(h)} + ib\phi^{\overline{h}} \\
&= \varepsilon BC\phi^{\overline{h}} + v_{\mathrm{out}(h)} \otimes j^{\mathrm{out}(h)} \phi^{\overline{h}}_{\mathrm{out}(h)} + \varepsilon C B^{\mathrm{out}(h)}\phi^{\overline{h}} + ib\phi^{\overline{h}} \\
&= \left(\beta_{\mathrm{out}(h)} \begin{bmatrix} v_{\mathrm{out}(h)} \\ C \\ b \end{bmatrix} \right) \phi^{\overline{h}},
\end{aligned}$$

where we have used (3.2) in the second equality. Thus $\operatorname{Ker}\beta_{\mathrm{out}(h)}$ is mapped to $\operatorname{Ker}\beta_{\mathrm{in}(h)}$. We also have

$$\begin{aligned}
\widetilde{B}'_h \alpha_{\mathrm{out}(h)}(\eta) &= \begin{bmatrix} -\eta_{\mathrm{in}(h)} B_h^{\mathrm{out}(h)} i^{\mathrm{out}(h)} \\ -\varepsilon(\overline{h})\eta_{\mathrm{out}(h)} i^{\mathrm{out}(h)} \otimes j^{\mathrm{in}(h)} + (B\eta - \eta B^{\mathrm{out}(h)})\phi^{\overline{h}} \\ j\eta\phi^{\overline{h}} \end{bmatrix} \\
&= \begin{bmatrix} -\eta_{\mathrm{in}(h)} \phi^{\overline{h}}_{\mathrm{in}(h)} i^{\mathrm{out}(h)} \\ B\eta\phi^{\overline{h}} - \eta\phi^{\overline{h}} B^{\mathrm{in}(h)} \\ j\eta\phi^{\overline{h}} \end{bmatrix} = \alpha_{\mathrm{in}(h)}(\eta\phi^{\overline{h}}),
\end{aligned}$$

where we have used (3.2) in the second equality. Thus $\operatorname{Im}\alpha_{\mathrm{out}(h)}$ is mapped to $\operatorname{Im}\alpha_{\mathrm{in}(h)}$. $\square$

By this lemma we have induced maps

$$B'_h \colon \operatorname{Ker}\beta_{\mathrm{out}(h)} / \operatorname{Im}\alpha_{\mathrm{out}(h)} \to \operatorname{Ker}\beta_{\mathrm{in}(h)} / \operatorname{Im}\alpha_{\mathrm{in}(h)},$$
$$i'_k \colon W_k \to \operatorname{Ker}\beta_k / \operatorname{Im}\alpha_k, \qquad j'_k \colon \operatorname{Ker}\beta_k / \operatorname{Im}\alpha_k \to W_k.$$

Under the isomorphism $\operatorname{Ker}\beta_k / \operatorname{Im}\alpha_k \cong \operatorname{Ker}\mathcal{D}_k^\dagger = V'_k$, it is a holomophic description of $(\mathcal{A}', \Psi')$.

Theorem 3.12. *The data $(\mathcal{A}', \Psi')$ satisfies the equation $\mu(\mathcal{A}', \Psi') = -\zeta'$.*

Proof. By the technique explained in §1(iv), it is enough to check $\mu_{\mathbb{C}}(B', i', j') = -\zeta'_{\mathbb{C}}$. Let $\left[\begin{smallmatrix} v_k \\ C \\ b \end{smallmatrix}\right] \in \widetilde{V}_k$. We have

$$\begin{aligned}
&\left(\sum_{\mathrm{in}(h)=k} \varepsilon(h)\widetilde{B}'_h\widetilde{B}'_{\overline{h}} + \widetilde{i}'_k\widetilde{j}'_k \right) \begin{bmatrix} v_k \\ C \\ b \end{bmatrix} \\
&= \begin{bmatrix} \sum_{\mathrm{in}(h)=k} \varepsilon(h) \left(B_h B_{\overline{h}} v_k + B_h C_{\overline{h}} i^k + C_h \phi^h_{\mathrm{out}(h)} i^{\mathrm{out}(h)} \right) + \langle j^{\mathrm{out}(h)}, i^{\mathrm{out}(h)} \rangle v_k + i_k j_k v_k + i_k b_k i^k \\ \sum_{\mathrm{in}(h)=k} -B_{\overline{h}} v_k \otimes j^k - C_{\overline{h}} i^k \otimes j^k + v_k \otimes j^{\mathrm{out}(h)}\phi^{\overline{h}} + \varepsilon(h) C\phi^h\phi^{\overline{h}} \\ \sum_{\mathrm{in}(h)=k} \varepsilon(h) b\phi^h\phi^{\overline{h}} - j_k v_k \otimes j^k - b_k i^k \otimes j^k \end{bmatrix}.
\end{aligned}$$

By (3.2), Lemma 3.4, (3.6) and the equation $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$, this is equal to

$$\begin{bmatrix} -\zeta'_{k,\mathbb{C}} v_k + 2 v_k \otimes j^k i^k + \sum_{\mathrm{in}(h)=k} \varepsilon(h) \left(B_h C_{\overline{h}} + C_h B^k_{\overline{h}} \right) i^k + i_k b_k i^k \\ -\zeta'_{k,\mathbb{C}} C - \sum_{\mathrm{in}(h)=k} \left(B_{\overline{h}} v_k \otimes j^k - v_k \otimes j^k B^k_h \right) \\ -\zeta'_{k,\mathbb{C}} b - j_k v_k \otimes j^k \end{bmatrix}.$$

If $\begin{bmatrix} v_k \\ C \\ b \end{bmatrix}$ is in $\operatorname{Ker}\beta_k$, then this is

$$-\zeta'_{k,\mathbb{C}}\begin{bmatrix} v_k \\ C \\ b \end{bmatrix} + \begin{bmatrix} v_k \otimes j^k i^k \\ -\sum_{\mathrm{in}(h)=k}\left(B_{\overline{h}} v_k \otimes j^k - v_k \otimes j^k B^k_h\right) \\ -j_k v_k \otimes j^k \end{bmatrix} = -\zeta'_{k,\mathbb{C}}\begin{bmatrix} v_k \\ C \\ b \end{bmatrix} - \alpha_k(v_k \otimes j^k).$$

This proves the k-component of $\mu_{\mathbb{C}}(B', i', j') = -\zeta'_{\mathbb{C}}$. Since k is arbitary, we get the conclusion. $\square$

Corollary 3.13. *(B', i', j') satisfies $\mu_{\mathbb{C}}(B', i', j') = -\zeta'_{\mathbb{C}}$ and the condition in Proposition 2.11(3) for $\zeta'_{\mathbb{R}}$.*

The above construction is equivariant under the action of G_V. Thus we have an induced map

$$\mathcal{F}_{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)} \colon \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\zeta'}(\mathbf{v}', \mathbf{w}).$$

The continuity of this map is clear from the definition. So far, this is just a continuous map between two topological spaces. We shall show that it is a hyper-Kähler isometry between open subsets $\mathfrak{M}^{\mathrm{reg}}_\zeta(\mathbf{v}, \mathbf{w})$, $\mathfrak{M}^{\mathrm{reg}}_{\zeta'}(\mathbf{v}', \mathbf{w})$, at least for reflection functors for simple reflections. The proof for holomorphicity with respect to any of three complex structures I, J, K is easy, and works for any $\mathcal{F}_{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)}$. (It implies that the map respects the Levi-Civita connection.)

Remark 3.14. The above definition depends on the choice of isomorphisms $W^k_k \cong \mathbb{C}$. We can avoid this ambiguity by the following modifications:

(a) Consider $\Phi^{\overline{h}}$ as an element of $S^+ \otimes \mathrm{L}(V^{\mathrm{in}(h)} \otimes (W^{\mathrm{in}(h)}_{\mathrm{in}(h)})^*, V^{\mathrm{out}(h)} \otimes (W^{\mathrm{out}(h)}_{\mathrm{out}(h)})^*)$, and consider (3.1) in an appropriate way.
(b) Replace $\mathrm{E}(V^k, V)$ and $\mathrm{L}(V^k, V)$ by $\mathrm{E}(V^k \otimes (W^k_k)^*, V)$ and $\mathrm{L}(V^k \otimes (W^k_k)^*, V)$ in (3.7) respectively,
(c) Replace $\mathrm{L}(V^k, V)$ by $\mathrm{L}(V^k \otimes (W^k_k)^*, V)$ in (3.8).

Conjecture 3.15. (1) Note that the proof of $\mu_{\mathbb{C}}(B', i', j') = -\zeta'_{\mathbb{C}}$ requires only the complex part (3.2) of the compatibility condition. It is natural to conjecture that the $\zeta'_{\mathbb{R}}$-semistability of (B', i', j') directly follows from the $\zeta_{\mathbb{R}}$-semistability of (B, i, j) and (B^k, i^k, j^k) (plus a similar condition for ϕ). Such a direct proof should also work for arbitary grand field. We shall prove this conjecture affirmatively for reflection functors for simple reflections later. (See §4(iii).)

(2) In §7 we shall see that for any element w of the Weyl group, there exists an admissible collection $\{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)\}$ with $\mathbf{v}^k = w *_{\mathbf{e}^k} 0$, where $\mathbf{e}^k \in \mathbb{Z}^I_{\ge 0}$ is the vector whose k-component is 1 and other components are 0, and $*_{\mathbf{e}^k}$ was defined in Definition 2.3. The cases $w = 1$ and $w = s_{k_0}$ correspond two examples given in §3(i). Are there any other admissible collections ?

(3) It is natural to conjecture that the composition of two reflection functors is again a reflection functor, where the corresponding admissible collection is given by Proposition 7.1. If this is true, we can give an another proof of results in §5, §6. For graphs of Dynkin types and reflection functors corresponding to elements of the Weyl group (as in (2)), this conjecture is true thanks to a description of the reflection functor in §9.

4. Simple reflections

In this section we study reflection functors for the second example for the admissible collection in more detail.

4(i). **Definition.** We change the notation for brevity. Fix a vertex $k \in I$. Let $\zeta_k \in \mathbb{R}^3 \otimes (i\mathbb{R})$ be the entry of ζ corresponding to the vertex k. We take an element $x \in S^+$ such that $\mu(x) = \zeta_k$, where μ is the hyper-Kähler moment map for the U(1)-action on S^+ studied in §1(ii),§1(iii). Such x is unique up to a multiplication by an element of U(1). Furthermore we assume $x \neq 0$ by the non-degeneracy assumption.

Suppose that collections of hermitian vector spaces V, W and a data $(\mathcal{A}, \Psi) \in \mathbf{M}(V, W)$ such that $\mu(\mathcal{A}, \Psi) = -\zeta$ is given. Set

$$\widetilde{V}_k \overset{\text{def.}}{=} V_k \oplus \bigoplus_{\substack{h \in H \\ \text{in}(h)=k}} V_{\text{out}(h)} \oplus W_k,$$

and let $\iota_{V_k} \colon V_k \to \widetilde{V}_k$, $\iota_h \colon V_{\text{out}(h)} \to \widetilde{V}_k$, $\iota_{W_k} \colon W_k \to \widetilde{V}_k$ be the inclusions, and let $\pi_{V_k} \colon \widetilde{V}_k \to V_k$, $\pi_h \colon \widetilde{V}_k \to V_{\text{out}(h)}$, $\pi_{W_k} \colon \widetilde{V}_k \to W_k$ be the projections.

Let us consider an operator

$$\mathcal{D}_k \overset{\text{def.}}{=} x^\dagger \otimes \iota_{V_k} + \sum_{\substack{h \in \Omega \\ \text{in}(h)=k}} \iota_h \mathcal{A}_h^\dagger + \sum_{\substack{h \in \overline{\Omega} \\ \text{in}(h)=k}} \iota_h \omega \mathcal{A}_{\overline{h}} + \iota_{W_k} \Psi_k^\dagger \colon S^+ \otimes V_k \to \widetilde{V}_k.$$

Assuming $\operatorname{Ker} \mathcal{D}_k = 0$, we define a new collection of hermitian vector spaces V' by

$$V'_l \overset{\text{def.}}{=} \begin{cases} V_l & \text{if } l \neq k, \\ \operatorname{Ker} \mathcal{D}_k^\dagger & \text{if } l = k, \end{cases} \tag{4.1}$$

where the hermitian structure on $\operatorname{Ker} \mathcal{D}_k^\dagger$ is the restriction of that on $\widetilde{V}_k$. Since $\mathcal{D}_k$ is injective by assumption, we have

$$\mathbf{v}' = s_k * \mathbf{v},$$

where $\mathbf{v}' = \dim V'$.

Let $I_k \colon V'_k = \operatorname{Ker} \mathcal{D}_k^\dagger \to \widetilde{V}_k$ and $P_k \colon \widetilde{V}_k \to V'_k = \operatorname{Ker} \mathcal{D}_k^\dagger$ be the inclusion and the orthogonal projection. We define a new data $(\mathcal{A}', \Psi') \in \mathbf{M}(V', W)$ by

$$\mathcal{A}'_h \overset{\text{def.}}{=} \begin{cases} \left(\mathcal{A}_h \pi_{V_k} + (\omega x)^\dagger \otimes \pi_{\overline{h}}\right) I_k & \text{if } h \in \Omega, \text{out}(h) = k \\ (\text{id}_{S^+} \otimes P_k) \left[(\text{id}_{S^+} \otimes \iota_{V_k}) \mathcal{A}_h - x \otimes \iota_h\right] & \text{if } h \in \Omega, \text{in}(h) = k \\ \mathcal{A}_h & \text{otherwise,} \end{cases}$$

$$\Psi'_l \overset{\text{def.}}{=} \begin{cases} (\text{id}_{S^+} \otimes P) \left[(\text{id}_{S^+} \otimes \iota_{V_k}) \Psi_k - x \otimes \pi_{W_k}\right] & \text{if } l = k, \\ \Psi_l & \text{otherwise.} \end{cases}$$

By Theorem 3.12 we have

Theorem 4.2. *The data $(\mathcal{A}', \Psi')$ satisfies the equation $\mu(\mathcal{A}', \Psi') = -s_k \zeta$.*

We denote the resulting map by

$$S_k \colon \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{s_k \zeta}(s_k * \mathbf{v}, \mathbf{w}).$$

We will also show that S_k maps $\mathfrak{M}_\zeta^{\text{reg}}(\mathbf{v}, \mathbf{w})$ to $\mathfrak{M}_{s_k \zeta}^{\text{reg}}(s_k * \mathbf{v}, \mathbf{w})$ later.

Remark 4.3. We can define the reflection functor S_k even when $\zeta_k = 0$. But in this case we have an orthogonal decomposition

$$V'_k = V_k \oplus \operatorname{Ker}\left(\sum_{\substack{h\in\Omega \\ \mathrm{in}(h)=k}} \mathcal{A}_h\pi_h + \sum_{\substack{h\in\overline{\Omega} \\ \mathrm{in}(h)=k}} (\omega\mathcal{A}_{\overline{h}})^\dagger\pi_h + \Psi_k\pi_{W_k}\right),$$

and $(\mathcal{A}', \Psi')$ is equal to $(\mathcal{A}, \Psi)$ extended to V'_k by 0.

The following is the main theorem of this paper.

Theorem 4.4. *The reflection functors S_k satisfy the defining relation* (2.1) *of the Weyl group. More precisely, we have*

$$S_kS_k = \mathrm{id}\colon \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}),$$
$$S_kS_l = S_lS_k\colon \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{s_ks_l\zeta}(s_k * s_l * \mathbf{v},\mathbf{w}) \quad \textit{if } \mathbf{A}_{kl} = 0,$$
$$S_kS_lS_k = S_lS_kS_l\colon \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{s_ks_ls_k\zeta}(s_k * s_l * s_k * \mathbf{v},\mathbf{w}) \quad \textit{if } \mathbf{A}_{kl} = 1,$$

where $\zeta_k \neq 0$, $\zeta_l \neq 0$, $\zeta_k + \zeta_l \neq 0$.

The second equality is trivial. The first and third equalities will be proved in §5 and §6 respectively.

As in [15, §9], we have the following application.

Theorem 4.5. *Suppose ζ satisfies* (2.7)*. Then there exists a Weyl group representation on $H^*(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}),\mathbb{R})$ if $\mathbf{w} - \mathbf{C}\mathbf{v} = 0$.*

This is an analogue of Slodowy's construction [19, IV] of Springer representation.

4(ii). **Holomorphic description of reflection functors.** Let us choose a particular complex strcture, say I, and rewrite the reflection functor S_k in the holomorphic description in §2(ii). Let $x = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ be the holomorphic description of x.

Let us consider the following:

$$V_k \xrightarrow{\alpha_k} \widetilde{V}_k = V_k \oplus \bigoplus_{h:\mathrm{in}(h)=k} V_{\mathrm{out}(h)} \oplus W_k \xrightarrow{\beta_k} V_k, \tag{4.6}$$

where

$$\alpha_k = \begin{bmatrix} z_1\,\mathrm{id}_{V_k} \\ \bigoplus B_{\overline{h}} \\ j_k \end{bmatrix}, \qquad \beta_k = \begin{bmatrix} z_2\,\mathrm{id}_{V_k} & \sum\varepsilon(h)B_h & i_k \end{bmatrix}.$$

The operator $\mathcal{D}_k$ can be identified with $(\alpha_k, \beta_k^\dagger)$.

We have an isomorphism $\operatorname{Ker}\mathcal{D}_k^\dagger \cong \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. We also have the followings by Lemma 3.10:

(1) (4.6) is a complex, i.e., $\beta_k\alpha_k = 0$.

(2) α_k is injective and β_k is surjective.

Note that these statements can also be proved directly by using $\mu_{\mathbb{C}}(B,i,j) = -\zeta_{\mathbb{C}}$, $z_1z_2 = \zeta_{k,\mathbb{C}}$, and the stability condition.

For $h \in H$ such that $\mathrm{in}(h) = k$, let

$$\widetilde{B}'_h \overset{\text{def.}}{=} \begin{bmatrix} B_h \\ -\varepsilon(h)z_2\iota_h \\ 0 \end{bmatrix} : V_{\mathrm{out}(h)} \to \widetilde{V}_k, \qquad \widetilde{B}'_{\overline{h}} \overset{\text{def.}}{=} \begin{bmatrix} B_{\overline{h}} & -z_1\pi_h & 0 \end{bmatrix} : \widetilde{V}_k \to V_{\mathrm{out}(h)}.$$

Since $\widetilde{B}'_{\overline{h}}\alpha_k = 0$, we have an induced homomorphism $\operatorname{Ker}\beta/\operatorname{Im}\alpha \to V_{\mathrm{out}(h)}$ from $\widetilde{B}'_{\overline{h}}$. Let us denote it by $B'_{\overline{h}}$. Since $\beta_k \widetilde{B}'_h = 0$, we have an induced homomorphism $V_{\mathrm{out}(h)} \to \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. Let us denote it by B'_h.

Let

$$\widetilde{i}'_k \overset{\text{def.}}{=} \begin{bmatrix} i_k \\ 0 \\ -z_2 \iota_{W_k} \end{bmatrix} : W_k \to \widetilde{V}_k, \qquad \widetilde{j}'_k \overset{\text{def.}}{=} \begin{bmatrix} j_k & 0 & -z_1 \pi_{W_k} \end{bmatrix} : \widetilde{V}_k \to W_k.$$

Since $\beta_k \widetilde{i}'_k = 0$, $\widetilde{j}'_k \alpha_k = 0$, we have induced maps $i'_k \colon W_k \to \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$ and $j'_k \colon \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k \to W_k$ as above.

We define a new data (B', i', j') by setting $B'_h = B_h$, $i'_l = i_l$, $j'_l = j_l$ for other edges h and vertices l. Then the data (B', i', j') is a holomorphic description of $(\mathcal{A}', \Psi')$ under the isomorphism $\operatorname{Ker}\beta/\operatorname{Im}\alpha \cong \operatorname{Ker}\mathcal{D}_k = V'_k$.

By Theorem 4.2 and Proposition 2.11 we have the following.

Corollary 4.7. *(B', i', j') satisfies $\mu_{\mathbb{C}}(B', i', j') = -s_k\zeta_{\mathbb{C}}$ and the the condition in Proposition 2.11(3) for $s_k\zeta_{\mathbb{R}}$.*

4(iii). **Another proof of Corollary 4.7.** We give another proof of Corollary 4.7 in this subsection. This is less conceptual, but works for arbitary grand field.

Note that we only need the equation $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$ to define (B', i', j') on $\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. The following proof also shows that (B', i', j') is $s_k\zeta_{\mathbb{R}}$-semistable (resp. $s_k\zeta_{\mathbb{R}}$-stable) if (B, i, j) is $\zeta_{\mathbb{R}}$-semistable (resp. $\zeta_{\mathbb{R}}$-stable).

Let $\widehat{V}_k = \bigoplus_{h:\mathrm{in}(h)=k} V_{\mathrm{out}(h)} \oplus W_k$. Let us modify (4.6) as follows:

$$V_k \xrightarrow{\sigma} \widehat{V}_k \xrightarrow{\tau} V_k, \tag{4.8}$$

where

$$\sigma = \begin{bmatrix} \bigoplus B_{\overline{h}} \\ j_k \end{bmatrix}, \qquad \tau = \begin{bmatrix} \sum \varepsilon(h) B_h & i_k \end{bmatrix}. \tag{4.9}$$

Note that the equations $\mu(\mathcal{A}, \Psi) = -\zeta$ and $\mu(x) = \zeta$ imply

$$\tau\sigma = -z_1 z_2 = -\zeta_{k,\mathbb{C}}, \qquad \sigma^\dagger\sigma - \tau\tau^\dagger = -|z_1|^2 + |z_2|^2 = 2i\zeta_{k,\mathbb{R}}. \tag{4.10}$$

In the following proof we shall use the following weaker condition instead of the right equalities:

$$\text{If } \zeta_{k,\mathbb{C}} = 0 \text{ and } i\zeta_{k,\mathbb{R}} < 0, \text{ then } \tau \text{ is surjective and } z_1 \neq 0,\ z_2 = 0. \tag{4.11}$$

This condition follows from the $\zeta_{\mathbb{R}}$-semistability of (B, i, j) and $\zeta_{\mathbb{R}}$-stability of x.

We also consider similar things for V', B', i', j' and denote them by

$$V'_k \xrightarrow{\sigma'} \widehat{V}'_k \xrightarrow{\tau'} V'_k.$$

Note that $\widehat{V}'_k = \widehat{V}_k$.

Assume $i\zeta_{k,\mathbb{R}} \le 0$ and consider

$$V'_k \xrightarrow{\sigma'} \widehat{V}'_k = \widehat{V}_k \xrightarrow{\tau} V_k.$$

(Exchange V and V' in the following discussion if $i\zeta_{k,\mathbb{R}} \ge 0$.)

Lemma 4.12 (cf. [11, 3.2])**.** (1) $\tau\sigma' = 0$

(2) *The complex is exact under* (4.11).

(3) $\sigma\tau = \sigma'\tau' - \zeta_{k,\mathbb{C}}\operatorname{id}_{\widehat{V}_k}$.

Proof. (1) We have

$$\begin{bmatrix}\sum \varepsilon(h)B_h & i_k\end{bmatrix}\begin{bmatrix}\widetilde{B}'_{\overline{h}}\\ \widetilde{j}'_k\end{bmatrix} = \begin{bmatrix}\sum_{h:\mathrm{in}(h)=k}\varepsilon(h)B_hB_{\overline{h}} + i_kj_k & -z_1\sum_{h:\mathrm{in}(h)=k}\varepsilon(h)B_h & -z_1 i_k\end{bmatrix} = -z_1\beta_k.$$

Since $V'_k = \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$, we have $\tau\sigma' = 0$.

(2) Since the Euler characteristic of the complex is zero, it is enough to show that σ' is injective and τ is surjective.

Suppose $\zeta_{k,\mathbb{C}} \neq 0$. Then $\tau\sigma = -\zeta_{k,\mathbb{C}}$ and $\tau'\sigma' = -\zeta'_{k,\mathbb{C}} = \zeta_{k,\mathbb{C}}$ imply the injectivity of σ' and the surjectivity of τ.

If $\zeta_{k,\mathbb{C}} = 0$, then we have $i\zeta_{k,\mathbb{R}} < 0$ by assumption. Then τ is surjective and $z_1 \neq 0$, $z_2 = 0$ by (4.11).

We want to show that $\operatorname{Ker}\sigma' = 0$. By the definition of (B', i', j'), the map σ' is induced from

$$\begin{bmatrix}\sigma & -z_1\operatorname{id}_{\widehat{V}_k}\end{bmatrix} : \widetilde{V}_k = V_k \oplus \widehat{V}_k \to \widehat{V}_k.$$

Suppose $\left[\begin{smallmatrix} v_k \\ \widehat{v}\end{smallmatrix}\right] \in V_k \oplus \widehat{V}_k$ is in the kernel of the above map. Then $\sigma(v_k) = z_1\widehat{v}$. Hence $\left[\begin{smallmatrix} v_k \\ \widehat{v}\end{smallmatrix}\right] = \alpha_k(v_k/z_1)$. This shows that σ' is injective.

(3) We have

$$\sigma'\tau' = \begin{bmatrix}\bigoplus \widetilde{B}'_{\overline{h}}\\ \widetilde{j}'_k\end{bmatrix}\begin{bmatrix}\sum\varepsilon(h')\widetilde{B}'_{h'} & \widetilde{i}'_k\end{bmatrix} = \begin{bmatrix}\sigma & -z_1\operatorname{id}_{\widehat{V}_k}\end{bmatrix}\begin{bmatrix}\tau\\ -z_2\operatorname{id}_{\widehat{V}_k}\end{bmatrix} = \sigma\tau + z_1z_2\operatorname{id}_{\widehat{V}_k}.$$

Thus we have the assertion by $z_1z_2 = \zeta_{k,\mathbb{C}}$. □

Now we begin the proof of Corollary 4.7. By Proposition 2.11(3) we may assume either (a) (B, i, j) is $\zeta_{\mathbb{R}}$-stable or (b) $i = 0 = j$ and B is $\zeta_{\mathbb{R}}$-stable in the sense of Definition 2.10. We prove that (B', i', j') (or B' in case (b)) is $\zeta'_{\mathbb{R}}$-stable. The proof for the case (b) is similar to that for (a). So we only give the proof for (a). Note also that the proof also shows that (B', i', j') is $\zeta'_{\mathbb{R}}$-semistable if (B, i, j) is $\zeta_{\mathbb{R}}$-semistable.

Suppose a collection $S = (S'_l)_{l\in I}$ of subspaces $S'_l \subset V'_l$ with $B'_{\mathrm{out}(h)}(S'_{\mathrm{out}(h)}) \subset S'_{\mathrm{in}(h)}$, $j'_l(S'_l) = 0$ is given. We have

$$\sigma'(S'_k) \subset \bigoplus S'_{\mathrm{out}(h)} \oplus 0, \qquad \tau'\left(\bigoplus S'_{\mathrm{out}(h)} \oplus 0\right) \subset S'_k. \tag{4.13}$$

Set

$$S_l \overset{\text{def.}}{=} \begin{cases} S'_l & \text{if } l \neq k,\\ \tau\left(\bigoplus_{h:\mathrm{in}(h)=k} S'_{\mathrm{out}(h)} \oplus 0\right) & \text{if } l = k.\end{cases}$$

Then $S = (S_l)_{l\in I}$ is a collection of subspaces $S_l \subset V_l$.

Claim. $B_{\mathrm{out}(h)}(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$, $j_l(S_l) = 0$.

Proof. If $\mathrm{in}(h) \neq k \neq \mathrm{out}(h)$, we have $B_{\mathrm{out}(h)}(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$. If $l \neq k$, we have $j_l(S_l) = 0$. If $\mathrm{in}(h) = k$, we have $B_{\mathrm{out}(h)}(S_{\mathrm{out}(h)}) \subset S_k$ by the definition of S_k.

We have

$$\begin{aligned}\sigma(S_k) = \sigma\tau\left(\bigoplus S_{\mathrm{out}(h)} \oplus 0\right) &= (\sigma'\tau' + \zeta_{k,\mathbb{C}}\operatorname{id})\left(\bigoplus S_{\mathrm{out}(h)} \oplus 0\right)\\ &\subset \sigma'(S'_k) + \left(\bigoplus S_{\mathrm{out}(h)} \oplus 0\right) = \bigoplus S_{\mathrm{out}(h)} \oplus 0.\end{aligned}$$

This proves the assertion. □

By the $\zeta_{\mathbb{R}}$-stability condition of (B, i, j) we have

$$0 > \zeta_{\mathbb{R}}(\dim S) = \sum_{l \neq k} i\zeta_{l,\mathbb{R}} \dim S_l + i\zeta_{k,\mathbb{R}} \dim S_k \tag{4.14}$$

unless $S = 0$. If $S = 0$, then $S'_l = 0$ for $l \neq k$. Then (4.13) and the injectivity of σ' (Lemma 4.12) imply $S'_k = 0$. Thus $S' = 0$.

So we may suppose (4.14). Consider a complex

$$S'_k \xrightarrow{\sigma'} \bigoplus S_{\mathrm{out}(h)} \oplus 0 \xrightarrow{\tau} S_k.$$

The left arrow is injective by Lemma 4.12. The right arrow is surjective by the definition of S_k. Hence we have

$$\dim S_k \leq \sum_{h:\mathrm{in}(h)=k} \dim S_{\mathrm{out}(h)} - \dim S'_k.$$

Noticing $i\zeta_{k,\mathbb{R}} \leq 0$, we substitute this inequality into (4.14). Then we get

$$0 > i \sum_{l \neq k} (\zeta_{l,\mathbb{R}} + \mathbf{A}_{kl}\zeta_{k,\mathbb{R}}) \dim S_l - i\zeta_{k,\mathbb{R}} \dim S'_k = \zeta'_{\mathbb{R}}(\dim S').$$

Next suppose a collection $T = (T'_l)_{l \in I}$ of subspaces $T'_l \subset V'_l$ with $B'_{\mathrm{out}(h)}(T'_{\mathrm{out}(h)}) \subset T'_{\mathrm{in}(h)}$, $\mathrm{Im}\, i'_l \subset T'_l$ is given. We set

$$T_l \overset{\text{def.}}{=} \begin{cases} T'_l & \text{if } l \neq k, \\ \tau\left(\bigoplus_{h:\mathrm{in}(h)=k} T'_{\mathrm{out}(h)} \oplus W_k\right) & \text{if } l = k. \end{cases}$$

By a similar argument as above we have $B_h(T_{\mathrm{out}(h)}) \subset T_{\mathrm{in}(h)}$ and $\mathrm{Im}\, i_l \subset T_l$. Thus we have

$$\zeta_{\mathbb{R}}(\dim V) \geq \zeta_{\mathbb{R}}(\dim T) = \sum_{l \neq k} i\zeta_{l,\mathbb{R}} \dim T_l + i\zeta_{k,\mathbb{R}} \dim T_k, \tag{4.15}$$

and we have the strict inequality unless $T = V$.

As above, we have

$$\dim T_k \leq \sum_{h:\mathrm{in}(h)=k} \dim T_{\mathrm{out}(h)} + \dim W_k - \dim T'_k. \tag{4.16}$$

Combining this with (4.15), we get

$$\zeta'_{\mathbb{R}}(\dim V') \geq \zeta'_{\mathbb{R}}(\dim T').$$

If we have the equality, we must have equalities in (4.15) and (4.16). The equality in (4.15) implies $T = V$. Substituting it into the equality of (4.16), we get $\dim T'_k = \dim V'_k$. Thus $T' = V'$. This shows that (B', i', j') is $\zeta'_{\mathbb{R}}$-stable. We have completed the proof.

Remark 4.17. Lemma 4.12 implies that S_k is the same as the map defined by Maffei [13]. Our two proofs of the $\zeta'_{\mathbb{R}}$-stability of (B', i', j') are totally different from his.

4(iv). **Reflection functor when $\zeta_{k,\mathbb{C}} \neq 0$.** In this subsection, we still use the holomorphic description as in the previous subsection. We further assume $\zeta_{k,\mathbb{C}} \neq 0$. Since we only need the complex equation $\mu_{\mathbb{C}}(x) = \zeta_{k,\mathbb{C}}$ in the holomorphic description, we may assume $z_1 = -1$ and $z_2 = -\zeta_{k,\mathbb{C}}$.

Let $\widehat{V}_k$, σ, τ as in §4(iii). Since $\tau\sigma = -\zeta_{k,\mathbb{C}}$ by (4.10),

$$\widehat{V}_k \ni x \longmapsto \left(x + \frac{1}{\zeta_{k,\mathbb{C}}}\sigma\tau x\right) \oplus -\frac{1}{\zeta_{k,\mathbb{C}}}\tau x \in \operatorname{Ker}\tau \oplus V_k \tag{4.18}$$

is an isomorphism between $\widehat{V}_k$ and $\operatorname{Ker}\tau \oplus V_k$.

Proposition 4.19. (1) *The isomorphism* (4.18) *induces an isomorphism* $\operatorname{Ker}\tau \xrightarrow{\cong} \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. *Combining it with the isomorphism* $V_k' \cong \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$, *we have an isomorphism* $V_k' \xrightarrow{\cong} \operatorname{Ker}\tau$.

(2) *Let $h \in H$ such that $\mathrm{in}(h) = k$. Under the above isomorphism $\operatorname{Ker}\tau \cong V_k'$, the map $B_{\overline{h}}'\colon V_k' \to V_{\mathrm{out}(h)}$ is identified with the composition of*

$$\operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{\text{projection}} V_{\mathrm{out}(h)}.$$

Similarly $j_k'\colon V_k' \to W_k$ is identified with the composition of $\operatorname{Ker}\tau \to \widehat{V}_k \to W_k$.

(3) *The map B_h' is identified with the composition of*

$$V_{\mathrm{out}(h)} \xrightarrow[\cong]{\varepsilon(h)\zeta_{k,\mathbb{C}}\otimes\mathrm{id}_{V_{\mathrm{out}(h)}}} V_{\mathrm{out}(h)} \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow[\cong]{(4.18)} \operatorname{Ker}\tau \oplus V_k \xrightarrow{\text{projection}} \operatorname{Ker}\tau.$$

Similarly i_k' is identified with the composition of

$$W_k \xrightarrow[\cong]{\zeta_{k,\mathbb{C}}\otimes\mathrm{id}_{W_k}} W_k \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow[\cong]{(4.18)} \operatorname{Ker}\tau \oplus V_k \xrightarrow{\text{projection}} \operatorname{Ker}\tau.$$

Remark 4.20. This proposition shows that the reflection functor S_k is same as the one defined by Crawley-Boevey and Hollands [2, §5] when $\zeta_{k,\mathbb{C}} \neq 0$.

Proof. (1) Since $\widetilde{V}_k = V_k \oplus \widehat{V}_k$, (4.18) induces the following commutative diagram:

$$\begin{tikzcd}
V_k \arrow[r, "\alpha_k"] \arrow[d, equal] & \widetilde{V}_k \arrow[r, "\beta_k"] \arrow[d, "\cong"] & V_k \arrow[d, equal] \\
V_k \arrow[r, "\left[\begin{smallmatrix}-1\\0\\1\end{smallmatrix}\right]"'] & V_k \oplus \operatorname{Ker}\tau \oplus V_k \arrow[r, "{[-\zeta_{k,\mathbb{C}}\ 0\ -\zeta_{k,\mathbb{C}}]}"'] & V_k
\end{tikzcd} \tag{4.21}$$

This induces an isomorphism $\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k \cong \operatorname{Ker}\tau$.

(2) Consider $B_{\overline{h}}'$. Note that (a) the restriction of the inverse of (4.18) to $\operatorname{Ker}\tau$ is the inclusion $\operatorname{Ker}\tau \hookrightarrow \widehat{V}_k$, and (b) the restriction of $\widetilde{B}_{\overline{h}}'$ to $\widehat{V}_k$ is $-z_1\pi_{V_{\mathrm{out}(h)}} = \pi_{V_{\mathrm{out}(h)}}$. Then the assertion for $B_{\overline{h}}'$ follows. The assertion for j_k' can be checked in the same way.

(3) Consider B_h'. The isomorphism $\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k \cong \operatorname{Ker}\tau$ is written as the composition of

$$\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k \to \widehat{V}_k/\operatorname{Im}\sigma \xrightarrow{(4.18)} (\operatorname{Ker}\tau \oplus V_k)\,/\,(0 \oplus V_k) \xrightarrow{\cong} \operatorname{Ker}\tau, \tag{4.22}$$

where the first arrow is the homomorphism induced from the natural projection $\widetilde{V}_k \to \widehat{V}_k$, and the last arrow is the homomorphism induced from the projection $\operatorname{Ker}\tau \oplus V_k \to \operatorname{Ker}\tau$. If we compose B'_h with the first arrow, then we get

$$\begin{bmatrix} -\varepsilon(h) z_2 \iota_{V_{\mathrm{out}(h)}} \\ 0 \end{bmatrix} \bmod \operatorname{Im}\sigma = \begin{bmatrix} \varepsilon(h) \zeta_{k,\mathbb{C}} \iota_{V_{\mathrm{out}(h)}} \\ 0 \end{bmatrix} \bmod \operatorname{Im}\sigma.$$

If we further compose maps in (4.22), we find the required identification of B'_h. The assertion for i'_k can be proved in a similar way. □

5. Relation (I)

In this section we check the relation $S_k S_k = \mathrm{id}$ under the assumption $\zeta_k \neq 0$. Rotating by an element of $\mathrm{Sp}(1)_L$ if necessary, we may assume $\zeta_{k,\mathbb{C}} \neq 0$. (Or by continuity with respect to $\zeta_{k,\mathbb{C}}$, we may assume $\zeta_{k,\mathbb{C}} \neq 0$.)

Suppose that data (B, i, j) for V, W is given. We apply the reflection functor S_k to get data (B', i', j') for V', W. Then we apply S_k again to obtain (B'', i'', j'') for V'', W.

Let us consider operators in (4.8) for (B', i', j') and denote them by σ', τ':

$$V'_k \xrightarrow{\sigma'} \bigoplus_{\substack{h\in H \\ \mathrm{in}(h)=k}} V'_{\mathrm{out}(h)} \oplus W_k, \xrightarrow{\tau'} V'_k$$

By Proposition 4.19, we can identify σ' and τ' with the following operators via $V'_{\mathrm{out}(h)} = V_{\mathrm{out}(h)}$ and the isomorphism $V'_k \cong \operatorname{Ker}\tau$ respectively:

$$\sigma' \colon \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k = \bigoplus_{\substack{h\in H \\ \mathrm{in}(h)=k}} V_{\mathrm{out}(h)} \oplus W_k,$$

(5.1)

$$\tau' \colon \widehat{V}_k \xrightarrow[\cong]{\zeta_{k,\mathbb{C}} \mathrm{id}_{\widehat{V}_k}} \widehat{V}_k \xrightarrow[\cong]{(4.18)} \operatorname{Ker}\tau \oplus V_k \xrightarrow{\text{projection}} \operatorname{Ker}\tau.$$

Hence we have $\operatorname{Ker}\tau' \cong \operatorname{Im}\sigma$. This induces an isomorphism

$$V_k \xrightarrow[\cong]{\sigma} \operatorname{Im}\sigma \cong \operatorname{Ker}\tau' \xrightarrow[\cong]{\text{Proposition 4.19(1)}} V''_k.$$

Our remaining tasks are to identify B'', i'', j'' with B, i, j under this isomorphism.

Let $h \in H$ such that $\mathrm{out}(h) = k$. First consider B''_h. By Proposition 4.19(2), it is identified with the composition of

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{\text{projection}} V_{\mathrm{in}(h)}.$$

If we compose $\sigma\colon V_k \to \operatorname{Ker}\tau'$, we get $V_k \xrightarrow{\sigma} \widehat{V}_k \xrightarrow{\text{projection}} V_{\mathrm{in}(h)}$. By the definition of σ, this is equal to $B_h \colon V_k \to V_{\mathrm{in}(h)}$. Similarly j''_k is identified with j_k.

Next consider $B''_{\overline{h}}$. By Proposition 4.19(3), $B''_{\overline{h}}$ is identified with the composition of

$$V_{\mathrm{in}(h)} \xrightarrow[\cong]{\varepsilon(\overline{h})(-\zeta_{k,\mathbb{C}})\otimes \mathrm{id}_{V_{\mathrm{in}(h)}}} V_{\mathrm{in}(h)} \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow[\text{(4.18) for } B',\, i',\, j']{\text{the first component of}} \operatorname{Ker}\tau'.$$

The last map is given by

$$\widehat{V}_k \ni x \longmapsto x - \frac{1}{\zeta_{k,\mathbb{C}}} \sigma'\tau' x \in \operatorname{Ker}\tau'.$$

By (5.1) and the definition of (4.18), this is equal to

$$-\frac{1}{\zeta_{k,\mathbb{C}}} \sigma\tau.$$

Therefore, $B''_{\overline{h}}$ can be identified with

$$-\frac{1}{\zeta_{k,\mathbb{C}}}\sigma\varepsilon(\overline{h})B_{\overline{h}}\left(\varepsilon(\overline{h})(-\zeta_{k,\mathbb{C}})\otimes \mathrm{id}_{V_{\mathrm{out}(h)}}\right) = \sigma B_{\overline{h}}.$$

Hence it can be identified with $B_{\overline{h}}$ if we compose $\sigma^{-1}\colon \operatorname{Im}\sigma \to V_k$. Similarly i''_k is identified with i_k. All other components are unchanged, hence (B'', i'', j'') is isomorphic to (B, i, j).

6. Relation (II)

Take two distinct vertices $k, l \in I$ such that $\mathbf{A}_{kl} = 1$. In this section we check the relation $S_k S_l S_k = S_l S_k S_l$ under the assumption $\zeta_k \neq 0$, $\zeta_l \neq 0$, $\zeta_k + \zeta_l \neq 0$. Rotating by an element of $\mathrm{Sp}(1)_L$ or using the continuity, we may assume $\zeta_{k,\mathbb{C}} \neq 0$, $\zeta_{l,\mathbb{C}} \neq 0$, $\zeta_{k,\mathbb{C}} + \zeta_{l,\mathbb{C}} \neq 0$.

Suppose that data (B, i, j) for V, W is given. Let us use the notation for data which are obtained by applying reflection functors successively to (B, i, j):

$$(B,i,j) \text{ for } V \overset{S_k}{\Longrightarrow} (B',i',j') \text{ for } V' \overset{S_l}{\Longrightarrow} (B'',i'',j'') \text{ for } V'' \overset{S_k}{\Longrightarrow} (B''',i''',j''') \text{ for } V'''$$

$$(B,i,j) \text{ for } V \overset{S_l}{\Longrightarrow} ('B,'i,'j) \text{ for } 'V \overset{S_k}{\Longrightarrow} (''B,''i,''j) \text{ for } ''V \overset{S_l}{\Longrightarrow} ('''B,'''i,'''j) \text{ for } '''V.$$

By the assumption, we have the unique oriented edge h_0 such that $\mathrm{out}(h_0) = k$, $\mathrm{in}(h_0) = l$. We also use the following notation:

$$\check{V}_k \overset{\text{def.}}{=} \bigoplus_{\substack{h\in H:\mathrm{in}(h)=k\\ h\neq \overline{h_0}}} V_{\mathrm{out}(h)} \oplus W_k, \qquad \check{V}_l \overset{\text{def.}}{=} \bigoplus_{\substack{h'\in H:\mathrm{in}(h')=l\\ h'\neq h_0}} V_{\mathrm{out}(h')} \oplus W_l. \tag{6.1}$$

So we have $\widehat{V}_k = \check{V}_k \oplus V_l$, $\widehat{V}_l = \check{V}_l \oplus V_k$.

6(i). **Isomorphisms for vector spaces.** Let us consider operators in (4.8) for (B', i', j') and denote them by σ', τ':

$$V'_l \xrightarrow{\sigma'} \widehat{V}'_l \overset{\text{def.}}{=} \bigoplus_{h':\mathrm{in}(h')=l} V'_{\mathrm{out}(h')} \oplus W_l \xrightarrow{\tau'} V'_l.$$

Note that $V'_l = V_l$ and $\widehat{V}'_l = \check{V}_l \oplus V'_k$. Let

$$\iota_{\check{V}_k}\colon \check{V}_k \to \widehat{V}_k, \qquad \iota_{\check{V}_l}\colon \check{V}_l \to \widehat{V}'_l$$

be the inclusions, and let

$$\pi_{\check{V}_k}\colon \widehat{V}_k \to \check{V}_k, \qquad \pi_{\check{V}_l}\colon \widehat{V}'_l \to \check{V}_l$$

be the projections.

Let $\mathfrak{I}$ be the composition of

$$\widehat{V}'_l = \check{V}_l \oplus V'_k \xrightarrow[\cong]{\text{Proposition 4.19(1)}} \check{V}_l \oplus \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \check{V}_l \oplus \widehat{V}_k = \check{V}_l \oplus \check{V}_k \oplus V_l$$

Lemma 6.2. (1) *Under the composition of*

$$\widehat{V}'_l \xrightarrow{\mathfrak{I}} \check{V}_l \oplus \check{V}_k \oplus V_l \xrightarrow{\text{projection}} \check{V}_l \oplus \check{V}_k,$$

the kernel of τ' is mapped isomorphically to the kernel of

$$\mathcal{H}\colon \check{V}_l \oplus \check{V}_k \to V_k,$$

where $\mathcal{H} \overset{\text{def.}}{=} \left[B_{\overline{h_0}}\tau'|_{\check{V}_l} \quad \tau|_{\check{V}_k}\right] = \left[B_{\overline{h_0}} \sum \varepsilon(h')B_{h'} \quad B_{\overline{h_0}} i_l \quad \sum \varepsilon(h)B_h \quad i_k\right].$

(2) *The composition of* $\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l = \breve{V}_l \oplus V'_k \xrightarrow{\text{projection}} \breve{V}_l$ *is identified with the composition of* $\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_l \oplus \breve{V}_k \xrightarrow{\text{projection}} \breve{V}_l$ *under the above isomorphism.*

(3) *The composition of*

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l \xrightarrow{\mathfrak{I}} \breve{V}_l \oplus \breve{V}_k \oplus V_l \xrightarrow{\text{projection}} \breve{V}_k \tag{6.3}$$

is identified with the composition of $\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_l \oplus \breve{V}_k \xrightarrow{\text{projection}} \breve{V}_k$ *under the above isomorphism.*

(4) *The composition of*

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l \xrightarrow{\mathfrak{I}} \breve{V}_l \oplus \breve{V}_k \oplus V_l \xrightarrow{\text{projection}} V_l$$

is identified with the composition of

$$\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l \xrightarrow{\text{projection}} \breve{V}_l \xrightarrow{-\varepsilon(h_0)\tau'\iota_{\breve{V}_l}} V'_l = V_l.$$

Proof. (1) Under the injective map $\mathfrak{I}\colon \widehat{V}'_l \to \breve{V}_l \oplus \widehat{V}_k$ the kernel of τ' is mapped to the kernel of

$$\begin{bmatrix} 0 & \tau \\ \tau'|_{\breve{V}_l} & \varepsilon(h_0)\pi_{\overline{h_0}} \end{bmatrix} : \breve{V}_l \oplus \widehat{V}_k \to V_k \oplus V_l,$$

where we have used Proposition 4.19(2) to rewrite the restriction of τ' to $V'_k \cong \operatorname{Ker}\tau$. Using $\widehat{V}_k = \breve{V}_k \oplus V_l$, we can rewrite the above as

$$\begin{bmatrix} 0 & \tau|_{\breve{V}_k} & \varepsilon(\overline{h_0})B_{\overline{h_0}} \\ \tau'|_{\breve{V}_l} & 0 & \varepsilon(h_0)\operatorname{id}_{V_l} \end{bmatrix} : \breve{V}_l \oplus \breve{V}_k \oplus V_l \to V_k \oplus V_l. \tag{6.4}$$

Then we eliminate the component V_l, and get the assertion.

(2),(3) Clear from the above discussion.

(4) The projection to the V_l-component from the kernel of (6.4) is given by

$$-\varepsilon(h_0)\tau'|_{\breve{V}_l}.$$

This is nothing but the assertion. □

Let us consider operators in (4.8) for (B'', i'', j'') and denote them by σ'', τ'':

$$V''_k \xrightarrow{\sigma''} \widehat{V}''_k \overset{\text{def.}}{=} \bigoplus_{h:\mathrm{in}(h)=k} V''_{\mathrm{out}(h)} \oplus W_k \xrightarrow{\tau''} V''_k.$$

Note that $\widehat{V}''_k = \breve{V}_k \oplus V''_l$.

Let $\mathcal{I}$ be the composition of

$$\widehat{V}''_k = \breve{V}_k \oplus V''_l \xrightarrow[\cong]{\text{Proposition 4.19(1) for } (B'',i'',j'')} \breve{V}_k \oplus \operatorname{Ker}\tau'$$

$$\xrightarrow{\text{inclusion}} \breve{V}_k \oplus \widehat{V}'_l \xrightarrow{\operatorname{id}_{\breve{V}_l}\oplus\mathfrak{I}} \breve{V}_k \oplus \breve{V}_l \oplus \breve{V}_k \oplus V_l.$$

Lemma 6.5. (1) *Under the composition of*

$$\widehat{V}''_k \xrightarrow{\mathcal{I}} \breve{V}_k \oplus \breve{V}_l \oplus \breve{V}_k \oplus V_l \xrightarrow{\text{projection to the first two factors}} \breve{V}_k \oplus \breve{V}_l$$

the kernel of τ'' *is mapped isomorphically to the kernel of*

$$\mathcal{G}\colon \breve{V}_k \oplus \breve{V}_l \to V_l,$$

where $\mathcal{G} \overset{\text{def.}}{=} \begin{bmatrix} B_{h_0}\tau|_{\breve{V}_k} & \tau'|_{\breve{V}_l} \end{bmatrix} = \begin{bmatrix} B_{h_0}\sum\varepsilon(h)B_h & B_{h_0}i_k & \sum\varepsilon(h')B_{h'} & i_l \end{bmatrix}.$

(2) *The composition of* $\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k = \check{V}_k \oplus V''_l \xrightarrow{\text{projection}} \check{V}_k$ *is identified with the composition of* $\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_k \oplus \check{V}_l \xrightarrow{\text{projection}} \check{V}_k$ *under the above isomorphism.*

(3) *The composition of*

$$\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow{\mathcal{I}} \check{V}_k \oplus \check{V}_l \oplus \check{V}_k \oplus V_l \xrightarrow{\text{projection}} \check{V}_l$$

is identified with the composition of $\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_k \oplus \check{V}_l \xrightarrow{\text{projection}} \check{V}_l$ *under the above isomorphism.*

(4) *The composition of*

$$\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow{\mathcal{I}} \check{V}_k \oplus \check{V}_l \oplus \check{V}_k \oplus V_l \xrightarrow{\text{projection to the third factor}} \check{V}_k \tag{6.6}$$

is identified with the composition of $\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_k \oplus \check{V}_l \xrightarrow{\text{projection}} \check{V}_k \to \check{V}_k$ *where the last map is*

$$-\varepsilon(\overline{h_0})\left(\zeta_{k,\mathbb{C}} \operatorname{id}_{\check{V}_k} + \pi_{\check{V}_k}\sigma\tau\iota_{\check{V}_k}\right).$$

Proof. We use Lemma 6.2 after replacing k, l, h_0, h', V, (B,i,j) by l, k, $\overline{h_0}$, h, V', (B',i',j') respectively.

(1) The kernel of τ'' is isomorphic to the kernel of

$$\left[B'_{h_0}\sum\varepsilon(h)B'_h \quad B'_{h_0}i'_k \quad \sum\varepsilon(h')B'_{h'} \quad i'_l\right] : \check{V}_k \oplus \check{V}_l \to V_l,$$

Although we should replace $V_{\operatorname{out}(h')}$, $V_{\operatorname{out}(h)}$ by $V'_{\operatorname{out}(h')}$, $V'_{\operatorname{out}(h)}$ in the definition of $\check{V}_k$, $\check{V}_l$ in (6.1), they are the same by definition. In the same way, the last V_l was a replacement of V'_l.

Also by definition, B'_h and i'_l appearing in the last two rows are equal to B_h and i_l respectively. Moreover, $B'_{h_0}B'_{h'}$ is equal to

$$\frac{1}{\zeta_{k,\mathbb{C}}}B_{h_0}(\varepsilon(h')B_{h'})\left(\varepsilon(h')\zeta_{k,\mathbb{C}} \otimes \operatorname{id}_{V_{\operatorname{out}(h')}}\right) = B_{h_0}B_{h'}$$

by Proposition 4.19(2),(3). Similarly, $B'_{h_0}i'_k$ is equal to $B_{h_0}i_k$. Thus the above operator is the same as $\mathcal{G}$.

(2),(3) It follows from Lemma 6.2(2),(3)

(4) Spelling out the definitions of $\mathcal{I}$, $\mathfrak{I}$, we can rewrite (6.6) as the composition of

$$\begin{aligned}\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k = \check{V}_k \oplus V''_l \xrightarrow{\cong} \check{V}_k \oplus \operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \check{V}_k \oplus \widehat{V}'_l = \check{V}_k \oplus \check{V}_l \oplus V'_k \\ \xrightarrow{\text{projection}} V'_k \xrightarrow{\cong} \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k = \check{V}_k \oplus V_l \xrightarrow{\text{projection}} \check{V}_k.\end{aligned} \tag{6.7}$$

By the replacement of Lemma 6.2(4) the composition of the first five maps of this is identified with

$$\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_l \oplus \check{V}_k \xrightarrow{\text{projection}} \check{V}_k \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow{-\varepsilon(\overline{h_0})\tau''} V''_k = V'_k \xrightarrow{\cong} \operatorname{Ker}\tau.$$

The composition of the last three maps is explicitly written as

$$\check{V}_k = \bigoplus_{\substack{h''\in H:\operatorname{in}(h'')=k \\ h''\neq\overline{h_0}}} V'_{\operatorname{out}(h'')} \oplus W_k \xrightarrow{-\varepsilon(\overline{h_0})\left[\sum\varepsilon(h'')B'_{h''} \ i'_k\right]} V'_k \xrightarrow{\cong} \operatorname{Ker}\tau.$$

By Proposition 4.19(3) this is equal to

$$\check{V}_k \xrightarrow{-\varepsilon(\overline{h_0})\zeta_{k,\mathbb{C}}\operatorname{id}_{\check{V}_k}} \check{V}_k \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{(4.18)} \operatorname{Ker}\tau \oplus V_k \xrightarrow{\text{projection}} \operatorname{Ker}\tau.$$

Composing the last two maps of (6.7) and using the definition of (4.18), we get the assertion. $\square$

By Lemmas 6.2, 6.5 we have

$$V_k''' \cong \operatorname{Ker}\mathcal{G}, \qquad V_l''' = V_l'' \cong \operatorname{Ker}\mathcal{H}. \tag{6.8}$$

Note that $\mathcal{G}$ is obtained from $\mathcal{H}$ by exchanging $k \leftrightarrow l$, $h_0 \leftrightarrow \overline{h_0}$. Thus we have isomorphisms

$$V_k''' \cong {}'''V_k, \qquad V_l''' \cong {}'''V_l.$$

The other components for V''' and $'''V$ are the same, so we have isomorphisms $V''' \cong {}'''V$.

6(ii). **Identification of data, Part (I).** Our remaining tasks are to identify (B''', i''', j''') with $({}'''B, {}'''i, {}'''j)$ under this isomorphism.

Take $h \in H$, $h' \in H$ such that $\operatorname{in}(h) = k$, $\operatorname{in}(h') = l$, $h \neq \overline{h_0}$, $h' \neq h_0$. By Proposition 4.19(2), $B'''_{\overline{h}}$ is identified with the composition of

$$\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}_k'' \xrightarrow{\text{projection}} V''_{\operatorname{out}(h)} = V_{\operatorname{out}(h)}.$$

By Lemma 6.5(2) it is identified with the composition of

$$\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_k \oplus \check{V}_l \xrightarrow{\text{projection}} V_{\operatorname{out}(h)}.$$

By Proposition 4.19(2), $B'''_{\overline{h'}} = B''_{\overline{h'}}$ is identified with the composition of

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}_l' \xrightarrow{\text{projection}} V'_{\operatorname{out}(h')} = V_{\operatorname{out}(h')}.$$

By Lemma 6.2(2) it is identified with the composition of

$$\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \check{V}_l \oplus \check{V}_k \xrightarrow{\text{projection}} V_{\operatorname{out}(h')}.$$

Noticing that the above identifications are interchanged under $k \leftrightarrow l$, $\mathcal{G} \leftrightarrow \mathcal{H}$, $h \leftrightarrow h'$, $h_0 \leftrightarrow \overline{h_0}$, we find that $B'''_{\overline{h}}$, $B'''_{\overline{h'}}$ are identified with ${}'''B_{\overline{h}}$, ${}'''B_{\overline{h'}}$ respectively. Similarly j_k''', j_l''' are identified with ${}'''j_k$, ${}'''j_l$ respectively.

6(iii). **Identification of data, Part (II).** By Proposition 4.19(3), $B'''_{h'} = B''_{h'}$ is identified with the composition of

$$V_{\operatorname{out}(h')} = V'_{\operatorname{out}(h')} \xrightarrow[\cong]{\varepsilon(h')(\zeta_{l,\mathbb{C}}+\zeta_{k,\mathbb{C}})\otimes\mathrm{id}_{V'_{\operatorname{out}(h')}}} V'_{\operatorname{out}(h')} \xrightarrow{\text{inclusion}} \widehat{V}_l' \xrightarrow[\text{(4.18) for } (B',i',j')]{\text{the first component of}} \operatorname{Ker}\tau'. \tag{6.9}$$

Let us consider the compositon of

$$V_{\operatorname{out}(h')} \xrightarrow{(6.9)} \operatorname{Ker}\tau' \xrightarrow[\cong]{\text{Lemma 6.2}} \operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \check{V}_l \oplus \check{V}_k. \tag{6.10}$$

Using Lemma 6.2(2) and the definition of (4.18), we find that the $\check{V}_l$-component of (6.10) is given by

$$\varepsilon(h')(\zeta_{k,\mathbb{C}} + \zeta_{l,\mathbb{C}})\iota_{h'} + \pi_{\check{V}_l}\sigma' B_{h'} \colon V_{\operatorname{out}(h')} \to \check{V}_l,$$

where $\iota_{h'} \colon V_{\operatorname{out}(h')} \to \check{V}_l$ is the inclusion.

Let us compute the $\check{V}_k$-component of (6.10). By Lemma 6.2(3) it is given by the composition of

$$V_{\operatorname{out}(h')} \xrightarrow{(6.9)} \operatorname{Ker}\tau' \xrightarrow{(6.3)} \check{V}_k.$$

Spelling out the definition of (6.9) and (6.3), we find that this is equal to

$$\varepsilon(h')(\zeta_{l,\mathbb{C}}+\zeta_{k,\mathbb{C}})\mathfrak{P}\frac{1}{\zeta_{k,\mathbb{C}}+\zeta_{l,\mathbb{C}}}B'_{\overline{h_0}}(\varepsilon(h')B_{h'}) = \mathfrak{P}B'_{\overline{h_0}}B_{h'},$$

where $\mathfrak{P}$ is the composition of

$$V'_k \xrightarrow[\cong]{\text{Proposition 4.19(1)}} \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{\text{projection}} \check{V}_k.$$

If we rewrite $B'_{\overline{h_0}}$ by Proposition 4.19(3), the above becomes

$$\frac{1}{\zeta_{k,\mathbb{C}}}\pi_{\check{V}_k}\sigma\left(\varepsilon(\overline{h_0})B_{\overline{h_0}}\right)\varepsilon(\overline{h_0})\zeta_{k,\mathbb{C}}B_{h'} = \pi_{\check{V}_k}\sigma B_{\overline{h_0}}B_{h'}.$$

Thus (6.10) is given by

$$\begin{bmatrix}\varepsilon(h')(\zeta_{k,\mathbb{C}}+\zeta_{l,\mathbb{C}})\iota_{h'}+\pi_{\check{V}_l}\sigma' B_{h'} \\ \pi_{\check{V}_k}\sigma B_{\overline{h_0}}B_{h'}\end{bmatrix} : V_{\mathrm{out}(h')} \to \check{V}_l\oplus\check{V}_k.$$

By Proposition 4.19(3), B'''_h is identified with the composition of

$$V_{\mathrm{out}(h)} = V''_{\mathrm{out}(h)} \xrightarrow[\cong]{\varepsilon(h)\zeta_{l,\mathbb{C}}\otimes\mathrm{id}_{V''_{\mathrm{out}(h)}}} V''_{\mathrm{out}(h)} \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow[\text{(4.18) for }(B'',i'',j'')]{\text{the first component of}} \operatorname{Ker}\tau''. \tag{6.11}$$

Let us consider the composition of

$$V_{\mathrm{out}(h)} \xrightarrow{(6.11)} \operatorname{Ker}\tau'' \xrightarrow[\cong]{\text{Lemma 6.5}} \operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \check{V}_k\oplus\check{V}_l. \tag{6.12}$$

Using the above calculation after replacing k, l, h_0, h', V, (B,i,j) by l, k, $\overline{h_0}$, h, V', (B',i',j') respectively, we find that (6.12) is given by

$$\begin{bmatrix}\varepsilon(h)\zeta_{l,\mathbb{C}}\iota_h+\pi_{\check{V}_k}\sigma'' B'_h \\ \pi_{\check{V}_l}\sigma' B'_{h_0}B'_h\end{bmatrix} : V_{\mathrm{out}(h)} \to \check{V}_k\oplus\check{V}_l,$$

where ι_h is the inclusion $V_{\mathrm{out}(h)} \to \check{V}_k$. By Proposition 4.19, we have

$$B'_{h_0}B'_h = B_{h_0}B_h,$$
$$\pi_{\check{V}_k}\sigma'' B'_h = \varepsilon(h)\zeta_{k,\mathbb{C}}\iota_h + \sigma B_h.$$

Thus the above is equal to

$$\begin{bmatrix}\varepsilon(h)(\zeta_{k,\mathbb{C}}+\zeta_{l,\mathbb{C}})\iota_h+\pi_{\check{V}_k}\sigma B_h \\ \pi_{\check{V}_l}\sigma' B_{h_0}B_h\end{bmatrix}.$$

Noticing that the above identifications are interchanged under $k\leftrightarrow l$, $\mathcal{G}\leftrightarrow\mathcal{H}$, $h\leftrightarrow h'$, $h_0\leftrightarrow\overline{h_0}$, we find that B'''_h, $B'''_{h'}$ are identified with $'''B_h$, $'''B_{h'}$ respectively.

Similarly, i'''_k, i'''_l are identified with $'''i_k$, $'''i_l$ respectively.

6(iv). **Rewrite B'''_{h_0}.** By Proposition 4.19(2), B'''_{h_0} is identified with the composition of

$$(6.13)\qquad \operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow{\text{projection}} V''_l.$$

We want to study the composition of

$$(6.14)\qquad \operatorname{Ker}\mathcal{G} \xrightarrow[\cong]{\text{Lemma 6.5(1)}} \operatorname{Ker}\tau'' \xrightarrow{(6.13)} V''_l$$
$$\xrightarrow[\cong]{\text{Proposition 4.19(1) for }(B',i',j')} \operatorname{Ker}\tau' \xrightarrow[\cong]{\text{Lemma 6.2(1)}} \operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_l \oplus \breve{V}_k.$$

First consider the $\breve{V}_l$-component. By Lemma 6.2(2), the $\breve{V}_l$-component of the composition of the last two maps of (6.14) is equal to the composition of

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l = \breve{V}_l \oplus V'_k \xrightarrow{\text{projection}} \breve{V}_l.$$

Substituting this back to (6.14), we find that the $\breve{V}_l$-component of (6.14) is equal to the composition of

$$\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l \xrightarrow{\text{projection}} \breve{V}_l$$

by Lemma 6.5(3).

Next consider the $\breve{V}_k$-component of (6.14). By Lemma 6.2(3), the $\breve{V}_k$-component of the composition of the last two maps of (6.14) is equal to the composition of

$$\operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l \xrightarrow{\mathfrak{I}} \breve{V}_l \oplus \breve{V}_k \oplus V_l \xrightarrow{\text{projection}} \breve{V}_k.$$

Substituting this back to (6.14), we find that the $\breve{V}_k$-component of (6.14) is equal to the composition of

$$\operatorname{Ker}\mathcal{G} \xrightarrow{\cong} \operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow{(6.6)} \breve{V}_k.$$

By Lemma 6.5(4) this is equal to

$$\operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l \xrightarrow{\text{projection}} \breve{V}_k \xrightarrow{-\varepsilon(\overline{h_0})\left(\zeta_{k,\mathbb{C}}\,\mathrm{id}_{\breve{V}_k} + \pi_{\breve{V}_k}\sigma\tau\iota_{\breve{V}_k}\right)} \breve{V}_k.$$

6(v). **Rewrite B''''_{h_0}.** By Proposition 4.19(3), B''''_{h_0} is identified with the composition of

$$(6.15)\qquad V''_l \xrightarrow[\cong]{\varepsilon(\overline{h_0})\zeta_{l,\mathbb{C}}\otimes \mathrm{id}_{V''_l}} V''_l \xrightarrow{\text{inclusion}} \widehat{V}''_k \xrightarrow[\text{(4.18) for }(B'',i'',j'')]{\text{the first component of}} \operatorname{Ker}\tau''.$$

We want to study the composition of

$$(6.16)\qquad \operatorname{Ker}\mathcal{H} \xrightarrow[\cong]{\text{Lemma 6.2(1)}} \operatorname{Ker}\tau' \xrightarrow[\cong]{\text{Proposition 4.19(1) for }(B',i',j')} V''_l$$
$$\xrightarrow{(6.15)} \operatorname{Ker}\tau'' \xrightarrow[\cong]{\text{Lemma 6.5}} \operatorname{Ker}\mathcal{G} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l.$$

First consider the $\breve{V}_k$-component. By Lemma 6.5(2), the $\breve{V}_k$-component of the composition of the last two maps of (6.16) is equal to the composition of

$$\operatorname{Ker}\tau'' \xrightarrow{\text{inclusion}} \widehat{V}''_k = \breve{V}_k \oplus V''_l \xrightarrow{\text{projection}} \breve{V}_k.$$

Spelling out the definition of maps in (6.15), we find that the $\check{V}_k$-component of the composition of the last three maps of (6.16) is equal to the composition of

$$V_l'' \xrightarrow{B''_{\overline{h_0}}} V_k'' \xrightarrow{\sigma''} \widehat{V}_k'' = \check{V}_k \oplus V_l'' \xrightarrow{\text{projection}} \check{V}_k. \tag{6.17}$$

The composition of the last two maps of (6.17) is equal to the composition of

$$V_k'' = V_k' \xrightarrow{\begin{bmatrix} \bigoplus B'_{\overline{h}} \\ j'_k \end{bmatrix}} \check{V}_k = \bigoplus_{\substack{h \in H : \mathrm{in}(h) = k \\ h \neq \overline{h_0}}} V_{\mathrm{out}(h)} \oplus W_k.$$

By Proposition 4.19(2), this can be rewritten as

$$V_k'' = V_k' \xrightarrow[\cong]{\text{Proposition 4.19(1)}} \operatorname{Ker} \tau \xrightarrow{\text{inclusion}} \widehat{V}_k = \check{V}_k \oplus V_l \xrightarrow{\text{projection}} \check{V}_k.$$

Let us substitute this back to (6.17), compose the map $\operatorname{Ker} \tau' \xrightarrow{\cong} V_l''$ in (6.16) and then rewrite $B''_{\overline{h_0}}$ by Proposition 4.19(2). We get

$$\operatorname{Ker} \tau' \xrightarrow{\text{inclusion}} \widehat{V}_l' \xrightarrow{\text{projection}} V_k' \xrightarrow[\cong]{\text{Proposition 4.19(1)}} \operatorname{Ker} \tau \xrightarrow{\text{inclusion}} \widehat{V}_k = \check{V}_k \oplus V_l \xrightarrow{\text{projection}} \check{V}_k.$$

This is nothing but (6.3). By Lemma 6.2(3) this is identified with

$$\operatorname{Ker} \mathcal{H} \xrightarrow{\text{inclusion}} \check{V}_k \oplus \check{V}_l \xrightarrow{\text{projection}} \check{V}_k$$

under the first map $\operatorname{Ker} \mathcal{H} \xrightarrow{\cong} \operatorname{Ker} \tau'$ in (6.16).

Next study the $\check{V}_l$-component of (6.16). By Lemma 6.5(3), the $\check{V}_l$-component of the composition of the last two maps of (6.16) is equal to the composition of

$$\begin{aligned} \operatorname{Ker} \tau'' \xrightarrow{\text{inclusion}} \widehat{V}_k'' \xrightarrow{\text{projection}} V_l'' & \\ \xrightarrow[\cong]{\text{Proposition 4.19(1)}} \operatorname{Ker} \tau' & \xrightarrow{\text{inclusion}} \widehat{V}_l' = \check{V}_l \oplus V_k' \xrightarrow{\text{projection}} \check{V}_l. \end{aligned} \tag{6.18}$$

If we compose (6.15) with the first two maps of above, we get

$$\varepsilon(\overline{h_0}) \zeta_{l,\mathbb{C}} \otimes \mathrm{id}_{V_l''} + B''_{h_0} B''_{\overline{h_0}} \colon V_l'' \to V_l''. \tag{6.19}$$

We study each summand separately. The first summand gives us

$$V_l'' \xrightarrow{\cong} \operatorname{Ker} \tau' \xrightarrow{\text{inclusion}} \widehat{V}_l' \xrightarrow{\text{projection}} \check{V}_l \xrightarrow{\varepsilon(\overline{h_0}) \zeta_{l,\mathbb{C}} \mathrm{id}_{\check{V}_l}} \check{V}_l.$$

If we compose first two maps $\operatorname{Ker} \mathcal{H} \to \operatorname{Ker} \tau' \to V_l''$ of (6.16) with above, we get

$$\operatorname{Ker} \mathcal{H} \xrightarrow{\text{inclusion}} \check{V}_l \oplus \check{V}_k \xrightarrow{\text{projection}} \check{V}_l \xrightarrow{\varepsilon(\overline{h_0}) \zeta_{l,\mathbb{C}} \mathrm{id}_{\check{V}_l}} \check{V}_l$$

by Lemma 6.2(2).

Next consider the second summand of (6.19). Let us rewrite B''_{h_0} by Proposition 4.19(3), and then compose with the last three maps in (6.18). The result is

$$\pi_{\check{V}_l} \sigma' B'_{h_0} \colon V_k' \to \check{V}_l.$$

By Proposition 4.19(2), this is equal to

$$V'_k \xrightarrow{\cong} \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{\text{projection}} V_l = V'_l \xrightarrow{\pi_{\breve{V}_l}\sigma'} \breve{V}_l.$$

If we rewrite $B''_{\overline{h_0}}$ by Proposition 4.19(2) and compose with above, we get

$$V''_l \xrightarrow{\cong} \operatorname{Ker}\tau' \xrightarrow{\text{inclusion}} \widehat{V}'_l \xrightarrow{\text{projection}} V'_k \xrightarrow{\cong} \operatorname{Ker}\tau \xrightarrow{\text{inclusion}} \widehat{V}_k \xrightarrow{\text{projection}} V_l = V'_l \xrightarrow{\pi_{\breve{V}_l}\sigma'} \breve{V}_l.$$

If we compose first two maps $\operatorname{Ker}\mathcal{H} \to \operatorname{Ker}\tau' \to V'_l$ of (6.16) with above, we get

$$\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l \xrightarrow{\text{projection}} \breve{V}_l \xrightarrow{-\varepsilon(h_0)\tau'\iota_{\breve{V}_l}} V'_l \xrightarrow{\pi_{\breve{V}_l}\sigma'} \breve{V}_l$$

by Lemma 6.2(4). Thus the $\breve{V}_l$-component of (6.16) is given by

$$\operatorname{Ker}\mathcal{H} \xrightarrow{\text{inclusion}} \breve{V}_k \oplus \breve{V}_l \xrightarrow{\text{projection}} \breve{V}_l \xrightarrow{\varepsilon(\overline{h_0})\zeta_{l,\mathbb{C}}\,\mathrm{id}_{\breve{V}_l} - \varepsilon(h_0)\pi_{\breve{V}_l}\sigma'\tau'\iota_{\breve{V}_l}} \breve{V}_l.$$

Since $\varepsilon(\overline{h_0}) = -\varepsilon(h_0)$, this shows that (6.16) and (6.14) is interchanged under $k \leftrightarrow l$, $\mathcal{G} \leftrightarrow \mathcal{H}$, $h_0 \leftrightarrow \overline{h_0}$. Thus B'''_{h_0}, $B'''_{\overline{h_0}}$ are identified with $'''B_{h_0}$, $'''B_{\overline{h_0}}$ respectively. Combining all identifications together, we find that (B''', i''', j''') is isomorphic to $('''B, '''i, '''j)$.

7. Composition

Suppose that we have another admissible collection $\left\{(\bar{\mathcal{A}}^l, \bar{\Psi}^l) \in \mathbf{M}(\bar{V}^l, W^l),\ \bar{\Phi}^{\overline{h}}\right\}_{l\in I, h\in\Omega}$. We apply the reflection functor $\mathcal{F}_{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)}$ to each $(\bar{\mathcal{A}}^l, \bar{\Psi}^l)$ to get data $(\bar{\mathcal{A}}'^l, \bar{\Psi}'^l)$ for $\bar{V}'^l$, W^l.

Let

$$\mathcal{D}^l_k \colon S^+ \otimes \mathrm{L}(V^k, \bar{V}^l) \to \bar{V}^l_k \oplus \mathrm{E}(V^k, \bar{V}^l) \oplus \left(V^k_l\right)^*$$

be the operator (3.8) for $(\bar{\mathcal{A}}^l, \bar{\Psi}^l)$. Here we use $W^l_m = \mathbb{C}$ for $m = l$ and 0 otherwise. Then $\bar{V}'^l$ was defined as $\operatorname{Ker}\mathcal{D}^{l\dagger}_k$. Let I^l_k and P^l_k be the inclusion and the orthogonal projection of $\operatorname{Ker}\mathcal{D}^{l\dagger}_k$ in $\bar{V}^l_k \oplus \mathrm{E}(V^k, \bar{V}^l) \oplus \left(V^k_l\right)^*$.

For each $h \in \Omega$ we define $\bar{\Phi}'^{\overline{h}} \in S^+ \otimes \mathrm{L}(\bar{V}'^{\,\mathrm{in}(h)}, \bar{V}'^{\,\mathrm{out}(h)})$ by

$$\left(\bar{\Phi}'^{\overline{h}}_k I^{\mathrm{in}(h)}_k\right) \begin{bmatrix} v \\ C \\ b \end{bmatrix} \overset{\text{def.}}{=} \left(\mathrm{id}_{S^+} \otimes P_{\mathrm{out}(h)}\right) \begin{bmatrix} \bar{\Phi}^{\overline{h}}_k v \\ \bar{\Phi}^{\overline{h}} C + \bar{\Psi}^{\mathrm{out}(h)}_{\mathrm{out}(h)} \otimes b \\ -\left(\omega\bar{\Psi}^{\mathrm{in}(h)}_{\mathrm{in}(h)}\right)^\dagger C_h + b\mathcal{A}^k_h \end{bmatrix}$$

$$\text{for } \begin{bmatrix} v \\ C \\ b \end{bmatrix} \in \bar{V}^{\mathrm{in}(h)}_k \oplus \mathrm{E}(V^k, \bar{V}^{\mathrm{in}(h)}) \oplus \left(V^k_{\mathrm{in}(h)}\right)^*.$$

Here $\bar{\Psi}^{\mathrm{out}(h)}_{\mathrm{out}(h)} \otimes b$ is considered as an element of $S^+ \otimes \mathrm{E}(V^k, \bar{V}^{\mathrm{out}(h)})$ via the inclusion $(V^k_{\mathrm{in}(h)})^* \otimes \bar{V}^{\mathrm{out}(h)}_{\mathrm{out}(h)} \subset \mathrm{E}(V^k, \bar{V}^{\mathrm{out}(h)})$.

Proposition 7.1. $\left\{(\bar{\mathcal{A}}'^l, \bar{\Psi}'^l),\ \bar{\Phi}'^{\overline{h}}\right\}_{l\in I, h\in\Omega}$ *is an admissible collection.*

Proof. Let us consider the complex (3.9) for $V = \bar{V}^l$, $W = W^l$ and denote the operators by α^l_k and β^l_k:

$$\mathrm{L}(V^k, \bar{V}^l) \xrightarrow{\alpha^l_k} \bar{V}^l_k \oplus \mathrm{E}(V^k, \bar{V}^l) \oplus \left(V^k_l\right)^* \xrightarrow{\beta^l_k} \mathrm{L}(V^k, \bar{V}^l).$$

Thus we have the isomorphism $\bar{V}_k^l \cong \operatorname{Ker} \beta_k^l / \operatorname{Im} \alpha_k^l$.

Our first aim is to give the holormorphic description of $\bar{\Phi}'^{\overline{h}}$. Let $\bar{B}^k, \bar{\imath}^k, \bar{\jmath}^k, \bar{\phi}^h$ be the holomorphic description of $\bar{\mathcal{A}}^k, \bar{\Psi}^k, \bar{\Phi}^{\overline{h}}$.

Let

$$\bar{\phi}_k'^h : \bar{V}_k^{\mathrm{out}(h)} \oplus \mathrm{E}(V^k, \bar{V}^{\mathrm{out}(h)}) \oplus \left(V_{\mathrm{out}(h)}^k\right)^* \to \bar{V}_k^{\mathrm{in}(h)} \oplus \mathrm{E}(V^k, \bar{V}^{\mathrm{in}(h)}) \oplus \left(V_{\mathrm{in}(h)}^k\right)^*,$$

$$\bar{\phi}_k'^h \begin{bmatrix} v \\ C \\ b \end{bmatrix} = \begin{bmatrix} \bar{\phi}_k^h v \\ \bar{\phi}^h C - \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}^{\mathrm{in}(h)} \otimes b \\ \bar{\jmath}_{\mathrm{out}(h)}^{\mathrm{out}(h)} C_{\overline{h}} + b B_{\overline{h}}^k \end{bmatrix}.$$

We claim that $\bar{\phi}_k'^h$ maps $\operatorname{Ker} \beta_k^{\mathrm{out}(h)}$ (resp. $\operatorname{Im} \alpha_k^{\mathrm{out}(h)}$) to $\operatorname{Ker} \beta_k^{\mathrm{in}(h)}$ (resp. $\operatorname{Im} \alpha_k^{\mathrm{in}(h)}$). In fact, we have

$$\begin{aligned} &\beta_k^{\mathrm{out}(h)} \bar{\phi}_k'^h \begin{bmatrix} v \\ C \\ b \end{bmatrix} \\ &= \varepsilon\left(\bar{B}^{\mathrm{in}(h)} \bar{\phi}^h C + \bar{\phi}^h C B^k\right) + \bar{B}_{\overline{h}}^{\mathrm{in}(h)} \bar{\imath}^{\mathrm{in}(h)} \otimes b - \bar{\imath}^{\mathrm{in}(h)} \otimes b B_{\overline{h}}^k + \bar{\phi}_k^h v \otimes j^k + \bar{\imath}^{\mathrm{out}(h)} \left(\bar{\jmath}^{\mathrm{out}(h)} C_{\overline{h}} + b B_{\overline{h}}^k\right) \\ &= \bar{\phi}^h \left(\varepsilon\left(\bar{B}^{\mathrm{out}(h)} C + C B^k\right) + v \otimes j^k + \bar{\imath}^{\mathrm{out}(h)} \otimes b\right) = \bar{\phi}^h \beta_k^{\mathrm{out}(h)} \begin{bmatrix} v \\ C \\ b \end{bmatrix}, \end{aligned}$$

where we have used (3.2) in the second equality. Similarly we have

$$\bar{\phi}_k'^h \alpha_k^{\mathrm{out}(h)} \eta = \alpha_k^{\mathrm{in}(h)} (\bar{\phi}^h \eta).$$

These show the claim.

Thus $\bar{\phi}_k'^h$ induces a homomorphism from $\bar{V}_k'^{\,\mathrm{out}(h)} = \operatorname{Ker} \beta_k^{\mathrm{out}(h)} / \operatorname{Im} \alpha_k^{\mathrm{out}(h)}$ to $\bar{V}_k'^{\,\mathrm{in}(h)} = \operatorname{Ker} \beta_k^{\mathrm{in}(h)} / \operatorname{Im} \alpha_k^{\mathrm{in}(h)}$. We denote the induced map by $\bar{\phi}_k'^h$ for brevity. It is clear that $\bar{\phi}_k'^h$ is the holomorphic description of $\bar{\Phi}_{\underline{k}}'^{\overline{h}}$. By the technique explained explained in §1(iv), our remaining task is to check (3.2) for $\bar{\phi}_k'^h$ and $(\bar{B}'^l, \bar{\imath}'^l, \bar{\jmath}'^l)$.

For ${}^t\!\begin{bmatrix} v & C & b \end{bmatrix} \in \bar{V}_{\mathrm{out}(h')}^{\mathrm{out}(h)} \oplus \mathrm{E}(V^{\mathrm{out}(h')}, \bar{V}^{\mathrm{out}(h)}) \oplus \left(V_{\mathrm{out}(h)}^{\mathrm{out}(h')}\right)^*$ we have

$$\begin{aligned} \bar{B}_{h'}'^{\,\mathrm{in}(h)} \bar{\phi}_{\mathrm{out}(h')}'^h \begin{bmatrix} v \\ C \\ b \end{bmatrix} &= \begin{bmatrix} \bar{B}_{h'}^{\mathrm{in}(h)} \bar{\phi}_{\mathrm{out}(h')}^h v + \left(\bar{\phi}_{\mathrm{in}(h')}^h C_{h'} - \delta_{hh'} \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}^{\mathrm{in}(h)} \otimes b\right) i_{\mathrm{out}(h')}^{\mathrm{out}(h')} \\ \varepsilon(\overline{h'}) \bar{\phi}_{\mathrm{out}(h')}^h v \otimes j_{\mathrm{in}(h')}^{\mathrm{in}(h')} + \left(\bar{\phi}^h C - \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}^{\mathrm{in}(h)} \otimes b\right) \phi^{\overline{h'}} \\ \left(\bar{\jmath}_{\mathrm{out}(h)}^{\mathrm{out}(h)} C_{\overline{h}} + b B_{\overline{h}}^{\mathrm{out}(h')}\right) \phi_{\mathrm{in}(h)}^{\overline{h'}} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\phi}_{\mathrm{in}(h')}^h \left(\bar{B}_{h'}^{\mathrm{out}(h)} v + C_{h'} i_{\mathrm{out}(h')}^{\mathrm{out}(h')}\right) - \delta_{hh'} \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}^{\mathrm{in}(h)} \otimes \left(b i_{\mathrm{out}(h)}^{\mathrm{out}(h)} + \bar{\jmath}_{\mathrm{out}(h)}^{\mathrm{out}(h)} v\right) \\ \varepsilon(\overline{h'}) \bar{\phi}_{\mathrm{out}(h')}^h v \otimes j_{\mathrm{in}(h')}^{\mathrm{in}(h')} + \left(\bar{\phi}^h C - \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}^{\mathrm{in}(h)} \otimes b\right) \phi^{\overline{h'}} \\ \bar{\jmath}_{\mathrm{out}(h)}^{\mathrm{out}(h)} C_{\overline{h}} \phi_{\mathrm{in}(h)}^{\overline{h'}} + b \phi_{\mathrm{out}(h)}^{\overline{h'}} B_{\overline{h}}^{\mathrm{in}(h')} + \delta_{hh'} \varepsilon(h) b i_{\mathrm{out}(h)}^{\mathrm{out}(h)} \otimes j_{\mathrm{in}(h)}^{\mathrm{in}(h)} \end{bmatrix} \\ &= \left(\bar{\phi}_{\mathrm{in}(h')}'^h \bar{B}_{h'}'^{\,\mathrm{out}(h)} - \delta_{hh'} \varepsilon(h) \bar{\imath}_{\mathrm{in}(h)}'^{\,\mathrm{in}(h)} \otimes \bar{\jmath}_{\mathrm{out}(h)}'^{\,\mathrm{out}(h)}\right) \begin{bmatrix} v \\ C \\ b \end{bmatrix}. \end{aligned}$$

We also have

$$\bar{B}'^{\,\mathrm{in}(h)}_{\overline{h}}\bar{i}'^{\,\mathrm{in}(h)}_{\mathrm{in}(h)} = \begin{bmatrix} \bar{B}^{\mathrm{in}(h)}_{\overline{h}}\bar{i}^{\mathrm{in}(h)}_{\mathrm{in}(h)} \\ \varepsilon(h)\bar{i}^{\mathrm{in}(h)}_{\mathrm{in}(h)} \otimes j^{\mathrm{out}(h)}_{\mathrm{out}(h)} \\ -j^{\mathrm{in}(h)}_{\mathrm{in}(h)}\phi^h_{\mathrm{in}(h)} \end{bmatrix} = \begin{bmatrix} \bar{\phi}^h_{\mathrm{out}(h)}\bar{i}^{\mathrm{out}(h)}_{\mathrm{out}(h)} \\ \varepsilon(h)\bar{i}^{\mathrm{in}(h)}_{\mathrm{in}(h)} \otimes j^{\mathrm{out}(h)}_{\mathrm{out}(h)} \\ -j^{\mathrm{out}(h)}_{\mathrm{out}(h)}B^{\mathrm{out}(h)}_{\overline{h}} \end{bmatrix} = \bar{\phi}'^h_{\mathrm{out}(h)}\bar{i}'^{\,\mathrm{out}(h)}_{\mathrm{out}(h)}.$$

For ${}^t[\, v\ C\ b\,] \in \bar{V}^{\mathrm{out}(h)}_{\mathrm{in}(h)} \oplus \mathrm{E}(V^{\mathrm{in}(h)}, \bar{V}^{\mathrm{out}(h)}) \oplus \left(V^{\mathrm{in}(h)}_{\mathrm{out}(h)}\right)^*$ we have

$$\begin{aligned} \bar{j}'^{\,\mathrm{out}(h)}_{\mathrm{out}(h)}\bar{B}'^{\,\mathrm{out}(h)}_{\overline{h}} \begin{bmatrix} v \\ C \\ b \end{bmatrix} &= \bar{j}^{\mathrm{out}(h)}_{\mathrm{out}(h)}\left(\bar{B}_{\overline{h}}v_{\mathrm{in}(h)} + C_{\overline{h}}i^{\mathrm{in}(h)}_{\mathrm{in}(h)}\right) + b\phi^h_{\mathrm{out}(h)}i^{\mathrm{out}(h)}_{\mathrm{out}(h)} \\ &= \bar{j}^{\mathrm{in}(h)}_{\mathrm{in}(h)}\bar{\phi}^h_{\mathrm{in}(h)}v + \bar{j}^{\mathrm{out}(h)}_{\mathrm{out}(h)}C_{\overline{h}}i^{\mathrm{in}(h)}_{\mathrm{in}(h)} + bB^{\mathrm{in}(h)}_{\overline{h}}i^{\mathrm{in}(h)}_{\mathrm{in}(h)} \\ &= \bar{j}'^{\,\mathrm{in}(h)}_{\mathrm{in}(h)}\bar{\phi}'^h_{\mathrm{in}(h)} \begin{bmatrix} v \\ C \\ b \end{bmatrix}. \end{aligned}$$

Thus we have checked (3.2). □

Let w be an element of the Weyl group and $s_{i_1} \cdots s_{i_n}$ its reduced expression. We consider the composition of reflection functors:

$$S_{i_1} \cdots S_{i_n} : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{w\zeta}(w * \mathbf{v}, \mathbf{w}),$$

where we assume

$$\zeta \notin \bigcup_{\theta \in R_+} \mathbb{R}^3 \otimes D_\theta$$

so that $\mathfrak{M}^{\mathrm{reg}}_\zeta(\mathbf{v}', \mathbf{w}) = \mathfrak{M}_\zeta(\mathbf{v}', \mathbf{w})$ for any $\mathbf{v}'$. By Theorem 4.4 the composite is independent of the choice of the reduced expression, so we may denote it by S_w.

We apply S_w to the trivial admissible collection (the first example) to get data $(\mathcal{A}^k, \varPsi^k) \in \mathfrak{M}_{w\zeta}(w *_{\mathbf{e}^k} 0, \mathbf{e}^k)$. By Proposition 7.1 (and induction), there exists $\Phi^{\overline{h}}$ such that $\left\{(\mathcal{A}^k, \varPsi^k, \Phi^{\overline{h}})\right\}_{k \in I, h \in \Omega}$ is an admissible collection. This is the admissible collection explained in Conjecture 3.15(2).

8. Reflection functors are Hyper-Kähler isometry

Theorem 8.1. *The reflection functor* $S_k \colon \mathfrak{M}^{\mathrm{reg}}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}^{\mathrm{reg}}_{s_k\zeta}(s_k * \mathbf{v}, \mathbf{w})$ *is a hyper-Kähler isometry, i.e., it respects the Riemannian metrics and three almost complex structures* I, J, K.

Proof. Take a point in $\mathfrak{M}^{\mathrm{reg}}_\zeta(\mathbf{v}, \mathbf{w})$ and fix its representative $(\mathcal{A}, \varPsi)$ in $\mu^{-1}(-\zeta)$. A tangent vector of $\mathfrak{M}^{\mathrm{reg}}_\zeta(\mathbf{v}, \mathbf{w})$ at $(\mathcal{A}, \varPsi)$ is represented by $(\delta\mathcal{A}, \delta\varPsi) \in \mathbf{M}(V, W)$ such that

$$d\mu_{(\mathcal{A},\varPsi)}(\delta\mathcal{A}, \delta\varPsi) = 0, \quad (\delta\mathcal{A}, \delta\varPsi) \text{ is orthogonal to the orbit } G_V \cdot (\mathcal{A}, \varPsi). \tag{8.2}$$

We take a family $(\mathcal{A}_t, \varPsi_t)$ $(-\varepsilon < t < \varepsilon)$ of data for V, W such that

$$\mathcal{A}|_{t=0} = \mathcal{A}, \quad \varPsi|_{t=0} = \varPsi, \quad \delta\mathcal{A} = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_t, \quad \delta\varPsi = \left.\frac{d}{dt}\right|_{t=0} \varPsi_t.$$

be the derivative of $\mathcal{A}_t$, Ψ_t at $t=0$. Let $(\mathcal{A}'_t, \Psi'_t)$ be the data for V', W obtained by applying the reflection functor S_k to $(\mathcal{A}, \Psi)$. Then $dS_k(\delta\mathcal{A}, \delta\Psi)$ is given

$$\delta\mathcal{A}' \overset{\text{def.}}{=} \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}'_t, \quad \delta\Psi' \overset{\text{def.}}{=} \left.\frac{d}{dt}\right|_{t=0} \Psi'_t.$$

We want to show

$$g\left((\delta\mathcal{A}, \delta\Psi), (\delta\mathcal{A}, \delta\Psi)\right) = g'\left((\delta\mathcal{A}', \delta\Psi'), (\delta\mathcal{A}', \delta\Psi')\right), \tag{8.3}$$

$$I(\delta\mathcal{A}, \delta\Psi) = I'(\delta\mathcal{A}', \delta\Psi'),\ J(\delta\mathcal{A}, \delta\Psi) = J'(\delta\mathcal{A}', \delta\Psi'),\ K(\delta\mathcal{A}, \delta\Psi) = K'(\delta\mathcal{A}', \delta\Psi'), \tag{8.4}$$

where (g, I, J, K) (resp. (g', I', J', K')) is the hyper-Kähler structure on $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ (resp. $\mathfrak{M}_{s_k\zeta}(s_k *\mathbf{v}, \mathbf{w})$).

Let $\mathcal{D}$ be as in (3.8). Let $\{e_\mu\}$ be an orthonormal basis of $V'_k = \operatorname{Ker} \mathcal{D}^\dagger$. The derivative $\delta(Ie_\mu)$ with respect to t satisfies $\mathcal{D}^\dagger \delta(Ie_\mu) = -(\delta\mathcal{D}^\dagger) Ie_\mu$. If we normalize e_μ by requiring $\delta(Ie_\mu) \perp \operatorname{Ker}\mathcal{D}^\dagger$, this equation implies

$$\delta(Ie_\mu) = -\mathcal{D}(1_{S^+} \otimes \Delta^{-1})(\delta D^\dagger) Ie_\mu. \tag{8.5}$$

Let us choose a complex structure, say I, and use the holomorphic description. We consider operators in (4.8) for $(\mathcal{A}, \Psi)$ and $(\mathcal{A}', \Psi')$ and denote them by σ, τ and σ', τ'. We have

$$\delta\tau\,\sigma + \tau\,\delta\sigma = 0, \quad \delta\tau\,\tau^\dagger - \sigma^\dagger\,\delta\sigma = 0. \tag{8.6}$$

Here the first equation comes from the differentiation of $\tau\sigma = -\zeta_{k,\mathbb{C}}$. The second equation is a consequence of $\tau\tau^\dagger - \sigma^\dagger\sigma = -\zeta_{\mathbb{R}}^{(k)}$ and the second condition in (8.2).

Let us consider (τ', e_μ) as an element of $\widehat{V}_k^*$. By (8.5), its derivative is given by

$$\begin{aligned}
\delta(\tau', e_\mu) &= \left(\delta \begin{bmatrix} \tau \\ -z_2 \end{bmatrix}, Ie_\mu\right) + \left(\begin{bmatrix} \tau \\ -z_2 \end{bmatrix}, \delta(Ie_\mu)\right) \\
&= \left(\begin{bmatrix} \delta\tau \\ 0 \end{bmatrix}, Ie_\mu\right) - \left(\begin{bmatrix} \tau \\ -z_2 \end{bmatrix}, \left\{\begin{bmatrix} z_1 & \overline{z_2} \\ \sigma & \tau^\dagger \end{bmatrix} \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & \Delta^{-1} \end{bmatrix} \begin{bmatrix} 0 & \delta\sigma^\dagger \\ 0 & \delta\tau \end{bmatrix}\right\} Ie_\mu\right) \\
&= \left(P\begin{bmatrix} \delta\tau \\ 0 \end{bmatrix}, e_\mu\right) - \left(P\begin{bmatrix} 0 \\ \delta\sigma \end{bmatrix} \Delta^{-1} \begin{bmatrix} \overline{z_1} & \sigma^\dagger \end{bmatrix} \begin{bmatrix} \tau \\ -z_2 \end{bmatrix}, e_\mu\right).
\end{aligned}$$

Similarly we have

$$\delta\left(\sigma'(e_\mu)\right) = \begin{bmatrix} \delta\sigma & 0 \end{bmatrix} Ie_\mu - \begin{bmatrix} \sigma & -z_1 \end{bmatrix} \begin{bmatrix} \overline{z_2} \\ \tau^\dagger \end{bmatrix} \Delta^{-1} \begin{bmatrix} 0 & \delta\tau \end{bmatrix} Ie_\mu.$$

Let us check that analogue of (8.6) for σ', τ':

$$\begin{aligned}
&\delta(\tau', e_\mu)\,\sigma'(e_\nu) + (\tau', e_\mu)\,\delta\left(\sigma'(e_\nu)\right) = 0, \\
&\delta(\tau', e_\mu)\,\tau'^\dagger(e_\nu) - (\sigma'^\dagger, e_\mu)\,\delta(\sigma'(e_\nu)) = 0.
\end{aligned} \tag{8.7}$$

The first equation is a consequence of $\tau'\sigma' = \zeta_{k,\mathbb{C}}$. The second equation is equivalent to

$$\begin{aligned}
P&\left(\begin{bmatrix} \delta\tau \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \delta\sigma \end{bmatrix} \Delta^{-1} \begin{bmatrix} \overline{z_1} & \sigma^\dagger \end{bmatrix} \begin{bmatrix} \tau \\ -z_2 \end{bmatrix}\right) \begin{bmatrix} \tau^\dagger & -\overline{z_2} \end{bmatrix} I \\
&\qquad - P\begin{bmatrix} \sigma^\dagger \\ -\overline{z_1} \end{bmatrix} \left(\begin{bmatrix} \delta\sigma & 0 \end{bmatrix} - \begin{bmatrix} \sigma & -z_1 \end{bmatrix} \begin{bmatrix} \overline{z_2} \\ \tau^\dagger \end{bmatrix} \Delta^{-1} \begin{bmatrix} 0 & \delta\tau \end{bmatrix}\right) I = 0.
\end{aligned}$$

By (4.10, 8.6), the left hand side is equal to

$$-P\begin{bmatrix}0 & 0\\ 0 & \overline{z_2}\,\delta\sigma\Delta^{-1}(-\overline{z_1}\tau+z_2\sigma^\dagger)\end{bmatrix} I + P\begin{bmatrix}0 & 0\\ 0 & \overline{z_1}(-\overline{z_2}\sigma+z_1\tau^\dagger)\Delta^{-1}\delta\tau\end{bmatrix} I.$$

If v' lies in $\operatorname{Ker}\mathcal{D}^\dagger = \operatorname{Ker}\left[\begin{smallmatrix}\overline{z_1} & \sigma^\dagger\\ z_2 & \tau\end{smallmatrix}\right]$, we have

$$\left(-\overline{z_1}\begin{bmatrix}0\\ \tau\end{bmatrix} + z_2\begin{bmatrix}0\\ \sigma^\dagger\end{bmatrix}\right) Iv' = \left(-\overline{z_1}\begin{bmatrix}0\\ z_2\end{bmatrix} + z_2\begin{bmatrix}0\\ \overline{z_1}\end{bmatrix}\right) Iv' = 0.$$

Hence the first term vanishes. Taking the hermitian adjoint, we have

$$\left(v', P\left(-\overline{z_2}\begin{bmatrix}0\\ \sigma\end{bmatrix} + z_1\begin{bmatrix}0\\ \tau^\dagger\end{bmatrix}\right)\right) = 0.$$

Hence the second term also vanishes. Thus we have checked (8.7).

Next we calculate the norm of $\delta(\tau', e_\mu)$ and $\delta(\sigma'(e_\mu))$. We have

$$\begin{aligned}
&\sum_\mu (\delta(\tau', e_\mu), \delta(\tau', e_\mu))\\
&= \operatorname{tr}\left(P\begin{bmatrix}\delta\tau\\ 0\end{bmatrix}\begin{bmatrix}\delta\tau^\dagger & 0\end{bmatrix} I\right) - \operatorname{tr}\left(P\begin{bmatrix}\delta\tau\\ 0\end{bmatrix}\begin{bmatrix}\tau^\dagger & -\overline{z_2}\end{bmatrix}\begin{bmatrix}z_1\\ \sigma\end{bmatrix}\Delta^{-1}\begin{bmatrix}0 & \delta\sigma^\dagger\end{bmatrix} I\right)\\
&\quad - \operatorname{tr}\left(P\begin{bmatrix}0\\ \delta\sigma\end{bmatrix}\Delta^{-1}\begin{bmatrix}\overline{z_1} & \sigma^\dagger\end{bmatrix}\begin{bmatrix}\tau\\ -z_2\end{bmatrix}\begin{bmatrix}\delta\tau^\dagger & 0\end{bmatrix} I\right)\\
&\qquad + \operatorname{tr}\left(P\begin{bmatrix}0\\ \delta\sigma\end{bmatrix}\Delta^{-1}\begin{bmatrix}\overline{z_1} & \sigma^\dagger\end{bmatrix}\begin{bmatrix}\tau\\ -z_2\end{bmatrix}\begin{bmatrix}\tau^\dagger & -\overline{z_2}\end{bmatrix}\begin{bmatrix}z_1\\ \sigma\end{bmatrix}\Delta^{-1}\begin{bmatrix}0 & \delta\sigma^\dagger\end{bmatrix} I\right).
\end{aligned}$$

By

$$IP = \begin{bmatrix}1-(|z_1|^2+|z_2|^2)\Delta^{-1} & -z_1\Delta^{-1}\sigma^\dagger - \overline{z_2}\Delta^{-1}\tau\\ -\overline{z_1}\sigma\Delta^{-1} - z_2\tau^\dagger\Delta^{-1} & 1-\sigma\Delta^{-1}\sigma^\dagger - \tau^\dagger\Delta^{-1}\tau\end{bmatrix}, \tag{8.8}$$

the first term is equal to

$$\operatorname{tr}\left((1-(|z_1|^2+|z_2|^2)\Delta^{-1})\delta\tau\,\delta\tau^\dagger\right).$$

The second term is

$$\begin{aligned}
&\operatorname{tr}\left((\overline{z_1}\sigma + z_2\tau^\dagger)\Delta^{-1}\delta\tau(z_1\tau^\dagger - \overline{z_2}\sigma)\Delta^{-1}\delta\sigma^\dagger\right)\\
&= |z_1|^2\operatorname{tr}\left(\sigma\Delta^{-1}\delta\tau\,\tau^\dagger\Delta^{-1}\delta\sigma^\dagger\right) + z_1z_2\operatorname{tr}\left(\tau^\dagger\Delta^{-1}\delta\tau\,\tau^\dagger\Delta^{-1}\delta\sigma^\dagger\right)\\
&\quad - \overline{z_1z_2}\operatorname{tr}\left(\sigma\Delta^{-1}\delta\tau\,\sigma\Delta^{-1}\delta\sigma^\dagger\right) - |z_2|^2\operatorname{tr}\left(\tau^\dagger\Delta^{-1}\delta\tau\,\sigma\Delta^{-1}\delta\sigma^\dagger\right).
\end{aligned}$$

The third term is

$$\begin{aligned}
&|z_1|^2\operatorname{tr}\left(\Delta^{-1}\sigma^\dagger\,\delta\sigma\Delta^{-1}\tau\,\delta\tau^\dagger\right) - z_1z_2\operatorname{tr}\left(\Delta^{-1}\sigma^\dagger\,\delta\sigma\Delta^{-1}\sigma^\dagger\,\delta\tau^\dagger\right)\\
&+\overline{z_1z_2}\operatorname{tr}\left(\Delta^{-1}\tau\,\delta\sigma\Delta^{-1}\tau\,\delta\tau^\dagger\right) - |z_2|^2\operatorname{tr}\left(\Delta^{-1}\tau\,\delta\sigma\Delta^{-1}\sigma^\dagger\,\delta\tau^\dagger\right).
\end{aligned}$$

The fourth term is

$$(|z_1|^2+|z_2|^2)\operatorname{tr}\left((1-\sigma\Delta^{-1}\sigma^\dagger - \tau^\dagger\Delta^{-1}\tau)(\delta\sigma\Delta^{-1}\delta\sigma^\dagger)\right).$$

Similarly we have

$$
\begin{aligned}
&\sum_\mu \left(\delta\left(\sigma'(e_\mu)\right), \delta\left(\sigma'(e_\mu)\right)\right) \\
&= \operatorname{tr}\left((1-(|z_1|^2+|z_2|^2)\Delta^{-1})\delta\sigma^\dagger\delta\sigma\right) \\
&\qquad - |z_1|^2 \operatorname{tr}\left(\sigma\Delta^{-1}\delta\sigma^\dagger\,\tau^\dagger\Delta^{-1}\delta\tau\right) - z_1 z_2 \operatorname{tr}\left(\tau^\dagger\Delta^{-1}\delta\sigma^\dagger\,\tau^\dagger\Delta^{-1}\delta\tau\right) \\
&\qquad\quad + \overline{z_1 z_2} \operatorname{tr}\left(\sigma\Delta^{-1}\delta\sigma^\dagger\,\sigma\Delta^{-1}\delta\tau\right) + |z_2|^2 \operatorname{tr}\left(\tau^\dagger\Delta^{-1}\delta\sigma^\dagger\,\sigma\Delta^{-1}\delta\tau\right) \\
&\qquad\qquad - |z_1|^2 \operatorname{tr}\left(\Delta^{-1}\sigma^\dagger\,\delta\tau^\dagger\Delta^{-1}\tau\,\delta\tau\right) + z_1 z_2 \operatorname{tr}\left(\Delta^{-1}\sigma^\dagger\,\delta\tau^\dagger\Delta^{-1}\sigma^\dagger\,\delta\sigma\right) \\
&\qquad\qquad\quad - \overline{z_1 z_2} \operatorname{tr}\left(\Delta^{-1}\tau\,\delta\tau^\dagger\Delta^{-1}\tau\,\delta\sigma\right) + |z_2|^2 \operatorname{tr}\left(\Delta^{-1}\tau\,\delta\tau^\dagger\Delta^{-1}\sigma^\dagger\delta\sigma\right) \\
&\qquad\qquad\qquad + (|z_1|^2+|z_2|^2) \operatorname{tr}\left((1-\sigma\Delta^{-1}\sigma^\dagger-\tau^\dagger\Delta^{-1}\tau)(\delta\tau^\dagger\Delta^{-1}\delta\tau)\right).
\end{aligned}
$$

Combining these terms and using (8.6), we get

$$\sum_\mu (\delta(\tau', e_\mu), \delta(\tau', e_\mu)) + \left(\delta\left(\sigma'(e_\mu)\right), \delta\left(\sigma'(e_\mu)\right)\right) = (\delta\tau, \delta\tau) + (\delta\sigma, \delta\sigma).$$

This formula implies (8.3) since other components of $(\mathcal{A}', \varPsi')$ are the same as those of $(\mathcal{A}, \varPsi)$.

Finally let us show (8.4). By the technique explained in §1(iv), it is enough to check (8.4) only for the complex structure I. By I, $\delta\sigma$ and $\delta\tau$ are multiplied by i. Then $\delta\sigma'$ and $\delta\tau'$ are also multiplied by i by the above formulas. These are the same as the action of I'. □

9. Identification with the action defined in [15]

In this section we identify our reflection functor with the action studied in [15, §9].

9(i). **Quick review of the ADHM description** [8]**.** (See also [15, §2].) Take and fix an affine Dynkin graph. Let I, H be as before, and let us choose an orientation Ω. Let $0 \in I$ be the vertex corresponding to the negative of the highest weight root of the corresponding simple Lie algebra. Let $\mathbf{n}$ be the vector in the kernel of the affine Cartan matrix whose 0-component is equal to 1. Such a vector is uniquely determined. Let $G_{\mathbf{n}}$ be the compact Lie group corresponding to $\mathbf{n}$ as in §2. Choose $\zeta \in \mathbb{R}^3 \otimes Z$, where $Z \subset \mathbb{R}^I \subset \mathfrak{g}_{\mathbf{n}}$ is the trace-free part of the center.

Let

$$X_\zeta \overset{\text{def.}}{=} \{\mathcal{A} \in \mathbf{M}(\mathbf{n}, 0) \mid \mu(\mathcal{A}) = -\zeta\} / (G_{\mathbf{n}}/\operatorname{U}(1)).$$

Note that the group $\operatorname{U}(1)$ of scalars in $G_{\mathbf{n}}$ acts trivially on $\mathbf{M}(\mathbf{n}, 0)$, so we can consider the action of the quotient group $G_{\mathbf{n}}/\operatorname{U}(1)$. Kronheimer [7] showed that if ζ is generic,

(a) X_ζ is a smooth 4-dimensional hyper-Kähler manifold,
(b) the metric is ALE (asymptotically locally Euclidean),
(c) X_ζ is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is the finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ associated to the affine Dynkin graph.

By the construction, $\mu^{-1}(-\zeta)$ can be considered as a principal $G_{\mathbf{n}}/\operatorname{U}(1)$-bundle over X_ζ. By defining the horizontal subspaces as the orthogonal complement of the tangent spaces to fibers, we have a natural connection on $\mu^{-1}(-\zeta)$. This is anti-self-dual [3] and has finite action [8, 2.2]. Let n_k be the k-component of $\mathbf{n}$. We identify $G_{\mathbf{n}}/\operatorname{U}(1)$ with $\prod_{k\neq 0}\operatorname{U}(n_k)$. For each $k \in I$, we have the associated vector bundle

$$\mathcal{R}_k \overset{\text{def.}}{=} \mu^{-1}(\zeta) \times_{G_{\mathbf{n}}/\operatorname{U}(1)} \mathbb{C}^{n_k},$$

where $G_{\mathbf{n}}/\operatorname{U}(1)$ acts on $\mathbb{C}^{n_k}$ through the projection $G_{\mathbf{n}}/\operatorname{U}(1) = \prod_{k\neq 0}\operatorname{U}(n_k) \to \operatorname{U}(n_k)$. When $k = 0$, we understand $\mathcal{R}_0$ as the trivial vector bundle. We call $\mathcal{R}_k$ a *tautological bundle*. It has an induced anti-self-dual connection and approximate an irreducible flat connection at infinity. This irreducible flat connection corresponds to an irreducible Γ-module R_k which corresponds to the vertex k by the McKay correspondence.

By the construction, there exists a bundle homomorphism

$$\xi_h \colon \mathcal{R}_{\mathrm{out}(h)} \to S^+ \otimes \mathcal{R}_{\mathrm{in}(h)}$$

for each $h \in \Omega$. This homomorphism is called a *tautological homomorphism*.

Suppose that collections of hermitian vector spaces V, W and a data $(\mathcal{A}, \Psi) \in \mathbf{M}(V, W)$ satisfying $\mu(\mathcal{A}, \Psi) = -\zeta$ is given. (ζ is the same as above.) Let us consider a vector bundle

$$\mathrm{E}(\mathcal{R}, V) \oplus \mathrm{L}(\mathcal{R}, W),$$

where (1) V and W are (collections of) trivial hermitian vector bundles, (2) $\mathrm{E}(\ ,\ )$, $\mathrm{L}(\ ,\ )$ are defined exactly as before by replacing vector spaces by vector bundles. Let $\iota_\Omega \colon \mathrm{E}_\Omega(\mathcal{R}, V) \to \mathrm{E}(\mathcal{R}, V)\oplus\mathrm{L}(\mathcal{R}, W)$, $\iota_{\overline{\Omega}} \colon \mathrm{E}_{\overline{\Omega}}(\mathcal{R}, V) \to \mathrm{E}(\mathcal{R}, V)\oplus\mathrm{L}(\mathcal{R}, W)$, $\iota_W \colon \mathrm{L}(\mathcal{R}, W) \to \mathrm{E}(\mathcal{R}, V)\oplus\mathrm{L}(\mathcal{R}, W)$ be the inclusions. We define an operator $\mathcal{D} \colon S^+ \otimes \mathrm{L}(\mathcal{R}, V) \to \mathrm{E}(\mathcal{R}, V) \oplus \mathrm{L}(\mathcal{R}, W)$ by

$$\mathcal{D}\eta \overset{\text{def.}}{=} \iota_\Omega\left(\omega\mathcal{A}\eta - \operatorname{tr}_{S^+}(\eta\otimes\omega\xi)\right) + \iota_{\overline{\Omega}}\left(\mathcal{A}^\dagger\eta - \operatorname{tr}_{S^+}(\eta\otimes\xi^\dagger)\right) + \iota_W\Psi^\dagger\eta.$$

This is an analogue of the operator in (3.8).

If $(\mathcal{A}, \Psi)$ is non-degenerate, then (1) $\mathcal{D}$ is injective [8, 9.2], and (2) the induced connection A on $\operatorname{Ker}\mathcal{D}^\dagger$ is anti-self-dual and has finite action [8, 4.1]. Moreover, vectors $\mathbf{v}$, $\mathbf{w}$ correspond to Chern classes of $\operatorname{Ker}\mathcal{D}^\dagger$ and the flat connection (representation of Γ) which A approximates at the end of X_ζ [8, §9].

The inverse map is constructed as follows (see [8, §5] for detail). Suppose an anti-self-dual connection A (satisfying an asymptotic condition) on a C^∞-vector bundle E with a hermitian metric is given. We define vector spaces V_k, W_k by

$$V_k \overset{\text{def.}}{=} L^2\text{-kernel of } D_A^- \colon \Gamma(S^-\otimes E\otimes\mathcal{R}_k) \to \Gamma(S^+\otimes E\otimes\mathcal{R}_k),$$
$$W_k \overset{\text{def.}}{=} \text{bounded harmonic sections of } E\otimes\mathcal{R}_k$$

Here $S^\pm$ is a positive/negative spinor bundle over X_ζ, and $D_A^\pm$ is the Dirac operator twisted by A and the connection on $\mathcal{R}_k$. Note that S^+ is a trivial bundle and the fiber is canonically identified with the vector space S^+ in §1. Those V, W have natural hermitian metrics. We define linear maps $\mathcal{A}_h \colon V_{\mathrm{out}(h)} \to S^+\otimes V_{\mathrm{in}(h)}$, $\Psi_k \colon W_k \to S^+\otimes V_k$ by

$$\mathcal{A}_h(v_{\mathrm{out}(h)}) = L^2\text{-projection of } (1_{S^-}\otimes 1_E\otimes\xi_h)v_{\mathrm{out}(h)},$$
$$(\omega\Psi_k)(s\otimes w_k) = D_A^+(s\otimes w_k).$$

Then $(\mathcal{A}, \Psi) \in \mathbf{M}(V, W)$ satisfies $\mu(\mathcal{A}, \Psi) = -\zeta$ and the non-degeneracy condition.

These maps are mutually converse and give a hyper-Kähler isometry between the framed moduli space $\mathfrak{M}_{X_\zeta}(E)$ of anti-self-dual connections on X_ζ and the hyper-Kähler quotient $\mathfrak{M}_\zeta^{\mathrm{reg}}(\mathbf{v}, \mathbf{w})$ [8, §8]. Here the framed moduli space $\mathfrak{M}_{X_\zeta}(E)$ is the quotient space of the space of anti-self-dual connections on a hermitian vector bundle E with finite action modulo the group of gauge transformations which converge to the identity at infinity. (See [14] for the definition of $\mathfrak{M}_{X_\zeta}(E)$.)

9(ii). **A reflection functor is the pull-back of connections.** Now suppose an element w of the finite Weyl group is given. Using the above ADHM description on $X_{w\zeta}$, we have a hyper-Kähler isometry between the framed moduli space $\mathfrak{M}_{X_{w\zeta}}(F)$ of anti-self-dual connections on a vector bundle F over $X_{w\zeta}$ and the hyper-Kähler quotient $\mathfrak{M}^{\mathrm{reg}}_{w\zeta}(\mathbf{v},\mathbf{w})$ [8, §8]. By a discussion in [7, §4], or by our reflection functor we have a hyper-Kähler isometry $f_w\colon X_\zeta \to X_{w\zeta}$. Hence the pull-back $(f_w^{-1})^*$ induces a hyper-Kähler isometry $\mathfrak{M}_{X_\zeta}(E) \to \mathfrak{M}_{X_{w\zeta}}((f_w^{-1})^*E)$.

Theorem 9.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathfrak{M}_{X_\zeta}(E) \arrow[r] \arrow[d, "(f_w^{-1})^*"'] & \mathfrak{M}^{\mathrm{reg}}_{\zeta}(\mathbf{v},\mathbf{w}) \arrow[d, "\mathcal{F}_{(\mathcal{A}^\bullet,\Psi^\bullet,\Phi^\bullet)}"] \\
\mathfrak{M}_{X_{w\zeta}}((f_w^{-1})^*E) \arrow[r] & \mathfrak{M}^{\mathrm{reg}}_{w\zeta}(w*\mathbf{v},\mathbf{w}),
\end{tikzcd}$$

where the horizontal arrows are the ADHM description, and $(\mathcal{A}^\bullet,\Psi^\bullet,\Phi^\bullet)$ *is the admissible collection corresponging to* w *given in §7.*

Let $\bar{\mathcal{R}}_k \overset{\text{def.}}{=} (f_w)^*\mathcal{R}_k$ where $\mathcal{R}_k$ is the tautological bundle on $X_{w\zeta}$. (Since we do not use it later, it is not confused with $\mathcal{R}_k$ on X_ζ.) We also pull-back the anti-self-dual connection on $\mathcal{R}_k$. Since $\bar{\mathcal{R}}_k$ has the unique anti-self-dual connection, we suppress the notation A. Pulling back a tautological homomorphism, we get a vector bundle homomorphism

$$\bar{\xi}_h\colon \bar{\mathcal{R}}_{\mathrm{out}(h)} \to S^+ \otimes \bar{\mathcal{R}}_{\mathrm{in}(h)}.$$

If we replace $\mathcal{R}_k$, ξ_h by $\bar{\mathcal{R}}_k$, $\bar{\xi}_h$ in the ADHM description, we get the composite of the bottom arrow and the left arrow in Theorem 9.1.

We apply the ADHM description to $\bar{\mathcal{R}}_k$ to get vector spaces and homomorphisms. For a later purpose, we slightly change the roles and consider the followings:

$$V_l^k \overset{\text{def.}}{=} \text{the dual space of the } L^2\text{-kernel of } D_A^-\colon \Gamma(S^-\otimes\bar{\mathcal{R}}_k\otimes\mathcal{R}_l^*) \to \Gamma(S^+\otimes\bar{\mathcal{R}}_k\otimes\mathcal{R}_l^*),$$
$$W_l^k \overset{\text{def.}}{=} \text{the dual space of the space of bounded harmonic sections of } \bar{\mathcal{R}}_k\otimes\mathcal{R}_l^*.$$

One can check $\dim V^k = w*_{\mathbf{e}^k}0$, $\dim W^k = \mathbf{e}^k$. In fact, $(W_l^k)^*$ is isomorphic to the space of the convariant constant sections at infinity by [8, 5.1]. In this case, it is equal to $\mathrm{Hom}_\Gamma(R_k, R_l)$. Thus $\dim W_l^k = 0$ if $k \neq l$ and $\dim W_l^k = 1$ if $k = l$. The former equality $\dim V^k = w*_{\mathbf{e}^k}0$ follows from the computation of the first Chern classes: Let us identify $H^2(X_\zeta,\mathbb{Z})$ with $\mathbb{Z}^I$ by $c_1(\mathcal{R}_k) \mapsto \mathbf{e}^k$. By the definition of the reflection functor for simple reflections and the induction on the length of w, we have $c_1(\bar{\mathcal{R}}_k) = wc_1(\mathcal{R}_k)$. Now $\dim V^k = w*_{\mathbf{e}^k}0$ follows from the formula of the first Chern class of a bundle constructed by the ADHM description (see [8, p.301 bottom]).

The tautological homomorphism $\xi_h\colon \mathcal{R}_{\mathrm{out}(h)} \to S^+\otimes\mathcal{R}_{\mathrm{in}(h)}$ induces a homomorphism

$${}^t\xi_h\colon \mathcal{R}^*_{\mathrm{in}(h)} \to S^+\otimes\mathcal{R}^*_{\mathrm{out}(h)}.$$

Using this homomorphism instead of ξ_h in the ADHM description, we get a homomorphism

$$(V^k_{\mathrm{in}(h)})^* \to S^+\otimes(V^k_{\mathrm{out}(h)})^*$$

as above. Then we take its transpose to get a homomorphism

$$\mathcal{A}_h^k\colon V^k_{\mathrm{out}(h)} \to S^+\otimes V^k_{\mathrm{in}(h)}.$$

We also have

$$(W_k^k)^* \to S^+\otimes(V_k^k)^*$$

as in the ADHM description. Taking the transpose and then taking $(\omega\cdot)^\dagger$, we get

$$\varPsi_k^k \colon W_k^k \to S^+ \otimes V_k^k.$$

Then $(\mathcal{A}^k, \varPsi^k)$ satisfies $\mu(\mathcal{A}^k, \varPsi^k) = -\zeta$.

We have a tautological bundle homomorphism $\bar{\xi}_h : \bar{\mathcal{R}}_{\mathrm{out}(h)} \to \bar{\mathcal{R}}_{\mathrm{in}(h)}$. Then we define $\Phi_k^{\overline{h}} \colon V_k^{\mathrm{in}(h)} \to S^+ \otimes V_k^{\mathrm{out}(h)}$ by

$${}^t\left(\Phi_k^{\overline{h}}\right)(v^*_{\mathrm{out}(h)}) \overset{\text{def.}}{=} L^2\text{-projection of } (1_{S^-} \otimes \bar{\xi}_h \otimes 1_{\mathcal{R}_k^*}) v^*_{\mathrm{out}(h)}$$

for $v^*_{\mathrm{out}(h)} \in (V_k^{\mathrm{out}(h)})^*$.

Proposition 9.2. *The collection* $\{(\mathcal{A}^\bullet, \varPsi^\bullet, \Phi^\bullet)\}$ *satisfies the compatibility condition* (3.1).

The proof is a straight-forward modification of that of [8, 5.6] and omitted.

As we can see by the reflection functor, $\mathfrak{M}(w *_{\mathbf{e}^k} 0, \mathbf{e}^k)$ consists of a single point. Combining this observation with Lemma 3.3, we conclude that $\{(\mathcal{A}^\bullet, \varPsi^\bullet, \Phi^\bullet)\}$ must be the same as the admissible collection constructed in §7.

Now the commutativity of the square follows from a straightforward modification of the argument in [8, §7] where the special case $w = 1$ was studied. We only write the relevant double complex ([8, §7b)]) here. The detail is left to the reader:

$$\begin{array}{ccccc}
\Omega^{0,2}_{-4}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k) & \xrightarrow{\alpha\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{-3}\left((\mathrm{E}(\mathcal{R},V)\oplus\mathrm{L}(\mathcal{R},W))\otimes\bar{\mathcal{R}}_k\right) & \xrightarrow{\beta\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{-2}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k) \\
\bar\partial\uparrow & & \uparrow\bar\partial & & \uparrow\bar\partial \\
\Omega^{0,2}_{-3}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k) & \xrightarrow{\alpha\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{-2}\left((\mathrm{E}(\mathcal{R},V)\oplus\mathrm{L}(\mathcal{R},W))\otimes\bar{\mathcal{R}}_k\right) & \xrightarrow{\beta\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{-1}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k) \\
\bar\partial\uparrow & & \uparrow\bar\partial & & \uparrow\bar\partial \\
\Omega^{0,2}_{-2}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k) & \xrightarrow{\alpha\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{-1}\left((\mathrm{E}(\mathcal{R},V)\oplus\mathrm{L}(\mathcal{R},W))\otimes\bar{\mathcal{R}}_k\right) & \xrightarrow{\beta\otimes 1_{\bar{\mathcal{R}}_k}} & \Omega^{0,2}_{0}(\mathrm{L}(\mathcal{R},V)\otimes\bar{\mathcal{R}}_k),
\end{array}$$

where $\mathcal{D} = \left[\begin{smallmatrix}\alpha & \beta^\dagger\end{smallmatrix}\right]$.

10. Identification with Lusztig's action

We shall compare our Weyl group action with Lusztig's one [11] in this section. We assume that the graph is of type ADE.

We recall results of Lusztig [10, 11].

Definition 10.1. (1) A *path* is either (a) a constant path e_k consisting a single vertex k, or (b) a sequence $(h_1, h_2, \dots, h_r)$ $(r \ge 1)$ such that $\mathrm{out}(h_1) = \mathrm{in}(h_2), \dots, \mathrm{out}(h_{r-1}) = \mathrm{in}(h_r)$.

(2) For a path f, we define $\mathrm{in}(f)$ and $\mathrm{out}(f)$ by

$$\mathrm{in}(f) \overset{\text{def.}}{=} \begin{cases} k & \text{if } f \text{ is a constant path } e_k, \\ \mathrm{in}(h_1) & \text{if } f = (h_1, \dots, h_r). \end{cases}$$

$$\mathrm{out}(f) \overset{\text{def.}}{=} \begin{cases} k & \text{if } f \text{ is a constant path } e_k, \\ \mathrm{out}(h_r) & \text{if } f = (h_1, \dots, h_r). \end{cases}$$

(3) Let $\mathcal{P}_l^k$ be the set of paths f with $\mathrm{in}(f) = k$, $\mathrm{out}(f) = l$. Let $\mathcal{F}_l^k$ be the $\mathbb{C}$-vector space with basis $\mathcal{P}_l^k$. The direct sum $\mathcal{F} \overset{\text{def.}}{=} \bigoplus_{k,l} \mathcal{F}_l^k$ forms an algebra in which the product is the composition of paths if they can be composed and is zero otherwise.

Let

$$\theta_{k,\zeta_{\mathbb{C}}} \overset{\text{def.}}{=} \sum_{\substack{h\in H\\ \mathrm{in}(h)=k}} \varepsilon(h)h\overline{h} + \zeta_{k,\mathbb{C}} e_k \in \mathcal{F}_k^k.$$

Let $Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}$ be the subset of $\prod_{k,l} \mathrm{Hom}(\mathcal{F}_l^k, \mathrm{Hom}(W_l, W_k))$ consisting elements π such that

$$\pi(f)\pi(g) + \pi(f\theta_{k,\zeta_{\mathbb{C}}} g) = 0$$

for all paths f, g with $\mathrm{out}(f) = \mathrm{in}(g) = k$.

If a point $(B, i, j) \in \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \subset \mathbf{M}(V, W)$ is given, we assign a point $\pi \in \prod_{k,l} \mathrm{Hom}(\mathcal{F}_l^k, \mathrm{Hom}(W_l, W_k))$ by

$$\pi(f) \overset{\text{def.}}{=} \begin{cases} j_k i_k & \text{if } f \text{ is a constant path } e_k,\\ j_{\mathrm{in}(h_1)} B_{h_1} \cdots B_{h_r} i_{\mathrm{out}(h_r)} & \text{if } f \text{ is a path } (h_1, h_2, \dots, h_r)\ (r \ge 1). \end{cases}$$

Then π is contained in $Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}$ by the equation $\mu_{\mathbb{C}}(B, i, j) = -\zeta_{\mathbb{C}}$. If we choose two points (B, i, j), $(B', i', j') \in \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$ such that $\overline{G^{\mathbb{C}}(B, i, j)} \cap \overline{G^{\mathbb{C}}(B', i', j')} \neq \emptyset$, then the corresponding point is the same. Thus we have a map

$$\vartheta \colon \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w}) \to Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}.$$

Remark that any point is $\zeta_{\mathbb{R}}$-semistable if $\zeta_{\mathbb{R}} = 0$, and $\mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w})$ is the affine algebro-geometric quotient, i.e., its coordinate ring is the invariant part of the coordinate ring of $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$.

Theorem 10.2 ([10, 1.3, 5.3], [11, 4.7])**.** *There exists a natural structure of an affine algebraic variety on $Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}$ such that ϑ is a finite, injective morphism. In particular, ϑ is a homeomorphism onto its image.*

For each vertex $k \in I$ let us define a map $S_k \colon \prod_{k,l} \mathrm{Hom}(\mathcal{F}_l^k, \mathrm{Hom}(W_l, W_k)) \to \prod_{k,l} \mathrm{Hom}(\mathcal{F}_l^k, \mathrm{Hom}(W_l, W_k))$ by

$$(S_k\pi)(f) \overset{\text{def.}}{=} \begin{cases} \pi(e_k) + \zeta_{k,\mathbb{C}}\, \mathrm{id}_{W_k} & \text{if } f \text{ is a constant path } e_k,\\ \sum_{J : J\subset J_0} \prod_{t\in J} \varepsilon(h_t)\zeta_{k,\mathbb{C}}\, (h_1, h_2, \dots, h_r)^{\wedge J} & \text{if } f \text{ is a path } (h_1, h_2, \dots, h_r)\ (r \ge 1). \end{cases}$$

Here $J_0 = \{t \in [2, r] \mid \mathrm{in}(h_t) = k, \mathrm{in}(h_{t-1}) = \mathrm{out}(h_t)\}$, and $(h_1, h_2, \dots, h_r)^{\wedge J}$ means the path obtained by omitting h_{t-1}, h_t for all $t \in J$. Then

(1) S_k's satisfy the defining relation of the Weyl group ([11, §1]),

(2) S_k maps $Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}$ to $Z_{\mathbf{w}}^{s_k\zeta_{\mathbb{C}}}$ ([11, 2.2]).

Now we compare our action with Lusztig's one.

Theorem 10.3. *The following diagram is commutative:*

$$\begin{CD} \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w}) @>{\vartheta}>> Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}} \\ @V{S_k}VV @VV{S_k}V \\ \mathfrak{M}_{(0,s_k\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w}) @>>{\vartheta}> Z_{\mathbf{w}}^{s_k\zeta_{\mathbb{C}}}, \end{CD}$$

where we set $S_k = \mathrm{id}$ in the left arrow when $\zeta_{k,\mathbb{C}} = 0$. (cf. Remark 4.3).

Proof. If $\zeta_{k,\mathbb{C}} = 0$, then both S_k's are identity. So we may assume $\zeta_{k,\mathbb{C}} \neq 0$.

Suppose that a data $[B, i, j] \in \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w})$ is given. Let $[B', i', j'] = S_k \cdot [B, i, j]$. The following follows from the holomorphic description of the reflection functor.

$$B'_{\overline{h}} B'_{h_1} = B_{\overline{h}} B_{h_1} + \delta_{hh_1} \varepsilon(h) \zeta_{k,\mathbb{C}} \, \mathrm{id}_{V_{\mathrm{out}(h)}} \quad \text{for } h, h_1 \in H \text{ such that } \mathrm{in}(h) = \mathrm{in}(h_1) = k,$$
$$j'_k B'_h = j_k B_h,$$
$$B'_{\overline{h}} i'_k = B_{\overline{h}} i_k,$$
$$j'_k i'_k = j_k i_k + \zeta_{k,\mathbb{C}} \, \mathrm{id}_{W_k} .$$

By induction we get

$$j'_{\mathrm{in}(h_1)} B'_{h_1} B'_{h_2} \cdots B'_{h_r} i'_{\mathrm{out}(h_r)} = \sum_{J : J \subset J_0} \prod_{t \in J} \varepsilon(h_t) \zeta_{k,\mathbb{C}} \left(j_{\mathrm{in}(h_1)} B_{h_1} \cdots B_{h_r} i_{\mathrm{out}(h_r)} \right)^{\wedge J}$$

if $r \geq 1$. Here J_0 is as above and $\left(j_{k_1} B_{h_1} \cdots B_{h_{s-1}} i_{k_s} \right)^{\wedge J}$ means that multiplication after omitting $B_{h_{t-1}} B_{h_t}$ for all $t \in J$. Now the commutativity is clear. □

Our S_k and Lusztig's one are almost the same, but have the following difference:

(1) it is clear that Lusztig's S_k is a morphism of an affine algebraic variety by definition, while our S_k is a homeomorphism which induces a hyper-Kähler isometry between open subsets $\mathfrak{M}^{\mathrm{reg}}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w})$ and $\mathfrak{M}^{\mathrm{reg}}_{(0,s_k\zeta_{\mathbb{C}})}(s_k * \mathbf{v}, \mathbf{w})$.

(2) our S_k is defined for all $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ while Lusztig's S_k is defined only for $\zeta_{\mathbb{R}} = 0$.

Note that if the singularities of $\mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w})$ are *normal* (a conjectural property of the quiver variety), then our S_k extends to the whole $\mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w})$.

Using S_k and the observation that a natural projective morphism (for fixed $\mathbf{w}$, $\zeta_{\mathbb{R}}$)

$$\pi \colon \bigsqcup_{\zeta_{\mathbb{C}} \in \mathbb{C}^I} \bigsqcup_{\mathbf{v}} \mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w}) \to \bigsqcup_{\zeta_{\mathbb{C}} \in \mathbb{C}^I} Z_{\mathbf{w}}^{\zeta_{\mathbb{C}}}$$

is small ([11, 6.5]), Lusztig constructed a Weyl group representation on $\bigoplus_{\mathbf{v}} H^*(\mathfrak{M}_{(\zeta_{\mathbb{R}},\zeta_{\mathbb{C}})}(\mathbf{v}, \mathbf{w}), \mathbb{R})$. This is an analogue of his construction [9] of Springer representation. Hotta [5] proved that Lusztig's Weyl group representations coincide with Slodowy's ones [19, IV]. The same proof works for our case thanks to Theorem 10.3. Thus Lusztig's Weyl group representations in [11] coincide with ones given in Theorem 4.5 and hence also with ones in [15, §9].

11. Lusztig's opposition

The purpose of this section is to relate Lusztig's *new symmetries* of quiver varieties [12] with our reflection functors. His symmetries were analogue of oppositions for Lie algebras and a necessary ingredient for his (conjectural) definition of canonical bases of finite dimensional modules of quantum affine algebras.

In this section we assume that the graph is of type ADE. Let w_0 be the longest element in the Weyl group. It induces an involution on I, which is denoted by $k \mapsto k^*$, by $w_0 \alpha_k = -\alpha_{k^*}$. Here α_k is the simple root corresponding to the vertex k. Let $h \mapsto h^*$ be an involution on H defined by

$$\mathrm{out}(h^*) = \mathrm{out}(h)^*, \quad \mathrm{in}(h^*) = \mathrm{in}(h)^*.$$

Let $(\mathcal{A}^k, \Psi^k, \Phi^{\overline{h}})$ be the holomorphic description of the admissible collection corresponding to the longest element w_0 by §7. Let

$$\mathcal{F}\colon \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{w_0\zeta}(w_0 * \mathbf{v},\mathbf{w}) \tag{11.1}$$

be the corresponding reflection functor.

The involution $k \mapsto k^*$ on I induces an isomorphism of hyper-Kähler manifolds

$$\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\zeta^*}(\mathbf{v}^*,\mathbf{w}^*), \tag{11.2}$$

where the k-component of ζ^* (resp. $\mathbf{v}^*$, $\mathbf{w}^*$) is the k^*-component of ζ (resp. $\mathbf{v}$, $\mathbf{w}$).

We define an isomorphism of hyper-Kähler manifolds

$$\mathfrak{M}_{-\zeta}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \tag{11.3}$$

by sending $(\mathcal{A},\Psi)$ to $((\omega\mathcal{A})^\dagger, (\omega\Psi)^\dagger)$. If we identify V_k, W_k with its dual space V_k^*, W_k^* via hermitian inner products, this isomorphism sends the holomorphic description (B,i,j) of $(\mathcal{A},\Psi)$ to

$$-\varepsilon(h)\,{}^tB_{\overline{h}}\colon V^*_{\mathrm{out}(h)} \to V^*_{\mathrm{in}(h)}, \quad (-1)\,{}^tj_k\colon W_k^* \to V_k^*, \quad {}^ti_k\colon V_k^* \to W_k^*.$$

Composing (11.1) with (11.2) (replacing ζ by $w_0\zeta$ and $\mathbf{v}$ by $w_0 * \mathbf{v}$) and (11.3) (replacing $\mathbf{v}$ by $w_0 * \mathbf{v}^*$ and $\mathbf{w}$ by $\mathbf{w}^*$), we get an isomorphism of hyper-Kähler manifolds

$$\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*), \tag{11.4}$$

where we have used $w_0\zeta^* = -\zeta$. Clearly this is an involution, i.e., the composition $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*) \to \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ is the identity. This (11.4) is our version of an *opposition*.

Now we turn to Lusztig's opposition. We choose the following orientation. Since the graph is of type ADE, we can assign $\pm$ to each vertex $k \in I$ so that there is no edges starting from a $(+)$-vertex (resp. $(-)$-vertex) and ending at a $(+)$-vertex (resp. $(-)$-vertex). Such a choice is unique up to overall exchange $+ \leftrightarrow -$. Then we choose the orientation so that $\varepsilon(h)$ is equal to the sign of the vertex $\mathrm{out}(h)$.

We choose and fix a parameter $\zeta_{\mathbb{R}}$ so that $i\zeta_{k,\mathbb{R}} < 0$ for all k. We then set $\zeta = (0,\zeta_{\mathbb{R}})$. By the holomorphic description of quiver varieties, we have natural morphisms,

$$\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w}), \qquad \mathfrak{M}_{w_0\zeta}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w}).$$

Let us denote by $\mathfrak{L}_\zeta(\mathbf{v},\mathbf{w})$ and $\mathfrak{L}_{w_0\zeta}(\mathbf{v},\mathbf{w})$ the inverse images of the origin $0 \in \mathfrak{M}_0(\mathbf{v},\mathbf{w})$. These are lagrangian subvarieties of $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})$ and $\mathfrak{M}_{w_0\zeta}(\mathbf{v},\mathbf{w})$ ([15, 5.8]).

Let

$$\mathfrak{L}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{L}_\zeta(w_0 * \mathbf{v},\mathbf{w}) \tag{11.5}$$

be Lusztig's new symmetry (opposition). (See below for the definition.)

Note that (11.2) (after replacing $\mathbf{v}$ by $w_0 * \mathbf{v}$) sends $\mathfrak{L}_\zeta(w_0 * \mathbf{v},\mathbf{w})$ to $\mathfrak{L}_{\zeta^*}(w_0 * \mathbf{v}^*,\mathbf{w}^*)$. Furthermore, we have an isomorphism of complex varieties $\mathfrak{L}_{\zeta^*}(w_0 * \mathbf{v}^*,\mathbf{w}^*) \to \mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*)$ since $\zeta_{\mathbb{R}}$-stability and $\zeta_{\mathbb{R}}^*$-stability are equivalent. (**NB**: The map $\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_{\zeta^*}(\mathbf{v},\mathbf{w})$ is an isomorphism of complex manifolds, but *not* of hyper-Kähler manifolds.)

We further compose a map $\mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*) \to \mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*)$ given by

$$\mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*) \ni (B_h, i_k, 0) \mapsto (\varepsilon(h)B_h, -i_k, 0) \in \mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*)$$

.

Composing (11.5) with these maps, we have an isomorphism of algebraic varieties

$$\mathfrak{L}_\zeta(\mathbf{v},\mathbf{w}) \to \mathfrak{L}_\zeta(w_0 * \mathbf{v}^*,\mathbf{w}^*). \tag{11.6}$$

Theorem 11.7. (1) *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{L}_\zeta(\mathbf{v},\mathbf{w}) & \longrightarrow & \mathfrak{M}_\zeta(\mathbf{v},\mathbf{w}) \\ {\scriptstyle (11.6)}\big\downarrow & & \big\downarrow{\scriptstyle (11.4)} \\ \mathfrak{L}_\zeta(w_0 * \mathbf{v}^*, \mathbf{w}^*) & \longrightarrow & \mathfrak{M}_\zeta(w_0 * \mathbf{v}^*, \mathbf{w}^*), \end{array}$$

where the horizontal arrows are natural inclusions.

(2) *The isomorphism* (11.4) *is identified with the map sending an instanton A defined on a vector bundle E to its dual A^* on E^*.*

Remark 11.8. In (11.4) W is changed to a collection of vector spaces whose k-component is $W^*_{k^*}$. On the other hand, it is changed to a collection of vector spaces whose k-component is $W^*_{k^*} \otimes \mathcal{L}^*_{k^*}$ for some 1-dimensional vector space $\mathcal{L}_{k^*}$ in (11.6) as we shall explain during the proof.

11(i). **Definition of Lusztig's new symmetry (opposition).** We first recall Lusztig's construction [12]. Let $\mathbf{c}$ be the Coxeter number of the Weyl group, and set $\mathbf{c}' = \mathbf{c} - 2 \in \mathbb{Z}_{\geq 0}$.

Let $\mathbf{P}$ be the preprojective algebra, i.e., the quotient algebra of the path algebra $\mathcal{F}$ (see Definition 10.1) by the two-sided ideal generated by elements

$$\sum_{\substack{h \in H \\ \mathrm{in}(h)=k}} h\overline{h}$$

(one for each $k \in I$).

There is unique algebra antiautomorphism $\iota\colon \mathbf{P} \to \mathbf{P}$ such that $\iota(e_k) = e_{k^*}$ and $\iota(h) = \overline{h}^*$. We write $\iota(x) = \overline{x}^*$ for $x \in \mathbf{P}$. If M is a $\mathbf{P}$-module, then the dual space M^* is naturally a $\mathbf{P}$-module, where $x \in \mathbf{P}$ acts on M^* by the transpose of the multiplication of $\overline{x}^*\colon M \to M$.

For each $n \in \mathbb{Z}_{\geq 0}$ let $\mathbf{P}^n$ be the subspace of $\mathbf{P}$ spanned by elements of length n. We regard $\mathbf{P}e_k$ as a collection of vector spaces with l-component $e_l\mathbf{P}e_k$. Similarly we consider $\mathbf{P}^n e_k$ as a collection of vector spaces. Then we have

$$\dim \mathbf{P}^0 e_k = \mathbf{e}^k, \tag{11.9.1}$$

$$\dim \mathbf{P}^{\mathbf{c}'} e_k = \mathbf{e}^{k^*}, \tag{11.9.2}$$

$$\dim \mathbf{P}^n e_k = 0 \quad \text{for } n > \mathbf{c}', \tag{11.9.3}$$

$$\dim \mathbf{P} e_k = w_0 *_{\mathbf{e}^k} 0. \tag{11.9.4}$$

(See [10, §4]. (11.9.4) is not stated, but can be deduced from, for example, [12, 1.11].) By (b) $e_{k^*}\mathbf{P}^{\mathbf{c}'}e_k$ is 1-dimensional. We set $\mathcal{L}_k \overset{\text{def.}}{=} e_{k^*}\mathbf{P}^{\mathbf{c}'}e_k$. Since ι maps $\mathcal{L}_k$ to $\mathcal{L}_k$, there exists a unique $\kappa_k \in \{-1, 1\}$ such that $\overline{x}^* = \kappa_k x$ for all $x \in \mathcal{L}_k$.

There exists a perfect bilinear pairing $(\ ,\ )\colon \mathbf{P}e_k \otimes \mathbf{P}e_k \to \mathcal{L}_k$ such that

$$(y, y') = 0 \quad \text{for } y \in \mathbf{P}^n e_k,\ y' \in \mathbf{P}^{n'} e_k \text{ with } n + n' \neq \mathbf{c}', \tag{11.10.1}$$

$$(y, y') = \overline{y}^* y' \quad \text{for } y \in \mathbf{P}^n e_k, y' \in \mathbf{P}^{\mathbf{c}'-n} e_k, \tag{11.10.2}$$

$$(\overline{x}^* y, y') = (y, xy') \quad \text{for } y, y' \in \mathbf{P}e_k,\ x \in \mathbf{P}, \tag{11.10.3}$$

$$(y, y') = \kappa_k (y', y) \quad \text{for } y, y' \in \mathbf{P}e_k. \tag{11.10.4}$$

(See [12, 1.14].) In particular, we have an isomorphism $(\mathbf{P}e_k)^* \cong \mathbf{P}e_k \otimes \mathcal{L}^*_k$. From (11.10.3) it is an isomorphism of $\mathbf{P}$-modules.

We have the canonical isomorphism $\mathbb{C}h \cong e_{\text{in}(h)}\mathbf{P}^1 e_{\text{out}(h)}$ of vector spaces. This induces an isomorphism

(11.11)
$$\mathcal{L}_{\text{out}(h)} \xrightarrow{\cong} e_{\text{in}(h)}\mathbf{P}^1 e_{\text{out}(h)} \otimes e_{\text{in}(h)^*}\mathbf{P}^{\mathbf{c}'-1} e_{\text{out}(h)} \xrightarrow{\cong} e_{\text{out}(h)^*}\mathbf{P}^{\mathbf{c}'-1} e_{\text{in}(h)} \otimes e_{\text{out}(h)}\mathbf{P}^1 e_{\text{in}(h)} \xrightarrow{\cong} \mathcal{L}_{\text{in}(h)},$$

where the first and the last isomorphisms are given by $(\ ,\ )$, and the middle isomorphism is given by

$$h \otimes y \mapsto \overline{y}^* \otimes \overline{h}.$$

Let us denote the composition by θ_h. By definition and (11.10) we have

$$(11.12)\qquad \begin{gathered} \theta_{\overline{h}} = \kappa_{\text{out}(h)}\kappa_{\text{in}(h)}\theta_h^{-1}, \\ \theta_h(yh, y') = (y, y'\overline{h}) \quad \text{for } y \in \mathbf{P}e_{\text{in}(h)},\ y' \in \mathbf{P}e_{\text{out}(h)}. \end{gathered}$$

Let $W = (W_k)_{k\in I}$ as before. We consider a projective $\mathbf{P}$-module

$$W^{\heartsuit} \overset{\text{def.}}{=} \bigoplus_{k\in I} \mathbf{P}e_k \otimes W_k,$$

where $\mathbf{P}$ acts trivially on W_k. Let $\text{Gr}_{\mathbf{P}}(W^{\heartsuit})$ be the projective variety of all $\mathbf{P}$-submodules of $W^{\heartsuit}$. If S is a $\mathbf{P}$-submodules of $W^{\heartsuit}$, then we define a collection of vector spaces $V = (V_k)_{k\in I}$ and data $(B, i, j) \in \mathbf{M}(V, W)$ by

(11.13)
$V_k \overset{\text{def.}}{=} e_k\left(W^{\heartsuit}/S\right)$,
B_h is the multiplication of $h\colon e_{\text{out}(h)}W^{\heartsuit}/S \to e_{\text{in}(h)}W^{\heartsuit}/S$,
i_k is the composition of the inclusion $W_k \subset e_kW^{\heartsuit}$ and the projection $e_kW^{\heartsuit} \to e_k\left(W^{\heartsuit}/S\right)$,
$j_k = 0$.

Then, by definition, we have $\mu_{\mathbb{C}}(B, i, j) = 0$. Also, there exists no proper $T = (T_k) \subset V$ which is B-invariant and contains $\operatorname{Im} i$. Hence (B, i, j) is $\zeta_{\mathbb{R}}$-stable. Moreover, it is contained in $\mathfrak{L}_\zeta(\mathbf{v}, \mathbf{w})$ ($\mathbf{v} = \dim W^{\heartsuit}/S$) by [15, 5.9, 5.11(3)]. This defines a map from $\text{Gr}_{\mathbf{P}}(W^{\heartsuit})$ to $\bigsqcup_{\mathbf{v}} \mathfrak{L}_\zeta(\mathbf{v}, \mathbf{w})$. This map is an isomorphism ([10, 2.26]).

If $S \in \text{Gr}_{\mathbf{P}}(W^{\heartsuit})$, then the annihilator $S^\perp$ of S in $(W^{\heartsuit})^*$ belongs to $\text{Gr}_{\mathbf{P}}((W^{\heartsuit})^*)$. As above, we can identify it with $\mathfrak{L}_\zeta(\mathbf{s}, \mathbf{w})$ ($\mathbf{s} = \dim S$). Note also that

$$\mathbf{w} - \mathbf{C}\mathbf{s} = \mathbf{w} - \mathbf{C}(\dim W^{\heartsuit} + w_0\mathbf{v}) = w_0(\mathbf{w} - \mathbf{C}\mathbf{v}).$$

Here we used (11.9.4).

Combining two isomorphisms, we have an isomorphism $\mathfrak{L}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{L}_\zeta(w_0 * \mathbf{v}, \mathbf{w})$. This is the *new symmetry* (11.5) defined in [12].

We further map $S^\perp$ to get $[\bar{B}, \bar{\imath}, \bar{\jmath}] \in \mathfrak{L}_\zeta(\mathbf{s}^*, \mathbf{w}^*)$ as in (11.6). Then it is explicitly given as follows:

(11.14)

$\bar{V}_k = (S_k)^*, \quad \bar{W}_k = \mathcal{L}_{k^*}^* \otimes W_{k^*}^*,$

$\bar{B}_h$ is the transpose of the multiplication of $\varepsilon(h)\overline{h}\colon (\bar{V}_{\mathrm{in}(h)})^* = S_{\mathrm{in}(h)} \to (\bar{V}_{\mathrm{out}(h)})^* = S_{\mathrm{out}(h)}$,

$\bar{\imath}_k$ is (-1) times the transpose of the composition of

$$(\bar{V}_k)^* = S_k = e_k S \xrightarrow{\text{inclusion}} e_k W^\heartsuit \xrightarrow{\text{projection}} e_k(\mathbf{P}^{\mathbf{c}'} e_{k^*} \otimes W_{k^*}) = (\bar{W}_k)^*,$$

$\bar{\jmath}_k = 0.$

Here we use the isomorphism via the pairing $(\ ,\ )$:

$$(W^\heartsuit)^* \cong \bigoplus_{k\in I} \mathbf{P}e_k \otimes \mathcal{L}_k^* \otimes W_k^*.$$

Then the map $(B, i, j) \mapsto (\bar{B}, \bar{\imath}, \bar{\jmath})$ is nothing but (11.6).

11(ii). **Proof of Theorem 11.7(1).** We first give a holomorphic description of $(\mathcal{A}^k, \Psi^k, \Phi^{\overline{h}})$ explicitly. Let

$$V^k \overset{\text{def.}}{=} \mathbf{P}e_k, \quad W_l^k \overset{\text{def.}}{=} \begin{cases} \mathbb{C} & \text{if } l = k, \\ 0 & \text{otherwise.} \end{cases}$$

As above we associate $(B^k, i^k, j^k) \in \mathbf{M}(V^k, W^k)$ to V^k. Explicitly it is given as follows:

(a) $B_h^k\colon V_{\mathrm{out}(h)}^k \to V_{\mathrm{in}(h)}^k$ is the multiplication of h from left: $fe_k \mapsto hfe_k$,
(b) $i_k^k\colon W_k^k \to V_k^k$ is the map $\mathbb{C} \ni 1 \mapsto e_k \in \mathbf{P}^0 e_k \subset \mathbf{P}e_k$,
(c) $j_k^k = 0$.

This defines a point in $\mathfrak{M}_\zeta(w_0 *_{\mathbf{e}^k} 0, \mathbf{e}^k)$. As $\mathfrak{M}_\zeta(w_0 *_{\mathbf{e}^k} 0, \mathbf{e}^k)$ is isomorphic to $\mathfrak{M}_{w_0\zeta}(0, \mathbf{e}^k)$ via $\mathcal{F}$, it consists of a single point. So the above (B^k, i^k, j^k) must be the holomorphic description of $(\mathcal{A}^k, \Psi^k)$. Furthermore, by (3.2), $\phi^h \in \mathrm{L}(V^{\mathrm{out}(h)}, V^{\mathrm{in}(h)})$ must be equal to

(11.15) $$\mathbf{P}e_{\mathrm{out}(h)} \ni y \mapsto y\overline{h} \in \mathbf{P}e_{\mathrm{in}(h)}.$$

Remark 11.16. More precisely, we proved the following: there exist elements $g^k \in G^{\mathbb{C}}(V^k)$ such that $g^k(B^k, i^k, j^k)$ and $g^{\mathrm{in}(h)}\phi^h(g^{\mathrm{out}(h)})^{-1}$ satisfy the equation $\mu_{\mathbb{R}}(g^k(B^k, i^k, j^k)) = -\zeta_{\mathbb{R}}$, the condition (3.1) and are holomorphic descriptions of $(\mathcal{A}^k, \Psi^k, \Phi^{\overline{h}})$. But we can use the reflection functor $\mathcal{F}_{(B^\bullet, i^\bullet, j^\bullet, \phi^\bullet)}$ instead of $\mathcal{F}_{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)}$ in the following proof. In fact, if we could prove directly that $\mathcal{F}_{(B^\bullet, i^\bullet, j^\bullet, \phi^\bullet)}$ preserves the stability condition (see Conjecture 3.15(1)), we do not need to invoke $\mathcal{F}_{(\mathcal{A}^\bullet, \Psi^\bullet, \Phi^\bullet)}$ and we could avoid the use of a result in §7.

Later we shall use the dual space of $\mathbf{P}e_k$. We consider the perfect pairing $\langle\ ,\ \rangle\colon e_{k^*}\mathbf{P} \otimes \mathbf{P}e_k \to \mathcal{L}_k$ given by

$$\langle y', y\rangle = (\overline{y'}^*, y).$$

Then we have an isomorphism of vector spaces

(11.17) $$(\mathbf{P}e_k)^* \cong e_{k^*}\mathbf{P} \otimes \mathcal{L}_k^*.$$

Lemma 11.18. *We have the following identification under* (11.17)*:*

(1) *The transpose of the multiplication map* $\mathbf{P}e_k \ni y \mapsto hy \in \mathbf{P}e_k$ *is identified with the map* $e_{k^*}\mathbf{P} \otimes \mathcal{L}_k^* \ni y' \otimes \chi \mapsto y'h \otimes \chi \in e_{k^*}\mathbf{P} \otimes \mathcal{L}_k^*$.

(2) *The transpose of* (11.15) *is identified with the map* $e_{\mathrm{in}(h)^*}\mathbf{P} \otimes \mathcal{L}_{\mathrm{in}(h)}^* \ni y' \otimes \chi \mapsto \overline{h}^* y \otimes {}^t\theta_h \chi \in e_{\mathrm{out}(h)^*}\mathbf{P} \otimes \mathcal{L}_{\mathrm{out}(h)}^*$.

The assertion follows from (11.10), (11.12).

Take a point $(B, i, j) \in \mathfrak{L}_\zeta(\mathbf{v}, \mathbf{w})$ corresponding to $S \in \mathrm{Gr}_{\mathbf{P}}(W^\heartsuit)$ as in (11.13). Consider the complex (3.9). We have isomorphisms

$$\begin{aligned} \mathrm{L}(V^k, W) &= \bigoplus_l (e_l \mathbf{P} e_k)^* \otimes W_l \cong \bigoplus_l e_{k^*}\mathbf{P}e_l \otimes \mathcal{L}_k^* \otimes W_l = e_{k^*} W^\heartsuit \otimes \mathcal{L}_k^*, \\ \mathrm{L}(V^k, V) &= \bigoplus_l (e_l \mathbf{P} e_k)^* \otimes V_l \cong \bigoplus_l e_{k^*}\mathbf{P}e_l \otimes \mathcal{L}_k^* \otimes e_l V, \end{aligned} \tag{11.19}$$

where we have used (11.17) in the middle (twice). Since $j = 0$, the $\mathrm{L}(V^k, W)$-component of α_k is 0. Thus the projection to $\mathrm{L}(V^k, W)$-component gives us a well-defined linear map

$$\operatorname{Ker} \beta_k / \operatorname{Im} \alpha_k \to e_{k^*} W^\heartsuit \otimes \mathcal{L}_k^*. \tag{11.20}$$

Lemma 11.21. *Let* $\vartheta_{k^*} \colon e_{k^*}W^\heartsuit \to e_{k^*}V$ *be the natural projection. The map* (11.20) *induces an isomorphism between* $\operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$ *and* $S_{k^*} \otimes \mathcal{L}_k^* = \operatorname{Ker}\vartheta_{k^*} \otimes \mathcal{L}_k^*$.

Proof. We have the multiplication map

$$\mathrm{L}(V^k, V) = \bigoplus e_{k^*}\mathbf{P}e_l \otimes \mathcal{L}_k^* \otimes e_l V \to e_{k^*}V \otimes \mathcal{L}_k^* = V_{k^*} \otimes \mathcal{L}_k^*$$

by the $\mathbf{P}$-module structure of V. Composing with β_k, we get a map

$$\widetilde{V}_k = V_k \oplus \mathrm{E}(V^k, V) \oplus \mathrm{L}(V^k, W) \to V_{k^*} \otimes \mathcal{L}_k^*.$$

By the definition of β_k, the V_k-component and the $\mathrm{E}(V^k, V)$-component are mapped to 0 (note $j^k = 0$) under this map. Moreover, the restriction to the $\mathrm{L}(V^k, W)$-component is nothing but the projection $\vartheta_{k^*} \otimes 1_{\mathcal{L}_k^*}$. Thus the kernel of β_k is mapped to $S_{k^*} \otimes \mathcal{L}_k^*$.

In order to prove that this map is an isomorphism, it is enough to show that (11.20) is surjective onto $S_{k^*} \otimes \mathcal{L}_k^*$ since $\dim S_{k^*} = \dim \operatorname{Ker}\beta_k/\operatorname{Im}\alpha_k$. We claim that for any $f \otimes w \in \mathrm{L}(V^k, W)$ there exists an element $f' \otimes v \in \mathrm{E}(V^k, V)$ such that

$$\beta_k \begin{bmatrix} 0 \\ f' \otimes v \\ f \otimes w \end{bmatrix} = e_{k^*} \otimes (\vartheta_{k^*} \otimes 1_{\mathcal{L}_k^*})(f \otimes w).$$

Here we consider $f \otimes w$ in the right hand side as an element of $e_{k^*}W^\heartsuit \otimes \mathcal{L}_k$ via (11.19) and $e_{k^*} \otimes (\vartheta_{k^*} \otimes 1_{\mathcal{L}_k^*})(f \otimes w)$ as an element of $\mathrm{L}(V^k, V)$ via (11.19) and the inclusion $e_{k^*}\mathbf{P}^0 e_{k^*} \otimes V_{k^*} \otimes \mathcal{L}_k^* \subset \bigoplus_l e_{k^*}\mathbf{P}e_l \otimes V_l \otimes \mathcal{L}_k^*$.

If $f \otimes w$ is of the form $e_{k^*} \otimes w \in e_{k^*}\mathbf{P}^0 e_{k^*} \otimes W_{k^*} \otimes \mathcal{L}_k^*$, then we take $f' \otimes v = 0$. If $f \otimes w$ is of the form $h_1 \cdots h_r \otimes w \in e_{k^*}\mathbf{P}e_{\mathrm{out}(h_r)} \otimes W_{\mathrm{out}(h)} \otimes \mathcal{L}_k^*$, we set

$$f' \otimes v = \pm \left(e_{k^*} \otimes (B_{h_2} \cdots B_{h_r}) i_{\mathrm{out}(h_r)} w + h_1 \otimes (B_{h_3} \cdots B_{h_r}) i_{\mathrm{out}(h_r)} w + \cdots + (h_1 \cdots h_{r-1}) \otimes i_{\mathrm{out}(h_r)} w \right).$$

If we choose the sign appropriately, we get

$$\beta_k(f' \otimes v) = e_{k^*} \otimes B_{h_1} \cdots B_{h_r} i_{\mathrm{out}(h_r)} w - h_1 \cdots h_r \otimes i_{\mathrm{out}(h_r)} w = e_{k^*} \otimes \vartheta_{k^*}(f \otimes w) - \beta_k(f \otimes w).$$

This shows the claim. If $\vartheta_{k^*}(f \otimes w) = 0$, then we have $\beta_k \begin{bmatrix} 0 \\ f'\otimes v \\ f\otimes w \end{bmatrix} = 0$. Hence (11.20) is a surjection onto $S_{k^*} \otimes \mathcal{L}_k^*$. □

Lemma 11.22. *Under the isomorphism* $\operatorname{Ker}\beta_k / \operatorname{Im}\alpha_k \to S_{k^*} \otimes \mathcal{L}_k^*$,

(a) B'_h *is given by*

$$S_{\mathrm{out}(h)^*} \otimes \mathcal{L}^*_{\mathrm{out}(h)} \ni y \otimes \chi \mapsto h^* y \otimes {}^t\theta_h \chi \in S_{\mathrm{in}(h)^*} \otimes \mathcal{L}^*_{\mathrm{in}(h)}$$

(b) i'_k *is zero,*

(c) j'_k *is the composition of*

$$S_{k^*} \otimes \mathcal{L}_k^* = e_{k^*} S \otimes \mathcal{L}_k^* \xrightarrow{\text{inclusion}} e_{k^*} W^{\heartsuit} \otimes \mathcal{L}_k^* \xrightarrow{\text{projection}} e_{k^*}(\mathbf{P}^{\mathbf{c}'} e_k \otimes W_k) \otimes \mathcal{L}_k^* = W_k.$$

The assertions follows from the definition of B'_h, i'_k, j'_k and Lemma 11.18.

Now we further map (B', i', j') into $\mathfrak{M}_\zeta(w_0 * \mathbf{v}^*, \mathbf{w}^*)$ as in (11.4). Furthermore, we identifying $\mathcal{L}^*_{\mathrm{out}(h)} \otimes \mathcal{L}_{\mathrm{in}(h)}$ with $\mathbb{C}$ by θ_h. Then the resulting data coincides with (11.14). This completes the proof of Theorem 11.7(1).

11(iii). **Proof of Theorem 11.7(2).** Let $\mathcal{R}_k^*$ be the dual bundle of $\mathcal{R}_k$. It has the dual of the connection of $\mathcal{R}$. It approximates a flat connection which is dual of the flat connection which $\mathcal{R}_k$ approximates. By the McKay correspondence, this flat connection corresponds to the vertex k^*. Moreover, we have a tautological homomorphism ${}^t(\omega\xi_h)^\dagger \colon \mathcal{R}^*_{\mathrm{out}(h)} \to S^+ \otimes \mathcal{R}^*_{\mathrm{in}(h)}$.

As in §9(ii) we get an admissible collection $(\mathcal{A}^k, \Psi^k, \Phi^{\overline{h}})$ by applying the ADHM description to $\mathcal{R}^*_{k^*}$. Comparing Chern classes we find that $(\mathcal{A}^k, \Psi^k)$ is isomorphic to the data corresponding to $(f_{w_0}^{-1})^* \mathcal{R}_k$. Thus the addmissible collection coincides with one corresponding to w_0 by Lemma 3.3.

Suppose that an anti-self-dual connection A corresponds to $(\mathcal{A}, \Psi) \in \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})$ via the ADHM description, where we use tautological bundles $\mathcal{R}_k$. Then its dual A^* corresponds to the data $(\mathcal{A}^*, \Psi^*) \in \mathfrak{M}_{-\zeta^*}(\mathbf{v}^*, \mathbf{w}^*)$, which is obtained by applying (11.2) and (11.3) (replacing ζ by $-\zeta^*$ and $\mathbf{v}$, $\mathbf{w}$ by $\mathbf{v}^*$, $\mathbf{w}^*$) to $(\mathcal{A}, \Psi)$ via the ADHM description, where we use $\mathcal{R}^*_{k^*}$, ${}^t(\omega\xi_{h^*})^\dagger$ instead of $\mathcal{R}_k$, ξ_h. If we describe the connection A by the original $\mathcal{R}_k$, ξ_h, then the corresponding data is given by the reflection of $(\mathcal{A}^*, \Psi^*)$ corresponding to w_0 by Theorem 9.1. This shows Theorem 11.7(2).

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