箙多様体と quantum affine algebras
(Quiver varieties and quantum affine algebras)

中島 啓 (Hiraku Nakajima)

ABSTRACT. この論文は, math.QA/0009231 (t-analogue of the q-characters of finite dimensional representations of quantum affine algebras) に Appendix として箙多様体の定義を付け加えたものである.

もともとの abstract: Frenkel-Reshetikhin introduced q-character of finite dimensional representations of quantum affine algebras [6]. We give a combinatorial algorithm to compute them for all simple modules. Our tool is t-analogue of the q-characters, which is similar to Kazhdan-Lusztig polynomials, and our algorithm has a resemblance with their definition.

We need the theory of quiver varieties for the definition of t-analogues and the proof. But it appear only in the last section. The rest of the paper is devoted to an explanation of the algorithm, which one can read without the knowledge about quiver varieties. A proof is given only in part. A full proof will appear elsewhere.

1. The quantum loop algebra

Let $\mathfrak{g}$ be a simple Lie algebra of type ADE over $\mathbb{C}$, $\mathbf{L}\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[z,z^{-1}]$ be its loop algebra, and $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e., without central extension and degree operator. Let I be the set of simple roots, P be the weight lattice, and P^* be its dual lattice (all for $\mathfrak{g}$). The algebra has the so-called Drinfeld's new realization: It is a $\mathbb{C}(q)$-algebra with generators q^h, $e_{k,r}$, $f_{k,r}$, $h_{k,n}$ ($h \in P^*$, $k \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z}\setminus\{0\}$) with certain relations (see e.g., [**1**, 12.2]).

The algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is a Hopf algebra, where the coproduct is defined using the Drinfeld-Jimbo realization of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. So a tensor product $M\otimes_{\mathbb{C}(q)} M'$ of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-modules M, M' has a structure of a $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-module.

Let $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ be its specialization at $q=\varepsilon\in\mathbb{C}^*$. For precise definition of the specialization, we first introduce an integral form $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})$ of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ and set $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) = \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})\otimes_{\mathbb{Z}[q,q^{-1}]}\mathbb{C}$, where $\mathbb{Z}[q,q^{-1}]\to\mathbb{C}$ is given by $q^{\pm1}\mapsto\varepsilon^{\pm1}$. See [**3**] for detail. But we assume ε is *not* a root of unity in this paper. So we just replace q by ε in the definition of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

Supported by the Grant-in-aid for Scientific Research (No.11740011), the Ministry of Education, Japan.

The quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ contains the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ for the finite dimensional Lie algebra $\mathfrak{g}$ as a subalgebra. The specialization $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ contains the specialization $\mathbf{U}_\varepsilon(\mathfrak{g})$ of $\mathbf{U}_q(\mathfrak{g})$.

1.1. Finite dimensional representations of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. The algebra $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ contains a commutative subalgebra generated by q^h, $h_{k,n}$ ($h \in P^*$, $k \in I$, $n \in \mathbb{Z} \setminus \{0\}$). Let us introduce generating functions $\psi_k^\pm(z)$ ($k \in I$) by

$$\psi_k^\pm(z) \overset{\text{def.}}{=} q^{\pm h_k} \exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty} h_{k,\pm m} z^{\mp m}\right).$$

A $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M is called of *type* 1 if M has a weight space decomposition as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module:

$$M = \bigoplus_{\lambda \in P} M(\lambda), \qquad M(\lambda) = \left\{ m \in M \,\middle|\, q^h * m = \varepsilon^{\langle h,\lambda\rangle} m \right\}.$$

We will only consider type 1 modules in this paper.

A type 1 module M is an *l-highest weight module* ('l' stands for the loop) if there exists a vector $m_0 \in M$ such that

$$e_{k,r} * m_0 = 0, \qquad \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})^- * m_0 = M,$$
$$\psi_k^\pm(z) * m_0 = \Psi_k^\pm(z) m_0 \quad \text{for } k \in I$$

for some $\Psi_k^\pm(z) \in \mathbb{C}[[z^\mp]]$. The pair of the I-tuple $(\Psi^+(z), \Psi^-(z)) = (\Psi_k^+(z), \Psi_k^-(z))_{k\in I} \in (\mathbb{C}[[z^\mp]]^I)^2$ is called the *l-highest weight* of M, and m_0 is called the *l-highest weight vector*.

Theorem 1.1.1 (Chari-Pressley [**2**]). (1) *Every finite-dimensional simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module of type* 1 *is an l-highest weight module, and its l-highest weight is given by*

$$\Psi_k^\pm(z) = \varepsilon^{\deg P_k} \left(\frac{P_k(\varepsilon^{-1}/z)}{P_k(\varepsilon/z)}\right)^\pm \tag{1.1.2}$$

for some polynomials $P_k(u) \in \mathbb{C}[u]$ with $P_k(0) = 1$. Here $(\ )^\pm \in \mathbb{C}[[z^\mp]]$ denotes the expansion at $z = \infty$ and 0 respectively.

(2) *Conversely, for given $P_k(u)$ as above, there exists a finite-dimensional simple l-highest weight $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M of type* 1 *such that the l-highest weight is given by the above formula.*

Assigning to M the I-tuple $P = (P_k)_{k\in I} \in \mathbb{C}[u]^I$ ($P_k(0) = 1$) defines a bijection between the set of all P's and the set of isomorphism classes of finite-dimensional simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules of type 1.

We denote by L_P the simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module associated to P. We call P the *Drinfeld polynomial*. For the abuse of terminology, we also say 'P is the l-highest weight of L_P'.

Since $\mathbb{C}\langle q^h, h_{k,n}\rangle$ is a commutative subalgebra of $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, any $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M decomposes into a direct sum $M = \bigoplus M(\Psi^+, \Psi^-)$ of generalized eigenspaces, where

$$M(\Psi^+, \Psi^-)$$
$$\overset{\text{def.}}{=} \left\{ m \in M \,\middle|\, (\psi_k^\pm(z) - \Psi_k^\pm(z)\,\mathrm{Id})^N * m = 0 \text{ for } k \in I \text{ and sufficiently large } N \right\},$$

for $\Psi_k^{\pm}(z) \in \mathbb{C}[[z^{\mp}]]$. The pair of the I-tuple $(\Psi^+, \Psi^-) = (\Psi_k^+, \Psi_k^-)_{k\in I}$ is called an *l-weight*, and $M(\Psi^+, \Psi^-)$ is called an *l-weight space* of M if $M(\Psi^+, \Psi^-) \neq 0$.

THEOREM 1.1.3 (Frenkel-Reshetikhin [**6**]). *Any l-weight of any finite dimensional* $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$*-module* M *of type* 1 *has the following form*

$$\Psi_k^{\pm}(z) = \varepsilon^{\deg Q_k - \deg R_k} \left(\frac{Q_k(\varepsilon^{-1}/z)R_k(\varepsilon/z)}{Q_k(\varepsilon/z)R_k(\varepsilon^{-1}/z)}\right)^{\pm} \tag{1.1.4}$$

for some polynomials

$$Q_k(u) = \prod_{i=1}^{s_k}(1 - a_{ki}u), \quad R_k(u) = \prod_{j=1}^{r_k}(1 - b_{kj}u).$$

Again for the abuse of terminology, we also say 'Q/R is an l-weight of M'. We denote the l-weight space $M(\Psi^+, \Psi^-)$ by $M(Q/R)$.

Frenkel-Reshetikhin [**6**] defined the q-character of M by

$$\chi_q(M) \stackrel{\text{def.}}{=} \sum_{Q/R} \dim M(Q/R) \prod_{k\in I}\prod_{i=1}^{s_k}\prod_{j=1}^{r_k} Y_{k,a_{ki}}Y_{k,b_{kj}}^{-1}.$$

THEOREM 1.1.5 (Frenkel-Reshetikhin [**6**]). (1) χ_q *defines an injective ring homomorphism from the Grothendieck ring* $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ *of finite dimensional* $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$*-modules of type* 1 *to* $\mathbb{Z}[Y_{k,a}^{\pm}]_{k\in I, a\in\mathbb{C}^*}$ *(a ring of Laurent polynomials in infinitely many variables).*

(2) *If we compose a map* $Y_{k,a}^{\pm} \mapsto y_k^{\pm}$ *(forgetting 'spectral parameters'), it gives the usual character of the restriction of* M *to a* $\mathbf{U}_\varepsilon(\mathfrak{g})$*-module.*

DEFINITION 1.1.6. A monomial $\prod_{k\in I}\prod_{i=1}^{s_k}\prod_{j=1}^{r_k} Y_{k,a_{ki}}Y_{k,b_{kj}}^{-1}$ appearing in the q-character χ_q is called *l-dominant* if $r_k = 0$ for all k, i.e., a product of positive powers of $Y_{k,c}$'s or 1.

If L_P is the simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module with l-highest weight P, its q-character contains an l-dominant monomial corresponding to the l-highest weight. We denote it by m_P. Its coefficient in $\chi_q(L_P)$ is 1.

Since $\{L_P\}_P$ forms a basis of $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, we have the following useful condition for the simplicity of a finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module M of type 1:

(1.1.7) If $\chi_q(M)$ contains only one l-dominant term, then M is simple.

1.2. Example. We give examples of q-characters.

If $\mathfrak{g} = A_n$, we have an evaluation homomorphism $\mathrm{ev}_a : \mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathbf{U}_\varepsilon(\mathfrak{g})$ corresponding to $\mathbf{L}\mathfrak{g} \to \mathfrak{g}$; $z \mapsto a$ (Jimbo). Hence pullbacks of simple $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules are simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules.

EXAMPLE 1.2.1. Let $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ and V be the 2-dimensional simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module. Then the q-character of $M_a = \mathrm{ev}_a(M)$ is given by[1]

$$\chi_q(M_a) = Y_{1,a} + Y_{1,a\varepsilon^2}^{-1}.$$

[1] This can be checked directly. But it also follows from Theorem 5.2.1 below.

Since χ_q is a ring homomorphism, we have

$$\begin{aligned}\chi_q(M_a \otimes M_b) &= \left(Y_{1,a} + Y_{1,a\varepsilon^2}^{-1}\right)\left(Y_{1,b} + Y_{1,b\varepsilon^2}^{-1}\right)\\ &= Y_{1,a}Y_{1,b} + Y_{1,a\varepsilon^2}^{-1}Y_{1,b} + Y_{1,a}Y_{1,b\varepsilon^2}^{-1} + Y_{1,a\varepsilon^2}^{-1}Y_{1,b\varepsilon^2}^{-1}.\end{aligned}$$

If $b \neq a\varepsilon^2, a\varepsilon^{-2}$, then $M_a \otimes M_b$ is simple by the criterion (1.1.7).

If $b = a\varepsilon^2$ or $a\varepsilon^{-2}$, then the second or third term becomes 1. In fact, it is known that $M_a \otimes M_{a\varepsilon^2}$ decomposes (in $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$) to a sum $M'_a \oplus M''$, where M' is the 3-dimensional simple $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-module, and M'' is the trivial module. Thus we have

$$\begin{aligned}\chi_q(M_a \otimes M_{a\varepsilon^2}) &= \chi_q(M'_a) + \chi_q(M'')\\ &= Y_{1,a}Y_{1,a\varepsilon^2} + Y_{1,a}Y_{1,a\varepsilon^4}^{-1} + Y_{1,a\varepsilon^2}^{-1}Y_{1,a\varepsilon^4}^{-1} + 1.\end{aligned}$$

See also Examples 4.1.6, 6.1.3, 7.2.1.

2. Standard modules

2.1. In [**15**] we defined a family of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules of type 1 and called them *standard modules*. They are parametrized by the I-tuples $P = (P_k)_{k\in I} \in \mathbb{C}[u]^I$ exactly as simple modules. We denote by M_P associated to P. The definition will be recalled in §8, but we give here their algebraic identification due to Varagnolo-Vasserot [**16**].

Definition 2.1.1. We say L_P an *l-fundamental representation* if

$$P_k(u) = \begin{cases} 1 - su & \text{if } k = k_0,\\ 1 & \text{otherwise,}\end{cases}$$

for some $s \in \mathbb{C}^*$ and $k_0 \in I$. We denote L_P by $L(\Lambda_{k_0})_s$. (Λ_k is the k-th fundamental weight of $\mathfrak{g}$.)

For $s \in \mathbb{C}^*$ and a finite sequence $(k_\alpha)_\alpha = (k_1, k_2, \dots)$ in I and a sequence $(n_\alpha)_\alpha = (n_1 \geq n_2 \geq \dots)$ of integers, we set

$$M(s;(k_\alpha)_\alpha,(n_\alpha)_\alpha) \overset{\text{def.}}{=} L(\Lambda_{k_1})_{\varepsilon^{n_1}s} \otimes L(\Lambda_{k_2})_{\varepsilon^{n_2}s} \otimes \cdots.$$

Note that $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$ is *not* cocommutative Hopf algebra, so the tensor product depends on the ordering of factors.

Theorem 2.1.2 (Varagnolo-Vasserot [**16**]). (1) *A standard module M is isomorphic to a module of the form*

$$\bigotimes_i M(s^i;(k^i_{\alpha_i})_{\alpha_i},(n^i_{\alpha_i})_{\alpha_i})$$
$$= M(s^1;(k^1_{\alpha_1})_{\alpha_1},(n^1_{\alpha_1})_{\alpha_1}) \otimes M(s^2;(k^2_{\alpha_2})_{\alpha_2},(n^2_{\alpha_2})_{\alpha_2}) \otimes \cdots \quad (\textit{finite tensor product})$$

such that $s^i/s^j \notin \varepsilon^{\mathbb{Z}}$ for $i \neq j$ and $n^i_1 \geq n^i_2 \geq \dots$ for each i.

(2) *The above tensor product is independent of the ordering of the factors $M(s^i;(k^i_{\alpha_i})_{\alpha_i},(n^i_{\alpha_i})_{\alpha_i})$.*

(3) *The I-tuple of polynomials P corresponding to M is the product of Drinfeld polynomials of l-fundamental representations appearing as factors of M.*

Note that if P is given, we can define a module M of the above form by decomposing P into a product of Drinfeld polynomials of l-fundamental representations. Thus we may denote the above module by M_P.

The following properties of M_P were shown in [**15**]:

(1) $\{M_P\}$ is a basis of $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$.
(2) M_P is an l-highest weight module with l-highest weight P (i.e., given by (1.1.2)).
(3) L_P is the unique simple quotient of M_P.
(4) M_P depends 'continuously' on P in a certain sense. For example, $\dim M_P$ is independent of P.
(5) for a generic P, $M_P \cong L_P$.

Conjecturally M_P is isomorphic to the specialization of the module $V^{\max}(\lambda)$, introduced by Kashiwara [**8**], and further studied by Chari-Pressley [**4**].

3. t-analogues of q-characters

A main tool in this paper is a t-analogue of the q-character:

$$\chi_{q,t}\colon \operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) \to \mathbb{Z}[t,t^{-1}][Y_{k,a}^{\pm}]_{k\in I, a\in\mathbb{C}^*}.$$

This is a homomorphism of additive groups, not of rings, and has the property $\chi_{q,t=1} = \chi_q$. We define $\chi_{q,t}$ for all standard modules M_P. Since $\{M_P\}_P$ is a basis of $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, we can extend it linearly to any finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules.

For the definition we need geometric constructions of standard modules, so we will postpone it to §8.3. We give an alternative definition, which is conjecturally the same as the geometric definition.

3.1. A conjectural definition. Let $M = M_P$ be a standard module, Q/R be an l-weight of M, $M(Q/R)$ be the corresponding l-weight space. Define a filtration on $M(Q/R)$ by

$$0 = M^{-1}(Q/R) \subset M^0(Q/R) \subset M^1(Q/R) \subset \cdots$$
$$M^n(Q/R) \stackrel{\text{def.}}{=} \bigcap_k \operatorname{Ker}(\psi_k^{\pm}(z) - \Psi_k^{\pm}(z)\,\mathrm{id})^{n+1}.$$

Conjecture 3.1.1. *The t-analogue $\chi_{q,t}(M_P)$, defined geometrically in §8.3, is equal to*

$$\chi_{q,t}(M_P) = \sum_{Q/R}\sum_n t^{2n-d(Q/R,P)} \dim\left(M^n(Q/R)/M^{n-1}(Q/R)\right)\ m_{Q/R},$$

where $d(Q/R,P)$ is an integer (determined explicitly from Q/R, P by (5.1.2) below), and $m_{Q/R}$ is a monomial in $Y_{k,a}^{\pm}$ corresponding to the l-weight space $M(Q/R)$.

This definition makes sense for any finite dimensional modules, but is *not* well-defined on the Grothendiek group $\operatorname{Rep}\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$. Thus the above does not hold for simple modules.

3.2. A main result of this paper is a combinatorial algorithm for computing $\chi_{q,t}(M_P)$ and $[M_P : L_Q]$. It is divided into three steps:

Step 1: Compute $\chi_{q,t}$ for all l-fundamental repsentations.
Step 2: Compute $\chi_{q,t}(M_P)$ for all standard modules M_P.
Step 3: Express the multiplicity $[M_P : L_Q]$ in terms of $\chi_{q,t}(M_R)$ for various R.

Step 1 is a modification of Frenkel-Mukhin's algorithm [**5**] for computing χ_q of l-fundamental representations. Step 2 is nothing but a study of $\chi_{q,t}$ of tensor products of l-fundamental representations. Although $\chi_{q,t}$ is not a ring homomorphism, $\chi_{q,t}$ of tensor products is given by a simply modified multiplication. For the proof we use an idea in [**13**]. Step 3 was essentially done in [**15**].

4. Step 3

We start with Step 3. The algorithm is similar to the definition of Kazhdan-Lusztig polynomials [**9**]. It is also similar to the algorithm for computing the transition matrix between the canonical basis and the PBW basis of type ADE [**10**].

4.1. Let

$$A_{k,a} \overset{\text{def.}}{=} Y_{k,a\varepsilon}Y_{k,a\varepsilon^{-1}}\prod_{l:l\neq k}Y_{l,a}^{c_{kl}},$$

where c_{kl} is the (k,l)-entry of the Cartan matrix.

Definition 4.1.1. (1) Let m, m' be monomials in $Y_{k,a}^{\pm}$ ($k\in I$, $a\in\mathbb{C}^*$). We define an ordering $\le$ among monomials by

$$m\le m' \Longleftrightarrow \frac{m'}{m} \text{ is a monimial in } A_{k,a}^{-1} \ (k\in I,\ a\in\mathbb{C}^*).$$

Here a monomial in $A_{k,a}^{-1}$ means a product of nonnegative powers of $A_{k,a}^{-1}$. It does not contain any factors $A_{k,a}$.

(2) If $\Psi^{\pm}$, $\Psi'^{\pm}$ are l-weights of finite dimensional $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$-modules, or $Q/R, Q'/R'$ are related to l-weights by (1.1.4), we write $\Psi^{\pm}\le\Psi'^{\pm}$, $Q/R\le Q'/R'$ if the corresponding monomials m, m' satisfy $m\le m'$.

Recall that $\chi_q(L_P)$ contains an l-dominant monomial m_P corresponding to the highest weight vector. It is known that any monomial m appearing $\chi_q(L_P)$, $\chi_q(M_P)$ satisfies $m\le m_P$ ([**5**, 4.1], [**15**, 13.5.2]).

Let

$$c_{QP}(t) \overset{\text{def.}}{=} \text{the coefficient of } m_Q \text{ in } \chi_{q,t}(M_P).$$

Then $(c_{QP}(t))_{P,Q}$ is upper-triangular and $c_{PP}(t)=1$ by the above mentioned result.

Let $(c^{QP}(t))$ be the inverse matrix $(c_{QP}(t))^{-1}$. Let

$$u_{RP}(t) \overset{\text{def.}}{=} \sum_Q c^{RQ}(t^{-1})c_{QP}(t).$$

Let $\overline{\ }$ be the involution on $\mathbb{Z}[t,t^{-1}]$ given by $t^{\pm1}\mapsto t^{\mp1}$.

Lemma 4.1.2 (Lusztig [**10**, 7.10]). *There exists a unique solution $Z_{QP}(t)\in\mathbb{Z}[t^{-1}]$ $(Q\le P)$ of*

$$Z_{RP}(t)=\sum_{Q:R\le Q\le P}\overline{Z_{RQ}(t)}u_{QP}(t), \tag{4.1.3}$$

$$Z_{PP}(t)=1,\quad Z_{QP}(t)\in t^{-1}\mathbb{Z}[t^{-1}] \text{ for } Q<P. \tag{4.1.4}$$

This lemma is proved by induction, and holds in a general setting. Lusztig has been using this (or its variant) in many places.

THEOREM 4.1.5. *The multiplicity $[M_P : L_Q]$ of a simple module L_Q in a standard module M_P is equal to $Z_{QP}(1)$.*

The proof will be given in §8.4.

EXAMPLE 4.1.6. Let $\mathfrak{g} = A_1$, $M_a = \mathrm{ev}_a^*(M)$ where M is the 2-dimensional simple $\mathbf{U}_\varepsilon(\mathfrak{g})$-module as before. By steps 1,2 explained below[2] we have

$$\chi_{q,t}(M_{a\varepsilon^2} \otimes M_a) = Y_{1,a}Y_{1,a\varepsilon^2} + Y_{1,a}Y_{1,a\varepsilon^4}^{-1} + Y_{1,a\varepsilon^2}^{-1}Y_{1,a\varepsilon^4}^{-1} + t^{-1}1. \tag{4.1.7}$$

Let $P(u) = (1-au)(1-a\varepsilon^2 u)$ (i.e., $M_P = M_{a\varepsilon^2} \otimes M_a$), $Q(u) = 1$ (i.e. $M_Q =$ trivial module). Then the above algorithm gives us $Z_{QP}(t) = t^{-1}$.

5. Step 1

5.1. Some definitions. Let M_P be a standard module. Let m_P be the monomial corresponding to the l-highest weight vector. Let $M_P(Q/R)$ be an l-weight space as before. We denote by $m_{Q/R}$ the corresponding monomial. We define $w_{k,a}(P), v_{k,a}(Q/R,P) \in \mathbb{Z}_{\geq 0}, u_{k,a}(Q/R) \in \mathbb{Z}$ by

$$m_P = \prod_{k\in I, a\in\mathbb{C}^*} Y_{k,a}^{w_{k,a}(P)},$$

$$m_{Q/R} = m_P \prod_{k\in I, a\in\mathbb{C}^*} A_{k,a}^{-v_{k,a}(Q/R,P)} = \prod_{k\in I, a\in\mathbb{C}^*} Y_{k,a}^{u_{k,a}(Q/R)}.$$

Suppose two standard modules M_{P^1}, M_{P^2} and l-weight spaces $M_{P^1}(Q^1/R^1) \subset M_{P^1}$, $M_{P^2}(Q^2/R^2) \subset M_{P^2}$ are given. We define

$$\begin{aligned} &d(Q^1/R^1, P^1; Q^2/R^2, P^2) \\ &\overset{\text{def.}}{=} \sum_{k,a} \left(v_{k,a}(Q^1/R^1, P^1) u_{k,a\varepsilon^{-1}}(Q^2/R^2) + w_{k,a\varepsilon}(P^1) v_{k,a}(Q^2/R^2, P^2) \right). \end{aligned} \tag{5.1.1}$$

We also define

$$d(Q/R, P) \overset{\text{def.}}{=} d(Q/R, P; Q/R, P). \tag{5.1.2}$$

We denote $d(Q^1/R^1, P^1; Q^2/R^2, P^2)$ also by $d(m_{Q^1/R^1}, m_{P^1}; m_{Q^2/R^2}, m_{P^2})$.

We need the following modification of $\chi_{q,t}$. Write $\chi_{q,t}(M_P) = \sum_m a_m(t)\ m$, where m is a monomial and $a_m(t)$ is its coefficient. Let

$$\widetilde{\chi_{q,t}}(M_P) \overset{\text{def.}}{=} \sum_m t^{d(m,m_P)} a_m(t)\, m, \tag{5.1.3}$$

where $d(m, m_P)$ is defined in (5.1.2).[3]

5.2. Frenkel-Mukhin [**5**, 5.1,5.2] proved that the image of the q-character χ_q is contained in

$$\bigcap_{k\in I} \left(\mathbb{Z}[Y_{l,a}^{\pm}]_{l\neq k, a\in\mathbb{C}^*} \otimes \mathbb{Z}[Y_{k,b}(1 + A_{k,b\varepsilon}^{-1})]_{b\in\mathbb{C}^*} \right).$$

[2] or direct calculation for the definition (8.3.1)

[3] In fact, $d(m, m_P)$ is determined from $a_m(t)$ so that $t^{d(m,m_P)}a_m(t)$ is a polynomial in t with nonzero constant term.

We have the t-analogue of this result, replacing $(1+A_{k,b\varepsilon}^{-1})^n$ by

$$\left(1+A_{k,b\varepsilon}^{-1}\right)_t^n \stackrel{\text{def.}}{=} \sum_{r=0}^{n} t^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_t A_{k,b\varepsilon}^{-r},$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_t$ is the t-binomial coefficient. More precisely, we have

THEOREM 5.2.1. (1) *For each $k \in I$, $\widetilde{\chi_{q,t}}(M_P)$ is expressed as a linear combination of*

$$\prod_i Y_{k,b_i}^{n_i} \left(1+A_{k,b_i\varepsilon}^{-1}\right)_t^{n_i} = Y_{k,b_1}^{n_1} \left(1+A_{k,b_1\varepsilon}^{-1}\right)_t^{n_1} Y_{k,b_2}^{n_2} \left(1+A_{k,b_2\varepsilon}^{-1}\right)_t^{n_2} \cdots$$

with coefficients in $\mathbb{Z}[t][Y_{l,a}^{\pm}]_{l\neq k, a\in\mathbb{C}^*}$, where $b_i \in \mathbb{C}^*$, $n_i \in \mathbb{Z}_{>0}$ with $b_i \neq b_j$ for $i \neq j$.

(2) *If L_P is an l-fundamental representation (and hence $M_P = L_P$), then $\chi_{q,t}(M_P)$ contains no l-dominant monomials other than m_P and the condition above uniquely determines $\chi_{q,t}(M_P)$.*

REMARK 5.2.2. The statement (1) for $t=1$ was proved by Frenkel-Mukhin [**5**]. And the proof of (2) is the same for $t=1$ and the general case, as illustrated in the following examples. In this sense, (2) should also be creditted to them.

5.3. Graph. We give few examples of $\chi_{q,t}$ of l-fundamental representations determined by the above theorem.

We attach to each standard module M_P, an oriented colored graph Γ_P. (It is a slight modification of the graph in [**6**, 5.3].) The vertices are monomials in $\chi_{q,t}(M_P)$. We draw an colored edge $\xrightarrow{k,a}$ from m_1 to m_2 if $m_2 = m_1 A_{k,a}^{-1}$. We also write the multiplicity of the monomials in $\chi_{q,t}(M_P)$.

EXAMPLE 5.3.1. Let $\mathfrak{g} = A_3 = \mathfrak{sl}_4$ and $M_P = L(\Lambda_2)_1$. Then the corresponding graph Γ_P is

$$\begin{array}{ccccc} Y_{2,1} & \xrightarrow{2,\varepsilon} & Y_{1,\varepsilon}Y_{2,\varepsilon^2}^{-1}Y_{3,\varepsilon} & \xrightarrow{1,\varepsilon^2} & Y_{1,\varepsilon^3}^{-1}Y_{3,\varepsilon} \\ & & \downarrow{\scriptstyle 3,\varepsilon^2} & & \downarrow{\scriptstyle 3,\varepsilon^2} \\ & & Y_{1,\varepsilon}Y_{3,\varepsilon^3}^{-1} & \xrightarrow{1,\varepsilon^2} & Y_{1,\varepsilon^3}^{-1}Y_{2,\varepsilon^2}Y_{3,\varepsilon^3}^{-1} \xrightarrow{2,\varepsilon^3} Y_{2,\varepsilon^4}^{-1}. \end{array}$$

Let us explain how we determine this graph inductively. We start with the l-highest weight $Y_{2,1}$. We know that its coefficient is 1. Applying Theorem 5.2.1(1) with $k=2$, we get $Y_{1,\varepsilon}Y_{2,\varepsilon^2}^{-1}Y_{3,\varepsilon}$ with coefficient 1. Then we apply Theorem 5.2.1(1) with $k=1$ to get $Y_{1,\varepsilon^3}^{-1}Y_{3,\varepsilon}$. And so on. All multiplicities are 1 in this case.

For $\mathfrak{g} = A_n$, it is known that the coefficients of $\chi_{q,t}(L(\Lambda_k)_a)$ are all 1.[4] Thus $\chi_{q,t}(L(\Lambda_k)_a) = \chi_{q,t=1}(L(\Lambda_k)_a)$.

EXAMPLE 5.3.2. Let $\mathfrak{g} = D_4$ and $M_P = L(\Lambda_2)_1$. The graph Γ_P is Figure 1. It is known that the restriction of M_P to a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module is a direct sum of the adjoint representation and the trivial representation. This fact is reflected in $\chi_{q,t}(M_P)$ where $Y_{2,\varepsilon^2}Y_{2,\varepsilon^4}^{-1}$ has the coefficient $[2]_t$ and all others has 1. Note that the number

[4] More generally, if the coefficients of α_k in the highest root is 1, then the same holds. This result easily follows from the theory of quiver varieties. **Exercise**: Check this using the above algorithm.

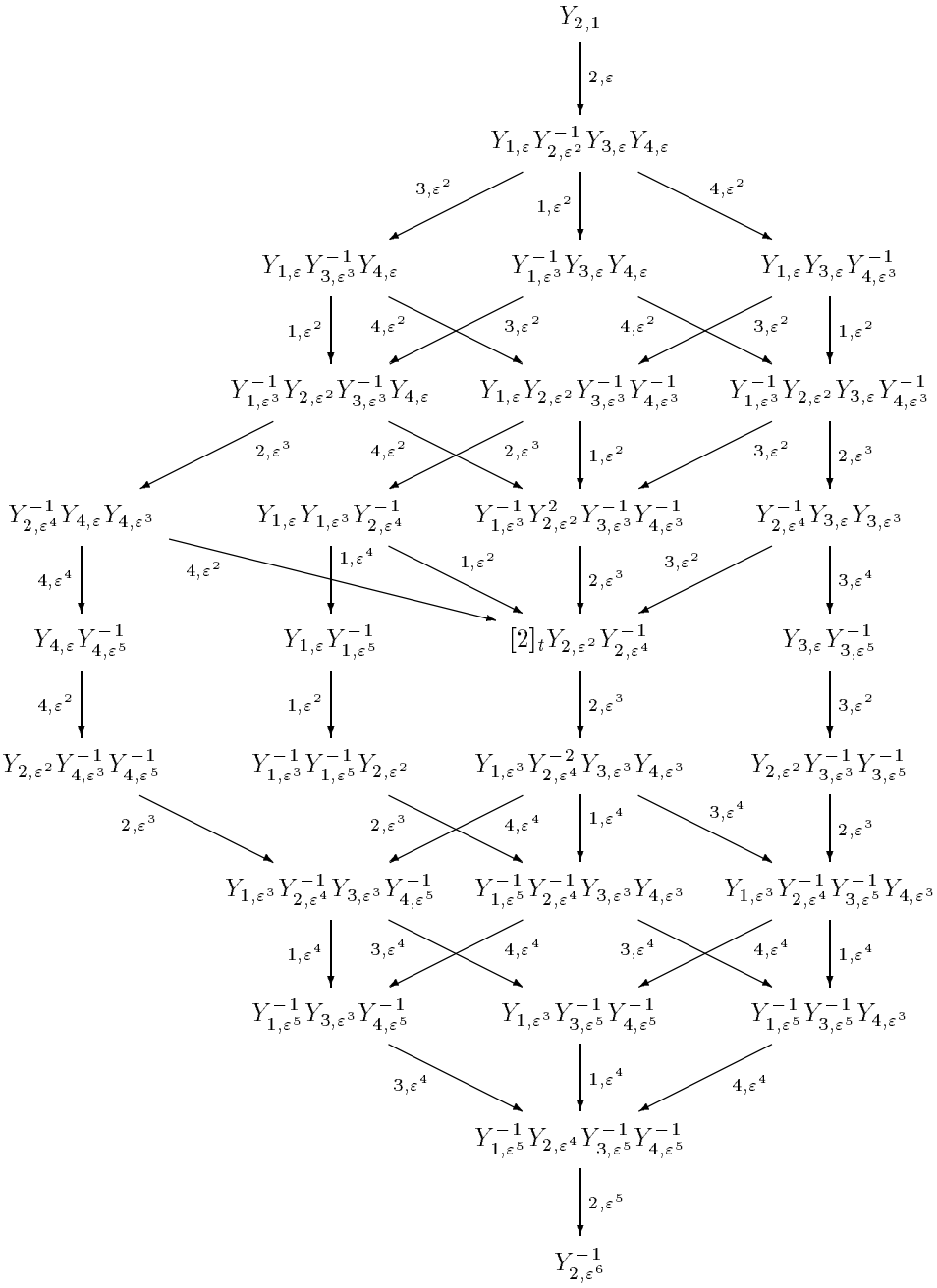

FIGURE 1. The graph for $L(\Lambda_2)_1$

of monomials is 28, which is the dimension of the adjoint representation. See also Example 7.2.3 below.

Let us give a more complicated example.

EXAMPLE 5.3.3. Let $\mathfrak{g} = A_2$ and $M_P = L(\Lambda_2)_\varepsilon^{\otimes 2} \otimes L(\Lambda_1)_1$. Although this is *not* an l-fundamental representation, $\chi_{q,t}(M_P)$ has no l-dominant terms other than m_P, so the condition Theorem 5.2.1(1) gives us $\chi_{q,t}$. The graph is Figure 2.

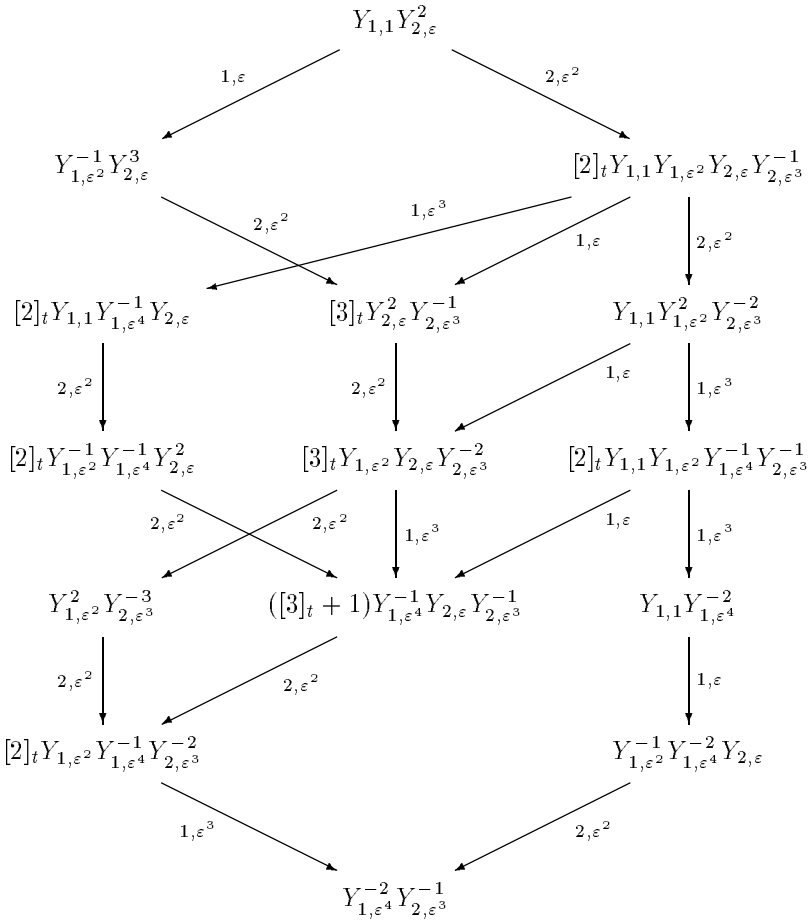

FIGURE 2. The graph for $L(\Lambda_2)_\varepsilon^{\otimes 2} \otimes L(\Lambda_1)_1$

REMARK 5.3.4. As we can see in above examples, the crystal graphs are subgraphs of Γ_P. The set of vertices is the same, but the set of arrows is smaller. We would like to discuss this further elsewhere.

6. Step 2

6.1. Let $M_P = M(s^1;(k_{\alpha_1}^1),(n_{\alpha_1}^1)) \otimes M(s^2;(k_{\alpha_2}^2),(n_{\alpha_2}^2)) \otimes \cdots$ be a standard module with $s^i/s^j \notin \varepsilon^{\mathbb{Z}}$ as in Theorem 2.1.2.

PROPOSITION 6.1.1. *We have*

$$\chi_{q,t}(M_P) = \chi_{q,t}(M(s^1;(k^1_{\alpha_1})_{\alpha_1},(n^1_{\alpha_1})_{\alpha_1}))\chi_{q,t}(M(s^2;(k^2_{\alpha_2})_{\alpha_2},(n^2_{\alpha_2})_{\alpha_2}))\cdots$$

if $s^i/s^j \notin \varepsilon^{\mathbb{Z}}$ *for* $i \neq j$.

Thus it is enough to study

$$\chi_{q,t}(M(s;(k_\alpha)_\alpha,(n_\alpha)_\alpha)) = \chi_{q,t}(L(\Lambda_{k_1})_{\varepsilon^{n_1}s} \otimes L(\Lambda_{k_2})_{\varepsilon^{n_2}s} \otimes \cdots).$$

Let

$$\chi_{q,t}(L(\Lambda_{k_\alpha})_{\varepsilon^{n_\alpha}s})) = \sum_{r_\alpha} a_{m_\alpha,r_\alpha}(t)m_{\alpha,r_\alpha},$$

where m_{α,r_α} is a monomial in $Y^{\pm}_{k,a}$ and $a_{m_\alpha,r_\alpha}(t) \in \mathbb{Z}[t,t^{-1}]$ is its coefficient.

If $t = 1$, $\chi_{q,1}$ is a ring homomorphism, hence we have

$$\chi_{q,1}(M(a;(k_\alpha)_\alpha,(n_\alpha)_\alpha)) = \sum_{r_1,r_2,\cdots} \prod_\alpha a_{m_r,r_\alpha}(1)m_{\alpha,r_\alpha}.$$

THEOREM 6.1.2. *Let* P^α *be the Drinfeld polynomial of* $L(\Lambda_{k_\alpha})_{\varepsilon^{n_\alpha}s}$. *Then we have*

$$\chi_{q,t}(M(a;(k_\alpha),(n_\alpha))) = \sum_{r_1,r_2,\cdots} t^{\sum_{\alpha,\beta}\pm d(m_{\alpha,r_\alpha},m_{P^\alpha};m_{\beta,r_\beta},m_{P^\beta})} \prod_\alpha a_{m_r,r_\alpha}(t)m_{\alpha,r_\alpha},$$

where the sign for $d(m_{\alpha,r_\alpha},m_{P^\alpha};m_{\beta,r_\beta},m_{P^\beta})$ *is* $-$ *if* $\alpha \leq \beta$ *and* $+$ *otherwise.*

EXAMPLE 6.1.3. For $\mathfrak{g} = A_1$, we have

$$d(Y^{-1}_{1,a\varepsilon^2},Y_{1,a};Y_{1,a},Y_{1,a}) = 1, \qquad d(Y_{1,a},Y_{1,a};Y^{-1}_{1,a},Y_{1,a\varepsilon^{-2}}) = 1$$

and all others are 0. Then we get (4.1.7).

If $P = (1-au)^n$, we get

$$\chi_{q,t}(M_P) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_t Y^{n-r}_{1,a} Y^{-r}_{1,a\varepsilon^2}$$

from $\chi_{q,t}(L(\Lambda_1)_a) = Y_{1,a} + Y^{-1}_{1,a\varepsilon^2}$. This also follows directly from the definition (8.3.1) below. The t-binomial coefficients appear as Poincaré polynomials of Grassmann manifolds.

7. Restrition to $\mathbf{U}_\varepsilon(\mathfrak{g})$

Finite dimensional simple $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules are classified by highest weights. Let $\operatorname{Res} M_P$ be the restriction of a standard module M_P to a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module. It decomposes into a sum of various simple modules. Once $\chi_q(M_P)$ is computed, the character of $\operatorname{Res} M_P$ is given by replacing $Y^{\pm}_{k,a}$ by $y^{\pm}_k$ (Theorem 1.1.5(2)). Combining with the knowledge of characters of simple finite dimensional $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules, we can determine the multiplicity of simple modules in $\operatorname{Res} M_P$.

Characters of simple finite dimensional $\mathbf{U}_\varepsilon(\mathfrak{g})$-modules are the same as that of simple $\mathfrak{g}$-modules, hence are known. However, we express them in terms of $\chi_{q,t}$ in this section.

7.1. For a dominant weight $\mathbf{w} = \sum w_k \Lambda_k$ we denote by $L_{\mathbf{w}}$ the simple highest weight $\mathbf{U}_\varepsilon(\mathfrak{g})$-module with the highest weight $\mathbf{w}$.

We consider a standard module M_P with $\deg P_k = w_k$. By the 'continuity' of M_P on P, $\operatorname{Res} M_P$ depends only on $w_k = \deg P_k$, and not on P itself. Let us denote the multiplicity of $L_{\mathbf{w}'}$ in $\operatorname{Res} M_P$ by $Z_{\mathbf{w}',\mathbf{w}}$, i.e.,

$$\operatorname{Res} M_P = \bigoplus_{\mathbf{w}'} L_{\mathbf{w}'}^{\oplus Z_{\mathbf{w}',\mathbf{w}}}.$$

We will give a formula expressing $Z_{\mathbf{w}',\mathbf{w}}$ in terms of $\chi_{q,t}(M_P)$. Although we can give algorithm for arbitary P in principle, the following choice will make the formula simple.

Choose and fix orientations of edges in the Dynkin diagram. We define integer $m(k)$ for each vertex k so that $m(k) - m(l) = 1$ if we have an oriented edge from k to l, i.e., $k \to l$. Then we define P by

$$P_k(u) = (1 - u\varepsilon^{m(k)})^{w_k}.$$

Let $\widetilde{\chi_{q,t}}(M_P)$ as in (5.1.3). Let $\widetilde{\chi}_t(M_P) \in \mathbb{Z}[t] \otimes \mathbb{Z}[t_k^{\pm}]_{k\in I}$ be a t-analogue of the ordinary character which is obtained from $\widetilde{\chi_{q,t}}(M_P)$ by sending $Y_{k,a}^{\pm}$ to $y_k^{\pm}$.

For another dominant weight $\mathbf{w}' = \sum w'_k \Lambda_k$, let

$$c_{\mathbf{w}',\mathbf{w}}(t) \overset{\text{def.}}{=} \text{the coefficient of } \prod y_k^{w'_k} \text{ in } \widetilde{\chi}_t(M_P).$$

The matrix $(c_{\mathbf{w}',\mathbf{w}}(t))_{\mathbf{w}',\mathbf{w}}$ is upper-triangular with respect to the usual order on weights, and diagonal entries are all 1.

THEOREM 7.1.1. *$c_{\mathbf{w}',\mathbf{w}}(0)$ is the weight multiplicity of $\mathbf{w}'$ in the highest weight module $L_{\mathbf{w}}$ with the highest weight $\mathbf{w}$.*

This is just a simple rephrasing of a main result in [**12, 14**]. The proof will be given in §8.5.

Note that $c_{\mathbf{w}',\mathbf{w}}(1)$ gives the weight multiplicity of $\mathbf{w}'$ in $\operatorname{Res} M_P$ since $\widetilde{\chi_{t=1}}$ is the ordinary character. Thus we have

$$c_{\mathbf{w}'',\mathbf{w}}(1) = \sum_{\mathbf{w}'} c_{\mathbf{w}'',\mathbf{w}'}(0)\, Z_{\mathbf{w}',\mathbf{w}}.$$

This equation determines the multiplicity $Z_{\mathbf{w}',\mathbf{w}}$ only from the knowledge of $\chi_{q,t}$.

According to a conjecture of Lusztig [**11**] together with a formula (8.5.1) below, $c_{\mathbf{w}',\mathbf{w}}(t)$ should be written by ferminonic form of Hatayama el al. [**7**]. More precisely, we should have $\sum_{\mathbf{w}'} c^{\mathbf{w}'',\mathbf{w}'}(0)\, c_{\mathbf{w}',\mathbf{w}}(t) = M(\mathbf{w},\mathbf{w}'',t^2)$, where $(c^{\mathbf{w}'',\mathbf{w}'}(0))$ is the inverse matrix of $(c_{\mathbf{w}'',\mathbf{w}'}(0))$. See [**11**] for the defintion of $M(\mathbf{w},\mathbf{w}',q)$. Although this formula can be checked in many examples, the complexity of the combinatorics prevent us from proving it in full generality. Conjecturally $M(\mathbf{w},\mathbf{w}',q=1)$ gives us the multiplicities of the restriction of M_P (Kirillov-Reshetikhin[5]). Thus the conjecture is compatible with our result in this section.

7.2.

EXAMPLE 7.2.1. Let $\mathfrak{g} = A_1$ and $\mathbf{w} = 2\Lambda_1$. We take $P = (1-u)^2$ by the above choice. By Example 6.1.3, we have

$$\widetilde{\chi}_t(M_P) = y_1^2 + (1+t^2) + y_1^{-2}.$$

[5] In fact, they consider more general modules, not necessarily standard modules.

Thus $Z_{0,\mathbf{w}} = 1$. Since $\mathrm{Res}(M_P) = L_{\Lambda_1} \otimes L_{\Lambda_1} = L_{2\Lambda_1} \oplus L_0$, this is the correct answer !

EXAMPLE 7.2.2. Let $\mathfrak{g} = A_3$ and $\mathbf{w} = \Lambda_2$. By Example 5.3.1 all the coefficients of $\chi_{q,t}(L(\Lambda_2)_1)$ are 1. Hence $\mathrm{Res}\, M_P = \mathrm{Res}\, L(\Lambda_2)_1$ is simple as a $\mathbf{U}_\varepsilon(\mathfrak{g})$-module.

EXAMPLE 7.2.3. Let $\mathfrak{g} = D_4$, $\mathbf{w} = \Lambda_2$, $\mathbf{w}' = 0$. By Example 5.3.2 we have $c_{\mathbf{w}',\mathbf{w}}(t) = 4 + t^2$. Thus $Z_{\mathbf{w}',\mathbf{w}} = 1$, i.e. $\mathrm{Res}(L(\Lambda_2)_1) = L_{\Lambda_2} \oplus L_0$.

8. Quiver varieties

In this section, we give the definition of $\chi_{q,t}$ and prove Theorems 4.1.5, 7.1.1. As we mentioned, those proofs are essentially given in [**12, 14**] and [**15**] respectively. The only things we do here are translation of results into the language of $\chi_{q,t}$. We believe that this section gives good introductions to [**12, 14, 15**].

8.1. Let $\mathbf{w} = \sum w_k \Lambda_k$ ($w_k \in \mathbb{Z}_{\geq 0}$) be a dominant weight of the finite dimensional Lie algebra $\mathfrak{g}$. In [**12, 14, 15**], we have attached to each $\mathbf{w}$, a map $\pi : \mathfrak{M}(\mathbf{w}) \to \mathfrak{M}_0(\infty, \mathbf{w})$ with the following properties:

(1) $\mathfrak{M}(\mathbf{w})$ is a finite disjoint union of nonsingular quasi-projective varieties of various dimensions.
(2) $\mathfrak{M}_0(\infty, \mathbf{w})$ is an affine algebraic variety.
(3) π is a projective morphism.
(4) There exist actions of $G_\mathbf{w} \times \mathbb{C}^*$ on $\mathfrak{M}(\mathbf{w})$ and $\mathfrak{M}_0(\infty, \mathbf{w})$ such that π is equivariant.
(5) $\mathfrak{M}_0(\infty, \mathbf{w})$ is a cone, and the vertex (denoted by 0) is the unique fixed point of the $\mathbb{C}^*$-action (restriction of $G_\mathbf{w} \times \mathbb{C}^*$-action to the second factor).

Here $G_\mathbf{w} = \prod_{k \in I} \mathrm{GL}(w_k, \mathbb{C})$.

We consider the fiber product

$$Z(\mathbf{w}) \stackrel{\text{def.}}{=} \mathfrak{M}(\mathbf{w}) \times_{\mathfrak{M}_0(\infty,\mathbf{w})} \mathfrak{M}(\mathbf{w}).$$

The convolution product makes the (Borel-Moore) homology group $H_*(Z(\mathbf{w}), \mathbb{C})$ into an associative (noncommutative) algebra. One of main results in [**14**] is a construction of a surjective algebra homomorphism

$$\mathbf{U}(\mathfrak{g}) \to H_{\mathrm{top}}(Z(\mathbf{w}), \mathbb{C}),$$

where $\mathbf{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (**NB**: not a 'quantum' version). Here $H_{\mathrm{top}}(\ )$ means the degree $= \dim_{\mathbb{R}} Z(\mathbf{w})$ part of the homology group. More precisely, we take degree = dimension part on each connected components of $Z(\mathbf{w})$, and then make the direct sum. Note that the the dimension differs on various components.

Let $\mathfrak{L}(\mathbf{w}) = \pi^{-1}(0)$. It is known that $\mathfrak{M}(\mathbf{w})$ has a holomorphic symplectic form such that $\mathfrak{L}(\mathbf{w})$ is a lagrangian subvariety. The convolution makes $H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{C})$ (the top degree part of the Borel-Moore homology group, in the same sense as above) into an $H_{\mathrm{top}}(Z(\mathbf{w}), \mathbb{C})$-module. It is a $\mathbf{U}(\mathfrak{g})$-module by the above homomorphism. By [**14**, 10.2] it is the simple finite dimensional $\mathbf{U}(\mathfrak{g})$-module $L_\mathbf{w}$ with highest weight $\mathbf{w}$. And connected components $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ of $\mathfrak{M}(\mathbf{w})$ are parametrized by vectors $\mathbf{v} = \sum v_k \alpha_k$ (α_k is the kth simple root of $\mathfrak{g}$) so that

$$H_{\mathrm{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{C}) = \bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v}, \mathbf{w}) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})$$

is the weight space decomposition of the simple highest weight module $L_{\mathbf{w}}$, where $H_{\mathrm{top}}(\mathfrak{M}(\mathbf{v},\mathbf{w}) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})$ has weight $\mathbf{w}-\mathbf{v}$. In particular, $\mathbf{v}=0$ corresponds to the highest weight vector. In fact, $\mathfrak{M}(0,\mathbf{w})$ is consisting of a single point.

The space $\mathfrak{M}_0(\infty,\mathbf{w})$ has a stratification

$$\mathfrak{M}_0(\infty,\mathbf{w}) = \bigcup \mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w}),$$

where $\mathbf{v}$ runs over the set of vectors such that $\mathbf{w}-\mathbf{v}$ is a weight of $L_{\mathbf{w}}$ which is *dominant* [**12**, §3].

8.2. Let us give the $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-version of the construction of the previous subsection.

We use the following notation: Let $R(G)$ denote the representation ring of a linear algebraic group G. If G acts a quasi-projective variety X, $K^G(X)$ denotes the Grothendieck group of G-equivariant coherent sheaves on X.

The representation ring $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ of $G_{\mathbf{w}} \times \mathbb{C}^*$ is isomorphic to the tensor product $R(G_{\mathbf{w}}) \otimes_{\mathbb{Z}} R(\mathbb{C}^*)$. Moreover, $R(\mathbb{C}^*)$ is isomorphic to $\mathbb{Z}[q,q^{-1}]$, where q is the canonical 1-dimensional representation of $\mathbb{C}^*$.

The convolution makes the Grothendieck group $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$ into a $R(G_{\mathbf{w}} \times \mathbb{C}^*) = R(G_{\mathbf{w}})[q,q^{-1}]$-algebra. One of main results in [**15**] is a construction of an algebra homomorphism

$$\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} R(G_{\mathbf{w}}) \to K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))/\text{ torsion}\,.$$

By the equivariance of π, $\mathfrak{L}(\mathbf{w}) = \pi^{-1}(0)$ is invariant under $G_{\mathbf{w}} \times \mathbb{C}^*$. The convolution makes $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ into a $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(Z(\mathbf{w}))$-module. Moreover, it is free of finite rank over $R(G_{\mathbf{w}} \times \mathbb{C}^*)$ [**15**, §7]. It is a $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} R(G_{\mathbf{w}})$-module by the above homomorphism. By [**15**, §13], it contains a vector m_0 such that

$$\begin{aligned} &e_{k,r} * m_0 = 0, \qquad \left(\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^- \otimes_{\mathbb{Z}} R(G_{\mathbf{w}})\right) * m_0 = K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})), \\ &\psi_k^{\pm}(z) * m_0 = q^{w_k} \left(\frac{\bigwedge_{-1/qz} q^{-1}W_k}{\bigwedge_{-q/z} q^{-1}W_k}\right)^{\pm} * m_0 \quad \text{for } k \in I. \end{aligned} \tag{8.2.1}$$

The right hand side of the third equation needs an explanation: First W_k is the vector representation of $\mathrm{GL}(w_k,\mathbb{C})$, considered as a $G_{\mathbf{w}} \times \mathbb{C}^*$-module. Then $\bigwedge_u V = \sum u^i \bigwedge^i V$. Since $\bigwedge_{-q/z} q^{-1}W_k$ is $1-(1/z)W_k+\dots$ (1 is the trivial module), we can define $\left(\bigwedge_{-q/z} q^{-1}W_k\right)^{-1}$ as a formal power series in $1/z$. This gives us the case $(\ )^+$ of the above formula. In the case $(\ )^-$, we expand as $\bigwedge_{-q/z} q^{-1}W_k = (-1/z)^{w_k}\left(\bigwedge^{w_k} W_k - z\bigwedge^{w_k-1} W_k + \cdots\right)$. Then $\bigwedge^{w_k} W_k$ is an invertible element, we can also define $\left(\bigwedge_{-q/z} q^{-1}W_k\right)^{-1}$. The vector m_0 is the canonical generator of $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{M}(0,\mathbf{w}))$. (Recall $\mathfrak{M}(0,\mathbf{w})$ is a point.)

The module $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ should be considered as a 'universal' standard module since standard modules are obtained from it by specializations as we explain now.

Let $a=(s,\varepsilon) \in G_{\mathbf{w}} \times \mathbb{C}^*$ be a semisimple element. It defines a homomorphism $\chi_a \colon R(G_{\mathbf{w}} \times \mathbb{C}^*) \to \mathbb{C}$ by sending a representation to the value of the character at a. Then

$$K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)} \mathbb{C} \tag{8.2.2}$$

is a module over $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g}) = \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$. By (8.2.1) it is a finite-dimensional l-highest weight module. This is the *standard module* M_P, where $P_k(u) = \chi_a(\bigwedge_{-u} q^{-1}W_k)$. Note that the set of conjugacy classes of $a = (s, \varepsilon)$ bijectively corresponds to the set of I-tuple of polynomials P with $\deg P_k = w_k$.

8.3. Let A be the Zariski closure of $a^{\mathbb{Z}}$ in $G_{\mathbf{w}} \times \mathbb{C}^*$. It is an abelian reductive group. We have $K^{G_{\mathbf{w}}\times\mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \otimes_{R(G_{\mathbf{w}}\times\mathbb{C}^*)} R(A) \cong K^A(\mathfrak{L}(\mathbf{w}))$ [**15**, §7]. Since χ_a factors through $R(A)$, the standard module M_P is isomorphic to $K^A(\mathfrak{L}(\mathbf{w})) \otimes_{R(A)} \mathbb{C}$. By Thomason's localization theorem, it is isomorphic to $K(\mathfrak{L}(\mathbf{w})^A) \otimes_{\mathbb{Z}} \mathbb{C}$, where $\mathfrak{L}(\mathbf{w})^A$ is the fixed point set. Furthermore, it is isomorphic to $H_*(\mathfrak{L}(\mathbf{w})^A, \mathbb{C})$ via the Chern character homomorphism [**15**, §7].

Let $\pi^A \colon \mathfrak{M}(\mathbf{w})^A \to \mathfrak{M}_0(\infty, \mathbf{w})^A$ be the restriction of the map $\pi \colon \mathfrak{M}(\mathbf{w}) \to \mathfrak{M}_0(\infty, \mathbf{w})$ to the fixed point set. Let $\mathfrak{M}(\mathbf{w})^A = \bigsqcup_\rho \mathfrak{M}(\rho)$ be the decomposition into connected components. Each $\mathfrak{M}(\rho)$ is a nonsingular quasi-projective variety. Then we have the direct sum decomposition

$$M_P \cong H_*(\mathfrak{L}(\mathbf{w})^A, \mathbb{C}) \cong \bigoplus_\rho H_*(\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C}).$$

In [**15**, §13, §14] we have shown that this is the l-weight space decomposition of M_P. In particular, the index ρ can be considered as an l-weight of M_P. Thus we have arrived at a geometric interpretation of χ_q:

$$\chi_q(M_P) = \sum_\rho \dim H_*(\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})\, m_\rho,$$

where m_ρ is the monomial corresponding to the l-weight ρ.

Now we define the t-analogue $\chi_{q,t}$ by

$$\chi_{q,t}(M_P) \overset{\text{def.}}{=} \sum_\rho \sum_k \dim H_k(\mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})\, t^{k - \dim_{\mathbb{C}} \mathfrak{M}(\rho)}\, m_\rho. \tag{8.3.1}$$

By [**15**, §14] we have a stratification

$$\mathfrak{M}_0(\infty, \mathbf{w})^A = \bigcup_\rho \mathfrak{M}_0^{\mathrm{reg}}(\rho),$$

consisting of nonsingular locally closed subvarieties. Here the index set $\{\rho\}$ is the subset of the above index set consisting of *l-dominant* l-weights.

8.4. Proof of Theorem 4.1.5. The l-highest weight P is fixed throughout the proof. Thus the dominant weight vector $\mathbf{w}$ and the element $a = (s, \varepsilon) \in G_{\mathbf{w}}$ are fixed.

We change the notation now. If ρ corresponds to an l-weight space $M_P(Q/R)$, we denote above $\mathfrak{M}(\rho)$ by $\mathfrak{M}(Q/R, P)$. We also denote by $\mathfrak{M}_0^{\mathrm{reg}}(Q, P)$ for above $\mathfrak{M}_0^{\mathrm{reg}}(\rho)$ if ρ corresponds to an l-dominant l-weight Q. Thus we have

$$\mathfrak{M}(\mathbf{w})^A = \bigsqcup_{Q/R} \mathfrak{M}(Q/R, P), \quad \mathfrak{M}_0(\infty, \mathbf{w})^A = \bigcup_Q \mathfrak{M}_0^{\mathrm{reg}}(Q, P).$$

In this notation $H_*(\mathfrak{M}(P,P) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})$ is the l-highest weight space. Since $\mathfrak{M}(0, \mathbf{w})$ is a single point as we explained, we have $\mathfrak{M}(P, P) = \mathfrak{M}(0, \mathbf{w})$. We also have $\mathfrak{M}_0^{\mathrm{reg}}(P, P) = \{0\}$.

LEMMA 8.4.1. (1) $\dim_{\mathbb{C}} \mathfrak{M}(Q/R,P) = d(Q/R,P)$. $\dim_{\mathbb{C}} \mathfrak{M}_0^{\mathrm{reg}}(Q,P) = d(Q,P)$.
(2) *If* $\mathfrak{M}_0^{\mathrm{reg}}(Q,P) \subset \overline{\mathfrak{M}_0^{\mathrm{reg}}(R,P)}$, *then* $R \le Q$.
(3) *Choose* $x \in \mathfrak{M}_0^{\mathrm{reg}}(Q,P)$. *Then* $(\pi^A)^{-1}(x) \cap \mathfrak{M}(S/T,P)$ *is isomorphic to* $\mathfrak{M}(S/T,Q) \cap \mathfrak{L}(\mathbf{w})$.

PROOF. (1) The first equation is the dimension formula [**15**, 4.1.6]. The second equation follows from $\dim_{\mathbb{C}} \mathfrak{M}_0^{\mathrm{reg}}(Q,P) = \dim_{\mathbb{C}} \mathfrak{M}(Q,P)$, which is clear from the definition [**15**, §4].
(2),(3) The results are known or trivial for $Q = P$. Now use the transversal slice at $x \in \mathfrak{M}_0^{\mathrm{reg}}(Q,P)$ [**15**, §3] to reduce a general case to this case. □

Let $D^b(\mathfrak{M}_0(\infty,\mathbf{w})^A)$ be the bounded derived category of complexes of sheaves such that cohomology sheaves are constant along each stratum $\mathfrak{M}_0^{\mathrm{reg}}(Q,P)$. Let $IC(\mathfrak{M}_0^{\mathrm{reg}}(Q,P))$ be the intersection homology complex associated with the constant local system $\mathbb{C}_{\mathfrak{M}_0^{\mathrm{reg}}(Q,P)}$ on $\mathfrak{M}_0^{\mathrm{reg}}(Q,P)$. By using the transversal slice [**15**, §3], one can check that it is an object in $D^b(\mathfrak{M}_0(\infty,\mathbf{w})^A)$. Let $\mathbb{C}_{\mathfrak{M}(Q/R,P)}$ be the constant local system on $\mathfrak{M}(Q/R,P)$. Then $\pi_*^A(\mathbb{C}_{\mathfrak{M}(Q/R,P)})$ is an object of $D^b(\mathfrak{M}_0(\infty,\mathbf{w})^A)$ again by the transversal slice argument. Using the decomposition theorem of Beilinson-Bernstein-Deligne, we have shown that there exists an isomorphism in $D^b(\mathfrak{M}_0(\infty,\mathbf{w})^A)$:

$$\pi_*^A(\mathbb{C}_{\mathfrak{M}(R,P)}[\dim_{\mathbb{C}} \mathfrak{M}(R,P)]) \cong \bigoplus_{Q,k} L_{Q,k}(R,P) \otimes IC(\mathfrak{M}_0^{\mathrm{reg}}(Q,P))[k] \tag{8.4.2}$$

for some vector space $L_{Q,k}(R,P)$ [**15**, 14.3.2]. Since $\pi^A(\mathfrak{M}(R,P)) \subset \overline{\mathfrak{M}_0^{\mathrm{reg}}(R,P)}$ by definition [**15**, §4], the summation runs over $Q \ge R$ by Lemma 8.4.1. Let

$$L_{RQ}(t) \overset{\mathrm{def.}}{=} \sum_k \dim L_{Q,k}(R,P)\, t^{-k}.$$

Applying the Verdier duality to the both hand side of (8.4.2) and using the self-duality of $\pi_*^A(\mathbb{C}_{\mathfrak{M}(R,P)}[\dim_{\mathbb{C}} \mathfrak{M}(R,P)])$ and $IC(\mathfrak{M}_0^{\mathrm{reg}}(Q,P))$, we find $L_{RQ}(t) = \overline{L_{RQ}(t)}$.

Choose a point x_Q from $\mathfrak{M}_0^{\mathrm{reg}}(Q,P)$ for each stratum. Let $i_{x_Q} : \{x_Q\} \to \mathfrak{M}_0(\infty,\mathbf{w})^A$ denote the inclusion. Consider

$$H^k(i_{x_Q}^! \pi_*^A \mathbb{C}_{\mathfrak{M}(R,P)}[\dim \mathfrak{M}(R,P)]) = H_{\dim_{\mathbb{C}} \mathfrak{M}(R,P)-k}((\pi^A)^{-1}(x_Q) \cap \mathfrak{M}(R,P), \mathbb{C}).$$

By Lemma 8.4.1(3) this is isomorphic to $H_{\dim_{\mathbb{C}} \mathfrak{M}(R,P)-k}(\mathfrak{M}(R,Q) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})$. Therefore we have

$$\begin{aligned} &\sum_k \dim H^k(i_{x_Q}^! \pi_*^A \mathbb{C}_{\mathfrak{M}(R,P)}[\dim \mathfrak{M}(R,P)])\, t^{\dim_{\mathbb{C}} \mathfrak{M}(Q,P)-k} \\ &= \sum_d \dim H_d(\mathfrak{M}(R,Q) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})\, t^{d+\dim_{\mathbb{C}} \mathfrak{M}(Q,P)-\dim_{\mathbb{C}} \mathfrak{M}(R,P)} = c_{RQ}(t), \end{aligned} \tag{8.4.3}$$

where we used $\dim_{\mathbb{C}} \mathfrak{M}(R,P) - \dim_{\mathbb{C}} \mathfrak{M}(Q,P) = \dim_{\mathbb{C}} \mathfrak{M}(R,Q)$ in the last equality.

By [**15**, 14.3.10], we have

$$[M_Q : L_R] = \dim H^*(i_{x_Q}^! IC(\mathfrak{M}_0^{\mathrm{reg}}(R,P))).$$

(In fact, we defined the standard module M_Q as $H_*((\pi^A)^{-1}(x_Q), \mathbb{C})$ in [**15**, §13], which apriori depends on P. Thus the definition coincides only when $Q = P$.

However, by using the transversal slice, we can show that the right hand side is the same for both definitions. cf. Lemma 8.4.1.)

Let

$$Z_{RQ}(t) \stackrel{\text{def.}}{=} \sum_k \dim H^k(i_{x_Q}^! IC(\mathfrak{M}_0^{\mathrm{reg}}(R,P)))\, t^{\dim \mathfrak{M}_0^{\mathrm{reg}}(Q,P)-k}.$$

We have $[M_Q : L_R] = Z_{RQ}(1)$. By the defining property of the intersection homology, $Z_{RQ}(t)$ satisfies (4.1.4).

Substituting (8.4.2) into (8.4.3), we get

$$c_{SQ}(t) = \sum_R L_{SR}(t) Z_{RQ}(t).$$

Now $L_{SR}(t) = \overline{L_{SR}(t)}$ implies (4.1.3). This completes the proof of Theorem 4.1.5.

8.5. Proof of Theorem 7.1.1. By the result explained in §8.1, the weight multiplicity of $\mathbf{w}'$ in $L_{\mathbf{w}}$ is equal to

$$\dim H_{\mathrm{top}}(\mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w}) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C}).$$

The assertion follows from more general formula

(8.5.1)

$$\widetilde{\chi}_t(M_P) = \sum_{\mathbf{w}'} \sum_d H_d(\mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w}) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C})\, t^{\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w})-d} \prod_k y_k^{w_k'}.$$

Note that $\mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w}) \cap \mathfrak{L}(\mathbf{w})$ is a lagrangian subvariety in $\mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w})$, so we have $\mathrm{top} = \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w})$.

In order to prove (8.5.1), we use [**12**, 5.7], where the Betti numbers are given in terms of those of fixed point components. It looks almost the same as above. However, there is one significant difference. The $\mathbb{C}^*$-action used there is different from our $\mathbb{C}^*$-action used here, defined in [**15**, §2]. This is the reason why we choose P and corresponding $a = (s,\varepsilon)$ as explained in §7. Then $A = \overline{a^{\mathbb{Z}}}$ is isomorphic to $\mathbb{C}^*$ and the action is the same as the $\mathbb{C}^*$-action considered in [**12**, §5].

We decompose $\mathfrak{M}(\mathbf{w})^A = \bigsqcup \mathfrak{M}(\rho)$ into connected components as before. By [**12**, 5.7] we have[6]

$$\dim H_d(\mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w}) \cap \mathfrak{L}(\mathbf{w}), \mathbb{C}) = \sum_\rho \dim H_{\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w})-d}(\mathfrak{M}(\rho), \mathbb{C}),$$

where the summation runs over the set of ρ such that the corresponding monomial m_ρ is sent to $\prod_k y_k^{w_k'}$ after $Y_{k,a} \to y_k$. The $\mathbb{C}^*$-action makes $\mathfrak{M}_0(\infty,\mathbf{w})^{\mathbb{C}^*} = \{0\}$, so $\mathfrak{M}(\rho) = \mathfrak{M}(\rho) \cap \mathfrak{L}(\mathbf{w})$. Hence the above expression coincides with the definition of the coefficients $\widetilde{\chi}_t$.

Acknowledgement. We would like to thank E. Frenkel and E. Mukhin for explanations of their algorithm computing q-characters of fundamental representations.

[6] In fact, this formula even holds for general P if we replace $\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{w}-\mathbf{w}',\mathbf{w}) - d$ by a suitable degree. However, this degree shift is given by a complicated expression in ρ. So our choice of P is most economical.

Appendix A. 箙多様体

A.1. 箙 (quiver). quiver Q は, 頂点の有限集合 I と (向きのついた) 辺の集合 Ω であって, 辺の始点と終点を対応させる写像 $\mathrm{out}:\Omega\to I$, $\mathrm{in}:\Omega\to I$ が与えられているもののことである. すなわち, 有限グラフの辺に向きを入れたものに他ならない. (図 3 参照)

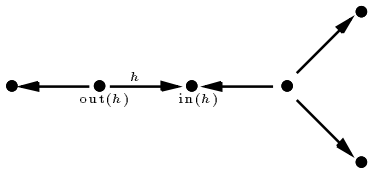

FIGURE 3. quiver の例

このとき道 **(path)** とは, 辺の有限列 $(h_1,h_2,\dots,h_N)$ で $\mathrm{in}(h_1)=\mathrm{out}(h_2),\ \dots,\ \mathrm{in}(h_{N-1})=\mathrm{out}(h_N)$ が成り立つもののことを言う. N を道の長さという. さらに長さ N が 0 の道, すなわちひとつの頂点 $k\in I$ だけからなるものも道であると約束する.

$\mathbb{F}$ を体とするとき, 道代数 **(path algebra)** $\mathbb{F}Q$ を道を基底とするベクトル空間に道の合成によって積を入れたものとする. もしも二つの道が合成できないとき, すなわちそれぞれの終点と始点とが一致しないときは, 積は 0 と定める. 例えば下の図 4 の定める道代数は, 一変数多項式環 $\mathbb{F}[x]$ に他ならない. 一般には, 道代数は非可換の環になる.

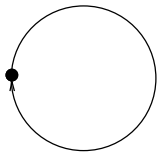

FIGURE 4. $\mathbb{F}[x]$ に対応する quiver

また $\mathfrak{I}$ を $\mathbb{F}Q$ の両側イデアルとするとき, 道代数を関係式 $\mathfrak{I}$ で割った環 $\mathbb{F}Q/\mathfrak{I}$ もよく考える. 例えば, 上の例で m 回まわるループ h^m を 0 とする関係式を考えれば, 対応する環は $\mathbb{F}[x]/x^m$ に他ならない.

quiver の表現とは, 道代数 $\mathbb{F}Q$ の左加群のことを言う. 言い直せば, 各頂点 $k\in I$ に対して $\mathbb{F}$-ベクトル空間 V_k が与えられ, 各辺 $h\in\Omega$ に対して線型写像 $B_h:V_{\mathrm{out}(h)}\to V_{\mathrm{in}(h)}$ が与えられているもののことである. また $\mathbb{F}Q/\mathfrak{I}$ の表現は, 上のような B_h に $\mathfrak{I}$ が定める条件を課したものに他ならない.

したがって, quiver の表現の同型類 (ただし次元が同じものだけを考える) の集合 (すなわち, 表現のモジュライ空間) は,

$$\bigoplus_{h\in\Omega}\mathrm{Hom}(V_{\mathrm{out}(h)},V_{\mathrm{in}(h)})\Big/\prod_{k\in I}\mathrm{GL}(V_k) \tag{A.1.1}$$

に他ならない. ベクトル空間を $\mathbf{E}_\Omega(V)$, $\prod_{k\in I}\mathrm{GL}(V_k)$ を G_V で表わす.

表現のモジュライ空間(A.1.1) は, Riemann 面上の正則ベクトル束のモジュライ空間と類似点がある. 実際後者は, (無限次元の) アファイン空間を (やはり無限次元の)Lie 群で割った空間になっている.

relation の例であとの箙多様体と深く関係するものがあるので説明する．$\mathcal{Q}^{\#}$ を $\mathcal{Q}=(I,\Omega)$ のダブルとする．すなわち，I は変えず，Ω の辺 h にその向きを逆にしたもの $\overline{h}$ を付け加えたものである．このとき，$\mathcal{Q}^{\#}$ の辺 h に対し，$\varepsilon(h)$ を $h\in\Omega$ のときに 1，$h\in\overline{\Omega}$ のときに -1 と定める．そこで，

$$\sum_{\mathrm{in}(h)=k}\varepsilon(h)h\overline{h}=0,\qquad (k\in I)$$

で生成される両側イデアルを考え $\mathfrak{I}$ とおく．$\mathbb{F}\mathcal{Q}^{\#}/\mathfrak{I}$ が preprojective algebra と呼ばれるものである．その表現の同型類は，

$$\left\{(B_h)\in\mathbf{E}_{\Omega\cup\overline{\Omega}}(V)\;\middle|\;\sum_{\mathrm{in}(h)=k}\varepsilon(h)B_hB_{\overline{h}}=0\right\}\Big/G_V \tag{A.1.2}$$

で与えられる．$\mathbf{E}_{\Omega\cup\overline{\Omega}}(V)=\mathbf{E}_{\Omega}(V)\oplus\mathbf{E}_{\overline{\Omega}}(V)$ は シンプレクティックベクトル空間[7]で，$\sum\varepsilon(h)B_hB_{\overline{h}}$ は G_V の作用に関するモーメント写像である．シンプレクティック形式やモーメント写像については，詳しい一般論がこの論説で必要になるわけではないが，あとに出てくる幾何学的不変式論や hyper-Kähler 構造との関連も含めて，[**19**, 第 3 章] を参照されればありがたい．よって上の空間は，もしも商空間が多様体の構造を持てばいわゆるシンプレクティック商と捉えることが出来る．またシンプレクティック商の一般論によれば，(A.1.1) の $\mathbf{E}_{\Omega}(V)/G_V$ が多様体になれば，その余接束 $T^*(\mathbf{E}_{\Omega}(V)/G_V)$ と同型になる．ところが，G_V のような代数群に関する商空間は，一般には位相空間として Hausdorff にならない．この困難を解消するためには幾何学的不変式論が必要で，のちに述べる箙多様体もその様に定義される．その様な技術的な相違点があるにせよ，preprojective algebra の幾何学的な意味は ‘余接束’ である．

ALE 空間 (より正確にはその下部構造の複素曲面) や箙多様体が(A.1.2) もしくはその modification として定義される．その意味で，(A.1.2) は有益なシンプレクティック多様体を構成するレシピと思うことが出来る．

A.2. 箙多様体 (quiver variety). 箙多様体の定義をする．まず(A.1.1) の $\mathbf{E}_{\Omega}(V)$ とその双対空間の直和を考える:

$$\mathbf{E}_{\Omega}(V)\oplus\mathbf{E}_{\Omega}(V)^*=\bigoplus_{h\in\Omega}\mathrm{Hom}(V_{\mathrm{out}(h)},V_{\mathrm{in}(h)})\oplus\mathrm{Hom}(V_{\mathrm{in}(h)},V_{\mathrm{out}(h)})$$

これは，E のダブル，すなわち E の辺に逆向きの辺も付け加えてできた quiver (これを $\mathcal{Q}^{\#}$ で表わす) に対応するベクトル空間と思える．E のダブルを H で表わす．さらに，各頂点ごとにベクトル空間 W_k を置いて，

$$\left(\bigoplus_{h\in H}\mathrm{Hom}(V_{\mathrm{out}(h)},V_{\mathrm{in}(h)})\right)\oplus\left(\bigoplus_{k\in I}\mathrm{Hom}(W_k,V_k)\oplus\mathrm{Hom}(V_k,W_k)\right) \tag{A.2.1}$$

というベクトル空間を考える．このベクトル空間を $\mathbf{M}(V,W)$ と書こう．$\mathbf{M}(V,W)$ には自然に $G_V=\prod_{k\in I}\mathrm{GL}(V_k)$ が作用する．

さて，(A.2.1) の点の意味するところは何であろうか？ 第一の成分は，quiver H の表現であるから意味は明らかであろう．一方，第二成分は [**17**] で現れたものの，quiver の表現論で扱われたことはなかったようである．前節のように表現論との関連から，第二成分の最高ウェイトベクトルとしての意味がはっきりしてくるのではあるが，ここで [**17**] での意味を説明しよう．箙多様体を ALE 空間上の正則ベクトル束のモジュライ空間として見るとき，$(W_k)_{k\in I}$ は，framing，すなわち無限遠でのベクトル束の自明

[7]この論説では，シンプレクティック形式はすべて複素数体上で考える．他の文献では，実のシンプレクティック形式と区別するために正則シンプレクティック形式と呼ぶ場合も多いので注意．

化に対応する. これを quiver の言葉に無理矢理に翻訳すると次のようになる. quiver $\mathcal{Q}^\#$ から一切の辺を取り除いた quiver を $\mathcal{Q}'$ とする. このとき, $\mathcal{Q}'$ の表現とは, 頂点の上にベクトル空間を置いたもの, そしてベクトル空間の間の線型写像は一切考えないものに他ならない. $\mathcal{Q}^\#$ の表現 $(V_k)_{k\in I}, (B_h)_{h\in H}$ に対し, B_h を忘れると, $\mathcal{Q}'$ の表現が与えられる. これは, 表現の間の関手である. (より一般に, $\mathcal{Q}^\#$ の部分グラフを考えれば, 同様の関手が定義される.) この関手を制限と呼ぼう. さて, 上の式で, $(W_k)_{k\in I}$ は, $\mathcal{Q}'$ の表現と考えることができる. すると

$$\bigoplus_{k\in I} \operatorname{Hom}(W_k, V_k)$$

は, $(W_k)_{k\in I}$ から $(V_k)_{k\in i}, (B_h)_{h\in I}$ の制限への quiver $\mathcal{Q}'$ での準同型の全体である.

$$\bigoplus_{k\in I} \operatorname{Hom}(V_k, W_k)$$

はもちろん, $(V_k)_{k\in I}, (B_h)_{h\in H}$ の制限から $(W_k)_{k\in i}$ への quiver $\mathcal{Q}'$ での準同型の全体である.

前に述べたような, quiver の表現とベクトル束の類似のもとでは, 次の対応があることは納得が行くであろう.

(1) 上の関手は, ベクトル束を部分多様体へ制限することに対応する.
(2) (A.2.1) の第二成分は, 部分多様体に与えられたベクトル束との間の準同型に対応する.

さて, (A.2.1) の各成分を B_h, i_k, j_k と書こう. このとき, 次の写像を考える.

$$\mu\colon \mathbf{M} \to \bigoplus_{k\in I} \operatorname{End}(V_k); \quad (B_h, i_k, j_k)_{h\in H, k\in I} \longmapsto \left(\sum_{h\in H:\mathrm{in}(h)=k} \varepsilon(h) B_h B_{\overline{h}} + i_k j_k \right)_{k\in I}.$$

$\mathbf{M}(V,W)$ は, その '半分'

$$\bigoplus_{h\in\Omega} \operatorname{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) \oplus \bigoplus_{k\in I} \operatorname{Hom}(W_k, V_k) \tag{A.2.2}$$

の余接束と見ることができ, 特に自然なシンプレクティック形式を持つ. 上のベクトル空間を $\mathbf{M}_\Omega(V,W)$ と書こう. G_V の作用は, $\mathbf{M}(V,W)$ のシンプレクティック形式を保つ. 上の写像 μ は, 運動量写像である. このあたりは, preprojective algebra の説明をしたときと同様である.

さて, シンプレクティック商

$$\mu^{-1}(0)/G_V$$

を考えたい. もしも商空間 $\mathbf{M}_\Omega(V,W)/G_V$ が多様体であれば, これはその余接束と同型である. しかし, 集合論的な商空間は一般には代数多様体になることが期待できない. 幾何学的不変式論 [**18**] の教えは, 商空間を考える代わりに商空間上の関数が何かを考えよ, と言うことである. 今の場合は, 二つの候補がある. まず第一のものは, $\mu^{-1}(0)$ をアファイン代数多様体と思って, 座標環の G_V 不変な部分環に対応するアファイン代数多様体を考えるものである. これは $\mu^{-1}(0)/\!\!/G_V$ と書かれるもので, その点は $\mu^{-1}(0)$ の閉軌道に他ならない.

もう一つは, $\mu^{-1}(0)$ 上の直線束 (今の場合, 自明なものを取る) に, G_V の作用を持ち上げ, 直線束のテンソル積の G_V 不変な切断のなす次数つき環に対応する準射影多様体を考えるものである. 自明な直線束への作用の持ち上げを, $\mathrm{GL}(V_k)$ の行列式の積と取ると, 以下のように定義される '安定 (stable)' な点の全体の商空間になる.

Definition A.2.3. $(B_h, i_k, j_k)_{h\in H, k\in I}$ が安定であるとは, 次の条件が成り立つときを言う: V_k の線型部分空間の集まり $(S_k)_{k\in I}$ であって,

- $S_k \subset \operatorname{Ker} j_k$ がすべての k について成り立つ
- $B_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$ がすべての h について成り立つ

という条件をみたすものは, すべての k について $S_k = 0$ となるものに限る.

安定な点の全体を $\mu^{-1}(0)^{\mathrm{s}}$ と書く. よって二つめの商は, $\mu^{-1}(0)^{\mathrm{s}}/G_V$ である.
上のようにして構成された二つの商空間をそれぞれ

$$\mathfrak{M}_0(\mathbf{v},\mathbf{w}), \qquad \mathfrak{M}(\mathbf{v},\mathbf{w})$$

で表わそう. ただし, $\mathbf{v}$, $\mathbf{w}$ はそれぞれ

$$\mathbf{v} = \sum_k \dim V_k\, \alpha_k\,, \quad \mathbf{w} = \sum_k \dim W_k\, \Lambda_k$$

で与えられる. 作り方から

$$\pi : \mathfrak{M}(\mathbf{v},\mathbf{w}) \to \mathfrak{M}_0(\mathbf{v},\mathbf{w})$$

という射影的な写像がある. 具体的には, $\mathfrak{M}(\mathbf{v},\mathbf{w})$ の点, すなわち安定な軌道に対して, その閉包に含まれるようなただ一つの閉軌道を対応させる写像である.
前節で $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ と書いたものは, stabilizer が自明である閉軌道の全体のなす $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$ の開部分多様体である.
ここで, なぜ $\mu^{-1}(0)$ だけでなく, $\mathbf{M}_\Omega(V,W)$ の幾何学的不変式論による商も考えないのか疑問に思わないだろうか？ 実は, その様な商を考えることは可能であるが,

- $\mathbf{M}_\Omega(V,W)$ の閉軌道は 0 しかない. すなわち $\mathbf{M}_\Omega(V,W)/\!\!/G_V$ は一点のみからなる空間である.
- 多くの場合, 安定な軌道は存在しない.

などの現象があり, あまり自然なものとは考えられないのである. したがって, $\mathbf{M}_\Omega(V,W)$ では商空間 $\mathbf{M}_\Omega(V,W)/\!\!/G_V$ を考えるのではなく, G_V-同変な $\mathbf{M}_\Omega(V,W)$ 上の対象を考えるのがいいようである.
また, 次のような現象もある. quiver$\mathcal{Q}$ は, グラフの辺に向きを入れたものであったが, 異なる向きを入れることを考える. このとき quiver $\mathcal{Q}$ は変わるが, その double $\mathcal{Q}^\#$ は変わらない. そして, $\mathfrak{M}_0(\mathbf{v},\mathbf{w})$, $\mathfrak{M}(\mathbf{v},\mathbf{w})$ は向きの取り方によらないが, $\mathbf{M}_\Omega(V,W)$ の向きの取り方によって, 安定な点を持ったり持たなかったり変わってしまうのである.
すなわち, 次の哲学が成り立っている.

商空間 $\mathbf{M}_\Omega(V,W)/G_V$ は意味を持たないが, その余接束は定義可能である.

しかし, そもそもなぜ商空間を考えることが必要なのか, G_V-同変な対象ではいけないのかという疑問はあろう. 筆者の答は, 次節以降で安定性が表現論的な意味を持つのを見ることで与えられる.
箙多様体の例を与えよう. グラフは A_1 型, すなわち頂点が一個で辺はないものとする. このとき, V, W はベクトル空間で, $i\colon W \to V$, $j\colon V \to W$ となる. $\mu = 0$ は, $ij = 0$ で, 安定であることは, j が単射であることと同値である. そこで, $\mathfrak{M}(\mathbf{v},\mathbf{w})$ からグラスマン多様体の余接束への写像を

$$\mathfrak{M}(\mathbf{v},\mathbf{w}) \ni G_V\cdot(i,j) \longmapsto (\operatorname{Image} j, ji) \in T^*\operatorname{Grass}(v,W)$$

で定める. ただし,

$$T^*\operatorname{Grass}(v,W) = \{(S,\xi) \mid S \text{ は } W \text{ の } v \text{ 次元部分空間で}, \xi\colon W/S \to S\}$$

である. 容易にチェックできるように, これは同型である. また, $\mathfrak{M}_0(V,W)$ の方は,

$$\mathfrak{M}_0(V,W) \ni G_V \cdot (i,j) \mapsto ji \in \{\xi \in \operatorname{End} W \mid \xi^2 = 0,\ \operatorname{rank}\xi \le \dim V\}$$

が同型写像を与える.

References

[1] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[2] ———, *Quantum affine algebras and their representations*, Representations of groups (Banff, AB, 1994), Amer. Math. Soc., Providence, RI, 1995, pp. 59–78.

[3] ———, *Quantum affine algebras at roots of unity*, Represent. Theory **1** (1997), 280–328 (electronic).

[4] ———, *Integrable and Weyl modules for quantum affine sl_2*, preprint, arXiv:math.QA/0007123.

[5] E. Frenkel and E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, preprint, arXiv:math.QA/9911112.

[6] E. Frenkel and N. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999, pp. 163–205.

[7] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and Y. Yamada, *Remarks on fermionic formula*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999, pp. 243–291.

[8] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413.

[9] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[10] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[11] ———, *Ferminonic form and Betti numbers*, preprint, arXiv:math.QA/0005010.

[12] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[13] ———, *Homology of moduli spaces instantons on ALE spaces I*, J. Differential Geom. **40** (1994), 105–127.

[14] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[15] ———, *Quiver varieties and finite dimensional representations of quantum affine algebras*, preprint, arXiv:math.QA/9912158, to appear in J. Amer. Math. Soc.

[16] M. Varagnolo and E. Vasserot, *Standard modules of quantum affine algebras*, preprint, arXiv:math.QA/0006084.

Reference for Appendix

[17] P.B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), 263–307.

[18] D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, Third Enlarged Edition, Springer-Verlag, 1994.

[19] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lect. Ser. **18**, AMS, 1999.

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: nakajima@kusm.kyoto-u.ac.jp

URL: http://www.kusm.kyoto-u.ac.jp/~nakajima