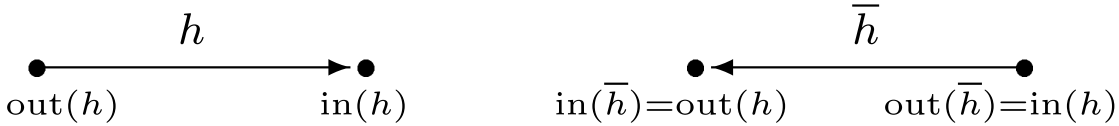
Notation for an Oriented Graph

- $I = \{1, 2, \ldots, n\}$: the set of vertices,
- E : the set of edges (assume no edge loops)
- H : the set of pairs consisting of an edge together with its orientation.
- for $h \in H$, let $\mathrm{in}(h)$, $\mathrm{out}(h)$, $\overline{h}$ be



- Ω : orientation of the graph, i.e., a subset $\Omega \subset H$ such that $\overline{\Omega} \cup \Omega = H$, $\Omega \cap \overline{\Omega} = \emptyset$. (assume that Ω has no cycle.)

Notation for Quiver Varieties

Put vector spaces V_k, W_k for each vertex $k \in I$.

$$\mathbf{M} \overset{\text{def.}}{=} \left(\bigoplus_{h \in H} \operatorname{Hom}(V_{\operatorname{out}(h)}, V_{\operatorname{in}(h)}) \right) \oplus \left(\bigoplus_{k=1}^{n} \operatorname{Hom}(W_k, V_k) \oplus \operatorname{Hom}(V_k, W_k) \right).$$

For $(B_h, i_k, j_k) \in \mathbf{M}$, let

$$\mu(B_h, i_k, j_k) \overset{\text{def.}}{=} \left(\sum_{h \in H : k = \operatorname{in}(h)} \varepsilon(h) B_h B_{\overline{h}} + i_k j_k \right)_k \in \bigoplus_k \operatorname{End}(V_k).$$

<u>Definition</u>.

$(B_h, i_k, j_k) \in \mu^{-1}(0)$ is <u>stable</u> if if a collection $S = (S_k)$ of subspaces is B-stable and contained in $\operatorname{Ker} j$, then $S = 0$.

<u>Definition</u>.

$$\mathfrak{M} \stackrel{\text{def.}}{=} \{\, (B_h, i_k, j_k) \in \mu^{-1}(0) \mid \text{stable}\,\} / \prod_k \mathrm{GL}(V_k)$$

$$\mathfrak{M}_0 \stackrel{\text{def.}}{=} \mu^{-1}(0) /\!\!/ \prod_k \mathrm{GL}(V_k) \quad \text{(GIT quotient)}$$

$$\left(\begin{aligned} &(B_h, i_k, j_k) \sim (B'_h, i'_k, j'_k) \iff \\ &\qquad\qquad \text{closures of } \prod_k \mathrm{GL}(V_k)\text{-orbits intersect.} \end{aligned} \right)$$

Theorem

1. The differential $d\mu$ is surjective at stable points.
2. The stabilizers are trivial at stable points.
3. $\mathfrak{M}$ is a nonsingular variety of dimension $2\,{}^t\mathbf{v}\mathbf{w} - {}^t\mathbf{v}\mathbf{C}\mathbf{v}$.
4. $\mathfrak{M}_0$ is an affine algebraic variety.
5. The natural map $\pi\colon \mathfrak{M} \to \mathfrak{M}_0$ is projective morphism.

Tautological Complex of vector bundles over $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$:

$$\bigoplus \mathrm{Hom}(V_k^1, V_k^2) \xrightarrow{\sigma} \begin{array}{c} \bigoplus \mathrm{Hom}(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}) \\ \oplus \\ \bigoplus \mathrm{Hom}(W_k, V_k^2) \\ \oplus \\ \bigoplus \mathrm{Hom}(V_k^1, W_k) \end{array} \xrightarrow{\tau} \bigoplus \mathrm{Hom}(V_k^1, V_k^2)$$

where

$$\sigma(\xi_k) = (B_h^2 \xi_{\mathrm{out}(h)} - \xi_{\mathrm{in}(h)} B_h^1) \oplus (-\xi_k i_k^1) \oplus j_k^2 \xi_k$$

$$\tau(C_h \oplus a_k \oplus b_k) = \left(\sum_{k=\mathrm{in}(h)} \varepsilon(h) B_h^2 C_{\overline{h}} + \varepsilon(h) C_h B_{\overline{h}}^1 + i_k^2 b_k + a_k j_k^1 \right)$$

Lemma

σ is injective and τ is surjective. Hence $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ is a vector bundle over $\mathfrak{M}(\mathbf{v}^1,\mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2,\mathbf{w})$.

Theorem (Kronheimer - N) (over $\mathbb{C}$)

When $\mathfrak{M}(\mathbf{v}^1,\mathbf{w})$ is an ALE space, we can construct the "moduli space" of holomorphic vector bundles over $\mathfrak{M}(\mathbf{v}^1,\mathbf{w})$, and can be identified with $\mathfrak{M}(\mathbf{v}^2,\mathbf{w})$. Moreover $\operatorname{Ker}\tau/\operatorname{Im}\sigma$ is identified with the universal bundle.

Convolution Algebra on Homology

$$Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$$

$$\overset{\text{def.}}{=} \{(x^1, x^2) \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \mid \pi(x^1) = \pi(x^2)\}$$

For $c \in H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$, $c' \in H_*(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w}))$, let

$$c * c' \overset{\text{def.}}{=} (p_{13})_*(p_{12}^{-1} c \cap p_{23}^{-1} c') \in H_*(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w}))$$

Fix $\mathbf{w}$ and consider the direct sum $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$. The convolution $*$ makes it a $\mathbb{Q}$-algebra.

For $x^0 \in \mathfrak{M}_0(\mathbf{v}^0, \mathbf{w})$, consider $\bigcup \pi^{-1}(x) \subset \bigcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$. Then $H_*(\bigcup_{\mathbf{v}} \pi^{-1}(x))$ is a $\bigoplus H_*(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$-module.

Convolution Algebra on Functions

Assume the base field is a finite field of p elements. Let $\mathfrak{F}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \stackrel{\text{def.}}{=} \{f : Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \to \mathbb{Q}\}$.

For $f \in \mathfrak{F}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$, $g \in \mathfrak{F}(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})$, let

$$(f * g)(x^1, x^3) \stackrel{\text{def.}}{=} \sum_{x^2} f(x^1, x^2) g(x^2, x^3) \in \mathfrak{F}(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w}).$$

Hecke correspondence

$$\mathfrak{P}_k(\mathbf{v},\mathbf{w}) \overset{\text{def.}}{=} \{(B,i,j,S) \mid \sum_{\mathrm{in}(h)=k} \operatorname{Im} B_h + \operatorname{Im} i_k \subset S \subset V_k$$

$$\text{codimension 1 subspace}\}/\prod \mathrm{GL}(V_l)$$

$$\xrightarrow{\pi_1 \times \pi_2} \mathfrak{M}(\mathbf{v}-\mathbf{e}^k,\mathbf{w}) \times \mathfrak{M}(\mathbf{v},\mathbf{w}).$$

Let $\Delta(\mathbf{v},\mathbf{w})$ be the diagonal in $\mathfrak{M}(\mathbf{v},\mathbf{w})$. Let consider

$$a_\lambda \mapsto [\Delta(\mathbf{v},\mathbf{w})] \quad \left(\lambda(\mathbf{v},\mathbf{w}) = \sum w_k \Lambda_k - \sum v_k \alpha_k\right),$$

$$e_k a_{\lambda(\mathbf{v},\mathbf{w})} \mapsto [\mathfrak{P}_k(\mathbf{v},\mathbf{w})],$$

$$a_{\lambda(\mathbf{v},\mathbf{w})} f_k \mapsto \pm\omega^{-1}[\mathfrak{P}_k(\mathbf{v},\mathbf{w})],$$

where ω is the exchange of the component.

<u>Theorem</u>

1. $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ is a Lagrangian subvariety.
2. The correspondence defines a homomorphism $\Phi\colon \widetilde{\mathbf{U}}_q\big|_{q=1} \to \bigoplus H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$.
3. (Dynkin graph) Φ is surjective and in fact we have
$$\bigoplus_\lambda \mathrm{End}(\Lambda_\lambda) \xrightarrow{\cong} \bigoplus H_{\mathrm{top}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$$
where the range where λ runs is determined (explicitly) by $\mathbf{w}$.
4. There is a q-analogue using $\mathfrak{F}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ instead of homology.

Suppose that the following data are given:

1. P : free $\mathbb{Z}$-module, $P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$
2. a natural pairing $\langle\ ,\ \rangle \colon P \otimes P^* \to \mathbb{Z}$,
3. $\alpha_k \in P$, $h_k \in P^*$ $(k = 1, 2, \ldots, n)$,

such that

(a) $\mathbf{C} = (\langle h_k, \alpha_l \rangle)_{k,l}$ is a symmetric generalized Cartan matrix, i.e., $\langle h_k, \alpha_k \rangle = 2$, and $\langle h_k, \alpha_l \rangle = \langle h_l, \alpha_k \rangle \leq 0$ for $k \neq l$,

(b) $\{\alpha_k\}_{k=1}^{n}$ is linearly independent,

(c) there exists $\Lambda_k \in P$ $(k = 1, 2, \ldots, n)$ such that $\langle h_l, \Lambda_k \rangle = \delta_{kl}$.

Let q be an indeterminate. Let introduce q-integers, q-factorials and q-binomial coefficients:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_! = [n][n-1]\cdots[1],$$

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]_!}{[n]_![m-n]_!} \ .$$

The **quantized enveloping algebra** $\mathbf{U}_q$ is the $\mathbb{Q}(q)$-algebra generated by e_k, f_k ($k = 1, 2, \ldots, n$), q^h ($h \in P^*$) with relations

$$q^h = 1 \quad \text{for } h = 0, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^*,$$

$$q^h e_k q^{-h} = q^{\langle h, \alpha_k \rangle} e_k, \quad q^h f_k q^{-h} = q^{-\langle h, \alpha_k \rangle} f_k,$$

$$e_k f_l - f_l e_k = \delta_{kl} \frac{q^{h_k} - q^{-h_k}}{q - q^{-1}},$$

$$\sum_{p=0}^{1-c_{kl}} (-1)^p \begin{bmatrix} 1 - c_{kl} \\ p \end{bmatrix} e_k^p e_l e_k^{1-c_{kl}-p} = 0 \quad (k \neq l),$$

(the same equation with $e_k \longleftrightarrow f_k$)

$$\mathbf{U}_q{}^+ \stackrel{\text{def.}}{=} \langle e_1, \ldots, e_n \rangle,\ \mathbf{U}_q{}^- \stackrel{\text{def.}}{=} \langle f_1, \ldots, f_n \rangle,\ \mathbf{U}_q{}^0 \stackrel{\text{def.}}{=} \langle q^h \mid h \in P^* \rangle$$

triangle decomposition

$$\mathbf{U}_q \cong \mathbf{U}_q{}^+ \otimes \mathbf{U}_q{}^0 \otimes \mathbf{U}_q{}^-$$

modified enveloping algebra

$$\widetilde{\mathbf{U}}_q \stackrel{\text{def.}}{=} \mathbf{U}_q{}^+ \otimes \left(\bigoplus_{\lambda \in P} \mathbb{Q}(q) a_\lambda \right) \otimes \mathbf{U}_q{}^-$$

where the multiplication is defined by the following rule:

$$a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda,$$

$$e_k a_\lambda = a_{\lambda+\alpha_k} e_k, \quad f_k a_\lambda = a_{\lambda-\alpha_k} f_k,$$

$$(e_k f_l - f_l e_k) a_\lambda = \delta_{kl} [\langle h_k, \lambda \rangle] a_\lambda.$$