Instanton Counting

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based on

Nekrasov : hep-th/0206161

Nekrasov + Okounkov : hep-th/0306238

N + Kota Yoshioka : math.AG/0306198, math.AG/0311058

and some other papers

Representations of Algebraic Groups, Quantum Groups, and Lie Algebras
July 13, 2004, Snowbird, Utah

Additional references

- Braverman : math.AG/0401409

  (affine) Whittaker modules

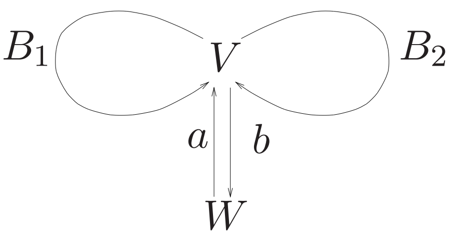
- Göttsche + N + Yoshioka : in preparation

  Donaldson invariants for projective toric surfaces

Quiver varieties and moduli spaces of instantons

- $r \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$
- $M(n,r)$, $M_0(n,r)$: quiver varieties for the Jordan quiver
  - $M(n,r)$: nonsingular of dimension $2nr$ (symplectic)
  - $M_0(n,r)$: affine algebraic variety
  - $\pi\colon M(n,r) \to M_0(n,r)$: projective morphism (resolution of singularities)
  - analog of $T^*\mathfrak{B} \to \mathcal{N}$ (Springer resolution)
  - $M(n,r)$, $M_0(n,r)$: Gieseker/Uhlenbeck (partial) compactification of moduli spaces of instantons

- V, W : cpx vector spaces with $\dim V = n$, $\dim W = r$
- $\mathbb{M}(n,r) = \operatorname{End} V \oplus \operatorname{End} V \oplus \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W)$
- $\mu\colon \mathbb{M}(n,r) \to \operatorname{End}(V)$; $\mu(B_1,B_2,a,b) = [B_1,B_2] + ab$



- $M_0(n,r) = \mu^{-1}(0) /\!\!/ \operatorname{GL}(V)$ (affine GIT quotient)
- $M(n,r) = \mu^{-1}(0)^{\text{stable}} / \operatorname{GL}(V)$
- stable $\overset{\text{def.}}{\Longleftrightarrow}$ $\nexists S \subsetneq V$ with $B_\alpha(S) \subset S$, $\operatorname{Im} a \subset S$

Example $r = 1$: Hilbert scheme of points

Theorem. $M(n,1) = \mathrm{Hilb}^n(\mathbb{A}^2), \quad M_0(n,1) = S^n(\mathbb{A}^2)$

$\mathrm{Hilb}^n(\mathbb{A}^2)$: Hilbert scheme of n points in the affine plane $\mathbb{A}^2$

$S^n(\mathbb{A}^2)$: symmetric product (unordered n points with mult.)

Sketch of Proof

- $\mathrm{Hilb}^n(\mathbb{A}^2) = \{I \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n\}$
- Set $V = \mathbb{C}[x,y]/I$
  $B_1, B_2 = \times x, \times y,\ a(1) = 1 \mod I,\ b = 0$
- ○ $S^n(\mathbb{A}^2) \to M_0(n,1)$ is induced by $\mathbb{A}^{2n} \to \mathbb{M}(n,1)$:
  $(B_1, B_2, a, b) = (\mathrm{diag}(x_1, \ldots, x_n), \mathrm{diag}(y_1, \ldots, y_n), 0, 0)$

Torus action

- $T = T^{r-1}$: maximal torus in $\mathrm{SL}(W)$
- $\widetilde{T} = \mathbb{C}^* \times \mathbb{C}^* \times T \curvearrowright M(n,r), M_0(n,r)$: torus action
  $(B_1, B_2, a, b) \longmapsto (t_1 B_1, t_2 B_2, ae^{-1}, t_1 t_2 eb)$
  $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*,\ e \in T$
- $\mathbb{C}[M_0(n,r)]$ (the coordinate ring of $M_0(n,r)$) is a $\widetilde{T}$-module.
- Similarly $H^i(M(n,r), \mathcal{O})$ and $H^i(M(n,r), E)$
  (E : $\widetilde{T}$-equivariant sheaf) : $\widetilde{T}$-modules

Lemma. *Weight spaces of $\mathbb{C}[M_0(n,r)]$ (and $H^i(M(n,r),E)$) are finite dimensional.*

Let $t_1 = e^{\varepsilon_1}$, $t_2 = e^{\varepsilon_2}$, $e = (e^{a_1}, \dots, e^{a_r}) \in T$ $(\sum_{\alpha=1}^{r} a_\alpha = 0)$.

(identified with characters)

Define

$$Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[M_0(n,r)]$$

$$\overset{\text{vanishing thm.}}{=} \sum_{n=0}^{\infty} \Lambda^{2nr} \sum_i (-1)^i \operatorname{ch}_{\widetilde{T}} H^i(M(n,r), \mathcal{O})$$

(**instanton part of Nekrasov's partition function**)

Problem. *Study* $Z^{\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$.

Example : $r = 1$

$$\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] = \exp\left(\sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{(1 - e^{d\varepsilon_1})(1 - e^{d\varepsilon_2})d} \right)$$

Thus

$$\varepsilon_1 \varepsilon_2 \log \left(\sum_{n=0}^{\infty} \Lambda^{2n} \operatorname{ch}_{\widetilde{T}} \mathbb{C}[S^n(\mathbb{A}^2)] \right) = \sum_{d=1}^{\infty} \frac{\Lambda^{2d}}{d^3} + \cdots \quad (\text{as } \varepsilon_1, \varepsilon_2 \to 0)$$

Nekrasov conjectured the same limiting bahaviour for $r \geq 2$. (Explained later.)

A reformulation in terms of equivariant K-theory

- $K^{\widetilde{T}}(M(n,r))$, $K^{\widetilde{T}}(M_0(n,r))$: Grothendieck groups of $\widetilde{T}$-equivariant sheaves (equivariant K-homology groups)
- They are $R = \mathbb{Z}[e^{\pm\varepsilon_1}, e^{\pm\varepsilon_2}, e^{\pm a_\alpha}]$-modules
- $\mathfrak{R}$: quotient field

Theorem (Localization). *Let ι_0 be the inclusion of the fixed point set $M_0(n,r)^{\widetilde{T}}$ in $M_0(n,r)$. Then*

$$K^{\widetilde{T}}(M_0(n,r)) \otimes_R \mathfrak{R} \xleftarrow[\cong]{\iota_{0*}} K^{\widetilde{T}}(M_0(n,r)^{\widetilde{T}}) \otimes_R \mathfrak{R}.$$

Observation. $M_0(n,r)^{\widetilde{T}} = \{0\}$, *so* RHS $= \mathfrak{R}$. *Moreover,*

$$\mathrm{ch}_{\widetilde{T}}\, \mathbb{C}[M_0(n,r)] = (\iota_{0*})^{-1}\left[\mathcal{O}_{M_0(n,r)}\right].$$

- $M(n,r)^{\widetilde{T}}$ is a finite set, parametrized by r-tuple of Young diagrams $\{\vec{Y} = (Y_1, \dots, Y_r)\}$ (explained below)
- $\iota\colon M(n,r)^{\widetilde{T}} \hookrightarrow M(n,r)$: inclusion

We have the commutative diagram

$$\begin{tikzcd}
{[\mathcal{O}_{M(n,r)}] \in K^{\widetilde{T}}(M(n,r)) \otimes_R \mathcal{R}} \arrow[r, "\cong", "(\iota_*)^{-1}"'] \arrow[d, "\pi_*"'] & \bigoplus_{\vec{Y}} \mathcal{R} \arrow[d, "\sum_{\vec{Y}}"] \\
{[\mathcal{O}_{M_0(n,r)}] \in K^{\widetilde{T}}(M_0(n,r)) \otimes_R \mathcal{R}} \arrow[r, "\cong", "(\iota_{0*})^{-1}"'] & \mathcal{R}
\end{tikzcd}$$

As $M(n,r)$ is smooth, we have explicit formula:

$$(\iota_*)^{-1}[\mathcal{O}_{M(r,n)}] = \bigoplus_{\vec{Y}} \frac{1}{\mathrm{ch}_{\widetilde{T}}\left(\bigwedge_{-1} T^*_{\vec{Y}}\right)}, \quad \bigwedge_{-1} T^*_{\vec{Y}} = \sum (-1)^k \bigwedge^k T^*_{\vec{Y}}$$

This formulation makes sense for equivariant homology groups:

$$\begin{tikzcd}
{[M(n,r)] \in H_*^{\widetilde{T}}(M(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_*)^{-1}"'] \arrow[d, "\pi_*"'] & \bigoplus_{\vec{Y}} \mathcal{S} \arrow[d, "\sum_{\vec{Y}}"] \\
{[M_0(n,r)] \in H_*^{\widetilde{T}}(M_0(n,r)) \otimes_S \mathcal{S}} \arrow[r, "\cong", "(\iota_{0*})^{-1}"'] & \mathcal{S}
\end{tikzcd}$$

and

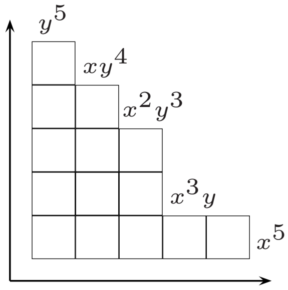
$$(\iota_*)^{-1}[M(n,r)] = \bigoplus_{\vec{Y}} \frac{1}{\mathrm{Euler}\,(T_{\vec{Y}})}$$

where

- S : symmetric power of $\mathrm{Lie}(\widetilde{T})^* = \mathbb{C}[\varepsilon_1, \varepsilon_2, a_\alpha] = H^*_{\widetilde{T}}(\mathrm{pt})$
- $\mathcal{S}$: quotient field

Fixed point set $M(n,r)^{\widetilde{T}}$

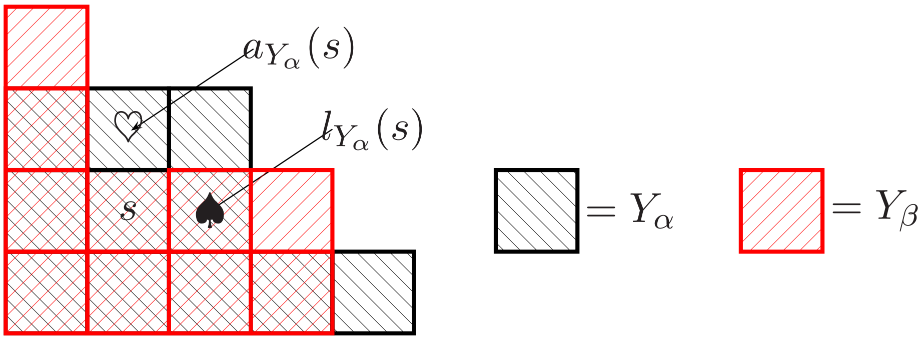
- $[B_1, B_2, a, b] \in M(n,r)$ is fixed by $T = T^{r-1}$
  $\iff$ a direct sum of $M(n_\alpha, 1)$ $(\sum n_\alpha = n)$
  ($\because W$ decomposes into 1-dim rep's of T)
- $M(n_\alpha, 1) = \mathrm{Hilb}^{n_\alpha}(\mathbb{A}^2) \ni I_\alpha$ is fixed by $\mathbb{C}^* \times \mathbb{C}^*$
  $\iff$ I_α is generated by monomials in x, y
  $\iff$ I_α corresponds to a Young diagram Y_α



- $M(n,r)^{\widetilde{T}} \cong \{\vec{Y} = (Y_1, \ldots, Y_r) \mid \sum |Y_\alpha| = n\}$
- formula for the character of the tangent space:

$$\begin{aligned} &\mathrm{ch}_{\widetilde{T}} \left(\bigwedge\nolimits_{-1} T^*_{\vec{Y}}\right) \\ &\quad = \prod_{\alpha,\beta} \prod_{s \in Y_\alpha} \left(1 - \exp(-l_{Y_\beta}(s)\varepsilon_1 + (1 + a_{Y_\alpha}(s))\varepsilon_2 + a_\beta - a_\alpha)\right) \\ &\qquad\qquad \times \prod_{t \in Y_\beta} \left(1 - \exp((1 + l_{Y_\alpha}(t))\varepsilon_1 - a_{Y_\beta}(t)\varepsilon_2 + a_\beta - a_\alpha)\right) \end{aligned}$$

where



Let $r = 1$. Put $\varepsilon_1 = -\varepsilon_2$. The homology version is

$$\sum_Y \left(-\frac{\Lambda}{\varepsilon_1}\right)^{2|Y|} \prod_{s \in Y} \frac{1}{h(s)^2}.$$

The hook length formula says

$$\prod_{s \in Y} \frac{1}{h(s)} = \frac{\dim R_Y}{n!},$$

where R_Y is the irreducible representation of S_n associated with Y. Note

$$\sum_{|Y|=n} \dim R_Y^2 = n!$$

Therefore

$$Z(\varepsilon_1, -\varepsilon_1; \Lambda) = \exp\left(-\frac{\Lambda^2}{\varepsilon_1^2}\right).$$

Recall that we have computed $\mathrm{ch}_{\widetilde{T}}\,\mathbb{C}[S^n(\mathbb{A}^2)]$.

Corollary.

$$\exp\left(\sum_{d=1}^{\infty}\frac{\Lambda^{2d}}{(1-e^{d\varepsilon_1})(1-e^{d\varepsilon_2})d}\right)$$
$$=\sum_{Y}\prod_{s\in Y}\frac{\Lambda^{2|Y|}}{(1-e^{-l_Y(s)\varepsilon_1+(1+a_Y(s))\varepsilon_2})(1-e^{(1+l_Y(s))\varepsilon_1-a_Y(s)\varepsilon_2})}$$

Purely combinatorial proof : Cauchy formula for Macdonald polynomials, i.e., a generalization of the proof in the previous transparency.

Remark. *Appearance of Macdonald polynomials are natural in view of Haiman's work.*

Perturbation Part

$$\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)\overset{\text{def.}}{=}\frac{1}{2\varepsilon_1\varepsilon_2}\left(-\frac{1}{6}\left(x+\frac{1}{2}(\varepsilon_1+\varepsilon_2)\right)^3+x^2\log(\Lambda)\right)$$
$$+\sum_{n\geq 1}\frac{1}{n}\frac{e^{-nx}}{(e^{n\varepsilon_1}-1)(e^{n\varepsilon_2}-1)},$$

$$\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(x;\Lambda)\overset{\text{def.}}{=}\gamma_{\varepsilon_1,\varepsilon_2}(x;\Lambda)+\frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{\pi^2 x}{6}-\zeta(3)\right)$$
$$+\frac{\varepsilon_1+\varepsilon_2}{2\varepsilon_1\varepsilon_2}\left(x\log(\Lambda)+\frac{\pi^2}{6}\right)+\frac{\varepsilon_1^2+\varepsilon_2^2+3\varepsilon_1\varepsilon_2}{12\varepsilon_1\varepsilon_2}\log(\Lambda),$$

$$Z^{\text{pert}}(\varepsilon_1,\varepsilon_2,\vec{a};\Lambda)\overset{\text{def.}}{=}\exp\left(-\sum_{\alpha\neq\beta}\widetilde{\gamma}_{\varepsilon_1,\varepsilon_2}(a_\alpha-a_\beta;\Lambda)\right)$$

Nekrasov Conjecture (2002) - Part 1

Define the full partition function by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) \stackrel{\text{def.}}{=} Z^{\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) Z^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda).$$

Conjecture. *Suppose $r \geq 2$.*

$$\varepsilon_1 \varepsilon_2 Z(\varepsilon_1, \varepsilon_2, \vec{a}; \mathfrak{q}) = F_0 + \dots,$$

where F_0 is the **Seiberg-Witten prepotential**, *given by the period integral of certain curves.*

Seiberg-Witten geometry

A family of curves (*Seiberg-Witten curves*) parametrized by $\vec{X} = (X_1, \dots, X_r)$ with $\prod X_\alpha = 1$:

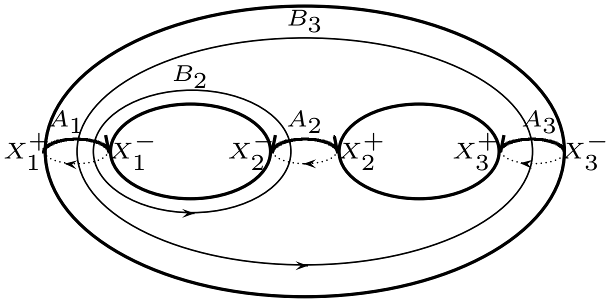
$$C_{\vec{u}} : y^2 = P(X)^2 - 4X^r \Lambda^{2r}, \qquad P(X) = \prod_{\alpha=1}^{r} (X - X_\alpha).$$

$C_{\vec{u}} \ni (y, X) \mapsto X \in \mathbb{P}^1$ gives a structure of hyperelliptic curves. The hyperelliptic involution ι is given by $\iota(y, X) = (-y, X)$.

Define the *Seiberg-Witten differential* (multivalued) by

$$dS = -\frac{1}{2\pi} \log X \frac{(XP'(X) - \frac{r}{2} P(X))dX}{yX}.$$

Find branched points $X_\alpha^\pm$ near X_α (Λ small). Choose cycles A_α, B_α ($\alpha = 2, \ldots, r$) as



Put

$$a_\alpha = \int_{A_\alpha} dS, \qquad a_\beta^D = 2\pi\sqrt{-1}\int_{B_\beta} dS$$

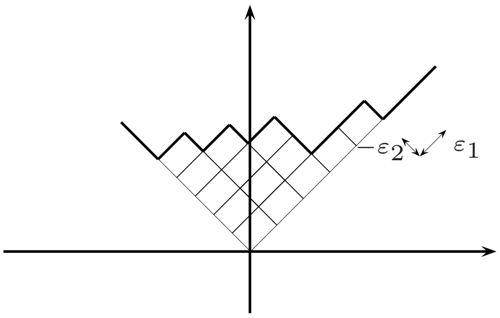
Then

$$\exists F_0 : \quad a_\beta^D = -\frac{\partial F_0}{\partial a_\beta}$$

Random partition

Theorem (Nekrasov-Okounkov). *Conjecture is true.*

Rotate the Young diagram 45° and consider it as a graph of a funtion.

$f_Y(x) =$ 

It satisfies

$$f(x) = |x| \quad \text{for } |x| \gg 0, \qquad |f(x) - f(y)| \leq |x - y|.$$

Put

$$f_{\vec{a},\vec{Y}}(x \mid \varepsilon_1, \varepsilon_2) = \sum_{\alpha=1}^{r} f_{Y_\alpha}(x - a_\alpha \mid \varepsilon_1, \varepsilon_2).$$

Then

- $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$ is a sum over the space of such functions.
- Consider it as a mesure on the space of Lipschitz functions.

As $\varepsilon_1, \varepsilon_2 \to 0$, these measure converge to the delta measure at a single function f_*.

- Analysis of $f_* \longrightarrow$ SW geometry

Blowup equation

Theorem (N+Yoshioka). *Assume $r \geq 2$. Then Z satisfies*

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{\vec{k} \in \mathbb{Z}^{r-1}} \exp\left[\left(-\frac{d - \frac{r}{2}}{12} - \frac{r}{48}\right)(r-1)(\varepsilon_1 + \varepsilon_2)\right]$$
$$\times Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}; \Lambda e^{-\varepsilon_1/4}) Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}; \Lambda e^{-\varepsilon_2/4}).$$

- $\mathbb{Z}^{r-1}$ = the weight lattice of $\mathrm{SU}(r)$
- the equation determines the coefficients of Λ^{2nr} recursively.
- the equation guarantees $\exists F_0 = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log Z$, and gives a differential equation for it. Conjecturally the Seiberg-Witten prepotential satisfies the same equation. (True for equivariant homology version.)

Nekrasov Conjecture (2002) - Part 2

Put $\varepsilon_1 = -\varepsilon_2 = ig_s$. ($g_s$: string coupling constant)

Conjecture. *Expand as*

$$\log Z^{\mathrm{inst}}(ig_s, -ig_s, \vec{a}; \Lambda) = F_0 g_s^{-2} + F_1 g_s^0 + \cdots + F_g g_s^{2g-2} + \cdots .$$

Then F_g is the genus g Gromov-Witten invariant for certain noncompact Calabi-Yau 3-fold.

e.g., $r = 2$, Calabi-Yau = canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$

- based on geometric engineering by Katz-Klemm-Vafa (1996)
- Physical proof by Iqbal+Kashani-Poor, hep-th/0212279, hep-th/0306032
- mathematical proof for $r = 2$ by Zhou, math.AG/0311237.

Vector bundles

The vector space V defines a vector bundle over $M(n,r)$ of rank n. It has a natural $\widetilde{T}$-equivariant structure.

Problem. *Study*

$$\sum(-1)^i \operatorname{ch}_{\widetilde{T}} H^i(M(n,r), V^{\otimes k}).$$

$\exists$ localization formula

? limit $\varepsilon_1, \varepsilon_2 \to 0$

? differential equation

Remark. Haiman : $r = 1$ case $H^i(\mathrm{Hilb}, V^{\otimes k}) = 0$ for $i > 0$.

$\Longrightarrow$ positivity result