D Remaining Proofs

Proof of Lemma 1

From the distributivity, we have $0_{A,B} \otimes 0_{C,C} = 0_{A\otimes C,B\otimes C}$. Therefore

$$\begin{aligned}
\mathrm{tr}^C_{A,B}(0_{A\otimes C,B\otimes C}) &= \mathrm{tr}^C_{A,B}(0_{A,B} \otimes 0_{C,C}) \\
\text{(superposing)} &= 0_{A,B} \otimes \mathrm{tr}^C_{\mathbf{I},\mathbf{I}}(0_{C,C}) \\
\text{(bilinearity)} &= 0_{A,B}.
\end{aligned}$$

We move to $\mathrm{tr}^C_{A,B}(f + g) = \mathrm{tr}^C_{A,B}(f) + \mathrm{tr}^C_{A,B}(g)$. First, we have (for each $i \in \{1, 2\}$)

$$\pi_i = (\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] : (A_1 \otimes C) \oplus (A_2 \otimes C) \to A_i \otimes C. \tag{1}$$

Proof:

$$\begin{aligned}
& \pi_i \\
(-\otimes C \text{ preserves biproducts}) &= \pi_i \circ \langle \pi_1 \otimes \mathrm{id}_C, \pi_2 \otimes \mathrm{id}_C \rangle \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \\
&= (\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C].
\end{aligned}$$

Second, we have

$$\langle \mathrm{tr}^C_{A,B}(f_1), \mathrm{tr}^C_{A,B}(f_2) \rangle = \mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle). \tag{2}$$

Proof: We calculate the first and second component of $\mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle)$.

$$\begin{aligned}
& \pi_i \circ \mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle) \\
\text{(naturality of trace)} &= \mathrm{tr}^C_{A,B}((\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle) \\
\text{(Equation 1)} &= \mathrm{tr}^C_{A,B}(\pi_i \circ \langle f_1, f_2 \rangle) \\
&= \mathrm{tr}^C_{A,B}(f_i)
\end{aligned}$$

(where $i \in \{1, 2\}$). From this, we conclude Equation 2. We now prove the lemma in question.

$$\begin{aligned}
& \mathrm{tr}^C_{A,B}(f) + \mathrm{tr}^C_{A,B}(g) \\
\text{(definition of +)} &= [\mathrm{id}_B, \mathrm{id}_B] \circ \langle \mathrm{tr}^C_{A,B}(f), \mathrm{tr}^C_{A,B}(g) \rangle \\
\text{(Equation 2)} &= [\mathrm{id}_B, \mathrm{id}_B] \circ \mathrm{tr}^C_{A,B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\
\text{(naturality of trace)} &= \mathrm{tr}^C_{A,B}(([\mathrm{id}_B, \mathrm{id}_B] \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\
&= \mathrm{tr}^C_{A,B}([([\mathrm{id}_B, \mathrm{id}_B] \circ \iota_1) \otimes \mathrm{id}_C, ([\mathrm{id}_B, \mathrm{id}_B] \circ \iota_2) \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\
&= \mathrm{tr}^C_{A,B}([\mathrm{id}_{B\otimes C}, \mathrm{id}_{B\otimes C}] \circ \langle f, g \rangle) \\
&= \mathrm{tr}^C_{A,B}(f + g).
\end{aligned}$$

Proof of Lemma 2

In this proof we omit the annotation of objects on trace operators.

(1) We have

$$\begin{aligned}
\operatorname{tr}(f) &= \operatorname{tr}\left(\sum_{1\leqslant i,j\leqslant 2} (\mathrm{id}_B \otimes \iota_i) \circ f_{ij} \circ (\mathrm{id}_A \otimes \pi_j)\right)\\
(\text{Lemma 1}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\Big((\mathrm{id}_B \otimes \iota_i) \circ f_{ij} \circ (\mathrm{id}_A \otimes \pi_j)\Big)\\
(\text{sliding}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\Big(f_{ij} \circ (\mathrm{id}_A \otimes (\pi_j \circ \iota_i)\Big)\\
\left(\pi_j \circ \iota_i = \begin{cases} 0 & (j \neq i) \\ \mathrm{id} & (j = i)\end{cases}\right) &= \sum_{1\leqslant i\leqslant 2} \operatorname{tr}\left(f_{ii}\right).
\end{aligned}$$

(2) We have

$$\begin{aligned}
\operatorname{tr}(f) &= \operatorname{tr}\left(\sum_{1\leqslant i,j\leqslant 2} (\iota_i \otimes \mathrm{id}_A) \circ f_{ij} \circ (\pi_j \otimes \mathrm{id}_A)\right)\\
(\text{Lemma 1}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\Big((\iota_i \otimes \mathrm{id}_A) \circ f_{ij} \circ (\pi_j \otimes \mathrm{id}_A)\Big)\\
(\text{naturality of trace}) &= \sum_{1\leqslant i,j\leqslant 2} \iota_i \circ \operatorname{tr}\left(f_{ij}\right) \circ \pi_j.
\end{aligned}$$

From this, we conclude that the matrix representation of $\operatorname{tr}(f)$ is $\begin{pmatrix} \operatorname{tr}(f_{11}) & \operatorname{tr}(f_{12}) \\ \operatorname{tr}(f_{21}) & \operatorname{tr}(f_{22}) \end{pmatrix}$.

Proof of Lemma 3

Let $f = \langle\!\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\!\rangle : A \to B_1 \oplus B_2$ be an $\mathbf{Int}(C)$-morphism where each component f_{ij} is in $\mathbf{Int}(C)(A, (B_i^+, B_j^-))$. This morphism corresponds to the following matrix of C-morphisms $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Similarly, the tuple $g = [[g_{11}, g_{12}, g_{21}, g_{22}]] : B_1 \oplus B_2 \to C$ with $g_{ij} \in \mathbf{Int}(C)((B_i^+, B_j^-), C)$ corresponds to the matrix $\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}$ (note the order of

g_{21} and g_{12}). The composite $g \circ f$ in $\mathbf{Int}(\mathcal{C})$ is the trace of the following $\mathcal{C}$-morphism:

$$\begin{aligned}
&(C^+ \otimes \sigma) \circ (g \otimes A^-) \circ ((B_1^+ \oplus B_2^+) \otimes \sigma) \circ (f \otimes C^-) \circ (A^+ \otimes \sigma) \\
=& \begin{pmatrix} C^+ \otimes \sigma & 0 \\ 0 & C^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} g_{11} \otimes A^- & g_{21} \otimes A^- \\ g_{12} \otimes A^- & g_{22} \otimes A^- \end{pmatrix} \circ \\
& \begin{pmatrix} B_1^+ \otimes \sigma & 0 \\ 0 & B_2^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} f_{11} \otimes C^- & f_{12} \otimes C^- \\ f_{21} \otimes C^- & f_{22} \otimes C^- \end{pmatrix} \circ \begin{pmatrix} A^+ \otimes \sigma & 0 \\ 0 & A^+ \otimes \sigma \end{pmatrix} \\
=& \begin{pmatrix} (C^+ \otimes \sigma) \circ (g_{11} \otimes A^-) & (C^+ \otimes \sigma) \circ (g_{21} \otimes A^-) \\ (C^+ \otimes \sigma) \circ (g_{12} \otimes A^-) & (C^+ \otimes \sigma) \circ (g_{22} \otimes A^-) \end{pmatrix} \circ \\
& \begin{pmatrix} (B_1^+ \otimes \sigma) \circ (f_{11} \otimes C^-) \circ (A^+ \otimes \sigma) & (B_1^+ \otimes \sigma) \circ (f_{12} \otimes C^-) \circ (A^+ \otimes \sigma) \\ (B_2^+ \otimes \sigma) \circ (f_{21} \otimes C^-) \circ (A^+ \otimes \sigma) & (B_2^+ \otimes \sigma) \circ (f_{22} \otimes C^-) \circ (A^+ \otimes \sigma) \end{pmatrix} \\
=& h
\end{aligned}$$

where h is the morphism whose matrix consists of the following elements:

$$\begin{aligned}
h_{ij} =& (C^+ \otimes \sigma) \circ (g_{1i} \otimes A^-) \circ (B_1^+ \otimes \sigma) \circ (f_{1j} \otimes C^-) \circ (A^+ \otimes \sigma) \\
&+ (C^+ \otimes \sigma) \circ (g_{2i} \otimes A^-) \circ (B_2^+ \otimes \sigma) \circ (f_{2j} \otimes C^-) \circ (A^+ \otimes \sigma).
\end{aligned}$$

Therefore

$$\mathrm{tr}^{B_1^- \oplus B_2^-}_{A^+, C^+}(h) = \sum_{1 \le i,j \le 2} \mathrm{tr}^{B_1^- \oplus B_2^-}_{A^+, C^+}(h_{ij}) = \sum_{1 \le i,j \le 2} g_{ij} \circ f_{ij}.$$

Thus we obtain $\langle\langle f_{ij} \rangle\rangle \circ [[g_{ij}]] = \sum g_{ij} \circ f_{ij}$.

We next calculate the composition of $h : C \to A$ and $f = \langle\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\rangle : A \to B_1 \oplus B_2$ in $\mathbf{Int}(\mathcal{C})$. The composition is the trace of the following morphism:

$$\begin{aligned}
&((B_1^+ \oplus B_2^+) \otimes \sigma) \circ (f \otimes C^-) \circ (A^+ \otimes \sigma) \circ (h \otimes (B_1^- \oplus B_2^-)) \circ (C^+ \otimes \sigma) \\
=& \begin{pmatrix} B_1^+ \otimes \sigma & 0 \\ 0 & B_2^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} f_{11} \otimes C^- & f_{12} \otimes C^- \\ f_{21} \otimes C^- & f_{22} \otimes C^- \end{pmatrix} \circ \\
& \begin{pmatrix} A^+ \otimes \sigma & 0 \\ 0 & A^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} h \otimes B_1^- & 0 \\ 0 & h \otimes B_2^- \end{pmatrix} \circ \begin{pmatrix} C^+ \otimes \sigma & 0 \\ 0 & C^+ \otimes \sigma \end{pmatrix} \\
=& \begin{pmatrix} (B_1^+ \otimes \sigma) \circ (f_{11} \otimes C^-) & (B_1^+ \otimes \sigma) \circ (f_{12} \otimes C^-) \\ (B_2^+ \otimes \sigma) \circ (f_{21} \otimes C^-) & (B_2^+ \otimes \sigma) \circ (f_{22} \otimes C^-) \end{pmatrix} \circ \\
& \begin{pmatrix} (A^+ \otimes \sigma) \circ (h \otimes B_1^-) \circ (C^+ \otimes \sigma) & 0 \\ 0 & (A^+ \otimes \sigma) \circ (h \otimes B_2^-) \circ (C^+ \otimes \sigma) \end{pmatrix} \\
=& h
\end{aligned}$$

where $h_{ij} = (B_i^+ \otimes \sigma) \circ (f_{ij} \otimes C^-) \circ (A^+ \otimes \sigma) \circ (h \otimes B_j^-) \circ (C^+ \otimes \sigma)$. Therefore we obtain

$$\mathrm{tr}^{A^-}_{C^+ \otimes (B_1^- \oplus B_2^-), (B_1^+ \oplus B_2^+) \otimes C_-}(h) = \begin{pmatrix} \mathrm{tr}(h_{11}) & \mathrm{tr}(h_{12}) \\ \mathrm{tr}(h_{21}) & \mathrm{tr}(h_{22}) \end{pmatrix} = \begin{pmatrix} f_{11} \circ h & f_{12} \circ h \\ f_{21} \circ h & f_{22} \circ h \end{pmatrix}.$$

This matrix corresponds to the tuple $\langle\langle f_{ij} \circ h \rangle\rangle$. Hence we obtain $\langle\langle f_{ij} \rangle\rangle \circ h = \langle\langle f_{ij} \circ h \rangle\rangle$.

Proof of Proposition 5

We show that the tupling, cotupling, projections and injections satisfy Condition B-1.

$$
\begin{aligned}
\pi_1 \circ \langle f, g\rangle &= [[\mathrm{id}, 0, 0, 0]] \circ \langle\!\langle f, 0, 0, g\rangle\!\rangle \\
&= \mathrm{id} \circ f + 0 \circ 0 + 0 \circ 0 + 0 \circ g \\
&= f. \\
&\quad \text{(we omit the proof for } \pi_2 \circ \langle f, g\rangle = g\text{)} \\
[f, g] \circ \iota_1 &= [[f, 0, 0, g]] \circ \langle\!\langle \mathrm{id}, 0, 0, 0\rangle\!\rangle \\
&= f \circ \mathrm{id} + 0 \circ 0 + 0 \circ 0 + g \circ 0 \\
&= f. \\
&\quad \text{(we omit the proof for } [f, g] \circ \iota_2 = g\text{)} \\
\langle f, g\rangle \circ h &= \langle\!\langle f, 0, 0, g\rangle\!\rangle \circ h \\
&= \langle\!\langle f \circ h, 0, 0, g \circ h\rangle\!\rangle \\
&= \langle f \circ h, g \circ h\rangle. \\
h \circ \langle f, g\rangle &= h \circ [[f, 0, 0, g]] \\
&= [[h \circ f, 0, 0, h \circ g]] \\
&= [h \circ f, h \circ g] \\
\pi_1 \circ \iota_1 &= [[\mathrm{id}, 0, 0, 0]] \circ \langle\!\langle \mathrm{id}, 0, 0, 0\rangle\!\rangle \\
&= \mathrm{id} \circ \mathrm{id} + 0 + 0 + 0 \\
&= \mathrm{id} \\
&\quad \text{(we omit the proof for } \pi_2 \circ \iota_2 = \mathrm{id}\text{)} \\
\langle f \circ \pi_1, g \circ \pi_2\rangle &= \langle\!\langle [[f, 0, 0, 0]], \mathbf{0}, \mathbf{0}, [[0, 0, 0, g]]\rangle\!\rangle \\
&= [[\langle\!\langle f, 0, 0, 0\rangle\!\rangle, \mathbf{0}, \mathbf{0}, \langle\!\langle 0, 0, 0, g\rangle\!\rangle]] \\
&= [\iota_1 \circ f, \iota_2 \circ g].
\end{aligned}
$$

Here $\mathbf{0} = [[0, 0, 0, 0]]$.

Proof of Theorem 3

That $\mathbf{Int}(C)$ has semibiproducts is already proved in Proposition 5. We show that N_C preserves semibiproducts. Below we just write N for N_C. We have seen that $N0$ is a zero object in $\mathbf{Int}(C)$. Next, we have (below we write $\mathbf{0}$ for $[[0, 0, 0, 0]]$)

$$
\begin{aligned}
\langle N\pi_1, N\pi_2\rangle \circ [N\iota_1, N\iota_2] &= \langle\!\langle N\pi_1, 0, 0, N\pi_2\rangle\!\rangle \circ [[N\iota_1, 0, 0, N\iota_2]] \\
&= \langle\!\langle [[N\,\mathrm{id}, 0, 0, 0]], \mathbf{0}, \mathbf{0}, [[0, 0, 0, N\,\mathrm{id}]]\rangle\!\rangle \\
&= N\,\mathrm{id} \oplus N\,\mathrm{id} \\
[N\iota_1, N\iota_2] \circ \langle N\pi_1, N\pi_2\rangle &= [[N\iota_1, 0, 0, N\iota_2]] \circ \langle\!\langle N\pi_1, 0, 0, N\pi_2\rangle\!\rangle \\
&= N\iota_1 \circ N\pi_1 + N\iota_2 \circ N\pi_2 \\
&= (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \otimes \mathbf{I} \\
&= N\,\mathrm{id}\,.
\end{aligned}
$$

Therefore N preserves semibiproducts.

Proof of Proposition 10

First, we note that

$$\langle\!\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\!\rangle^* = [[f_{11}^*, f_{21}^*, f_{12}^*, f_{22}^*]].$$

In particular, $\iota_i^* = \pi_i$ for $i = 1, 2$.

We only consider the case where the cut elimination happens on an &-rule and $\oplus_0$-rule.

$$\begin{aligned}
&[\![\mathrm{Cut}(\mathrm{And}(\Pi_1, \Pi_2), \mathrm{Or}_0(\Pi))]\!] \\
= &(\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ([\![\mathrm{And}(\Pi_1, \Pi_2)]\!] \otimes [\![\mathrm{Or}_0(\Pi)]\!]) \\
= &(\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ (((\Gamma \otimes \alpha_1) \circ [\![\Pi_1]\!]) + ((\Gamma \otimes \alpha_2) \circ [\![\Pi_2]\!])) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!])) \\
&\text{(tensor products preserve semibiproducts)} \\
= &(\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ (((\Gamma \otimes \alpha_1) \circ [\![\Pi_1]\!]) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!]) + \\
&((\Gamma \otimes \alpha_2) \circ [\![\Pi_2]\!]) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!]))) \\
= &(\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ((\Gamma \otimes \alpha_1 \otimes \alpha_1 \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + ((\Gamma \otimes \alpha_2 \otimes \alpha_1 \otimes \Delta) \circ [\![\Pi_2]\!] \otimes [\![\Pi]\!])) \\
= &(\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\alpha_1 \otimes \alpha_1)) \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + (\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\alpha_2 \otimes \alpha_1)) \otimes \Delta) \circ ([\![\Pi_2]\!] \otimes [\![\Pi]\!]) \\
= &(\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\mathcal{U} \otimes (\alpha_1^* \circ \alpha_1))) \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + \\
&(\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\mathcal{U} \otimes (\alpha_2^* \circ \alpha_1))) \otimes \Delta) \circ ([\![\Pi_2]\!] \otimes [\![\Pi]\!]) + \\
&(\alpha_1^* \circ \alpha_1 = \iota_1^* \circ a^* \circ a \circ \iota_1 = \pi_1 \circ \iota_1 = \mathrm{id}, \alpha_2^* \circ \alpha_1 = \iota_2^* \circ a^* \circ a \circ \iota_1 = \pi_2 \circ \iota_1 = 0) \\
= &(\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) \\
= &[\![\mathrm{Cut}(\Pi_1, \Pi)]\!] .
\end{aligned}$$

Proof of Proposition 11

In order to prove Proposition 11, we introduce *coherence* $\frown_A \subset \mathbb{N}^2 \times \mathbb{N}^2$ for each formula A.

Definition 8. *For a reflective relation $\frown \subset X \times X$, we define $\asymp, \smile, \frown \subset X \times X$ by*

$$\begin{aligned}
x \asymp y &\Leftrightarrow x = y \vee \neg(x \frown y) \\
x \frown y &\Leftrightarrow x \neq y \wedge x \frown y \\
x \smile y &\Leftrightarrow \neg(x \frown y)
\end{aligned}$$

Definition 9. *For a formula A, we define a reflective relation $\frown_A \subset \mathbb{N}^2 \times \mathbb{N}^2$ by*

- $(n, m) \frown_\alpha (a, b) \Longleftrightarrow n = a \wedge m = b$
- $(n, m) \frown_{\alpha^\perp} (a, b) \Longleftrightarrow (m, n) \asymp_\alpha (b, a)$
- $(\lceil n, k \rceil, \lceil m, l \rceil) \frown_{A \otimes B} (\lceil a, c \rceil, \lceil b, d \rceil)$ *iff*

$$(n, m) \frown_A (a, b) \wedge (k, l) \frown_B (c, d)$$

– $(\lceil n,k\rceil,\lceil m,l\rceil)\smallsmile_{A\wp B}(\lceil a,c\rceil,\lceil b,d\rceil)$ *iff*

$$(n,m)\frown_A(a,b)\vee(k,l)\frown_B(c,d)$$

or

$$(\lceil n,k\rceil,\lceil m,l\rceil)=(\lceil a,c\rceil,\lceil b,d\rceil)$$

– $(n,m)\smallsmile_{A\oplus B}(a,b)$ *iff*

$$(n,m)=(\overline{k},\overline{l})\wedge(a,b)=(\overline{c},\overline{d})\wedge(k,l)\frown_A(c,d)$$

or

$$(n,m)=(\underline{k},\underline{l})\wedge(a,b)=(\underline{c},\underline{d})\wedge(k,l)\frown_B(c,d)$$

or

$$(n,m)=(a,b)$$

– $(n,m)\smallsmile_{A\& B}(a,b)$ *iff* $(m,n)\asymp_{A^\perp\oplus B^\perp}(b,a)$

We can extend $(-)^\perp$ to all formulae

$$(n,m)\smallsmile_{A^\perp}(a,b)\Longleftrightarrow(m,n)\asymp_A(b,a).$$

This is consistent to the extention of the negation of the MALL since we have:

$$\begin{aligned}\smallsmile_{(A^\perp)^\perp}&=\smallsmile_A\\ \smallsmile_{(A\otimes B)^\perp}&=\smallsmile_{A^\perp\wp B^\perp}\\ \smallsmile_{(A\wp B)^\perp}&=\smallsmile_{A^\perp\otimes B^\perp}\\ \smallsmile_{(A\& B)^\perp}&=\smallsmile_{A^\perp\oplus B^\perp}\\ \smallsmile_{(A\oplus B)^\perp}&=\smallsmile_{A^\perp\& B^\perp}\end{aligned}$$

For a k-tuple of formulae $A_1,\cdots,A_k$ and k-tuples of natural numbers $\boldsymbol{n},\boldsymbol{m},\boldsymbol{a},\boldsymbol{b}$, we write $(\boldsymbol{n},\boldsymbol{m})\smallsmile_{A_1,\cdots,A_k}(\boldsymbol{a},\boldsymbol{b})$ when

$$(n_i,m_i)\asymp_{A_i}(a_i,b_i)\text{ for each }1\le i\le k\text{ implies }(\boldsymbol{n},\boldsymbol{m})=(\boldsymbol{a},\boldsymbol{b}).$$

We define $(\boldsymbol{n},\boldsymbol{m})\asymp_{A_1,\cdots,A_k}(\boldsymbol{a},\boldsymbol{b})$ by the negation of $(\boldsymbol{n},\boldsymbol{m})\smallsmile_{A_1,\cdots,A_k}(\boldsymbol{a},\boldsymbol{b})$.

$$(\boldsymbol{n},\boldsymbol{m})\asymp_{A_1,\cdots,A_k}(\boldsymbol{a},\boldsymbol{b})\Longleftrightarrow\forall i.(n_i,m_i)\asymp_{A_i}(a_i,b_i).$$

Proposition 13. *Let $\Pi\vdash A_1,\cdots,A_n$ be a MALL proof. For $(\boldsymbol{n},\boldsymbol{m}),(\boldsymbol{a},\boldsymbol{b})\in\|\Pi\|$,*

$$(\boldsymbol{n},\boldsymbol{m})\smallsmile_{A_1,\cdots,A_k}(\boldsymbol{a},\boldsymbol{b})$$

Proof. We prove by the induction of Π.
• Ax $\vdash A,A^\perp$
For $(nm,mn),(ab,ba)\in\|\,\mathrm{Ax}_A\,\|$,

$$\begin{aligned}(n,m)\asymp_A(a,b)\wedge(m,n)\asymp_{A^\perp}(b,a)&\Leftrightarrow(n,m)\asymp_A(a,b)\wedge(n,m)\smallsmile_A(a,b)\\&\Rightarrow(n,m)=(a,b)\end{aligned}$$

• $\mathrm{Cut}(\Pi_0, \Pi_1) \vdash \Gamma, \Delta$
Let $(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-)$ and $(\boldsymbol{a}^+\boldsymbol{b}^+, \boldsymbol{a}^-\boldsymbol{b}^-)$ be elements of $\|\mathrm{Cut}(\Pi_0, \Pi_1)\|$ such that

$$(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \asymp_{\Gamma,\Delta} (\boldsymbol{a}^+\boldsymbol{b}^+, \boldsymbol{a}^-\boldsymbol{b}^-).$$

From the definition of $\| - \|$, there are natural numbers i, j, p, q such that

$$\begin{array}{ll} (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_0\| & (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in \|\Pi_1\| \\ (\boldsymbol{a}^+ p, \boldsymbol{a}^- q) \in \|\Pi_0\| & (q\boldsymbol{b}^+, p\boldsymbol{b}^-) \in \|\Pi_1\| \end{array}$$

We show $(i, j) \asymp_A (p, q)$: If $(i, j) \frown_A (p, q)$ then $(j, i) \asymp_{A^\perp} (q, p)$. We have

$$(j\boldsymbol{m}^+, i\boldsymbol{m}^-) \asymp_{A^\perp,\Delta} (q\boldsymbol{b}^+, p\boldsymbol{b}^-)$$

since $(\boldsymbol{m}^+, \boldsymbol{m}^-) \asymp_\Delta (\boldsymbol{b}^+, \boldsymbol{b}^-)$. Then by the I.H., $(j, i) = (q, p)$. This contradicts to $(i, j) \frown_A (p, q)$. Hence $(i, j) \asymp_A (p, q)$.

Since $(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-)$, we have $(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \asymp_{\Gamma,A} (\boldsymbol{a}^+ p, \boldsymbol{a}^- q)$. Then by the I.H., $(i, j) = (p, q)$ and we see $(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-)$ and $(\boldsymbol{m}^+, \boldsymbol{m}^-) = (\boldsymbol{b}^+, \boldsymbol{b}^-)$.
• $\mathrm{Ten}(\Pi_0, \Pi_1) \vdash \Gamma, A \otimes B, \Delta$
For

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \in \|\mathrm{Ten}(\Pi_0, \Pi_1)\|$$

and

$$(\boldsymbol{a}^+\lceil p^+, q^+\rceil \boldsymbol{b}^+, \boldsymbol{a}^-\lceil p^-, q^-\rceil \boldsymbol{b}^-) \in \|\mathrm{Ten}(\Pi_0, \Pi_1)\|,$$

such that

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \asymp_{\Gamma,A\otimes B,\Delta} (\boldsymbol{a}^+\lceil p^+, q^+\rceil \boldsymbol{b}^+, \boldsymbol{a}^-\lceil p^-, q^-\rceil \boldsymbol{b}^-)$$

we have

$$(i^+, i^-) \smile_A (p^+, p^-)$$

or

$$(j^+, j^-) \smile_B (q^+, q^-)$$

or

$$i^+ = p^+, i^- = p^-, j^+ = q^+, j^- = q^-.$$

For the first case, since $(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-)$, we see

$$(\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \asymp_\Gamma (\boldsymbol{a}^+ p^+, \boldsymbol{a}^- p^-)$$

and by the I.H., $(i^+, i^-) = (p^+, p^-)$. This contradicts to $(i^+, i^-) \smile_A (p^+, p^-)$. The second case is similar. Hence $i^+ = p^+, i^- = p^-, j^+ = q^+, j^- = q^-$ stand. Then by I.H., we have $(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-)$ and $(\boldsymbol{m}^+, \boldsymbol{m}^-) = (\boldsymbol{b}^+, \boldsymbol{b}^-)$.
• $\mathrm{Par}(\Pi) \vdash \Gamma, A \wp B$
Let $(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil)$ and $(\boldsymbol{a}^+\lceil p^+, q^+\rceil, \boldsymbol{a}^-\lceil p^-, q^-\rceil)$ be elements of $\|\mathrm{Par}(\Pi)\|$ such that

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \asymp_{\Gamma,A\wp B} (\boldsymbol{a}^+\lceil p^+, q^+\rceil, \boldsymbol{a}^-\lceil p^-, q^-\rceil).$$

By the definition, this is equivalent to

$$(\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \asymp_{\Gamma,A,B} (\boldsymbol{a}^+ p^+ q^+, \boldsymbol{a}^- p^- q^-).$$

Then by the I.H.,

$$(\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) = (\boldsymbol{a}^+ p^+ q^+, \boldsymbol{a}^- p^- q^-)$$

• $\mathrm{Perm}_\sigma(\Pi)$
For $(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)), (\sigma(\boldsymbol{a}^+), \sigma(\boldsymbol{b}^-)) \in \| \mathrm{Perm}_\sigma(\Pi)\|$,

$$\begin{aligned}(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \asymp_{\sigma(\Gamma)} (\sigma(\boldsymbol{a}^+), \sigma(\boldsymbol{b}^-)) &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{b}^-) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{b}^-)\end{aligned}$$

• $\mathrm{And}(\Pi_0, \Pi_1)$
For $(\boldsymbol{n}^+ i, \boldsymbol{n}^- j), (\boldsymbol{a}^+ p, \boldsymbol{a}^- q) \in \| \mathrm{And}(\Pi_0, \Pi_1)\|$, if

$$(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \asymp_{\Gamma,A\&B} (\boldsymbol{a}^+ p, \boldsymbol{a}^- q)$$

then from the definition of $\asymp_{A\&B}$ and $\| \mathrm{And}(\Pi_0, \Pi_1)\|$, there are two cases:

$$i = \overline{i'}, j = \overline{j'}, p = \overline{p'}, q = \overline{q'}$$

or

$$i = \underline{i'}, j = \underline{j'}, p = \underline{p'}, q = \underline{q'}.$$

For the first case,

$$\begin{aligned}&(\boldsymbol{n}^+ \bar{i}, \boldsymbol{n}^- \bar{j}) \asymp_{\Gamma,A\&B} (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (\bar{i}, \bar{j}) \frown_{A^\perp \oplus B^\perp} (\overline{p}, \overline{q}) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) \frown_{A^\perp} (p, q) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (j, i) \asymp_A (q, p) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (j, i) = (q, p).\end{aligned}$$

The second case is similar.
• $\mathrm{Or}_0(\Pi) \vdash A \oplus B, \Gamma$: For $(\boldsymbol{n}^+ \bar{i}, \boldsymbol{n}^- \bar{j}), (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) \in \| \mathrm{Or}_0(\Pi)\|$,

$$\begin{aligned}(\boldsymbol{n}^+ \bar{i}, \boldsymbol{n}^- \bar{j}) \asymp_{\Gamma,A\oplus B} (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (\bar{i}, \bar{j}) \asymp_{A\oplus B} (\overline{p}, \overline{q}) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) \asymp_A (p, q) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) = (p, q)\end{aligned}$$

• $\mathrm{Or}_1(\Pi) \vdash A \oplus B, \Gamma$: Similar to Or_0.

Corollary 2. *Let Π be a proof of MALL. For every $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in \|\Pi\|$,*

$$[\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-}$$

is a singleton or the empty set.

Proof. We show by the induction of Π. We only prove the case of Cut rule. We have

$$[\![\mathrm{Cut}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-} = \{\mathrm{cut}(f, g) \mid \exists i, j.\ f \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} \wedge g \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+, i\boldsymbol{m}^-}\}$$

where

$$\mathrm{cut} : \mathbf{Pfn}((k+1)\mathbb{N}, (k+1)\mathbb{N}) \times \mathbf{Pfn}((h+1)\mathbb{N}, (h+1)\mathbb{N}) \to \mathbf{Pfn}((k+h)\mathbb{N}, (k+h)\mathbb{N})$$

is given by

$$\mathrm{cut}(f, g) = \mathrm{tr}^{\mathbb{N}}_{(k+h)\mathbb{N}, (k+h)\mathbb{N}}((k\mathbb{N} \otimes \sigma_{h\mathbb{N}, \mathbb{N}}) \circ (k\mathbb{N} \otimes g) \circ (f \otimes h\mathbb{N}) \circ (k\mathbb{N} \otimes \sigma_{\mathbb{N}, h\mathbb{N}}))$$

If there are i, j and p, q such that

$$\begin{array}{ll} \exists f \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} & \exists g \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+, i\boldsymbol{m}^-} \\ \exists u \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ p, \boldsymbol{n}^- q} & \exists v \in [\![\Pi_1]\!]_{q\boldsymbol{m}^+, p\boldsymbol{m}^-} \end{array}$$

then

$$(\boldsymbol{n}^+ i, \boldsymbol{n}^- j), (\boldsymbol{n}^+ p, \boldsymbol{n}^- q) \in \|\Pi_0\|$$

and

$$(j\boldsymbol{m}^+, i\boldsymbol{m}^-), (q\boldsymbol{m}^+, p\boldsymbol{m}^-) \in \|\Pi_1\|.$$

By Proposition 13, we have $(i, j) \frown_A (p, q)$ and $(j, i) \frown_{A^\perp} (q, p)$. Hence $(i, j) = (p, q)$ and $\mathrm{Cut}(\Pi_0, \Pi_1)$ is a singleton or the empty set.

We write $\mathrm{W}(\Pi)$ for the set of weights of Π.

Proposition 14. *For a weighted MALL proof* (Π, w) *and* $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w$*, we have*

(1) $\|\Pi\| = \bigcup_{w \in \mathrm{W}(\Pi)} |\Pi|_w$
(2) $[\Pi]_w \in [\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-}$

Proof. We prove (1) and (2) simultaneously by the induction of Π.
• Ax_A
(1) By the definition.
(2)$[\![\mathrm{Ax}_A]\!]_{nm, mn} = \{\sigma_{\mathbb{N}, \mathbb{N}}\} = \{[\mathrm{Ax}_A]_w\}$ where $\sigma_{\mathbb{N}, \mathbb{N}} : \mathbb{N} + \mathbb{N} \rightharpoonup \mathbb{N} + \mathbb{N}$ is the swapping map.
• $\mathrm{Cut}(\Pi_0, \Pi_1)$
(1)

$$\begin{aligned} \bigcup_{w \in \mathrm{W}(\mathrm{Cut}(\Pi_0, \Pi_1))} |\mathrm{Cut}(\Pi_0, \Pi_1)|_w &= \bigcup_{w \in \mathrm{W}(\mathrm{Cut}(\Pi_0, \Pi_1))} \left\{ (\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i, j. \begin{array}{l} (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_0|_w \\ (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in |\Pi_1|_w \end{array} \right\} \\ &= \left\{ (\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i, j. \begin{array}{l} (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w \in \mathrm{W}(\Pi_0)} |\Pi_0|_w \\ (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in \bigcup_{w \in \mathrm{W}(\Pi_1)} |\Pi_1|_w \end{array} \right\} \\ &= \left\{ (\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i, j. \begin{array}{l} (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_0\| \\ (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in \|\Pi_1\| \end{array} \right\} \\ &= \|\mathrm{Cut}(\Pi_0, \Pi_1)\| \end{aligned}$$

(2) For $(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \in |\operatorname{Cut}(\Pi_0, \Pi_1)|_w$, there is i, j such that

$$(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_0|_w \qquad (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in |\Pi_1|_w$$

By the definition, $[\operatorname{Cut}(\Pi_0, \Pi_1)]_w$ is $\operatorname{cut}([\Pi_0]_w, [\Pi_1]_w)$. By the I.H., $[\Pi_0]_w \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j}$ and $[\Pi_1]_w \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+, i\boldsymbol{m}^-}$. Hence

$$[\operatorname{Cut}(\Pi_0, \Pi_1)]_w = \operatorname{cut}([\Pi_0]_w, [\Pi_1]_w) \in [\![\operatorname{Cut}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-}$$

• $\operatorname{Perm}_\sigma(\Pi)$
(1)

$$\begin{aligned}
\bigcup_{w \in \mathrm{W}(\operatorname{Perm}_\sigma(\Pi))} |\operatorname{Perm}_\sigma(\Pi)|_w &= \bigcup_{w \in \mathrm{W}(\operatorname{Perm}_\sigma(\Pi))} \{(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \mid (\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w\} \\
&= \left\{(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \,\middle|\, (\boldsymbol{n}^+, \boldsymbol{n}^-) \in \bigcup_{w \in \mathrm{W}(\Pi)} |\Pi|_w \right\} \\
&= \left\{(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \,\middle|\, (\boldsymbol{n}^+, \boldsymbol{n}^-) \in \|\Pi\| \right\} \\
&= \|\operatorname{Perm}_\sigma(\Pi)\|
\end{aligned}$$

(2) For $(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \in |\Pi|_w$

$$[\operatorname{Perm}_\sigma(\Pi)]_w = \theta_\sigma \circ [\Pi]_w \circ \theta_\sigma^{-1} \in \theta_\sigma \circ [\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-} \circ \theta_\sigma^{-1} = [\![\Pi]\!]_{\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)}$$

where θ_σ is the permutation of coproducts of $\mathbb{N}$ following σ.
• $\operatorname{Ten}(\Pi_0, \Pi_1)$
(1)

$$\begin{aligned}
&\bigcup_{w \in \mathrm{W}(\operatorname{Ten}(\Pi_0, \Pi_1))} |\operatorname{Ten}(\Pi_0, \Pi_1)|_w \\
&= \bigcup_{w \in \mathrm{W}(\operatorname{Ten}(\Pi_0, \Pi_1))} \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l}(\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in |\Pi_0|_w \\ (j^+\boldsymbol{m}^+, j^-\boldsymbol{m}^-) \in |\Pi_1|_w\end{array} \right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l}(\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in \bigcup_{w \in \mathrm{W}(\Pi_0)} |\Pi_0|_w \\ (j^+\boldsymbol{m}^+, j^-\boldsymbol{m}^-) \in \bigcup_{w \in \mathrm{W}(\Pi_1)} |\Pi_1|_w\end{array} \right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l}(\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in \|\Pi_0\| \\ (j^+\boldsymbol{m}^+, j^-\boldsymbol{m}^-) \in \|\Pi_1\|\end{array} \right\} \\
&= \|\operatorname{Ten}(\Pi_0, \Pi_1)\|
\end{aligned}$$

(2) For $(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \in |\operatorname{Ten}(\Pi_0, \Pi_1)|_w$,

$$\begin{aligned}
[\operatorname{Ten}(\Pi_0, \Pi_1)]_w &= (\mathrm{id} \otimes c \otimes \mathrm{id}) \circ ([\Pi_0]_w \otimes [\Pi_1]_w) \circ (\mathrm{id} \otimes c^{-1} \otimes \mathrm{id}) \\
&\in [\![\operatorname{Ten}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-}
\end{aligned}$$

• Par(Π)
(1)

$$\begin{aligned}
&\bigcup_{w\in \mathrm{W}(\mathrm{Par}(\Pi))} |\,\mathrm{Par}(\Pi)|_w \\
&= \bigcup_{w\in \mathrm{W}(\mathrm{Par}(\Pi))} \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in |\Pi|_w \right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in \textstyle\bigcup_{w\in \mathrm{W}(\Pi)} |\Pi|_w \right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in \|\Pi\| \right\} \\
&= \|\,\mathrm{Par}(\Pi)\|
\end{aligned}$$

(2) For $(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \in |\,\mathrm{Par}(\Pi)|_w$,

$$[\mathrm{Par}(\Pi)]_w = (\mathrm{id}\otimes c)\circ[\Pi]_w\circ(\mathrm{id}\otimes c^{-1}) \in [\![\mathrm{Par}(\Pi)]\!]_{\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil}$$

• And(Π_0, Π_1)
(1)

$$\begin{aligned}
&\bigcup_{w\in \mathrm{W}(\mathrm{And}(\Pi_0,\Pi_1))} [\mathrm{And}(\Pi_0,\Pi_1)]_w \\
&= \left(\bigcup_{w\in \mathrm{W}(\Pi_0)} \{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in [\Pi_0]_w\}\right) \\
&\quad \cup \left(\bigcup_{w\in \mathrm{W}(\Pi_1)} \{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in [\Pi_1]_w\}\right) \\
&= \left\{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w\in \mathrm{W}(\Pi_0)} [\Pi_0]_w \right\} \\
&\quad \cup \left\{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w\in \mathrm{W}(\Pi_1)} [\Pi_1]_w \right\} \\
&= \left\{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_0\| \right\} \cup \left\{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_1\| \right\} \\
&= \|\,\mathrm{And}(\Pi_0,\Pi_1)\|
\end{aligned}$$

(2) For a weight $w(\mathrm{And}) = 0$ and $(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \in |\,\mathrm{And}(\Pi_0,\Pi_1)|_w$,

$$[\mathrm{And}(\Pi_0,\Pi_1)]_w = [\Pi_0]_w \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} = [\![\mathrm{And}(\Pi_0,\Pi_1)]\!]_{\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}}$$

The case $w(\mathrm{And}) = 1$ is similar.
• $\mathrm{Or}_0(\Pi)$

(1)

$$\bigcup_{w \in \mathrm{W}(\mathrm{Or}_0(\Pi))} |\,\mathrm{Or}_0(\Pi)|_w = \bigcup_{w \in \mathrm{W}(\mathrm{Or}_0(\Pi))} \{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi|_w\}$$

$$= \left\{ (\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \;\middle|\; (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w \in \mathrm{W}(\Pi)} |\Pi|_w \right\}$$

$$= \left\{ (\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \;\middle|\; (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi\| \right\}$$

$$= \|\,\mathrm{Or}_0(\Pi)\|$$

(2) For $(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \in |\,\mathrm{Or}_0(\Pi)|_w$,

$$[\mathrm{Or}_0(\Pi)]_w = [\Pi]_w \in [\![\Pi]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} = [\![\mathrm{Or}_0(\Pi)]\!]_{\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}}$$

• $\mathrm{Or}_1(\Pi)$: Similar to Or_0.

Corollary 3 (Proposition 11).

(1) For any proof Π, $\|\Pi\| = \bigcup_{w:\text{weight of } \Pi} |\Pi|_w$.
(2) For any proof Π, well-behaved weight w of Π and $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w$, we have

$$[\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-} = \{[\Pi]_w\}.$$

Proof. (1) is exactly (1) in Proposition 14.
(2) From Proposition 14, we see $\{[\Pi]_w\} \subset [\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-}$. Then by Proposition 13, $[\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-}$ is a singleton set and the inclusion is equality.