Int Construction and Semibiproducts

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Abstract. We study a relationship between the **Int** construction of Joyal et al. and a weakening of biproducts called *semibiproducts*. We then provide an application of geometry of interaction interpretation for the multiplicative additive linear logic (MALL for short) of Girard. We consider not biproducts but semibiproducts because in general the **Int** construction does not preserve biproducts. We show that **Int** construction is left biadjoint to the forgetful functor from the 2-category of compact closed categories with semibiproducts to the 2-category of traced symmetric monoidal categories with semibiproducts. We then illustrate a traced distributive symmetric monoidal category with biproducts $\mathcal{B}(\mathbf{Pfn})$ and relate the interpretation of MALL in $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ to token machines defined over weighted MALL proofs.

1 Introduction

Traced monoidal categories introduced in [19] provide a convenient mathematical tool to study feedback, interactive computation, fixed point operators and so on. In [19], the structure theorem for traced monoidal categories is shown; the 2-category of traced monoidal categories is freely embedded to the 2-category of tortile monoidal categories, which arises as **Int** construction (also called $\mathcal{G}$ construction in [1]). **Int** appears in studies related to bidirectional / interactive computation such as geometry of interaction (GoI) [7], context semantics [10], game semantics [4] and attribute grammars [20].

We are interested in the categorical structures that are preserved by **Int** construction. In this paper, we study the case of biproducts and see if the structure theorem holds under the presence of biproducts. We found a counterexample to the preservation of biproducts by **Int** (see Appendix A), but still the pairwise biproducts $(A^+, A^-) \oplus (B^+, B^-) := (A^+ \oplus B^+, A^- \oplus B^-)$ in $\mathbf{Int}(\mathcal{C})$ behave almost like biproducts; they satisfy the axioms of biproducts except η-equalities. We characterise such a weak biproduct structure as *semibiproducts*. The main theorems of this paper are that $\mathbf{Int}(\mathcal{C})$ has semibiproducts when $\mathcal{C}$ is a traced distributive symmetric monoidal category with semibiproducts (Theorem 4 in Section 4.1), and that the structure theorem holds under the presence of semibiproducts (Theorem 5 in Section 4.2).

We then give an application of the above results to GoI interpretation of multiplicative additive linear logic (MALL). We construct an example of a traced distributive symmetric monoidal category $\mathcal{B}(\mathbf{Pfn})$ with biproducts and relate the interpretation of MALL in $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ to token machines defined over weighted MALL proofs. Semibiproducts in $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ are sufficient for this GoI interpretation because only β-equalities play a role.

2 Categorical Preliminary

Traced Symmetric Monoidal Categories and Int Construction

We recall the concept of traced symmetric monoidal categories and **Int** construction by Joyal et al [19] (also called $\mathcal{G}$-construction in [1]). Below we mainly consider *strict* symmetric monoidal categories for legibility. A *trace operator* on a symmetric monoidal category $(\mathcal{C}, \mathbf{I}, \otimes, \sigma)$ is a mapping $\mathrm{tr}^A_{B,C} : \mathcal{C}(B \otimes A, C \otimes A) \to \mathcal{C}(B, C)$ satisfying the following equations:

$$
\begin{array}{ll}
\text{(Naturality)} & h \circ \mathrm{tr}^A_{B,C}(f) \circ g = \mathrm{tr}^A_{B',C'}((h \otimes A) \circ f \circ (g \otimes A)) \\
\text{(Dinaturality)} & \mathrm{tr}^A_{B,C}((C \otimes g) \circ f) = \mathrm{tr}^{A'}_{B,C}(f \circ (B \otimes g)) \\
\text{(Vanishing I)} & \mathrm{tr}^{\mathbf{I}}_{A,B}(f) = f \\
\text{(Vanishing II)} & \mathrm{tr}^{A \otimes B}_{C,D}(g) = \mathrm{tr}^A_{C,D}(\mathrm{tr}^B_{C \otimes A, D \otimes A}(g)) \\
\text{(Superposing)} & \mathrm{tr}^A_{B \otimes C, B \otimes D}(B \otimes f) = B \otimes \mathrm{tr}^A_{C,D}\, f \\
\text{(Yanking)} & \mathrm{tr}^A_{A,A}(\sigma_{A,A}) = \mathrm{id}\,.
\end{array}
$$

We simplified the original superposing axiom in [19] using naturality and dinaturality [13]. A *traced symmetric monoidal category* (*TSMC*) is a pair of a symmetric monoidal category (SMC) and a trace operator on it.

Joyal et al's **Int** construction freely constructs tortile monoidal categories from traced monoidal categories. In this paper, we restrict this construction to TSMCs. Let $\mathcal{C}$ be a TSMC. We define the category $\mathbf{Int}(\mathcal{C})$ by the following data. An object is a pair (A^+, A^-) of $\mathcal{C}$-objects, and a morphism from (A^+, A^-) to (B^+, B^-) is a $\mathcal{C}$-morphism $f : A^+ \otimes B^- \to B^+ \otimes A^-$. The composition of $\mathbf{Int}(\mathcal{C})$-morphisms $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$ is define by the following trace:

$$g \circ f = \mathrm{tr}^{B^-}_{A^+ \otimes C^-, C^+ \otimes A^-}((\mathrm{id} \otimes \sigma) \circ (g \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma)).$$

The category $\mathbf{Int}(\mathcal{C})$ is compact closed, whose structure on objects is given as follows. For more detail, see [19].

$$
\begin{array}{l}
\mathbf{I}_{\mathbf{Int}(\mathcal{C})} = (\mathbf{I}, \mathbf{I}), \qquad (A^+, A^-) \otimes_{\mathbf{Int}(\mathcal{C})} (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-), \\
(A^+, A^-)^* = (A^-, A^+).
\end{array}
$$

Below we write **CptCl** for the 2-category of compact closed categories, strong symmetric monoidal functors and monoidal natural isomorphisms, and **TSMC** for the 2-category of TSMCs, traced strong symmetric monoidal functors and monoidal natural isomorphisms. Every compact closed category has a unique trace called *canonical trace* [19, 12], and this gives rise to the forgetful 2-functor $\mathfrak{U} : \mathbf{CptCl} \to \mathbf{TSMC}$.

Theorem 1 ([19, 14]). **Int** *construction can be extended to a pseudo-functor* $\mathbf{Int} : \mathbf{TSMC} \to \mathbf{CptCl}$, *and it is a left biadjoint of* $\mathfrak{U}$.

The unit $N_{\mathcal{C}} : \mathcal{C} \to \mathbf{Int}(\mathcal{C})$ of this biadjunction is full and faithful, and defined by $N_{\mathcal{C}} C = C \otimes \mathrm{id}_{\mathbf{I}}$ and $N_{\mathcal{C}} f = f \otimes \mathrm{id}_{\mathbf{I}}$.

Semifunctors, Seminatural Transformations and Karoubi Envelope

The material in this section is from [15–17]. A *semifunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of a mapping from $\mathcal{C}$-objects to $\mathcal{D}$-objects and a mapping from $\mathcal{C}$-morphisms to $\mathcal{D}$-morphisms, and they satisfy the conditions of functors except the preservation of identity morphisms. A *seminatural transformation* $\alpha : F \to G$ between semifunctors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\alpha_A : FA \to GA$ satisfying the naturality condition *plus* an additional condition $\alpha_A \circ F(\mathrm{id}_A) = \alpha_A$ (or $G(\mathrm{id}_A) \circ \alpha_A = \alpha_A$). [1] We note that this extra condition is redundant when one of F, G is an ordinary functor. An instance of a seminatural transformation is the identity $\{F(\mathrm{id}_A)\}_{A \in |\mathcal{C}|}$ on a semifunctor $F : \mathcal{C} \to \mathcal{D}$. Small categories, semifunctors and seminatural transformations form the 2-category $\mathbf{Cat}_{\mathrm{semi}}$. An adjunction $(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_{\mathrm{semi}}$ is called *semiadjunction*, and it specifies the following natural isomorphism (and vice versa):

$$\{f \in \mathcal{C}(FA, B) \mid f \circ F(\mathrm{id}_A) = f\} \to \{g \in \mathcal{D}(A, GB) \mid G(\mathrm{id}_B) \circ g = g\}.$$

Let $\mathcal{C}$ be a category. *Karoubi envelope* $\mathfrak{K}(\mathcal{C})$ of $\mathcal{C}$ (also called Cauchy completion) is the category defined as follows. An object is a pair (A, f) of a $\mathcal{C}$-object A and an idempotent f over A (i.e. a morphism $f : A \to A$ such that $f \circ f = f$). A morphism $\varphi : (A, f) \to (B, g)$ is a $\mathcal{C}$-morphism $\varphi : A \to B$ of $\mathcal{C}$ such that $g \circ \varphi \circ f = \varphi$. We can extend $\mathfrak{K}$ to a 2-functor $\mathfrak{K} : \mathbf{Cat}_{\mathrm{semi}} \to \mathbf{Cat}$ as follows (Theorem 7.3, [16]):

$$\mathfrak{K}(F)(A, f) = (FA, Ff), \qquad \mathfrak{K}(\alpha)_{(A,f)} = Gf \circ \alpha_A \quad (\alpha : F \to G).$$

There are two major effects of Karoubi envelope.

1. It turns semifunctors and seminatural transformations to ordinary ones in a universal way; precisely speaking, it is a right 2-adjoint of the forgetful 2-functor from $\mathbf{Cat}$ to $\mathbf{Cat}_{\mathrm{semi}}$. [2]
2. It freely adds a splitting to every idempotent in a category; that is, it is a left biadjoint of the forgetful functor $U : \mathbf{Cat}_{\mathrm{split}} \to \mathbf{Cat}$, where $\mathbf{Cat}_{\mathrm{split}}$ is the full sub 2-category of $\mathbf{Cat}$ consisting of the small categories where all idempotents split. The unit of this biadjunction is a full and faithful functor $H_{\mathcal{C}} : \mathcal{C} \to \mathfrak{K}(\mathcal{C})$ defined by $H_{\mathcal{C}}A = (A, \mathrm{id}_A)$ and $H_{\mathcal{C}}f = f$. We note that $H_{\mathfrak{K}\mathcal{C}}$ is an equivalence; the functor $P : \mathfrak{K}\,\mathfrak{K}(\mathcal{C}) \to \mathfrak{K}(\mathcal{C})$ defined by $P((A, f), f') = (A, f')$ and $P(\varphi) = \varphi$ is an inverse of $H_{\mathfrak{K}\mathcal{C}}$.

By cutting down the 2-adjunction between $\mathbf{Cat}$ and $\mathbf{Cat}_{\mathrm{semi}}$, we obtain:

Theorem 2 ([17]). *Karoubi envelope* $\mathfrak{K} : \mathbf{Cat}_{\mathrm{semi}} \to \mathbf{Cat}$ *induces the biequivalence between* $\mathbf{Cat}_{\mathrm{semi}}$ *and* $\mathbf{Cat}_{\mathrm{split}}$.

[1] This condition makes the category of small categories and semifunctors Cartesian closed; see [16] for detail.

[2] In [5, 16], Karoubi envelope $\mathfrak{K}$ is shown to be an ordinary right adjoint. This can easily be extended to right 2-adjoint.

Karoubi Enverope of Symmetric Monoidal Categories

Let $\mathcal{C}$ be an SMC. The following data equip $\mathfrak{K}\,\mathcal{C}$ with a SMC structure:

$$\mathbf{I}_{\mathfrak{K}\,\mathcal{C}} = (\mathbf{I}, \mathrm{id}_{\mathbf{I}}), \qquad (A, f) \otimes_{\mathfrak{K}\,\mathcal{C}} (B, g) = (A \otimes_{\mathcal{C}} B, f \otimes_{\mathcal{C}} g).$$

We take this as the default symmetric monoidal structure on $\mathfrak{K}\,\mathcal{C}$. The functor $H_{\mathcal{C}} : \mathcal{C} \to \mathfrak{K}\,\mathcal{C}$ is strict symmetric monoidal w.r.t. the above structure. We also note that if $F : \mathcal{C} \to \mathcal{D}$ is strong symmetric monoidal, then so is $\mathfrak{K}\,F$.

Proposition 1. *1. If $\mathcal{C}$ is a TSMC, then so is $\mathfrak{K}\,\mathcal{C}$.*
2. If $\mathcal{C}$ is a compact closed category, then so is $\mathfrak{K}\,\mathcal{C}$.

Proof. 1. We give the trace of $\varphi : (B, f) \otimes (A, h) \to (C, g) \otimes (A, h)$ by $\mathrm{tr}^{A}_{B,C}(\varphi)$.
2. We define the duality by $(A, f)^* = (A^*, f^*)$. We also give the unit and counit in $\mathfrak{K}\,\mathcal{C}$ by $(f \otimes \mathrm{id}) \circ \eta_A$ and $\epsilon_A \circ (\mathrm{id} \otimes f)$, where η_A and ϵ_A are the unit and counit in $\mathcal{C}$.

Biproducts and Semibiproducts

The biproduct $A \oplus B$ of A and B is the structure which is simultaneously the binary product and coproduct of A and B. Typical categories having biproducts are the category of Abelian groups, the category of vector spaces and the category of sets and relations. Here we propose a definition of biproducts that is more friendly to 2-category theory. We write $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ for the diagonal functor.

Definition 1. *A category $\mathcal{C}$ has* binary biproducts *if there is a functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and adjunctions $(\oplus \dashv \Delta, \eta, \epsilon)$ and $(\Delta \dashv \oplus, \eta', \epsilon')$ such that $\epsilon' \circ \eta = \mathrm{id}$ (we call this equation (*)).*

We omit the word "binary" when it is obvious from the context. In this paper we speak about *chosen* biproducts. We write $\langle -, - \rangle, [-, -], \pi_1, \pi_2, \iota_1, \iota_2$ for tupling, cotupling, projections and injections associated to $\oplus \dashv \Delta \dashv \oplus$. The equation (*) is the conjunction of $\pi_1 \circ \iota_1 = \mathrm{id}$ and $\pi_2 \circ \iota_2 = \mathrm{id}$.

Recall that a zero object 0 is an object that is simultaniously initial and terminal. We say that a category $\mathcal{C}$ has finite biproducts if it has biproducts and a zero object.

We next define the preservation of biproducts.

Definition 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with biproducts. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves* biproducts *if the following canonical maps form an isomorphism:*

$$FA \oplus FB \underset{\langle F\pi_1, F\pi_2 \rangle}{\overset{[F\iota_1, F\iota_2]}{\rightleftarrows}} F(A \oplus B).$$

A symmetric monoidal category $\mathcal{C}$ with biproducts is called distributive *if $A \otimes - : \mathcal{C} \to \mathcal{C}$ preserves biproducts for any $\mathcal{C}$-object A.*

It is not difficult to see that $G \circ F$ preserves biproducts when $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve biproducts, and that any equivalence preserves biproducts.

In this paper we deal with a weakening of biproducts called *semibiproducts* as well. We will see that semibiproducts arise in $\mathbf{Int}(\mathcal{C})$ when a TSMC $\mathcal{C}$ has semibiproducts (Theorem 4). We take the following as the definition of semibiproducts.

Definition 3. *A category $\mathcal{C}$ has* (binary) semibiproducts *if there is a semifunctor* $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *and semiadjunctions* $(\oplus \dashv \Delta, \eta, \epsilon)$ *and* $(\Delta \dashv \oplus, \eta', \epsilon')$ *such that* $\epsilon' \circ \eta = \mathrm{id}$ *(we call this equation (*)).*

The above abstract definition can be expanded in two ways: one using the operations on morphisms and the other using seminatural transformations.

B-1 There exists a mapping $\oplus : |\mathcal{C}| \times |\mathcal{C}| \to |\mathcal{C}|$ and tupling, projections, cotupling and injections

$$\langle -, - \rangle : \mathcal{C}(A, B) \times \mathcal{C}(A, C) \to \mathcal{C}(A, B \oplus C), \qquad (\pi_i)_{A_1, A_2} \in \mathcal{C}(A_1 \oplus A_2, A_i),$$
$$[-, -] : \mathcal{C}(B, A) \times \mathcal{C}(C, A) \to \mathcal{C}(B \oplus C, A), \qquad (\iota_i)_{A_1, A_2} \in \mathcal{C}(A_i, A_1 \oplus A_2)$$

(where $i \in \{1, 2\}$) subject to the following equalities:

$$\pi_i \circ \langle f_1, f_2 \rangle = f_i, \qquad [f_1, f_2] \circ \iota_i = f_i, \qquad \langle f \circ \pi_1, g \circ \pi_2 \rangle = [\iota_1 \circ f, \iota_2 \circ g],$$
$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle, \quad h \circ [f, g] = [h \circ f, h \circ g], \quad \pi_i \circ \iota_i = \mathrm{id}\,.$$

B-2 There exists a semifunctor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and seminatural transformations

$$\delta_A : A \to A \oplus A, \qquad \gamma_A : A \oplus A \to A,$$
$$(\pi_i)_{A_1, A_2} : A_1 \oplus A_2 \to A_i, \qquad (\iota_i)_{A_1, A_2} : A_i \to A_1 \oplus A_2$$

(where $i \in \{1, 2\}$) subject to the following equalities:

$$\pi_i \circ \delta = \mathrm{id}, \qquad \gamma \circ \iota_i = \mathrm{id}, \qquad \pi_i \circ \iota_i = \mathrm{id},$$
$$(\pi_1 \oplus \pi_2) \circ \delta = \mathrm{id} \oplus \mathrm{id}, \qquad \gamma \circ (\iota_1 \oplus \iota_2) = \mathrm{id} \oplus \mathrm{id}\,.$$

From Theorem 2, one can easily check that a category $\mathcal{C}$ has semibiproducts if and only if $\Re\,\mathcal{C}$ has biproducts.

Definition 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with semibiproducts. A functor $F : \mathcal{C} \to \mathcal{D}$ preserves* semibiproducts *if we have: the following equations:*

$$F(\mathrm{id}_A \oplus \mathrm{id}_B) = F(A \oplus B) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} FA \oplus FB \xrightarrow{[F\iota_1, F\iota_2]} F(A \oplus B)$$

$$\mathrm{id}_{FA} \oplus \mathrm{id}_{FB} = FA \oplus FB \xrightarrow{[F\iota_1, F\iota_2]} F(A \oplus B) \xrightarrow{\langle F\pi_1, F\pi_2 \rangle} FA \oplus FB.$$

A symmetric monoidal category $\mathcal{C}$ with semibiproducts is called distributive *if $A \otimes - : \mathcal{C} \to \mathcal{C}$ preserves semibiproducts.*

This is a generalisation of Definition 2. Another equivalent definition is that the canonical seminatural transformations between $F(-\oplus+)$ and $F(-)\oplus F(+)$ form an isomorphism in $\mathbf{Cat}_{\mathrm{semi}}$. From Theorem 2, a semifunctor $F : \mathcal{C} \to \mathcal{D}$ preserves semibiproducts if and only if $\Re\,F : \Re\,\mathcal{C} \to \Re\,\mathcal{D}$ preserves biproducts. In compact closed categories tensor products always distribute over semibiproducts.

Commutative Monoid Enrichment by (Semi) Biproducts

We show that binary biproducts on a category $\mathcal{C}$ induces a commutative-monoid enrichment on $\mathcal{C}$. This is a slight improvement of the well-known fact that a category with finite biproducts is commutative monoid enriched. [3]

[3] The unicity of the enrichment is discussed in [26].

Proposition 2. *1. Let $\mathcal{C}$ be a category with binary biproducts. Then there is a commutative monoid enrichment on $\mathcal{C}$ (which we call the* canonical enrichment*).*

2. Let $\mathcal{C}, \mathcal{D}$ be categories with binary biproducts. Then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves biproducts if and only if it is enriched w.r.t. the canonical enrichments on $\mathcal{C}$ and $\mathcal{D}$.

Proof. We define the unit $0_{A,B} \in \mathcal{C}(A, B)$ and multiplication $+ \in \mathcal{C}(A, B)^2 \to \mathcal{C}(A, B)$ by $0_{A,B} = A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B = A \xrightarrow{\iota_2} B \oplus A \xrightarrow{\pi_1} B$ and $f + g = [\mathrm{id}_A, \mathrm{id}_A] \circ \langle f, g \rangle$. See Appendix C for the proof.

We also have $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}$.

We note that in a SMC $(\mathcal{C}, \mathbf{I}, \otimes)$ with biproducts, tensor products are ditributive if and only if they are bilinear:

$$0 \otimes f = f \otimes 0 = 0, \qquad (f + g) \otimes h = f \otimes h + g \otimes h, \qquad h \otimes (f + g) = h \otimes f + h \otimes g.$$

The next fact is probably less known. We weaken Proposition 2 by replacing biproducts with semibiproducts.

Proposition 3. *1. Let $\mathcal{C}$ be a category with binary semibiproducts. Then there is a commutative monoid enriched on $\mathcal{C}$ (which we also call the* canonical enrichment*).*

2. Let $\mathcal{C}, \mathcal{D}$ be categories with binary semibiproducts. Then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves semibiproducts if and only if it is enriched w.r.t. the canonical enrichments on $\mathcal{C}$ and $\mathcal{D}$.

Comparison with Other Definitions of biproducts

In [22], the concept of binary biproducts is defined in Abelian categories (which are Abelian-group enriched categories with extra properties). This definition reliese on the enrichment, hence is not suitable for extending it to general categeories. In [18], Houston adopted the following definition: a category has finite biproducts if it has finite products and finite coproducts such that the following two canonical maps are invertible:

$$?_1 : 0 \to 1, \qquad m_{A,B} = [\langle \mathrm{id}_A, 0_{A,B} \rangle, \langle 0_{BA}, \mathrm{id}_B \rangle] : A + B \to A \times B,$$

where $0_{A,B}$ is the *zero morphism* defined to be $?_B \circ (?_1)^{-1} \circ !_A$. This definition is independent from the enrichment. On the other hand, $m_{A,B}$ refers to zero morphisms that are defined through a zero object. The definition of binary biproducts in this paper is independent from zero object and enrichment, and is written in the 2-categorical language. The following proposition shows that our definition of binary biproducts is compatible with Houston's definition:

Proposition 4. *A category $\mathcal{C}$ has finite biproducts in the sense of Houston if and only if $\mathcal{C}$ has a zero object and binary biproducts in the sense of Definition 1.*

Proof. See Appendix C.

The separation of zero objects and binary biproducts also revealed that the commutative monoid enrichment by finite biproducts relies only on binary biproducts.

3 Categorical Structure of Int($\mathcal{C}$) for a Traced Distributive Symmetric Monoidal Category $\mathcal{C}$ with Biproducts

We show that **Int**($\mathcal{C}$) has semibiproducts if $\mathcal{C}$ is a traced distributive SMC with biproducts. Motivation of this setting comes from the fact that if a compact closed category $\mathcal{A}$ has a zero object and binary products or coproducts then $\mathcal{A}$ has biproducts [18]. In general, **Int**($\mathcal{C}$) does not have biproducts for traced distributive SMCs $\mathcal{C}$ with biproducts; in Appendix A we give such an example.

3.1 Matrix of Morphisms

Let $\mathcal{C}$ be a traced distributive SMC with biproducts. The trace operator preserves the unit and multiplication on each homset:

Lemma 1. *We have* $\mathrm{tr}^C_{A,B}(0) = 0$ *and* $\mathrm{tr}^C_{A,B}(f + g) = \mathrm{tr}^C_{A,B}\, f + \mathrm{tr}^C_{A,B}\, g$.

We next associate to a morphism $f : A \otimes (B_1 \oplus B_2) \otimes A' \to C \otimes (D_1 \oplus D_2) \otimes C'$ a matrix

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \qquad \text{(where } f_{ij} = (A \otimes \pi_i \otimes A') \circ f \circ (C \otimes \iota_j \otimes C')\text{)}.$$

The original f can be recovered from the matrix by the following sum:

$$f = \sum_{1 \le i,j \le 2} (C \otimes \iota_i \otimes C') \circ f_{ij} \circ (A \otimes \pi_j \otimes A').$$

Below we identify morphisms and matrices associated to them. We show some useful equations that hold for matrix representations of morphisms. They are very much like matrix calculations in linear algebra.

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \circ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} (g_{11} \circ f_{11} + g_{12} \circ f_{21}) & (g_{11} \circ f_{12} + g_{12} \circ f_{22}) \\ (g_{21} \circ f_{11} + g_{22} \circ f_{21}) & (g_{21} \circ f_{22} + g_{22} \circ f_{22}) \end{pmatrix}$$

$$g \otimes \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \otimes h = \begin{pmatrix} (g \otimes f_{11} \otimes h) & (g \otimes f_{12} \otimes h) \\ (g \otimes f_{21} \otimes h) & (g \otimes f_{22} \otimes h) \end{pmatrix}$$

$$A \otimes (f \oplus g) \otimes C = \begin{pmatrix} A \otimes f \otimes B & 0 \\ 0 & A \otimes g \otimes B \end{pmatrix}$$

$$\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} : A \otimes (B \oplus C) \to (B \oplus C) \otimes A.$$

Lemma 2. *1. For any $\mathcal{C}$-morphism $f : A \otimes (B_1 \oplus B_2) \to C \otimes (B_1 \oplus B_2)$, we have*

$$\mathrm{tr}^{B_1 \oplus B_2}_{A,C} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \mathrm{tr}^{B_1}_{A,C}(f_{11}) + \mathrm{tr}^{B_2}_{A,C}(f_{22}).$$

2. For any $\mathcal{C}$-morphism $f : (B_1 \oplus B_2) \otimes A \to (C_1 \oplus C_2) \otimes A$, we have

$$\mathrm{tr}^{A}_{B_1 \oplus B_2, C_1 \oplus C_2} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{tr}^A_{B_1,C_1}(f_{11}) & \mathrm{tr}^A_{B_1,C_2}(f_{12}) \\ \mathrm{tr}^A_{B_2,C_1}(f_{21}) & \mathrm{tr}^A_{B_2,C_2}(f_{22}) \end{pmatrix}.$$

3.2 Semibiproducts in $\mathbf{Int}(C)$

Our interest is whether we can construct biproducts in $\mathbf{Int}(C)$ from those in C. The example in Appendix A shows that in general $\mathbf{Int}(C)$ may not have biproducts. Instead, we show that *semibiproducts* exist in $\mathbf{Int}(C)$.

We define a binary operator $\oplus$ on $\mathbf{Int}(C)$-objects by

$$(A^+, A^-) \oplus (B^+, B^-) = (A^+ \oplus B^+, A^- \oplus B^-).$$

We show that this becomes the object part of the binary semibiproducts in $\mathbf{Int}(C)$. First, the following isomorphism:

$$\begin{aligned}\mathbf{Int}(C)(A, B_1 \oplus B_2) &= C(A^+ \otimes (B_1^- \oplus B_2^-), (B_1^+ \oplus B_2^+) \otimes A^-) \\ \simeq \prod_{1\le i,j\le 2} C(A^+ \otimes B_i^-, B_j^+ \otimes A^-) &= \prod_{1\le i,j\le 2} \mathbf{Int}(C)(A, (B_i^+, B_j^-))\end{aligned}$$

allows us to identify an $\mathbf{Int}(C)$-morphism $f : A \to B_1 \oplus B_2$ and the tuple $\langle\!\langle f_{11}, f_{12}, f_{21}, f_{22}\rangle\!\rangle$ of $\mathbf{Int}(C)$-morphisms $f_{ij} : A \to (B_i^+, B_j^-)$. Similarly, we identify $g : B_1 \oplus B_2 \to C$ and the tuple $[[g_{11}, g_{12}, g_{21}, g_{22}]]$ of morphisms $g_{ij} : (B_i^+, B_j^-) \to C$. The composition of $\mathbf{Int}(C)$-morphisms involving $B_1 \oplus B_2$ is calculated like inner-product of vectors.

Lemma 3. *We consider the following diagram in* $\mathbf{Int}(C)$*:*

$$C \xrightarrow{h} A \xrightarrow{\langle\!\langle f_{ij}\rangle\!\rangle} B_1 \oplus B_2 \xrightarrow{[[g_{ij}]]} C \xrightarrow{i} D\,.$$

Then we have

$$[[g_{ij}]] \circ \langle\!\langle f_{ij}\rangle\!\rangle = \sum g_{ij} \circ f_{ij}, \qquad \langle\!\langle f_{ij}\rangle\!\rangle \circ h = \langle\!\langle f_{ij} \circ h\rangle\!\rangle, \qquad i \circ [[g_{ij}]] = [[i \circ g_{ij}]],$$

where the big sum means the addition of $\mathbf{Int}\,C$*-morphisms as* C*-morphisms.*

We are now ready to give binary semibiproducts in $\mathbf{Int}(C)$.

Proposition 5. *The assignment* $(B_1, B_2) \mapsto B_1 \oplus B_2$ *for* B_1, B_2 *in* $\mathbf{Int}(C)$*, together with the following morphisms:*

$$\begin{aligned}\langle f, g\rangle &= \langle\!\langle f, 0, 0, g\rangle\!\rangle, & \pi_1 &= [[\mathrm{id}, 0, 0, 0]], & \pi_2 &= [[0, 0, 0, \mathrm{id}]] \\ [f, g] &= [[f, 0, 0, g]], & \iota_1 &= \langle\!\langle \mathrm{id}, 0, 0, 0\rangle\!\rangle, & \iota_2 &= \langle\!\langle 0, 0, 0, \mathrm{id}\rangle\!\rangle\end{aligned}$$

satisfy Condition B-1 (which is equivalent to Condition B in Definition 3).

Theorem 3. *For any traced distributive SMC C with biproducts,* $\mathbf{Int}(C)$ *is a compact closed category with semibiproducts. Moreover,* $\mathbf{Int}(C)$ *is distributive as an SMC and the unit functor* $N_C : C \to \mathbf{Int}(C)$ *preserves semibiproducts.*

Proposition 6. *Let C be a traced distributive SMC C with biproducts. We write* $(0_C, +_C)$ *and* $(0_{\mathbf{Int}(C)}, +_{\mathbf{Int}(C)})$ *for the canonical enrichments over C and* $\mathbf{Int}(C)$*, respectively. Then we have*

$$0_{\mathbf{Int}(C)} = 0_C, \qquad f +_{\mathbf{Int}(C)} g = f +_C g.$$

The preservation of zero object by $\mathbf{Int}$ is easy: one can easily show that if C has a zero object 0, then for any C-object A, the pair $(A, 0)$ and $(0, A)$ are both zero object in $\mathbf{Int}(C)$; in particular, $N(0)$ is a zero object.

4 Int Construction and Semibiproducts

4.1 Preservation of Semibiproducts

We Give an extension of Theorem 3. We show that we can construct semibiproducts in $\mathbf{Int}(C)$ from semibiproducts in a traced distributive SMC C, and that the unit N_C of biadjunction $\mathbf{Int} \dashv \mathfrak{U}$ preserves semibiproducts. We observe that it is enough to show that N_C preserves semibiproducts when C has biproducts. Then by Theorem 3, we see that N_C preserves semibiproducts for the general case.

Proposition 7. *For a traced SMC C, $\mathfrak{K}\,\mathbf{Int}(C)$ is equivalent to $\mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)$ as SMCs.*

Proof. We define $\Phi : \mathfrak{K}\,\mathbf{Int}(C) \to \mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)$ by

$$\Phi(((A^+, A^-), f)) = ((A^+, \mathrm{id}_{A^+}), (A^-, \mathrm{id}_{A^-}), f) \qquad \Phi(\varphi) = \varphi$$

for $\varphi : ((A^+, A^-), f) \to ((B^+, B^-), g)$ and $\Psi : \mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C) \to \mathfrak{K}\,\mathbf{Int}(C)$ by

$$\Psi(((A^+, f^+), (A^-, f^-)), h) = ((A^+, A^-), h) \qquad \Psi(\varphi) = \varphi$$

for $\varphi : (((A^+, f^+), (A^-, f^-)), h) \to (((B^+, g^+), (B^-, g^-)), k)$. Obviously $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{K}\,\mathbf{Int}(C)}$. We also have a monoidal natural isomorphism $\alpha : \Phi \circ \Psi \xrightarrow{\cong} \mathrm{id}_{\mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)}$ given by $\alpha_{(((A^+, f^+),(A^-, f^-)),h)} = h$. Hence $\mathfrak{K}\,\mathbf{Int}(C)$ and $\mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)$ are equivalent as SMCs.

Proposition 8. *If a traced distributive SMC C has semibiproducts then $\mathfrak{K}\,\mathbf{Int}(C)$ has biproducts and is distributive as an SMC.*

Proof. By Theorem 7, $\mathfrak{K}\,\mathbf{Int}(C)$ is equivalent to $\mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)$ as SMCs. This equivalence induces biproducts on $\mathfrak{K}\,\mathbf{Int}(C)$, and Φ and Ψ preserves these biproducts. Since $\mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C)$ is distributive, $\mathfrak{K}\,\mathbf{Int}(C)$ is also distributive.

Theorem 4. *Let C be a traced distributive SMC with semibiproducts. Then $\mathbf{Int}(C)$ has semibiproducts and is distributive as an SMC. The canonical functor $N_C : C \to \mathbf{Int}(C)$ preserves semibiproducts.*

Proof. By Theorem 3, $N_{\mathfrak{K}(C)} : \mathfrak{K}(C) \to \mathbf{Int}\,\mathfrak{K}(C)$ preserves semibiproducts. Hence $\mathfrak{K}(N_{\mathfrak{K}(C)})$ preserves biproducts. Since $H_{\mathfrak{K}(C)} : \mathfrak{K}(C) \to \mathfrak{K}\,\mathfrak{K}(C)$ and $\Phi : \mathfrak{K}\,\mathbf{Int}\,\mathfrak{K}(C) \to \mathfrak{K}\,\mathbf{Int}(C)$ are equivalences, they preserve biproducts. Hence $\Phi \circ \mathfrak{K}(N_{\mathfrak{K}(C)}) \circ H_{\mathfrak{K}(C)}$ preserves biproducts. By the definition of these functors, $\Phi \circ \mathfrak{K}(N_{\mathfrak{K}(C)}) \circ H_{\mathfrak{K}(C)}$ is equal to $\mathfrak{K}(N_{\mathfrak{K}(C)})$. Hence $N_{\mathfrak{K}(C)} : C \to \mathbf{Int}(C)$ preserves semibiproducts.

4.2 The Structure Theorem

Let $\mathbf{TSMC}_{\oplus_s}$ be the sub 2-category of $\mathbf{TSMC}$ whose 0-cells are traced distributive SMCs with semibiproducts and whose 1-cells preserve semibiproducts. Let $\mathbf{CptCl}_{\oplus_s}$ be the sub 2-category of $\mathbf{CptCl}$ whose 0-cells are distributive compact closed categories with semibiproducts and whose 1-cells preserve semibiproducts. By Theorem 4,

$\mathbf{Int}(\mathcal{C})$ is a $\mathbf{CptCl}_{\oplus_s}$-object and $N_{\mathcal{C}} : \mathcal{C} \to \mathbf{Int}(\mathcal{C})$ is a $\mathbf{TSMC}_{\oplus_s}$-morphism. Hence $N^*_{\mathcal{C}} : \mathbf{CptCl}(\mathbf{Int}(\mathcal{C}), \mathcal{D}) \to \mathbf{TSMC}(\mathcal{C}, \mathcal{D})$ is restricted to a full and faithful functor

$$N^*_{\mathcal{C}} : \mathbf{CptCl}_{\oplus_s}(\mathbf{Int}(\mathcal{C}), \mathcal{D}) \to \mathbf{TSMC}_{\oplus_s}(\mathcal{C}, \mathcal{D})$$

for $\mathcal{C} \in \mathbf{TSMC}_{\oplus_s}$ and $\mathcal{D} \in \mathbf{CptCl}_{\oplus_s}$. As in the proof of the biadjunction $\mathbf{Int} \dashv \mathfrak{U}$ [19, 14], there is $F' : \mathbf{Int}(\mathcal{C}) \to \mathcal{D}$ such that $N^*_{\mathcal{C}}(F') \cong F$ for any $F \in \mathbf{TSMC}(\mathcal{C}, \mathcal{D})$. Here F' is defined by $F'(A^+, A^-) = FA^+ \otimes (FA^-)^*$ and $F'(f : (A^+, A^-) \to (B^+, B^-))$ is defined by

$$FA^+ \otimes (FA^-)^* \xrightarrow{1\otimes\eta\otimes 1} FA^+ \otimes FB^- \otimes (FB^-)^* \otimes (FA^-)^*$$
$$\xrightarrow{(m^{-1}\circ Ff\circ m)\otimes 1} FB^+ \otimes FA^- \otimes (FB^-)^* \otimes (FA^-)^*$$
$$\xrightarrow{\cong} FB^+ \otimes (FB^-)^* \otimes FA^- \otimes (FA^-)^* \xrightarrow{1\otimes\epsilon'} FB^+ \otimes (FB^-)^*.$$

Theorem 5. $(\mathbf{Int}, N)$ *is a left biadjoint of the forgetful functor* $\mathbf{CptCl}_{\oplus_s} \to \mathbf{TSMC}_{\oplus_s}$.

Proof. We show that $\mathfrak{R}(F')$ preserves biproducts. By the definitions of functors $P_{\mathfrak{R}(\mathcal{D})} : \mathfrak{R}\mathfrak{R}(\mathcal{D}) \to \mathfrak{R}(\mathcal{D})$ and $\Phi : \mathfrak{R}\,\mathbf{Int}(\mathcal{C}) \to \mathfrak{R}\,\mathbf{Int}\,\mathfrak{R}(\mathcal{C})$, we see $\mathfrak{R}(F') = P_{\mathfrak{R}(\mathcal{D})} \circ \mathfrak{R}((\mathfrak{R}F)') \circ \Phi$. Here $(\mathfrak{R}F)'$ makes sense since $\mathfrak{R}(\mathcal{D})$ is a compact closed category when $\mathcal{D}$ is a compact closed category. Since $P_{\mathfrak{R}(\mathcal{D})}$ and Φ are equivalences, they preserve biproducts. By Lemma 4, $(\mathfrak{R}\,F)' : \mathbf{Int}(\mathfrak{R}(\mathcal{C})) \to \mathfrak{R}(\mathcal{D})$ preserves semibiproducts and especially $\mathfrak{R}((\mathfrak{R}\,F)')$ also preserves biproducts. Hence $\mathfrak{R}(F')$ preserves biproducts. This is equivalent to the preservation of semibiproducts by F'. Then, as in [19, 14], we see $N^*_{\mathcal{C}}$ is essentially surjective on objects and full and faithful.

Lemma 4. *For a traced distributive SMC $\mathcal{C}$ with biproducts and distributive compact closed category $\mathcal{D}$ with biproducts, $F' : \mathbf{Int}(\mathcal{C}) \to \mathcal{D}$ preserves semibiproducts.*

Proof. The canonical seminatural transformations

$$\theta : F'((A^+, A^-) \oplus (B^+, B^-)) \leftrightarrows F'(A^+, A^-) \oplus F'(B^+, B^-) : \theta'$$

are represented by (a) and (b)

$$(a)\ \begin{array}{c|cccc} \theta & Z_{00} & Z_{01} & Z_{10} & Z_{11} \\ \hline Z_{00} & \mathrm{id}_{Z_{00}} & 0 & 0 & 0 \\ Z_{11} & 0 & 0 & 0 & \mathrm{id}_{Z_{11}} \end{array} \quad (b)\ \begin{array}{c|cc} \theta' & Z_{00} & Z_{11} \\ \hline Z_{00} & \mathrm{id}_{Z_{00}} & 0 \\ Z_{01} & 0 & 0 \\ Z_{10} & 0 & 0 \\ Z_{11} & 0 & \mathrm{id}_{Z_{11}} \end{array} \quad (c)\ \begin{array}{c|cccc} \theta' \circ \theta & Z_{00} & Z_{01} & Z_{10} & Z_{11} \\ \hline Z_{00} & \mathrm{id}_{Z_{00}} & 0 & 0 & 0 \\ Z_{01} & 0 & 0 & 0 & 0 \\ Z_{10} & 0 & 0 & 0 & 0 \\ Z_{11} & 0 & 0 & 0 & \mathrm{id}_{Z_{11}} \end{array}$$

via isomorphisms $F'(A^+, A^-) \oplus F'(B^+, B^-) \cong Z_{00} \oplus Z_{11}$ and $F'((A^+, A^-) \oplus (B^+, B^-)) \cong Z_{00} \oplus Z_{01} \oplus Z_{10} \oplus Z_{11}$ where $Z_{00} = (FA^+) \otimes (FA^-)^*$, $Z_{01} = (FA^+) \otimes (FB^-)^*$, $Z_{10} = (FB^+) \otimes (FA^-)^*$ and $Z_{11} = (FB^+) \otimes (FB^-)^*$. Then $\theta \circ \theta' = \mathrm{id}$ and $\theta' \circ \theta$ is represented by (c). Hence θ and θ' are seminatural isomorphisms since (c) corresponds to $F'(\mathrm{id}_{(A^+,A^-)} \oplus \mathrm{id}_{(B^+,B^-)})$.

5 Application to GoI Interpretation of MALL

We apply semibiproducts in the categories constructed by **Int** to GoI-style interpretation of the multiplicative additive linear logic (MALL for short) [6]; its proof system is described in Appendix B. The interpretation given here extends the multiplicative fragment of categorical GoI interpretation [3, 11] with additives. [4]

In Section 5.1, we introduce the matrix construction $\mathcal{B}$ that adds small biproducts to a given category. Roughly, an object in $\mathcal{B}(C)$ is a set-indexed family of C-objects, and morphisms between such families are matrices of sets of C-morphisms. This construction sends traced SMCs to traced distributive SMCs with biproducts.

Category $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ is a compact closed category with semibiproducts. In Section 5.2, we give an $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-object $\mathcal{U}$ equipped with equality $\mathcal{U} = \mathcal{U}^*$ and isomorphisms $\mathcal{U} \otimes \mathcal{U} \cong \mathcal{U}$ and $\mathcal{U} \oplus \mathcal{U} \cong \mathcal{U}$. With this structure we give an interpretation of a MALL proof $\Pi \vdash A_1, \cdots, A_k$ as an $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-morphism $[\![\Pi]\!] : \mathbf{I} \to \mathcal{U}^{\otimes k}$, where $\mathcal{U}^{\otimes k}$ is the k-fold tensor of $\mathcal{U}$. This interpretation is sound with respect to cut eliminations.

We then introduce a token machine that computes denotations of *weighted* proofs (Section 5.3); a weight is a decoration of &-rules in a proof, and it tells the direction to proceed to the machine. We then show that the contents of the morphism $[\![\Pi]\!]$ (which is a matrix of sets of partial functions) consists of the denotation $[\Pi]_w$ of Π by the token machine, with w ranging over all possible weights on Π (Section 5.4).

5.1 Adding Small Biproducts

We first give the *matrix construction*, which adds small biproducts to a given category.

Definition 5. *For a category C, we define the category $\mathcal{B}(C)$ by the following data:*

- **object***: a family $A = \{A_i\}_{i \in |A|}$ of C-objects indexed by a set $|A|$*
- **morphism***: $\varphi : A \to B$ is a $|A| \times |B|$-indexed family of sets of C-morphisms $\{\varphi_{i,j} \subset C(A_i, B_j)\}_{i \in |A|, j \in |B|}$. The identity morphism on A is*

$$\mathrm{id}_{i,j} = \begin{cases} \{\mathrm{id}_{A_i}\} & (i = j) \\ \phi & (i \neq j) \end{cases}$$

and the composition of $\varphi : A \to B$ and $\psi : B \to C$ is defined by

$$(\psi \circ \varphi)_{i,k} = \{g \circ f \mid \exists j \in |B|. g \in \psi_{j,k} \wedge f \in \varphi_{i,j}\} \qquad (i \in |A|, k \in |C|).$$

The small biproduct of a family of $\mathcal{B}(C)$-objects $\{A_l\}_{l \in \Lambda}$: is given as follows:

$$\left| \bigoplus_{l \in \Lambda} A_l \right| = \sum_{l \in \Lambda} |A_l|, \qquad \left(\bigoplus_{l \in \Lambda} A_l \right)_{(l,i)} = (A_l)_i.$$

The matrix construction preserves traced symmetric monoidal structures.

[4] Our interpretation eagerly applies cuts to the denotation of proofs; the original GoI suspends the application of cuts until the execution formula is applied.

Proposition 9. *Let $\mathcal{C}$ be a traced SMC. Then $\mathcal{B}(\mathcal{C})$ is a traced distributive SMC with biproducts.*

We equip $\mathcal{B}(\mathcal{C})$ with the following symmetric monoidal structure: the unit is $\{\mathbf{I}\}_{*\in 1}$ and the tensor product of A and B is $\{A_i \otimes B_j\}_{(i,j)\in|A|\times|B|}$. The trace of $\varphi : B \otimes A \to C \otimes A$ is given by $\mathrm{tr}^A_{B,C}(\varphi)_{i,j} = \{\mathrm{tr}^{A_k}_{B_i,C_j}(f) \mid \exists k \in |A|. f \in \varphi_{(i,k),(j,k)}\}$ $(i \in |B|, j \in |C|)$. It is easy to show that the tensor product of $\mathcal{B}(\mathcal{C})$ distributes over biproducts.

5.2 GoI Interpretation of MALL Proofs

We next extend the categorical GoI interpretation of MLL to MALL. Let **Pfn** be the traced SMC of sets and partial functions [11]. We set-up an $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-object $\mathcal{U}$ with two isomorphisms and one equality:

$$\mathcal{U} \otimes \mathcal{U} \cong \mathcal{U}, \qquad \mathcal{U} \oplus \mathcal{U} \cong \mathcal{U}, \qquad \mathcal{U}^* = \mathcal{U}$$

then interpret a MALL proof $\Pi \vdash A_1, \ldots, A_k$ as an $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-morphism $[\![\Pi]\!] : \mathbf{I} \to \mathcal{U}^{\otimes k}$. We note that the above isomorphism can be weaken to retracts.

The object $\mathcal{U}$ and the above isomorphisms are given as follows. We fix two bijections $\ulcorner -, - \urcorner : \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ and $c : \mathbb{N} + \mathbb{N} \cong \mathbb{N}$, then define a $\mathcal{B}(\mathcal{C})$-object U to be the $\mathbb{N}$-fold copy of $\mathbb{N}$, that is, $|U| = \mathbb{N}$ and $U_i = \mathbb{N}$ $(i \in \mathbb{N})$. There are two isomorphisms $f : U \oplus U \to U$ and $g : U \otimes U \to U$ defined by

$$f_{x,y} = \begin{cases} \{\mathrm{id}_{\mathbb{N}}\} & (y = c(x)) \\ \emptyset & (\text{otherwise}) \end{cases}, \quad g_{(x,x'),y} = \begin{cases} \{c\} & (y = \ulcorner x, x' \urcorner) \\ \emptyset & (\text{otherwise}) \end{cases}.$$

These give rise to an $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-object $\mathcal{U} = (U, U)$ such that $\mathcal{U}^* = \mathcal{U}$ and two isomorphisms:

$$a = f \otimes f^{-1} : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}, \qquad m = g \otimes g^{-1} : \mathcal{U} \otimes \mathcal{U} \to \mathcal{U}.$$

Note that $a^{-1} = a^*$ and $m^{-1} = m^*$. Below we write $\alpha_i : \mathcal{U} \to \mathcal{U}$ for $\alpha_i = a \circ \iota_{i+1}$.

We move on to the interpretation of proofs. We identify a context consisting of k formulae and $\mathcal{U}^{\otimes k}$. The interpretation employs the compact closed structure (unit $\eta_{\mathcal{U}} : \mathbf{I} \to \mathcal{U} \otimes \mathcal{U}$ and counit $\varepsilon_{\mathcal{U}} : \mathcal{U} \otimes \mathcal{U} \to \mathbf{I}$; see [19] for their definition) and the semibiproduct structure on $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$:

$$\begin{aligned}
[\![\mathrm{Ax}_A]\!] &= \eta_{\mathcal{U}} \\
[\![\mathrm{Cut}(\Pi_0, \Pi_1)]\!] &= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ([\![\Pi_0]\!] \otimes [\![\Pi_1]\!]) \\
[\![\mathrm{Ten}(\Pi_0, \Pi_1)]\!] &= (\Gamma \otimes m \otimes \Delta) \circ ([\![\Pi_0]\!] \otimes [\![\Pi_1]\!]) \\
[\![\mathrm{Par}(\Pi)]\!] &= (\Gamma \otimes m) \circ [\![\Pi]\!] \\
[\![\mathrm{Perm}_\sigma(\Pi)]\!] &= f_\sigma \circ [\![\Pi]\!] \qquad (f_\sigma \text{ is a morphism corresponding to } \sigma) \\
[\![\mathrm{And}(\Pi_0, \Pi_1)]\!] &= (\Gamma \otimes \alpha_0) \circ [\![\Pi_0]\!] + (\Gamma \otimes \alpha_1) \circ [\![\Pi_1]\!] \\
[\![\mathrm{Or}_i(\Pi)]\!] &= (\alpha_i \otimes \Gamma) \circ [\![\Pi]\!] .
\end{aligned}$$

In the above definition + in And-rule is the canonical enrichment given by semibiproducts on $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$.

Proposition 10. *If a cut elimination in Π yields Π', then $[\![\Pi]\!] = [\![\Pi']\!]$.*

5.3 The Token Machine for Weighted MALL Proofs

We define a token machine that computes denotations of *weighted* proofs in [23]. A weight assigns left or right to each &-rule in a proof, and it tells the direction to proceed to the token machine. Since a proof can have different weights, the token machine may compute different denotations of a proof depending on the weight. We formulate a weight as a mapping w from the set of occurrences of &-rules in Π to $\{0, 1\}$, which denotes left and right.

For a proof Π and a weight w of Π, we define a machine whose state is a triple $(A, n, \uparrow)$ or $(A, n, \downarrow)$ where A is a formula in Π and n is a natural number. Our presentation is from [21]. The transition rules of the machine are shown below. There, we distinguish the same formulae that appear in different places by superscription, and we treat contexts Γ and Δ as formulae for simplicity. The expressions $\overline{n}$ and $\underline{n}$ stand for $c(\mathrm{inl}(n))$ and $c(\mathrm{inr}(n))$ respectively.

$$-\frac{}{\vdash A, A^{\perp}} \qquad \begin{array}{ll}(A, n, \uparrow) & \mapsto (A^{\perp}, n, \downarrow)\\ (A^{\perp}, n, \uparrow) & \mapsto (A, n, \downarrow)\end{array}$$

$$-\frac{\vdash \Gamma^1, A \quad \vdash A^{\perp}, \Delta^1}{\vdash \Gamma^0, \Delta^0} \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow)\\ (\Delta^0, n, \uparrow) & \mapsto (\Delta^1, n, \uparrow)\\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow)\\ (\Delta^1, n, \downarrow) & \mapsto (\Delta^0, n, \downarrow)\\ (A, n, \downarrow) & \mapsto (A^{\perp}, n, \uparrow)\\ (A^{\perp}, n, \downarrow) & \mapsto (A, n, \uparrow)\end{array}$$

$$-\frac{\vdash A^1_{\sigma 0}, \cdots, A^1_{\sigma n}}{\vdash A^0_0, \cdots, A^0_n} \quad (\sigma \text{ is a permutation}) \qquad \begin{array}{ll}(A^0_i, n, \uparrow) & \mapsto (A^1_i, n, \uparrow)\\ (A^1_i, n, \downarrow) & \mapsto (A^1_i, n, \downarrow)\end{array}$$

$$-\frac{\vdash \Gamma^1, A \quad \vdash B, \Delta^1}{\vdash \Gamma^0, A \otimes B, \Delta^0} \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) \mapsto (\Gamma^1, n, \uparrow) & (A \otimes B, \overline{n}, \uparrow) \mapsto (A, n, \uparrow)\\ (\Delta^0, n, \uparrow) \mapsto (\Delta^1, n, \uparrow) & (A \otimes B, \underline{n}, \uparrow) \mapsto (B, n, \uparrow)\\ (\Gamma^1, n, \downarrow) \mapsto (\Gamma^0, n, \downarrow) & (A, n, \downarrow) \mapsto (A \otimes B, \overline{n}, \downarrow)\\ (\Delta^1, n, \downarrow) \mapsto (\Delta^0, n, \downarrow) & (B, n, \downarrow) \mapsto (A \otimes B, \underline{n}, \downarrow)\end{array}$$

$$-\frac{\vdash \Gamma^1, A, B}{\vdash \Gamma^0, A \wp B} \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow)\\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow)\\ (A \wp B, \overline{n}, \uparrow) & \mapsto (A, n, \uparrow)\\ (A \wp B, \underline{n}, \uparrow) & \mapsto (B, n, \uparrow)\\ (A, n, \downarrow) & \mapsto (A \wp B, \overline{n}, \downarrow)\\ (B, n, \downarrow) & \mapsto (A \wp B, \underline{n}, \downarrow)\end{array}$$

$$-\frac{\vdash \Gamma^1, A \quad \vdash \Gamma^2, B}{\vdash \Gamma^0, A \& B} \quad (w(\&) = 0) \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow)\\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow)\\ (A \& B, n, \uparrow) & \mapsto (A, n, \uparrow)\\ (A, n, \downarrow) & \mapsto (A \& B, n, \downarrow)\end{array}$$

$$-\frac{\vdash \Gamma^1, A \quad \vdash \Gamma^2, B}{\vdash \Gamma^0, A \& B} \quad (w(\&) = 1) \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^2, n, \uparrow)\\ (\Gamma^2, n, \downarrow) & \mapsto (\Gamma^0, n, \uparrow)\\ (A \& B, n, \uparrow) & \mapsto (B, n, \uparrow)\\ (B, n, \downarrow) & \mapsto (A \& B, n, \downarrow)\end{array}$$

$$-\frac{\vdash A, \Gamma^1}{\vdash A \oplus B, \Gamma^0} \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow)\\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow)\\ (A \oplus B, n, \uparrow) & \mapsto (A, n, \uparrow)\\ (A, n, \downarrow) & \mapsto (A \oplus B, n, \downarrow)\end{array}$$

$$-\frac{\vdash B, \Gamma^1}{\vdash A \oplus B, \Gamma^0} \qquad \begin{array}{ll}(\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow)\\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow)\\ (A \oplus B, n, \uparrow) & \mapsto (B, n, \uparrow)\\ (B, n, \downarrow) & \mapsto (A \oplus B, n, \downarrow)\end{array}$$

This machine is essentially the same as the one given in [23], with a minor difference that tokens are not altered when passing through $\&, \oplus_0, \oplus_1$-rules. This is because our token machine is defined so that it corresponds to our categorical interpretation given in Section 5.2 (c.f. Proposition 11). Especially, how it passes tokens depends on our choice of retracts $f : U \oplus U \to U$ and $g : U \otimes U \to U$. For example, if we take another retraction $f : U \oplus U \to U$

$$f_{x,y} = \begin{cases} \{\lambda n.2n\} & (y = c(x), x = \mathrm{inl}(x')) \\ \{\lambda n.2n+1\} & (y = c(x), x = \mathrm{inr}(x')) \\ \phi & (\text{otherwise}) \end{cases}$$

then we still have Proposition 11 by changing the definition of &-rule as follows.

$$-\frac{\vdash \Gamma^1, A \quad \vdash \Gamma^2, B}{\vdash \Gamma^0, A\&B}\ (w(\&) = 0) \qquad -\frac{\vdash \Gamma^1, A \quad \vdash \Gamma^2, B}{\vdash \Gamma^0, A\&B}\ (w(\&) = 1)$$

$$\begin{array}{ll|ll} (\Gamma^0, n, \uparrow) & \mapsto (\Gamma^1, n, \uparrow) & (\Gamma^0, n, \uparrow) & \mapsto (\Gamma^2, n, \uparrow) \\ (\Gamma^1, n, \downarrow) & \mapsto (\Gamma^0, n, \downarrow) & (\Gamma^2, n, \downarrow) & \mapsto (\Gamma^0, n, \uparrow) \\ (A\&B, n, \uparrow) & \mapsto (A, 2n, \uparrow) & (A\&B, n, \uparrow) & \mapsto (B, 2n+1, \uparrow) \\ (A, 2n, \downarrow) & \mapsto (A\&B, n, \downarrow) & (B, 2n+1, \downarrow) & \mapsto (A\&B, n, \downarrow) \end{array}$$

It is straight forward to modify our proofs of Proposition 11.

For a proof $\Pi \vdash A_1, \ldots, A_k$ and a weight w of Π, we define a partial function $[\Pi]_w : k\mathbb{N} \rightharpoonup k\mathbb{N}$ (here $k\mathbb{N}$ is the k-fold coproduct of $\mathbb{N}$) by

$$[\Pi]_w(i, n) = \begin{cases} (j, m) & ((A_i, n, \uparrow) \mapsto^* (A_j, m, \downarrow)) \\ \text{undefined} & (\text{otherwise}) \end{cases}$$

where $(A_i, n, \uparrow) \mapsto^* (A_j, m, \downarrow)$ means that the many-step $\mapsto$ transitions from the initial state $(A_i, n, \uparrow)$ terminates at $(A_j, m, \downarrow)$.

5.4 Calculation of Weights from Indices

We show that the categorical GoI in Section 5.2 compiles the computation of the token machine over a proof and all possible *weights* on it.

From the equation

$$\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))(\mathbf{I}, \mathcal{U}^{\otimes k}) = \mathcal{B}(\mathbf{Pfn})(U^{\otimes k}, U^{\otimes k}) = \mathbb{N}^k \times \mathbb{N}^k \to 2^{\mathbf{Pfn}(k\mathbb{N}, k\mathbb{N})},$$

every interpretation of a proof Π determines a family $\{[\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-} \subseteq \mathbf{Pfn}(k\mathbb{N}, k\mathbb{N})\}_{\boldsymbol{n}^+ \in \mathbb{N}^k, \boldsymbol{n}^- \in \mathbb{N}^k}$ of sets of **Pfn**-morphisms. We write $||\Pi|| \subseteq \mathbb{N}^k \times \mathbb{N}^k$ for the set of indeces giving non-empty sets, that is, $||\Pi|| = \{(\boldsymbol{n}^+, \boldsymbol{n}^-) | \, [\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-} \neq \emptyset\}$. The categorical interpretation $[\![\Pi]\!]$ is a compilation of the denotations of Π with all the possible weights on it. We can actually compute the index $(\boldsymbol{n}^+, \boldsymbol{n}^-)$ from w such that $[\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-}$ contains the denotation of Π with weight w by the token machine.

For a proof $\Pi \vdash A_1, \ldots, A_k$ with a weight w, we define a relation $|\Pi|_w \subset \mathbb{N}^k \times \mathbb{N}^k$ as follows:

$$
\begin{aligned}
|\,\mathrm{Ax}_A\,|_w &= \{((n,m),(m,n)) | n, m \in \mathbb{N}\} \\
|\,\mathrm{Cut}(\Pi_0,\Pi_1)|_w &= \{(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \mid \exists i, j \in \mathbb{N}.(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_0|_w, (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in |\Pi_1|_w\} \\
|\,\mathrm{Ten}(\Pi_0,\Pi_1)|_w &= \{(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) | \\
&\qquad (\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in |\Pi_0|_w, (j^+\boldsymbol{m}^+, j^-\boldsymbol{m}^-) \in |\Pi_1|_w\} \\
|\,\mathrm{Par}(\Pi)|_w &= \{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) | (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in |\Pi_1|_w\} \\
|\,\mathrm{Perm}_\sigma(\Pi)|_w &= \{(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) | (\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w\} \\
|\,\mathrm{And}(\Pi_0,\Pi_1)|_w &= \begin{cases} \{(\boldsymbol{n}^+\overline{i}, \boldsymbol{n}^-\overline{j}) | (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_0|_w\} \ (w(\mathrm{And}) = 0) \\ \{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) | (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_1|_w\} \ (w(\mathrm{And}) = 1) \end{cases} \\
|\,\mathrm{Or}_0(\Pi)|_w &= \{(\overline{i}\boldsymbol{n}^+, \overline{j}\boldsymbol{n}^-) | (i\boldsymbol{n}^+, j\boldsymbol{n}^-) \in |\Pi_0|_w\} \\
|\,\mathrm{Or}_1(\Pi)|_w &= \{(\underline{i}\boldsymbol{n}^+, \underline{j}\boldsymbol{n}^-) | (i\boldsymbol{n}^+, j\boldsymbol{n}^-) \in |\Pi_1|_w\}.
\end{aligned}
$$

where we write a list of natural numbers by $n_1 n_2 \cdots n_k$. Hence for $\boldsymbol{n} = n_1 n_2 \cdots n_k$ and $\boldsymbol{m} = m_1 m_2 \cdots m_l$, a concatenation $\boldsymbol{n}i\boldsymbol{m}$ is a list $n_1 n_2 \cdots n_k i m_1 m_2 \cdots m_l$.

Definition 6. *A weight w of Π is well-behaved when $|\Pi|_w \neq \phi$.*

Proposition 11. *(1) For any proof Π, $\|\Pi\| = \bigcup_{w:\text{weight of } \Pi} |\Pi|_w$.*
(2) For any proof Π with a well-behaved weight w and $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w$, we have

$$[\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-} = \{[\Pi]_w\}.$$

Corollary 1. *The set $\{[\Pi]_w | w$: well-behaved weights of $\Pi\}$ is an invariant under cut eliminations.*

6 Related Work

In recent studies on the axiomatic / categorical quantum mechanics, compact closed categories with biproducts and *dagger structure* are employed [2, 24, 25]; the dagger structure is an axiomatisation of adjoints of linear maps. Among such studies, our work is strongly influenced by Selinger's result on *CPM construction* [25]. Selinger showed that for a dagger-biproduct dagger-compact closed $\mathcal{C}$, the dagger-Karoubi envelope of **CPM**($\mathcal{C}$) has biproducts. CPM construction may be regarded as the realisation of the computation with bidirectional information flow, which is reminiscent to **Int** construction. This observation is the starting point of this paper.

One of the potential application field of this work is the geometry of interaction (GoI) [7–9]. In [3], Abramsky, Haghverdi and Scott captured the underlying categorical structure of GoI, and presented a passage from GoI to combinatory algebras. In [11], Haghverdi and Scott gave another categorical analysis of GoI I that treats the concept of execution formula. Extending GoI with additives was considered in GoI III [9], and later more elementary approaches, such as Mairson and Rival's context semantics [23] and Laurent's token machine [21] (which also covers exponentials) are proposed. In particular, Mairson and Rival's context semantics for weighted proofs is almost the same one that we gave in Section 5.

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A An Example of a Traced Distributive SMC $\mathcal{C}$ with Biproducts such that Int($\mathcal{C}$) does not have Biproducts

We show that $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ does not have biproducts (see Section 5 for the definition of $\mathcal{B}(\mathbf{Pfn})$). Note that $\mathcal{B}(\mathbf{Pfn})$ is a traced distributive smc with biproducts. Let $(\{A^+\},\{A^-\})$ and $(\{B^+\},\{B^-\})$ be $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$-objects such that A^+ and B^+ and A^- and B^- are finite sets and $|A^+| > |B^+|$ and $|A^-| > |B^-|$. We suppose $\mathbf{Int}(\mathcal{B}(\mathbf{Pfn}))$ has biproducts and we write $(\{C_i^+\}_{i\in I},\{C_j^-\}_{j\in J})$ for the biproduct of $(\{A^+\},\{A^-\})$ and $(\{B^+\},\{B^-\})$. Then there should be following bijection.

$$\mathcal{P}(\mathbf{Pfn}(X^+ + A^-, A^+ + X^-) + \mathbf{Pfn}(X^+ + B^-, B^+ + X^-)) \cong \mathcal{P}\left(\sum_{i\in I, j\in J} \mathbf{Pfn}(X^+ + C_j^-, C_i^+ + X^-)\right)$$

for any sets X^+ and X^-. Because of cardinarity, I and J and C_i^+ and C_j^- should be finite sets. Hence we have

$$(a^+ + x^- + 1)^{(x^+ + a^-)} + (b^+ + x^- + 1)^{(x^+ + b^-)} = \sum_{i\in I, j\in J} (c_i^+ + x^- + 1)^{(x^+ + c_j^-)} \qquad \cdots (*)$$

for any natural number x^+, x^- where a^+, a^-, b^+, b^-, c_i^+ and c_j^- are cardinarities of A^+, A^-, B^+, B^-, C_i^+ and C_j^- respectively.

Lemma 5. *There are $i_0 \in I$ and $j_0 \in J$ such that $c_{i_0}^+ = a^+$ and $c_{j_0}^- = a^-$.*

Proof. By letting $x^+ = 0$ in $(*)$, we have

$$\begin{aligned}
\lim_{x^-\to\infty} \frac{\sum_{i\in I, j\in J}(c_i^+ + x^- + 1)^{c_j^-}}{(a^+ + x^- + 1)^{a^-}} &= \lim_{x^-\to\infty} \frac{(a^+ + x^- + 1)^{a^-} + (b^+ + x^- + 1)^{b^-}}{(a^+ + x^- + 1)^{a^-}} && \text{by } (*) \\
&= 1 + \lim_{x^-\to\infty} \frac{(b^+ + x^- + 1)^{b^-}}{(a^+ + x^- + 1)^{a^-}} = 1 && (a^- > b^-).
\end{aligned}$$

Since

$$\lim_{x^-\to\infty} \frac{(c_i^+ + x^- + 1)^{c_j^-}}{(a^+ + x^- + 1)^{a^-}} = \begin{cases} \infty & (c_j^- > a^-) \\ 1 & (c_j^- = a^-) \\ 0 & (c_j^- < a^-) \end{cases},$$

there is j_0 such that $c_{j_0}^- = a^-$. Similarly, by letting $x^- = 0$ in $(*)$, we have

$$\begin{aligned}
\lim_{x^+\to\infty} \frac{\sum_{i\in I, j\in J}(c_i^+ + 1)^{(x^+ + c_j^-)}}{(a^+ + 1)^{(x^+ + a^-)}} &= \lim_{x^+\to\infty} \frac{(a^+ + 1)^{(x^+ + a^-)} + (b^+ + 1)^{(x^+ + b^-)}}{(a^+ + 1)^{(x^+ + a^-)}} && \text{by } (*) \\
&= 1 + \lim_{x^+\to\infty} \frac{(b^+ + 1)^{(x^+ + b^-)}}{(a^+ + 1)^{(x^+ + a^-)}} = 1 && (a^- > b^-, a^+ > b^+).
\end{aligned}$$

Since

$$\lim_{x^+\to\infty} \frac{(c_i^+ + 1)^{(x^+ + c_j^-)}}{(a^+ + 1)^{(x^+ + a^-)}} = \begin{cases} \infty & (c_i^+ > a^+) \\ (a^+ + 1)^{c_j^- - a^-} & (c_i^+ = a^+) \\ 0 & (c_i^+ < a^+) \end{cases},$$

there is i_0 such that $c_{i_0}^+ = a^+$.

We show $(*)$ implies contradiction. By $(*)$, both I and J can not be empty sets. If $|I \times J| = 1$ then the RHS of $(*)$ is $(a^+ + x^- + 1)^{(x^+ + a^-)}$ by this lemma, that is less than the LHS of $(*)$. Hence $|I| \geq 2$ or $|J| \geq 2$. However, if $|I| \geq 2$ then

$$\begin{aligned}
1 &= \lim_{x^- \to \infty} \frac{(a^+ + x^- + 1)^{(x^+ + a^-)} + (b^+ + x^- + 1)^{(x^+ + b^-)}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} \\
&= \lim_{x^- \to \infty} \frac{\sum_{i \in I, j \in J} (c_i^+ + x^- + 1)^{(x^+ + c_j^-)}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} \\
&\geq \lim_{x^- \to \infty} \frac{2(x^- + 1)^{(x^+ + a^-)}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} = 2,
\end{aligned}$$

if $|J| \geq 2$ then

$$\begin{aligned}
1 &= \lim_{x^+ \to \infty} \frac{(a^+ + x^- + 1)^{(x^+ + a^-)} + (b^+ + x^- + 1)^{(x^+ + b^-)}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} \\
&= \lim_{x^+ \to \infty} \frac{\sum_{i \in I, j \in J} (c_i^+ + x^- + 1)^{(x^+ + c_j^-)}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} \\
&\geq \lim_{x^+ \to \infty} \frac{(a^+ + x^- + 1)^{(x^+ + a^-)} + (a^+ + x^- + 1)^{x^+}}{(a^+ + x^- + 1)^{(x^+ + a^-)}} > 1.
\end{aligned}$$

Hence **Int**($\mathcal{B}$(**Pfn**)) does not have birproducts.

B Multiplicative Additive Linear Logic

Here we give a short description of MALL [6]. The set of formulae is defined by the following BNF:

$$\text{(Formula) } A ::= \alpha \mid \alpha^\perp \mid A \wp A \mid A \otimes A \mid A \& A \mid A \oplus A.$$

We extend the negation to all formulae as follows:

$$(\alpha)^\perp = \alpha^\perp, \qquad (\alpha^\perp)^\perp = \alpha$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp, \qquad (A \otimes B)^\perp = A^\perp \wp B^\perp,$$
$$(A \& B)^\perp = A^\perp \oplus B^\perp, \qquad (A \oplus B)^\perp = A^\perp \& B^\perp.$$

The inference rules are given as follows:

$$\frac{}{\mathrm{Ax}_A \vdash A, A^\perp}\,(axiom) \qquad \frac{\Pi \vdash \Gamma, A \qquad \Pi' \vdash A^\perp, \Delta}{\mathrm{Cut}(\Pi, \Pi') \vdash \Gamma, \Delta}\,(cut) \qquad \frac{\Pi \vdash \Gamma}{\mathrm{Perm}_\sigma(\Pi) \vdash \sigma(\Gamma)}$$

$$\frac{\Pi \vdash \Gamma, A \qquad \Pi \vdash B, \Delta}{\mathrm{Ten}(\Pi, \Pi') \vdash \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \frac{\Pi \vdash \Gamma, A, B}{\mathrm{Par}(\Pi) \vdash \Gamma, A \wp B}\,(\wp)$$

$$\frac{\Pi \vdash \Gamma, A \qquad \Pi' \vdash \Gamma, B}{\mathrm{And}(\Pi, \Pi') \vdash \Gamma, A \& B}\,(\&) \qquad \frac{\Pi \vdash A, \Gamma}{\mathrm{Or}_0(\Pi) \vdash A \oplus B, \Gamma}\,(\oplus_0) \qquad \frac{\Pi \vdash B, \Gamma}{\mathrm{Or}_1(\Pi) \vdash A \oplus B, \Gamma}\,(\oplus_1)$$

C Proofs

(This part will be removed in the final version)

On the Definition of Biproducts

We compare the definition of biproducts in Definition 1 and the one in [18], which is given in terms of the invertibility of certain canonical maps in bicartesian categories.

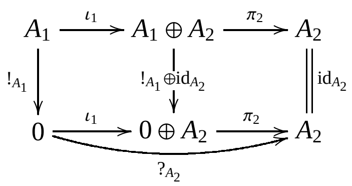
Definition 7. *[18] A category C has* H-biproducts *if it is bicartesian such that X) the canonical morphism* $?_1 : 0 \to 1$ *is invertible, and Y) the following canonical natural transformation is invertible:*

$$m_{A,B} = [\langle \mathrm{id}_A, 0\rangle, \langle 0, \mathrm{id}_B\rangle] : A + B \to A \times B$$

where 0 *is the zero map defined by* $0_{A,B} = ?_B \circ (?_1)^{-1} \circ !_A$.

Proposition 12. *A category C has H-biproducts if and only if C has a zero object and binary biproducts in the sense of Definition 1.*

Proof: (if) In this part, the symbols $\langle -, -\rangle, \pi_1, \pi_2, [-, -], \iota_1, \iota_2$ denote the tupling, projections, cotupling and injections associated to $\oplus \dashv \Delta \dashv \oplus$. Category C is clearly bicartesian and the canonical morphism $?_1 : 0 \to 1$ is id_0. We therefore show that the canonical natural transformation $m_{A,B}$ is invertible. Since C has a zero object, we have $\pi_1 \circ \iota_2 = \pi_2 \circ \iota_1 = 0$ (below we proved $\pi_1 \circ \iota_2 = 0$).



Then the canonical natural transformation is equal to the identity map on $A \oplus B$:

$$m_{A,B} = [\langle \mathrm{id}, 0\rangle, \langle 0, \mathrm{id}\rangle] = [\langle \pi_1 \circ \iota_1, \pi_2 \circ \iota_1\rangle, \langle \pi_1 \circ \iota_2, \pi_2 \circ \iota_2\rangle] = \mathrm{id}_{A\oplus B}\,.$$

Therefore C has H-biproducts.

(only if) Let C be a bicartesian category satisfying Condition X and Y. In this part, the symbols $\langle -, -\rangle, \pi_1, \pi_2$ denote the tupling and projections of binary products, and $[-, -], \iota, \iota'$ denote the cotupling and injections of binary coproducts. We show that the coproduct functor $+ : C \times C \to C$ is also a right adjoint of Δ. We define the unit of this adjunction by $\delta_A = m^{-1}_{A,A} \circ \langle \mathrm{id}, \mathrm{id}\rangle : A \to A + A$, and counit (=projections) by

$p_{A,B} = \pi_1 \circ m_{A,B} : A + B \to A$ and $p'_{A,B} = \pi_2 \circ m_{A,B} : A + B \to B$. Then we have

$$\begin{aligned}
p_{A,A} \circ \delta_A &= \pi_1 \circ m_{A,A} \circ m^{-1}_{A,A} \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \mathrm{id}_A \,. \\
p'_{A,A} \circ \delta_A &= \pi_2 \circ m_{A,A} \circ m^{-1}_{A,A} \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \mathrm{id}_A \,. \\
(p_{A,B} + p'_{A,B}) \circ \delta_{A+B} &= (\pi_1 \circ m_{A,B} + \pi_2 \circ m_{A,B}) \circ m^{-1}_{A+B,A+B} \circ \langle \mathrm{id}_{A+B}, \mathrm{id}_{A+B} \rangle \\
&\quad \text{(naturality of } m^{-1}) \\
&= m^{-1}_{A,B} \circ (\pi_1 \circ m_{A,B} \times \pi_2 \circ m_{A,B}) \circ \langle \mathrm{id}_{A+B}, \mathrm{id}_{A+B} \rangle \\
&= m^{-1}_{A,B} \circ \langle \pi_1 \circ m_{A,B}, \pi_2 \circ m_{A,B} \rangle \\
&= m^{-1}_{A,B} \circ \langle \pi_1, \pi_2 \rangle \circ m_{A,B} \\
&= \mathrm{id} \,.
\end{aligned}$$

We next show that $p \circ \iota_1 = \mathrm{id}$ and $p' \circ \iota_2 = \mathrm{id}$. We only show the former.

$$p_{A,B} \circ \iota_1 = \pi_1 \circ m_{A,B} \circ \iota_1 = \pi_1 \circ \langle \mathrm{id}_A, 0 \rangle = \mathrm{id}_A \,.$$

Proof of Proposition 2 and 3

Proposition 2-1 Let $\mathcal{C}$ be a category with binary biproducts. We define $0_{A,B}$ and $0'_{A,B}$ by

$$0_{A,B} = \; A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B \,, \qquad 0'_{A,B} = \; A \xrightarrow{\iota_2} B \oplus A \xrightarrow{\pi_1} B \,.$$

We also define a binary operator $+$ on $\mathcal{C}(A, B)$ by

$$f + g = [\mathrm{id}, \mathrm{id}] \circ \langle f, g \rangle.$$

Lemma 6. *For any $f : A \to B$, we have $0_{B,C} \circ f = 0_{A,C}$ and $f \circ 0_{C,A} = 0_{C,B}$.*

All squares below commute by naturality:

$$\begin{array}{ccccc}
A & \xrightarrow{\iota_1} & A \oplus C & \xrightarrow{\pi_2} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \oplus \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}_C} \\
B & \xrightarrow[\iota_1]{} & B \oplus C & \xrightarrow[\pi_2]{} & C
\end{array}
\qquad
\begin{array}{ccccc}
C & \xrightarrow{\iota_1} & C \oplus A & \xrightarrow{\pi_2} & A \\
{\scriptstyle \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \oplus f} & & \downarrow{\scriptstyle f} \\
C & \xrightarrow[\iota_1]{} & C \oplus B & \xrightarrow[\pi_2]{} & B
\end{array}$$

Hence we obtain $0_{B,C} \circ f = 0_{A,C}$ and $g \circ 0_{B,C} = 0_{B,D}$.

Lemma 7. *We have $0_{A,B} = 0'_{A,B}$.*

We have $0'_{B,C} \circ f = 0'_{A,C}$ and $f \circ 0'_{C,A} = 0'_{C,B}$ for any $f : A \to B$. Then the above equation is immediate.

Lemma 8. *We have $\langle \pi_2, \pi_1 \rangle = [\iota_2, \iota_1]$.*

We have

$$\pi_1 \circ [\iota_2, \iota_1] = [\pi_1 \circ \iota_2, \pi_1 \circ \iota_1] = [0', \mathrm{id}] = [0, \pi_2 \circ \iota_2] = \pi_2 \circ [\iota_1, \iota_2] = \pi_2.$$

Similarly we have $\pi_2 \circ [\iota_2, \iota_1] = \pi_1$. Hence $[\iota_2, \iota_1] = \langle \pi_2, \pi_1 \rangle$.

Lemma 9. *We have* $\langle [a, b], [c, d]\rangle = [\langle a, c\rangle, \langle b, d\rangle]$.

We calculate the first and second projections of the r.h.s.:

$$\pi_1 \circ [\langle a, c\rangle, \langle b, d\rangle] = [a, b], \;\; \pi_2 \circ [\langle a, c\rangle, \langle b, d\rangle] = [c, d].$$

Hence we obtain the equation in question.

Lemma 10. *We have* $\langle \mathrm{id}, 0\rangle = \iota_1, \langle 0, \mathrm{id}\rangle = \iota_2$.

We have

$$\langle \mathrm{id}, 0\rangle = \langle \pi_1 \circ \iota_1, \pi_2 \circ \iota_1\rangle = \langle \pi_1, \pi_2\rangle \circ \iota_1 = \iota_1.$$

Similarly we have $\langle 0, \mathrm{id}\rangle = \iota_2$.

Lemma 11. *We have* $\langle 0, \iota_1\rangle = \iota_2 \circ \iota_1$ *and* $\langle 0, \iota_2\rangle = \iota_2 \circ \iota_2$.

From Lemma 9 and 10, we have

$$[\langle 0, \iota_1\rangle, \langle 0, \iota_2\rangle] = \langle [0, 0], [\iota_1, \iota_2]\rangle = \langle 0, \mathrm{id}\rangle = \iota_2.$$

Therefore $\iota_2 \circ \iota_1 = \langle 0, \iota_1\rangle$ and $\iota_2 \circ \iota_2 = \langle 0, \iota_2\rangle$.

Lemma 12. *The following morphism:*

$$[[\iota_1, \iota_2 \circ \iota_1], \iota_2 \circ \iota_2] : (A \oplus B) \oplus C \to A \oplus (B \oplus C).$$

is the inverse of the following associativity:

$$a = \langle\langle \pi_1, \pi_1 \circ \pi_2\rangle, \pi_2 \circ \pi_2\rangle : A \oplus (B \oplus C) \to (A \oplus B) \oplus C.$$

There is a canonical inverse of a:

$$a^{-1} = \langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2\rangle\rangle : (A \oplus B) \oplus C \to A \oplus (B \oplus C).$$

We inspect its contents by composing injections:

$$\begin{aligned} a^{-1} \circ \iota_1 \circ \iota_1 &= \langle \mathrm{id}, \langle 0, 0\rangle\rangle = \langle \mathrm{id}, 0\rangle = \iota_1 \\ a^{-1} \circ \iota_1 \circ \iota_2 &= \langle 0, \iota_1\rangle = \iota_2 \circ \iota_1 \\ a^{-1} \circ \iota_2 &= \langle 0, \langle 0, \mathrm{id}\rangle\rangle = \langle 0, \iota_2\rangle = \iota_2 \circ \iota_2. \end{aligned}$$

Therefore we obtain

$$a^{-1} = [[\iota_1, \iota_2 \circ \iota_1], \iota_2 \circ \iota_2].$$

We are now ready to prove that C is commutative-monoid enriched. Below we show that the morphism 0 and the binary operator + form a commutative monoid.

$$f + 0 = [\mathrm{id}, \mathrm{id}] \circ \langle f, 0\rangle = [\mathrm{id}, \mathrm{id}] \circ \langle \mathrm{id}, 0\rangle \circ f = [\mathrm{id}, \mathrm{id}] \circ \iota_1 \circ f = f,$$

$$0 + f = [\mathrm{id}, \mathrm{id}] \circ \langle 0, f\rangle = [\mathrm{id}, \mathrm{id}] \circ \langle 0, \mathrm{id}\rangle \circ f = [\mathrm{id}, \mathrm{id}] \circ \iota_2 \circ f = f.$$

$$
\begin{aligned}
f + (g + h) &= [\mathrm{id}, \mathrm{id}] \circ \langle f, [\mathrm{id}, \mathrm{id}] \circ \langle g, h\rangle\rangle \\
&= [\mathrm{id}, \mathrm{id}] \circ \langle f, [\mathrm{id}, \mathrm{id}] \circ \langle g, h\rangle\rangle \\
&= [\mathrm{id}, [\mathrm{id}, \mathrm{id}]] \circ \langle f, \langle g, h\rangle\rangle \\
&= [\mathrm{id}, [\mathrm{id}, \mathrm{id}]] \circ a^{-1} \circ a \circ \langle f, \langle g, h\rangle\rangle \\
&= [[\mathrm{id}, \mathrm{id}], \mathrm{id}] \circ \langle\langle f, g\rangle, h\rangle \\
&= (f + g) + h.
\end{aligned}
$$

We next show that $-\circ h$ and $h \circ -$ are both monoid homomorphisms. From Lemma 6 we have $0 \circ h = h \circ 0 = 0$. Next, by definition of $+$ we have

$$(f + g) \circ h = [\mathrm{id}, \mathrm{id}] \circ \langle f, g\rangle \circ h = [\mathrm{id}, \mathrm{id}] \circ \langle f \circ h, g \circ h\rangle = f \circ h + g \circ h,$$

$$h \circ (f + g) = [h, h] \circ \langle f, g\rangle = [\mathrm{id}, \mathrm{id}] \circ (h \oplus h) \circ \langle f, g\rangle = h \circ f + h \circ g.$$

Proposition 2-2 (if)

$$
\begin{aligned}
[F\iota_1, F\iota_2] \circ \langle F\pi_1, F\pi_2\rangle &= F\iota_1 \circ F\pi_1 + F\iota_2 \circ F\pi_2 \\
(F \text{ enriched}) &= F(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \\
&= \mathrm{id}\,. \\
\langle F\pi_1, F\pi_2\rangle \circ [F\iota_1, F\iota_2] &= [\langle F\pi_1, F\pi_2\rangle \circ F\iota_1, \langle F\pi_1, F\pi_2\rangle \circ F\iota_2] \\
&= [\langle F\pi_1 \circ F\iota_1, F\pi_2 \circ F\iota_1\rangle, \langle F\pi_1 \circ F\iota_2, F\pi_2 \circ F\iota_2\rangle] \\
(F \text{ enriched}) &= [\langle \mathrm{id}, 0\rangle, \langle 0, \mathrm{id}\rangle] \\
(\text{Lemma 10}) &= [\iota_1, \iota_2] \\
&= \mathrm{id}\,.
\end{aligned}
$$

(only if)

$$
\begin{aligned}
F(f + g) &= F[\mathrm{id}, \mathrm{id}] \circ F\langle f, g\rangle \\
(\text{Canonical iso}) &= F[\mathrm{id}, \mathrm{id}] \circ [F\iota_1, F\iota_2] \circ \langle F\pi_1, F\pi_2\rangle \circ F\langle f, g\rangle \\
&= [F\,\mathrm{id}, F\,\mathrm{id}] \circ \langle Ff, Fg\rangle \\
&= Ff + Fg \\
F0_{A,B} &= F\pi_2 \circ F\iota_1 \\
&= \pi_2 \circ \langle F\pi_1, F\pi_2\rangle \circ [F\iota_1, F\iota_2] \circ \iota_1 \\
(\text{Canonical iso}) &= \pi_2 \circ \iota_1 \\
&= 0_{FA,FB}.
\end{aligned}
$$

Proposition 3-1 Category $\Re C$ has binary biproducts, hence is canonically enriched by Proposition 2-1. Since the functor $H_C : C \to \Re C$ fully faithfully embeds C into $\Re C$, the canonical enrichment can be restricted to C by the embedding. This enrichment can be explicitly described using the isomorphism $H_{C(A,B)} : C(A, B) \to \Re C(H_C A, H_C B)$ on homsets:

$$(0_C)_{A,B} = H^{-1}_{C(A,B)}((0_{\Re C})_{H_C A, H_C B}), \qquad f +_C g = H^{-1}_{C(A,B)}(H_{C(A,B)} f +_{\Re C} H_{C(A,B)} g),$$

and by expanding the definition of H and the canonical enrichment, we obtain

$$(0_{\mathcal{C}})_{A,B} = \pi_1 \circ \iota_2 = \pi_2 \circ \iota_1, \qquad (f +_{\mathcal{C}} g) = [\mathrm{id}, \mathrm{id}] \circ \langle f, g \rangle,$$

that is, the equations defining the canonical enrichment for binary biproducts in Proposition 2 also determine that for binary semibiproducts.

Proposition 3-2 (if) The functor $\Re F : \Re\mathcal{C} \to \Re\mathcal{D}$ is enriched by Proposition 2-2.

$$\begin{aligned}
F(f +_{\mathcal{C}} g) &= F(H^{-1}_{\mathcal{C}(A,B)}(H_{\mathcal{C}(A,B)} f +_{\Re\mathcal{C}} H_{\mathcal{C}(A,B)} g)) \\
\text{(naturality of } H) &= H^{-1}_{\mathcal{D}(FA,FB)}(\Re F(H_{\mathcal{C}(A,B)} f +_{\Re\mathcal{C}} H_{\mathcal{C}(A,B)} g)) \\
(\Re F \text{ is enriched}) &= H^{-1}_{\mathcal{D}(FA,FB)}(\Re F(H_{\mathcal{C}(A,B)} f) +_{\Re\mathcal{D}} \Re F(H_{\mathcal{C}(A,B)} g))) \\
\text{(naturality of } H) &= H^{-1}_{\mathcal{D}(FA,FB)}(H_{\mathcal{D}(FA,FB)} F f +_{\Re\mathcal{D}} H_{\mathcal{D}(FA,FB)} F g) \\
&= F f +_{\mathcal{D}} F g.
\end{aligned}$$

(only if) We show the equations in Definition 4. First, note that

$$\begin{aligned}
\mathrm{id} \oplus \mathrm{id} &= (\mathrm{id} \oplus \mathrm{id}) \circ (\mathrm{id} \oplus \mathrm{id}) \\
&= [\iota_1, \iota_2] \circ \langle \pi_1, \pi_2 \rangle \\
&= [\mathrm{id}, \mathrm{id}] \circ \langle \iota_1 \circ \pi_1, \iota_2 \circ \pi_2 \rangle \\
&= \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
[F\iota_1, F\iota_2] \circ \langle F\pi_1, F\pi_2 \rangle &= F(\iota_1 \circ \pi_1) + F(\iota_2 \circ \pi_2) \\
(F \text{ enriched}) &= F(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \\
&= F(\mathrm{id}_A \oplus \mathrm{id}_B).
\end{aligned}$$

For the other equation,

$$\begin{aligned}
\langle F\pi_1, F\pi_2 \rangle \circ [F\iota_1, F\iota_2] &= \langle [F(\pi_1 \circ \iota_1), F(\pi_1 \circ \iota_2)], [F(\pi_2 \circ \iota_1), F(\pi_2 \circ \iota_2)] \rangle \\
(F \text{ enriched}) &= \langle [\mathrm{id}_{FA}, 0], [0, \mathrm{id}_{FB}] \rangle \\
&= \langle \pi_1 \circ (\mathrm{id}_{FA} \oplus \mathrm{id}_{FB}), \pi_2 \circ (\mathrm{id}_{FA} \oplus \mathrm{id}_{FB}) \rangle \\
&= \langle \pi_1, \pi_2 \rangle \circ (\mathrm{id}_{FA} \oplus \mathrm{id}_{FB}) \\
&= (\mathrm{id}_{FA} \oplus \mathrm{id}_{FB}).
\end{aligned}$$

Remaining proofs can be found at

`http://www.kurims.kyoto-u.ac.jp/˜naophiko/paper/app.pdf`

D Remaining Proofs

Proof of Lemma 1

From the distributivity, we have $0_{A,B} \otimes 0_{C,C} = 0_{A\otimes C,B\otimes C}$. Therefore

$$\begin{aligned} \mathrm{tr}^C_{A,B}(0_{A\otimes C,B\otimes C}) &= \mathrm{tr}^C_{A,B}(0_{A,B} \otimes 0_{C,C}) \\ \text{(superposing)} &= 0_{A,B} \otimes \mathrm{tr}^C_{\mathbf{I},\mathbf{I}}(0_{C,C}) \\ \text{(bilinearity)} &= 0_{A,B}. \end{aligned}$$

We move to $\mathrm{tr}^C_{A,B}(f+g) = \mathrm{tr}^C_{A,B}(f) + \mathrm{tr}^C_{A,B}(g)$. First, we have (for each $i \in \{1,2\}$)

$$\pi_i = (\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] : (A_1 \otimes C) \oplus (A_2 \otimes C) \to A_i \otimes C. \tag{1}$$

Proof:

$$\begin{aligned} &\pi_i \\ (-\otimes C \text{ preserves biproducts}) &= \pi_i \circ \langle \pi_1 \otimes \mathrm{id}_C, \pi_2 \otimes \mathrm{id}_C \rangle \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \\ &= (\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C]. \end{aligned}$$

Second, we have

$$\langle \mathrm{tr}^C_{A,B}(f_1), \mathrm{tr}^C_{A,B}(f_2) \rangle = \mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle). \tag{2}$$

Proof: We calculate the first and second component of $\mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle)$.

$$\begin{aligned} &\pi_i \circ \mathrm{tr}^C_{A,B\oplus B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle) \\ \text{(naturality of trace)} &= \mathrm{tr}^C_{A,B}((\pi_i \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f_1, f_2 \rangle) \\ \text{(Equation 1)} &= \mathrm{tr}^C_{A,B}(\pi_i \circ \langle f_1, f_2 \rangle) \\ &= \mathrm{tr}^C_{A,B}(f_i) \end{aligned}$$

(where $i \in \{1,2\}$). From this, we conclude Equation 2. We now prove the lemma in question.

$$\begin{aligned} &\mathrm{tr}^C_{A,B}(f) + \mathrm{tr}^C_{A,B}(g) \\ \text{(definition of +)} &= [\mathrm{id}_B, \mathrm{id}_B] \circ \langle \mathrm{tr}^C_{A,B}(f), \mathrm{tr}^C_{A,B}(g) \rangle \\ \text{(Equation 2)} &= [\mathrm{id}_B, \mathrm{id}_B] \circ \mathrm{tr}^C_{A,B}([\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\ \text{(naturality of trace)} &= \mathrm{tr}^C_{A,B}(([\mathrm{id}_B, \mathrm{id}_B] \otimes \mathrm{id}_C) \circ [\iota_1 \otimes \mathrm{id}_C, \iota_2 \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\ &= \mathrm{tr}^C_{A,B}([([\mathrm{id}_B, \mathrm{id}_B] \circ \iota_1) \otimes \mathrm{id}_C, ([\mathrm{id}_B, \mathrm{id}_B] \circ \iota_2) \otimes \mathrm{id}_C] \circ \langle f, g \rangle) \\ &= \mathrm{tr}^C_{A,B}([\mathrm{id}_{B\otimes C}, \mathrm{id}_{B\otimes C}] \circ \langle f, g \rangle) \\ &= \mathrm{tr}^C_{A,B}(f+g). \end{aligned}$$

Proof of Lemma 2

In this proof we omit the annotation of objects on trace operators.

(1) We have

$$\begin{aligned}
\operatorname{tr}(f) &= \operatorname{tr}\left(\sum_{1\leqslant i,j\leqslant 2} (\mathrm{id}_B \otimes \iota_i) \circ f_{ij} \circ (\mathrm{id}_A \otimes \pi_j)\right) \\
(\text{Lemma 1}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\left((\mathrm{id}_B \otimes \iota_i) \circ f_{ij} \circ (\mathrm{id}_A \otimes \pi_j)\right) \\
(\text{sliding}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\left(f_{ij} \circ (\mathrm{id}_A \otimes (\pi_j \circ \iota_i)\right) \\
\left(\pi_j \circ \iota_i = \begin{cases} 0 & (j \neq i) \\ \mathrm{id} & (j = i) \end{cases}\right) &= \sum_{1\leqslant i\leqslant 2} \operatorname{tr}(f_{ii})\,.
\end{aligned}$$

(2) We have

$$\begin{aligned}
\operatorname{tr}(f) &= \operatorname{tr}\left(\sum_{1\leqslant i,j\leqslant 2} (\iota_i \otimes \mathrm{id}_A) \circ f_{ij} \circ (\pi_j \otimes \mathrm{id}_A)\right) \\
(\text{Lemma 1}) &= \sum_{1\leqslant i,j\leqslant 2} \operatorname{tr}\left((\iota_i \otimes \mathrm{id}_A) \circ f_{ij} \circ (\pi_j \otimes \mathrm{id}_A)\right) \\
(\text{naturality of trace}) &= \sum_{1\leqslant i,j\leqslant 2} \iota_i \circ \operatorname{tr}\left(f_{ij}\right) \circ \pi_j.
\end{aligned}$$

From this, we conclude that the matrix representation of $\operatorname{tr}(f)$ is $\begin{pmatrix} \operatorname{tr}(f_{11}) & \operatorname{tr}(f_{12}) \\ \operatorname{tr}(f_{21}) & \operatorname{tr}(f_{22}) \end{pmatrix}$.

Proof of Lemma 3

Let $f = \langle\!\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\!\rangle : A \to B_1 \oplus B_2$ be an $\mathbf{Int}(\mathcal{C})$-morphism where each component f_{ij} is in $\mathbf{Int}(\mathcal{C})(A, (B_i^+, B_j^-))$. This morphism corresponds to the following matrix of $\mathcal{C}$-morphisms $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Similarly, the tuple $g = [[g_{11}, g_{12}, g_{21}, g_{22}]] : B_1 \oplus B_2 \to C$ with $g_{ij} \in \mathbf{Int}(\mathcal{C})((B_i^+, B_j^-), C)$ corresponds to the matrix $\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}$ (note the order of

g_{21} and g_{12}). The composite $g \circ f$ in $\mathbf{Int}(\mathcal{C})$ is the trace of the following $\mathcal{C}$-morphism:

$$\begin{aligned}
&(C^+ \otimes \sigma) \circ (g \otimes A^-) \circ ((B_1^+ \oplus B_2^+) \otimes \sigma) \circ (f \otimes C^-) \circ (A^+ \otimes \sigma) \\
&= \begin{pmatrix} C^+ \otimes \sigma & 0 \\ 0 & C^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} g_{11} \otimes A^- & g_{21} \otimes A^- \\ g_{12} \otimes A^- & g_{22} \otimes A^- \end{pmatrix} \circ \\
&\quad \begin{pmatrix} B_1^+ \otimes \sigma & 0 \\ 0 & B_2^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} f_{11} \otimes C^- & f_{12} \otimes C^- \\ f_{21} \otimes C^- & f_{22} \otimes C^- \end{pmatrix} \circ \begin{pmatrix} A^+ \otimes \sigma & 0 \\ 0 & A^+ \otimes \sigma \end{pmatrix} \\
&= \begin{pmatrix} (C^+ \otimes \sigma) \circ (g_{11} \otimes A^-) & (C^+ \otimes \sigma) \circ (g_{21} \otimes A^-) \\ (C^+ \otimes \sigma) \circ (g_{12} \otimes A^-) & (C^+ \otimes \sigma) \circ (g_{22} \otimes A^-) \end{pmatrix} \circ \\
&\quad \begin{pmatrix} (B_1^+ \otimes \sigma) \circ (f_{11} \otimes C^-) \circ (A^+ \otimes \sigma) & (B_1^+ \otimes \sigma) \circ (f_{12} \otimes C^-) \circ (A^+ \otimes \sigma) \\ (B_2^+ \otimes \sigma) \circ (f_{21} \otimes C^-) \circ (A^+ \otimes \sigma) & (B_2^+ \otimes \sigma) \circ (f_{22} \otimes C^-) \circ (A^+ \otimes \sigma) \end{pmatrix} \\
&= h
\end{aligned}$$

where h is the morphism whose matrix consists of the following elements:

$$\begin{aligned}
h_{ij} &= (C^+ \otimes \sigma) \circ (g_{1i} \otimes A^-) \circ (B_1^+ \otimes \sigma) \circ (f_{1j} \otimes C^-) \circ (A^+ \otimes \sigma) \\
&\quad + (C^+ \otimes \sigma) \circ (g_{2i} \otimes A^-) \circ (B_2^+ \otimes \sigma) \circ (f_{2j} \otimes C^-) \circ (A^+ \otimes \sigma).
\end{aligned}$$

Therefore

$$\mathrm{tr}^{B_1^- \oplus B_2^-}_{A^+, C^+}(h) = \sum_{1 \le i,j \le 2} \mathrm{tr}^{B_1^- \oplus B_2^-}_{A^+, C^+}(h_{ij}) = \sum_{1 \le i,j \le 2} g_{ij} \circ f_{ij}.$$

Thus we obtain $\langle\!\langle f_{ij} \rangle\!\rangle \circ [[g_{ij}]] = \sum g_{ij} \circ f_{ij}$.

We next calculate the composition of $h : C \to A$ and $f = \langle\!\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\!\rangle : A \to B_1 \oplus B_2$ in $\mathbf{Int}(\mathcal{C})$. The composition is the trace of the following morphism:

$$\begin{aligned}
&((B_1^+ \oplus B_2^+) \otimes \sigma) \circ (f \otimes C^-) \circ (A^+ \otimes \sigma) \circ (h \otimes (B_1^- \oplus B_2^-)) \circ (C^+ \otimes \sigma) \\
&= \begin{pmatrix} B_1^+ \otimes \sigma & 0 \\ 0 & B_2^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} f_{11} \otimes C^- & f_{12} \otimes C^- \\ f_{21} \otimes C^- & f_{22} \otimes C^- \end{pmatrix} \circ \\
&\quad \begin{pmatrix} A^+ \otimes \sigma & 0 \\ 0 & A^+ \otimes \sigma \end{pmatrix} \circ \begin{pmatrix} h \otimes B_1^- & 0 \\ 0 & h \otimes B_2^- \end{pmatrix} \circ \begin{pmatrix} C^+ \otimes \sigma & 0 \\ 0 & C^+ \otimes \sigma \end{pmatrix} \\
&= \begin{pmatrix} (B_1^+ \otimes \sigma) \circ (f_{11} \otimes C^-) & (B_1^+ \otimes \sigma) \circ (f_{12} \otimes C^-) \\ (B_2^+ \otimes \sigma) \circ (f_{21} \otimes C^-) & (B_2^+ \otimes \sigma) \circ (f_{22} \otimes C^-) \end{pmatrix} \circ \\
&\quad \begin{pmatrix} (A^+ \otimes \sigma) \circ (h \otimes B_1^-) \circ (C^+ \otimes \sigma) & 0 \\ 0 & (A^+ \otimes \sigma) \circ (h \otimes B_2^-) \circ (C^+ \otimes \sigma) \end{pmatrix} \\
&= h
\end{aligned}$$

where $h_{ij} = (B_i^+ \otimes \sigma) \circ (f_{ij} \otimes C^-) \circ (A^+ \otimes \sigma) \circ (h \otimes B_j^-) \circ (C^+ \otimes \sigma)$. Therefore we obtain

$$\mathrm{tr}^{A^-}_{C^+ \otimes (B_1^- \oplus B_2^-), (B_1^+ \oplus B_2^+) \otimes C_-}(h) = \begin{pmatrix} \mathrm{tr}(h_{11}) & \mathrm{tr}(h_{12}) \\ \mathrm{tr}(h_{21}) & \mathrm{tr}(h_{22}) \end{pmatrix} = \begin{pmatrix} f_{11} \circ h & f_{12} \circ h \\ f_{21} \circ h & f_{22} \circ h \end{pmatrix}.$$

This matrix corresponds to the tuple $\langle\!\langle f_{ij} \circ h \rangle\!\rangle$. Hence we obtain $\langle\!\langle f_{ij} \rangle\!\rangle \circ h = \langle\!\langle f_{ij} \circ h \rangle\!\rangle$.

Proof of Proposition 5

We show that the tupling, cotupling, projections and injections satisfy Condition B-1.

$$\begin{aligned}
\pi_1 \circ \langle f, g\rangle &= [[\mathrm{id}, 0, 0, 0]] \circ \langle\!\langle f, 0, 0, g\rangle\!\rangle \\
&= \mathrm{id} \circ f + 0 \circ 0 + 0 \circ 0 + 0 \circ g \\
&= f. \\
&\quad \text{(we omit the proof for } \pi_2 \circ \langle f, g\rangle = g\text{)} \\
[f, g] \circ \iota_1 &= [[f, 0, 0, g]] \circ \langle\!\langle \mathrm{id}, 0, 0, 0\rangle\!\rangle \\
&= f \circ \mathrm{id} + 0 \circ 0 + 0 \circ 0 + g \circ 0 \\
&= f. \\
&\quad \text{(we omit the proof for } [f, g] \circ \iota_2 = g\text{)} \\
\langle f, g\rangle \circ h &= \langle\!\langle f, 0, 0, g\rangle\!\rangle \circ h \\
&= \langle\!\langle f \circ h, 0, 0, g \circ h\rangle\!\rangle \\
&= \langle f \circ h, g \circ h\rangle. \\
h \circ \langle f, g\rangle &= h \circ [[f, 0, 0, g]] \\
&= [[h \circ f, 0, 0, h \circ g]] \\
&= [h \circ f, h \circ g] \\
\pi_1 \circ \iota_1 &= [[\mathrm{id}, 0, 0, 0]] \circ \langle\!\langle \mathrm{id}, 0, 0, 0\rangle\!\rangle \\
&= \mathrm{id} \circ \mathrm{id} + 0 + 0 + 0 \\
&= \mathrm{id} \\
&\quad \text{(we omit the proof for } \pi_2 \circ \iota_2 = \mathrm{id}\text{)} \\
\langle f \circ \pi_1, g \circ \pi_2\rangle &= \langle\!\langle [[f, 0, 0, 0]], \mathbf{0}, \mathbf{0}, [[0, 0, 0, g]]\rangle\!\rangle \\
&= [[\langle\!\langle f, 0, 0, 0\rangle\!\rangle, \mathbf{0}, \mathbf{0}, \langle\!\langle 0, 0, 0, g\rangle\!\rangle]] \\
&= [\iota_1 \circ f, \iota_2 \circ g].
\end{aligned}$$

Here $\mathbf{0} = [[0, 0, 0, 0]]$.

Proof of Theorem 3

That $\mathbf{Int}(C)$ has semibiproducts is already proved in Proposition 5. We show that N_C preserves semibiproducts. Below we just write N for N_C. We have seen that $N0$ is a zero object in $\mathbf{Int}(C)$. Next, we have (below we write $\mathbf{0}$ for $[[0, 0, 0, 0]]$)

$$\begin{aligned}
\langle N\pi_1, N\pi_2\rangle \circ [N\iota_1, N\iota_2] &= \langle\!\langle N\pi_1, 0, 0, N\pi_2\rangle\!\rangle \circ [[N\iota_1, 0, 0, N\iota_2]] \\
&= \langle\!\langle [[N\,\mathrm{id}, 0, 0, 0]], \mathbf{0}, \mathbf{0}, [[0, 0, 0, N\,\mathrm{id}]]\rangle\!\rangle \\
&= N\,\mathrm{id} \oplus N\,\mathrm{id} \\
[N\iota_1, N\iota_2] \circ \langle N\pi_1, N\pi_2\rangle &= [[N\iota_1, 0, 0, N\iota_2]] \circ \langle\!\langle N\pi_1, 0, 0, N\pi_2\rangle\!\rangle \\
&= N\iota_1 \circ N\pi_1 + N\iota_2 \circ N\pi_2 \\
&= (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \otimes \mathbf{I} \\
&= N\,\mathrm{id}\,.
\end{aligned}$$

Therefore N preserves semibiproducts.

Proof of Proposition 10

First, we note that

$$\langle\!\langle f_{11}, f_{12}, f_{21}, f_{22} \rangle\!\rangle^* = [[f_{11}^*, f_{21}^*, f_{12}^*, f_{22}^*]].$$

In particular, $\iota_i^* = \pi_i$ for $i = 1, 2$.

We only consider the case where the cut elimination happens on an &-rule and $\oplus_0$-rule.

$$\begin{aligned}
&[\![\mathrm{Cut}(\mathrm{And}(\Pi_1, \Pi_2), \mathrm{Or}_0(\Pi))]\!] \\
&= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ([\![\mathrm{And}(\Pi_1, \Pi_2)]\!] \otimes [\![\mathrm{Or}_0(\Pi)]\!]) \\
&= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ (((\Gamma \otimes \alpha_1) \circ [\![\Pi_1]\!]) + ((\Gamma \otimes \alpha_2) \circ [\![\Pi_2]\!])) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!])) \\
&\quad \text{(tensor products preserve semibiproducts)} \\
&= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ (((\Gamma \otimes \alpha_1) \circ [\![\Pi_1]\!]) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!]) + \\
&\quad ((\Gamma \otimes \alpha_2) \circ [\![\Pi_2]\!]) \otimes ((\alpha_1 \otimes \Delta) \circ [\![\Pi]\!]))) \\
&= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ((\Gamma \otimes \alpha_1 \otimes \alpha_1 \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + ((\Gamma \otimes \alpha_2 \otimes \alpha_1 \otimes \Delta) \circ [\![\Pi_2]\!] \otimes [\![\Pi]\!])) \\
&= (\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\alpha_1 \otimes \alpha_1)) \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + (\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\alpha_2 \otimes \alpha_1)) \otimes \Delta) \circ ([\![\Pi_2]\!] \otimes [\![\Pi]\!]) \\
&= (\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\mathcal{U} \otimes (\alpha_1^* \circ \alpha_1))) \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) + \\
&\quad (\Gamma \otimes (\varepsilon_{\mathcal{U}} \circ (\mathcal{U} \otimes (\alpha_2^* \circ \alpha_1))) \otimes \Delta) \circ ([\![\Pi_2]\!] \otimes [\![\Pi]\!]) + \\
&\quad (\alpha_1^* \circ \alpha_1 = \iota_1^* \circ a^* \circ a \circ \iota_1 = \pi_1 \circ \iota_1 = \mathrm{id}, \alpha_2^* \circ \alpha_1 = \iota_2^* \circ a^* \circ a \circ \iota_1 = \pi_2 \circ \iota_1 = 0) \\
&= (\Gamma \otimes \varepsilon_{\mathcal{U}} \otimes \Delta) \circ ([\![\Pi_1]\!] \otimes [\![\Pi]\!]) \\
&= [\![\mathrm{Cut}(\Pi_1, \Pi)]\!] \,.
\end{aligned}$$

Proof of Proposition 11

In order to prove Proposition 11, we introduce *coherence* $\frown\!\!\!\!\smile_A \subset \mathbb{N}^2 \times \mathbb{N}^2$ for each formula A.

Definition 8. *For a reflective relation* $\frown\!\!\!\!\smile \subset X \times X$*, we define* $\asymp, \smile, \frown \subset X \times X$ *by*

$$\begin{aligned}
x \asymp y &\Leftrightarrow x = y \lor \neg(x \frown\!\!\!\!\smile y) \\
x \frown y &\Leftrightarrow x \neq y \land x \frown\!\!\!\!\smile y \\
x \smile y &\Leftrightarrow \neg(x \frown\!\!\!\!\smile y)
\end{aligned}$$

Definition 9. *For a formula A, we define a reflective relation* $\frown\!\!\!\!\smile_A \subset \mathbb{N}^2 \times \mathbb{N}^2$ *by*

- $(n, m) \frown\!\!\!\!\smile_\alpha (a, b) \Longleftrightarrow n = a \land m = b$
- $(n, m) \frown\!\!\!\!\smile_{\alpha^\perp} (a, b) \Longleftrightarrow (m, n) \asymp_\alpha (b, a)$
- $(\lceil n, k \rceil, \lceil m, l \rceil) \frown\!\!\!\!\smile_{A \otimes B} (\lceil a, c \rceil, \lceil b, d \rceil)$ *iff*

$$(n, m) \frown\!\!\!\!\smile_A (a, b) \land (k, l) \frown\!\!\!\!\smile_B (c, d)$$

– $(\lceil n,k\rceil,\lceil m,l\rceil) \smile_{A\wp B} (\lceil a,c\rceil,\lceil b,d\rceil)$ *iff*

$$(n,m) \frown_A (a,b) \vee (k,l) \frown_B (c,d)$$

or

$$(\lceil n,k\rceil,\lceil m,l\rceil) = (\lceil a,c\rceil,\lceil b,d\rceil)$$

– $(n,m) \smile_{A\oplus B} (a,b)$ *iff*

$$(n,m) = (\overline{k},\overline{l}) \wedge (a,b) = (\overline{c},\overline{d}) \wedge (k,l) \frown_A (c,d)$$

or

$$(n,m) = (\underline{k},\underline{l}) \wedge (a,b) = (\underline{c},\underline{d}) \wedge (k,l) \frown_B (c,d)$$

or

$$(n,m) = (a,b)$$

– $(n,m) \smile_{A\& B} (a,b)$ *iff* $(m,n) \asymp_{A^\perp \oplus B^\perp} (b,a)$

We can extend $(-)^\perp$ to all formulae

$$(n,m) \smile_{A^\perp} (a,b) \Longleftrightarrow (m,n) \asymp_A (b,a).$$

This is consistent to the extention of the negation of the MALL since we have:

$$\begin{aligned}
\smile_{(A^\perp)^\perp} &= \smile_A \\
\smile_{(A\otimes B)^\perp} &= \smile_{A^\perp \wp B^\perp} \\
\smile_{(A\wp B)^\perp} &= \smile_{A^\perp \otimes B^\perp} \\
\smile_{(A\& B)^\perp} &= \smile_{A^\perp \oplus B^\perp} \\
\smile_{(A\oplus B)^\perp} &= \smile_{A^\perp \& B^\perp}
\end{aligned}$$

For a k-tuple of formulae $A_1,\cdots,A_k$ and k-tuples of natural numbers $\boldsymbol{n},\boldsymbol{m},\boldsymbol{a},\boldsymbol{b}$, we write $(\boldsymbol{n},\boldsymbol{m}) \smile_{A_1,\cdots,A_k} (\boldsymbol{a},\boldsymbol{b})$ when

$$(n_i,m_i) \asymp_{A_i} (a_i,b_i) \text{ for each } 1 \le i \le k \text{ implies } (\boldsymbol{n},\boldsymbol{m}) = (\boldsymbol{a},\boldsymbol{b}).$$

We define $(\boldsymbol{n},\boldsymbol{m}) \asymp_{A_1,\cdots,A_k} (\boldsymbol{a},\boldsymbol{b})$ by the negation of $(\boldsymbol{n},\boldsymbol{m}) \smile_{A_1,\cdots,A_k} (\boldsymbol{a},\boldsymbol{b})$.

$$(\boldsymbol{n},\boldsymbol{m}) \asymp_{A_1,\cdots,A_k} (\boldsymbol{a},\boldsymbol{b}) \Longleftrightarrow \forall i.(n_i,m_i) \asymp_{A_i} (a_i,b_i).$$

Proposition 13. *Let $\Pi \vdash A_1,\cdots,A_n$ be a MALL proof. For $(\boldsymbol{n},\boldsymbol{m}),(\boldsymbol{a},\boldsymbol{b}) \in \|\Pi\|$,*

$$(\boldsymbol{n},\boldsymbol{m}) \smile_{A_1,\cdots,A_k} (\boldsymbol{a},\boldsymbol{b})$$

Proof. We prove by the induction of Π.
• $\mathrm{Ax} \vdash A, A^\perp$
For $(nm,mn),(ab,ba) \in \|\,\mathrm{Ax}_A\,\|$,

$$\begin{aligned}
(n,m) \asymp_A (a,b) \wedge (m,n) \asymp_{A^\perp} (b,a) &\Leftrightarrow (n,m) \asymp_A (a,b) \wedge (n,m) \smile_A (a,b) \\
&\Rightarrow (n,m) = (a,b)
\end{aligned}$$

• $\mathrm{Cut}(\Pi_0, \Pi_1) \vdash \Gamma, \Delta$
Let $(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-)$ and $(\boldsymbol{a}^+\boldsymbol{b}^+, \boldsymbol{a}^-\boldsymbol{b}^-)$ be elements of $\|\mathrm{Cut}(\Pi_0, \Pi_1)\|$ such that

$$(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \asymp_{\Gamma,\Delta} (\boldsymbol{a}^+\boldsymbol{b}^+, \boldsymbol{a}^-\boldsymbol{b}^-).$$

From the definition of $\| - \|$, there are natural numbers i, j, p, q such that

$$\begin{array}{ll} (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_0\| & (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in \|\Pi_1\| \\ (\boldsymbol{a}^+ p, \boldsymbol{a}^- q) \in \|\Pi_0\| & (q\boldsymbol{b}^+, p\boldsymbol{b}^-) \in \|\Pi_1\| \end{array}$$

We show $(i, j) \asymp_A (p, q)$: If $(i, j) \frown_A (p, q)$ then $(j, i) \asymp_{A^\perp} (q, p)$. We have

$$(j\boldsymbol{m}^+, i\boldsymbol{m}^-) \asymp_{A^\perp,\Delta} (q\boldsymbol{b}^+, p\boldsymbol{b}^-)$$

since $(\boldsymbol{m}^+, \boldsymbol{m}^-) \asymp_\Delta (\boldsymbol{b}^+, \boldsymbol{b}^-)$. Then by the I.H., $(j, i) = (q, p)$. This contradicts to $(i, j) \frown_A (p, q)$. Hence $(i, j) \asymp_A (p, q)$.

Since $(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-)$, we have $(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \asymp_{\Gamma,A} (\boldsymbol{a}^+ p, \boldsymbol{a}^- q)$. Then by the I.H., $(i, j) = (p, q)$ and we see $(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-)$ and $(\boldsymbol{m}^+, \boldsymbol{m}^-) = (\boldsymbol{b}^+, \boldsymbol{b}^-)$.
• $\mathrm{Ten}(\Pi_0, \Pi_1) \vdash \Gamma, A \otimes B, \Delta$
For

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \in \|\mathrm{Ten}(\Pi_0, \Pi_1)\|$$

and

$$(\boldsymbol{a}^+\lceil p^+, q^+\rceil \boldsymbol{b}^+, \boldsymbol{a}^-\lceil p^-, q^-\rceil \boldsymbol{b}^-) \in \|\mathrm{Ten}(\Pi_0, \Pi_1)\|,$$

such that

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil \boldsymbol{m}^+, \boldsymbol{n}^-\lceil i^-, j^-\rceil \boldsymbol{m}^-) \asymp_{\Gamma,A\otimes B,\Delta} (\boldsymbol{a}^+\lceil p^+, q^+\rceil \boldsymbol{b}^+, \boldsymbol{a}^-\lceil p^-, q^-\rceil \boldsymbol{b}^-)$$

we have

$$(i^+, i^-) \smile_A (p^+, p^-)$$

or

$$(j^+, j^-) \smile_B (q^+, q^-)$$

or

$$i^+ = p^+, i^- = p^-, j^+ = q^+, j^- = q^-.$$

For the first case, since $(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-)$, we see

$$(\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \asymp_\Gamma (\boldsymbol{a}^+ p^+, \boldsymbol{a}^- p^-)$$

and by the I.H., $(i^+, i^-) = (p^+, p^-)$. This contradicts to $(i^+, i^-) \smile_A (p^+, p^-)$. The second case is similar. Hence $i^+ = p^+, i^- = p^-, j^+ = q^+, j^- = q^-$ stand. Then by I.H., we have $(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-)$ and $(\boldsymbol{m}^+, \boldsymbol{m}^-) = (\boldsymbol{b}^+, \boldsymbol{b}^-)$.
• $\mathrm{Par}(\Pi) \vdash \Gamma, A \wp B$
Let $(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil)$ and $(\boldsymbol{a}^+\lceil p^+, q^+\rceil, \boldsymbol{a}^-\lceil p^-, q^-\rceil)$ be elements of $\|\mathrm{Par}(\Pi)\|$ such that

$$(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \asymp_{\Gamma,A\wp B} (\boldsymbol{a}^+\lceil p^+, q^+\rceil, \boldsymbol{a}^-\lceil p^-, q^-\rceil).$$

By the definition, this is equivalent to

$$(\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \asymp_{\Gamma,A,B} (\boldsymbol{a}^+ p^+ q^+, \boldsymbol{a}^- p^- q^-).$$

Then by the I.H.,

$$(\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) = (\boldsymbol{a}^+ p^+ q^+, \boldsymbol{a}^- p^- q^-)$$

• $\mathrm{Perm}_\sigma(\Pi)$
For $(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)), (\sigma(\boldsymbol{a}^+), \sigma(\boldsymbol{b}^-)) \in \|\mathrm{Perm}_\sigma(\Pi)\|$,

$$\begin{aligned}(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \asymp_{\sigma(\Gamma)} (\sigma(\boldsymbol{a}^+), \sigma(\boldsymbol{b}^-)) &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{b}^-) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{b}^-)\end{aligned}$$

• $\mathrm{And}(\Pi_0, \Pi_1)$
For $(\boldsymbol{n}^+ i, \boldsymbol{n}^- j), (\boldsymbol{a}^+ p, \boldsymbol{a}^- q) \in \|\mathrm{And}(\Pi_0, \Pi_1)\|$, if

$$(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \asymp_{\Gamma,A\&B} (\boldsymbol{a}^+ p, \boldsymbol{a}^- q)$$

then from the definition of $\asymp_{A\&B}$ and $\|\mathrm{And}(\Pi_0, \Pi_1)\|$, there are two cases:

$$i = \overline{i'}, j = \overline{j'}, p = \overline{p'}, q = \overline{q'}$$

or

$$i = \underline{i'}, j = \underline{j'}, p = \underline{p'}, q = \underline{q'}.$$

For the first case,

$$\begin{aligned}&(\boldsymbol{n}^+ \overline{i}, \boldsymbol{n}^- \overline{j}) \asymp_{\Gamma,A\&B} (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (\overline{i}, \overline{j}) \frown_{A^\perp \oplus B^\perp} (\overline{p}, \overline{q}) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) \frown_{A^\perp} (p, q) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (j, i) \asymp_A (q, p) \\ \Rightarrow\ &(\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (j, i) = (q, p).\end{aligned}$$

The second case is similar.
• $\mathrm{Or}_0(\Pi) \vdash A \oplus B, \Gamma$: For $(\boldsymbol{n}^+ \overline{i}, \boldsymbol{n}^- \overline{j}), (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) \in \|\mathrm{Or}_0(\Pi)\|$,

$$\begin{aligned}(\boldsymbol{n}^+ \overline{i}, \boldsymbol{n}^- \overline{j}) \asymp_{\Gamma,A\oplus B} (\boldsymbol{a}^+ \overline{p}, \boldsymbol{a}^- \overline{q}) &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (\overline{i}, \overline{j}) \asymp_{A\oplus B} (\overline{p}, \overline{q}) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) \asymp_\Gamma (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) \asymp_A (p, q) \\ &\Rightarrow (\boldsymbol{n}^+, \boldsymbol{n}^-) = (\boldsymbol{a}^+, \boldsymbol{a}^-) \wedge (i, j) = (p, q)\end{aligned}$$

• $\mathrm{Or}_1(\Pi) \vdash A \oplus B, \Gamma$: Similar to Or_0.

Corollary 2. *Let Π be a proof of MALL. For every $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in \|\Pi\|$,*

$$[\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-}$$

is a singleton or the empty set.

Proof. We show by the induction of Π. We only prove the case of Cut rule. We have

$$[\![\mathrm{Cut}(\Pi_0,\Pi_1)]\!]_{\boldsymbol{n}^+\boldsymbol{m}^+,\boldsymbol{n}^-\boldsymbol{m}^-} = \{\mathrm{cut}(f,g) \mid \exists i,j.\ f \in [\![\Pi_0]\!]_{\boldsymbol{n}^+i,\boldsymbol{n}^-j} \wedge g \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+,i\boldsymbol{m}^-}\}$$

where

$$\mathrm{cut} : \mathbf{Pfn}((k+1)\mathbb{N},(k+1)\mathbb{N}) \times \mathbf{Pfn}((h+1)\mathbb{N},(h+1)\mathbb{N}) \to \mathbf{Pfn}((k+h)\mathbb{N},(k+h)\mathbb{N})$$

is given by

$$\mathrm{cut}(f,g) = \mathrm{tr}^{\mathbb{N}}_{(k+h)\mathbb{N},(k+h)\mathbb{N}}((k\mathbb{N}\otimes\sigma_{h\mathbb{N},\mathbb{N}})\circ(k\mathbb{N}\otimes g)\circ(f\otimes h\mathbb{N})\circ(k\mathbb{N}\otimes\sigma_{\mathbb{N},h\mathbb{N}}))$$

If there are i,j and p,q such that

$$\begin{array}{ll}\exists f \in [\![\Pi_0]\!]_{\boldsymbol{n}^+i,\boldsymbol{n}^-j} & \exists g \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+,i\boldsymbol{m}^-}\\ \exists u \in [\![\Pi_0]\!]_{\boldsymbol{n}^+p,\boldsymbol{n}^-q} & \exists v \in [\![\Pi_1]\!]_{q\boldsymbol{m}^+,p\boldsymbol{m}^-}\end{array}$$

then

$$(\boldsymbol{n}^+i,\boldsymbol{n}^-j),(\boldsymbol{n}^+p,\boldsymbol{n}^-q) \in \|\Pi_0\|$$

and

$$(j\boldsymbol{m}^+,i\boldsymbol{m}^-),(q\boldsymbol{m}^+,p\boldsymbol{m}^-) \in \|\Pi_1\|.$$

By Proposition 13, we have $(i,j) \frown_A (p,q)$ and $(j,i) \frown_{A^\perp} (q,p)$. Hence $(i,j)=(p,q)$ and $\mathrm{Cut}(\Pi_0,\Pi_1)$ is a singleton or the empty set.

We write $\mathrm{W}(\Pi)$ for the set of weights of Π.

Proposition 14. *For a weighted MALL proof* (Π,w) *and* $(\boldsymbol{n}^+,\boldsymbol{n}^-) \in |\Pi|_w$*, we have*

(1) $\|\Pi\| = \bigcup_{w\in\mathrm{W}(\Pi)} |\Pi|_w$
(2) $[\Pi]_w \in [\![\Pi]\!]_{\boldsymbol{n}^+,\boldsymbol{n}^-}$

Proof. We prove (1) and (2) simultaneously by the induction of Π.
• Ax_A
(1) By the definition.
(2)$[\![\mathrm{Ax}_A]\!]_{nm,mn} = \{\sigma_{\mathbb{N},\mathbb{N}}\} = \{[\mathrm{Ax}_A]_w\}$ where $\sigma_{\mathbb{N},\mathbb{N}} : \mathbb{N}+\mathbb{N} \rightharpoonup \mathbb{N}+\mathbb{N}$ is the swapping map.
• $\mathrm{Cut}(\Pi_0,\Pi_1)$
(1)

$$\begin{aligned}\bigcup_{w\in\mathrm{W}(\mathrm{Cut}(\Pi_0,\Pi_1))} |\mathrm{Cut}(\Pi_0,\Pi_1)|_w &= \bigcup_{w\in\mathrm{W}(\mathrm{Cut}(\Pi_0,\Pi_1))} \left\{(\boldsymbol{n}^+\boldsymbol{m}^+,\boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i,j.\begin{array}{l}(\boldsymbol{n}^+i,\boldsymbol{n}^-j)\in|\Pi_0|_w\\(j\boldsymbol{m}^+,i\boldsymbol{m}^-)\in|\Pi_1|_w\end{array}\right\}\\ &= \left\{(\boldsymbol{n}^+\boldsymbol{m}^+,\boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i,j.\begin{array}{l}(\boldsymbol{n}^+i,\boldsymbol{n}^-j)\in\bigcup_{w\in\mathrm{W}(\Pi_0)}|\Pi_0|_w\\(j\boldsymbol{m}^+,i\boldsymbol{m}^-)\in\bigcup_{w\in\mathrm{W}(\Pi_1)}|\Pi_1|_w\end{array}\right\}\\ &= \left\{(\boldsymbol{n}^+\boldsymbol{m}^+,\boldsymbol{n}^-\boldsymbol{m}^-) \,\middle|\, \exists i,j.\begin{array}{l}(\boldsymbol{n}^+i,\boldsymbol{n}^-j)\in\|\Pi_0\|\\(j\boldsymbol{m}^+,i\boldsymbol{m}^-)\in\|\Pi_1\|\end{array}\right\}\\ &= \|\mathrm{Cut}(\Pi_0,\Pi_1)\|\end{aligned}$$

(2) For $(\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-) \in |\operatorname{Cut}(\Pi_0, \Pi_1)|_w$, there is i, j such that

$$(\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\Pi_0|_w \qquad (j\boldsymbol{m}^+, i\boldsymbol{m}^-) \in |\Pi_1|_w$$

By the definition, $[\operatorname{Cut}(\Pi_0, \Pi_1)]_w$ is $\operatorname{cut}([\Pi_0]_w, [\Pi_1]_w)$. By the I.H., $[\Pi_0]_w \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j}$ and $[\Pi_1]_w \in [\![\Pi_1]\!]_{j\boldsymbol{m}^+, i\boldsymbol{m}^-}$. Hence

$$[\operatorname{Cut}(\Pi_0, \Pi_1)]_w = \operatorname{cut}([\Pi_0]_w, [\Pi_1]_w) \in [\![\operatorname{Cut}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+\boldsymbol{m}^+, \boldsymbol{n}^-\boldsymbol{m}^-}$$

• $\operatorname{Perm}_\sigma(\Pi)$
(1)

$$\begin{aligned}
\bigcup_{w \in \mathrm{W}(\operatorname{Perm}_\sigma(\Pi))} |\operatorname{Perm}_\sigma(\Pi)|_w &= \bigcup_{w \in \mathrm{W}(\operatorname{Perm}_\sigma(\Pi))} \{(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \mid (\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\Pi|_w\} \\
&= \left\{ (\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \,\middle|\, (\boldsymbol{n}^+, \boldsymbol{n}^-) \in \bigcup_{w \in \mathrm{W}(\Pi)} |\Pi|_w \right\} \\
&= \left\{ (\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \,\middle|\, (\boldsymbol{n}^+, \boldsymbol{n}^-) \in \|\Pi\| \right\} \\
&= \| \operatorname{Perm}_\sigma(\Pi) \|
\end{aligned}$$

(2) For $(\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)) \in |\Pi|_w$

$$[\operatorname{Perm}_\sigma(\Pi)]_w = \theta_\sigma \circ [\Pi]_w \circ \theta_\sigma^{-1} \in \theta_\sigma \circ [\![\Pi]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-} \circ \theta_\sigma^{-1} = [\![\Pi]\!]_{\sigma(\boldsymbol{n}^+), \sigma(\boldsymbol{n}^-)}$$

where θ_σ is the permutation of coproducts of $\mathbb{N}$ following σ.
• $\operatorname{Ten}(\Pi_0, \Pi_1)$
(1)

$$\begin{aligned}
&\bigcup_{w \in \mathrm{W}(\operatorname{Ten}(\Pi_0, \Pi_1))} |\operatorname{Ten}(\Pi_0, \Pi_1)|_w \\
&= \bigcup_{w \in \mathrm{W}(\operatorname{Ten}(\Pi_0, \Pi_1))} \left\{ (\boldsymbol{n}^+ \lceil i^+, j^+ \rceil \boldsymbol{m}^+, \boldsymbol{n}^- \lceil i^-, j^- \rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l} (\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in |\Pi_0|_w \\ (j^+ \boldsymbol{m}^+, j^- \boldsymbol{m}^-) \in |\Pi_1|_w \end{array} \right\} \\
&= \left\{ (\boldsymbol{n}^+ \lceil i^+, j^+ \rceil \boldsymbol{m}^+, \boldsymbol{n}^- \lceil i^-, j^- \rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l} (\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in \bigcup_{w \in \mathrm{W}(\Pi_0)} |\Pi_0|_w \\ (j^+ \boldsymbol{m}^+, j^- \boldsymbol{m}^-) \in \bigcup_{w \in \mathrm{W}(\Pi_1)} |\Pi_1|_w \end{array} \right\} \\
&= \left\{ (\boldsymbol{n}^+ \lceil i^+, j^+ \rceil \boldsymbol{m}^+, \boldsymbol{n}^- \lceil i^-, j^- \rceil \boldsymbol{m}^-) \,\middle|\, \begin{array}{l} (\boldsymbol{n}^+ i^+, \boldsymbol{n}^- i^-) \in \|\Pi_0\| \\ (j^+ \boldsymbol{m}^+, j^- \boldsymbol{m}^-) \in \|\Pi_1\| \end{array} \right\} \\
&= \| \operatorname{Ten}(\Pi_0, \Pi_1) \|
\end{aligned}$$

(2) For $(\boldsymbol{n}^+ \lceil i^+, j^+ \rceil \boldsymbol{m}^+, \boldsymbol{n}^- \lceil i^-, j^- \rceil \boldsymbol{m}^-) \in |\operatorname{Ten}(\Pi_0, \Pi_1)|_w$,

$$\begin{aligned}
[\operatorname{Ten}(\Pi_0, \Pi_1)]_w &= (\mathrm{id} \otimes c \otimes \mathrm{id}) \circ ([\Pi_0]_w \otimes [\Pi_1]_w) \circ (\mathrm{id} \otimes c^{-1} \otimes \mathrm{id}) \\
&\in [\![\operatorname{Ten}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+ \lceil i^+, j^+ \rceil \boldsymbol{m}^+, \boldsymbol{n}^- \lceil i^-, j^- \rceil \boldsymbol{m}^-}
\end{aligned}$$

• Par(Π)
(1)

$$
\begin{aligned}
&\bigcup_{w\in \mathrm{W}(\mathrm{Par}(\Pi))} |\,\mathrm{Par}(\Pi)|_w \\
&= \bigcup_{w\in \mathrm{W}(\mathrm{Par}(\Pi))} \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in |\Pi|_w\right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in \textstyle\bigcup_{w\in \mathrm{W}(\Pi)} |\Pi|_w\right\} \\
&= \left\{(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \,\middle|\, (\boldsymbol{n}^+ i^+ j^+, \boldsymbol{n}^- i^- j^-) \in \|\Pi\|\right\} \\
&= \|\,\mathrm{Par}(\Pi)\|
\end{aligned}
$$

(2) For $(\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil) \in |\,\mathrm{Par}(\Pi)|_w$,

$$
[\mathrm{Par}(\Pi)]_w = (\mathrm{id} \otimes c) \circ [\Pi]_w \circ (\mathrm{id} \otimes c^{-1}) \in [\![\mathrm{Par}(\Pi)]\!]_{\boldsymbol{n}^+\lceil i^+, j^+\rceil, \boldsymbol{n}^-\lceil i^-, j^-\rceil}
$$

• And(Π_0, Π_1)
(1)

$$
\begin{aligned}
&\bigcup_{w\in \mathrm{W}(\mathrm{And}(\Pi_0,\Pi_1))} [\mathrm{And}(\Pi_0, \Pi_1)]_w \\
&= \left(\bigcup_{w\in \mathrm{W}(\Pi_0)} \{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in [\Pi_0]_w\}\right) \\
&\quad \cup \left(\bigcup_{w\in \mathrm{W}(\Pi_1)} \{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in [\Pi_1]_w\}\right) \\
&= \left\{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w\in \mathrm{W}(\Pi_0)} [\Pi_0]_w\right\} \\
&\quad \cup \left\{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w\in \mathrm{W}(\Pi_1)} [\Pi_1]_w\right\} \\
&= \left\{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_0\|\right\} \cup \left\{(\boldsymbol{n}^+\underline{i}, \boldsymbol{n}^-\underline{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\Pi_1\|\right\} \\
&= \|\,\mathrm{And}(\Pi_0, \Pi_1)\|
\end{aligned}
$$

(2) For a weight $w(\mathrm{And}) = 0$ and $(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \in |\,\mathrm{And}(\Pi_0, \Pi_1)|_w$,

$$
[\mathrm{And}(\Pi_0, \Pi_1)]_w = [\Pi_0]_w \in [\![\Pi_0]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} = [\![\mathrm{And}(\Pi_0, \Pi_1)]\!]_{\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}}
$$

The case $w(\mathrm{And}) = 1$ is similar.
• $\mathrm{Or}_0(\Pi)$

(1)

$$
\begin{aligned}
\bigcup_{w \in \mathrm{W}(\mathrm{Or}_0(\Pi))} |\,\mathrm{Or}_0(\mathit{\Pi})|_w &= \bigcup_{w \in \mathrm{W}(\mathrm{Or}_0(\Pi))} \{(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \mid (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in |\mathit{\Pi}|_w\} \\
&= \left\{ (\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \bigcup_{w \in \mathrm{W}(\mathit{\Pi})} |\mathit{\Pi}|_w \right\} \\
&= \left\{ (\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \,\middle|\, (\boldsymbol{n}^+ i, \boldsymbol{n}^- j) \in \|\mathit{\Pi}\| \right\} \\
&= \|\,\mathrm{Or}_0(\mathit{\Pi})\|
\end{aligned}
$$

(2) For $(\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}) \in |\,\mathrm{Or}_0(\mathit{\Pi})|_w$,

$$
[\mathrm{Or}_0(\mathit{\Pi})]_w = [\mathit{\Pi}]_w \in [\![\mathit{\Pi}]\!]_{\boldsymbol{n}^+ i, \boldsymbol{n}^- j} = [\![\mathrm{Or}_0(\mathit{\Pi})]\!]_{\boldsymbol{n}^+\bar{i}, \boldsymbol{n}^-\bar{j}}
$$

• $\mathrm{Or}_1(\mathit{\Pi})$: Similar to Or_0.

Corollary 3 (Proposition 11).

(1) For any proof $\mathit{\Pi}$, $\|\mathit{\Pi}\| = \bigcup_{w:\text{weight of }\mathit{\Pi}} |\mathit{\Pi}|_w$.
(2) For any proof $\mathit{\Pi}$, well-behaved weight w of $\mathit{\Pi}$ and $(\boldsymbol{n}^+, \boldsymbol{n}^-) \in |\mathit{\Pi}|_w$, we have

$$
[\![\mathit{\Pi}]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-} = \{[\mathit{\Pi}]_w\}.
$$

Proof. (1) is exactly (1) in Proposition 14.
(2) From Proposition 14, we see $\{[\mathit{\Pi}]_w\} \subset [\![\mathit{\Pi}]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-}$. Then by Proposition 13, $[\![\mathit{\Pi}]\!]_{\boldsymbol{n}^+, \boldsymbol{n}^-}$ is a singleton set and the inclusion is equality.