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A Representation Theorem for Unique Decomposition Categories

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Abstract

Haghverdi introduced the notion of unique decomposition categories as a foundation for categorical study of Girard's Geometry of Interaction (GoI). The execution formula in GoI provides a semantics of cut-elimination process, and we can capture the execution formula in every unique decomposition category: each hom-set of a unique decomposition category comes equipped with a partially defined countable summation, which captures the countable summation that appears in the execution formula. The fundamental property of unique decomposition categories is that if the execution formula in a unique decomposition category is always defined, then the unique decomposition category has a trace operator that is given by the execution formula. In this paper, we introduce a subclass of unique decomposition categories, which we call strong unique decomposition categories, and we prove the fundamental property for strong unique decomposition categories as a corollary of a representation theorem for strong unique decomposition categories: we show that for every strong unique decomposition category $\mathcal{C}$, there is a faithful strong symmetric monoidal functor from $\mathcal{C}$ to a category with countable biproducts, and the countable biproducts characterize the structure of the strong unique decomposition category via the faithful functor.

Keywords: Geometry of interaction, unique decomposition category, traced monoidal category, representation theorem

1 Introduction

Girard introduced Geometry of Interaction (GoI) [3], which aims to capture semantics of cut-elimination process rather than invariant under cut-elimination like usual denotational semantics. GoI interprets proofs as square matrices, and if a proof reduces to another proof via cut-elimination, then the execution

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formula

$$\mathrm{Ex}\begin{pmatrix} A & B \\ C & D \end{pmatrix} := A + \sum_{n=0}^{\infty} BD^nC$$

provides an invariant under the cut-elimination.

Work by Hyland, Abramsky, Haghverdi and Scott [4,1] showed that traced symmetric monoidal categories [11] play important roles in modeling the execution formula. Especially, in [4,5], Haghverdi and Scott got much closer to the original execution formula by using unique decomposition categories. The notion of unique decomposition categories introduced by Haghverdi is a generalization of partially additive categories [15]. The main point of unique decomposition categories is that in a unique decomposition category, we can uniquely decompose a morphism $f : X \otimes Z \to Y \otimes Z$ into four components

$$\begin{pmatrix} f_{XY} : X \to Y & f_{ZY} : Z \to Y \\ f_{XZ} : X \to Z & f_{ZZ} : Z \to Z \end{pmatrix},$$

and each hom-set comes equipped with a partially defined countable summation. For example, we can partially define the *standard trace formula* [5]:

$$f_{XY} + \sum_{n=0}^{\infty} f_{ZY} \circ f_{ZZ}^n \circ f_{XZ} : X \to Y.$$

The following fundamental property of unique decomposition categories connects the standard trace formula with categorical trace operators.

Proposition 1.1 ([4,5]) *If the standard trace formula is defined for any morphism of the form $f : X \otimes Z \to Y \otimes Z$, then the standard trace formula provides a trace operator of the unique decomposition category.*

In the proof of the proposition, there are certain implicit assumptions aside from the definition of unique decomposition categories (see Appendix B in [8]), and a sufficient condition would be to require quasi projections and quasi injections, which is a part of data of unique decomposition categories, to be "natural" and "compatible with monoidal structural isomorphisms". The main motivation of this paper is to explicitly describe a subclass of unique decomposition categories that enjoys the fundamental property. Our idea is to find a subclass of unique decomposition categories that provides "good" embedding of unique decomposition categories in the subclass into categories with countable biproducts. We consider categories with countable biproducts because countable biproducts always provide a trace operator given by the execution formula (see Section 5). Although we found a subclass of unique decomposition categories, namely strong unique decomposition categories, in this paper by trial and error, organization of this paper is top-down:

(i) In Section 2, we recall Kleene equality, biproducts and categorical traces.

(ii) In Section 3, we recall the definition of Σ-monoids and embed each Σ-monoid into a total Σ-monoid.

(iii) In Section 4, we introduce strong unique decomposition categories, and we embed a strong unique decomposition category into a total strong unique decomposition category via the embedding in (ii). We give examples of strong unique decomposition categories.

(iv) In Section 5, we embed a total strong unique decomposition category into a category with countable biproducts by matrix construction [13]. Then, we give a representation theorem for strong unique decomposition categories (Theorem 5.3). The fundamental property for strong unique decomposition categories is a corollary of the representation theorem.

Consequences of the representation theorem are:

- A proof of Proposition 1.1 in which we do not need to be careful with partiality of summations on hom-sets of strong unique decomposition categories.
- We show that all strong unique decomposition categories are *partially traced*.

Related work

The paper by Malherbe, Scott and Selinger [14] is closely related to our work. They gave an embedding of partially traced symmetric monoidal categories introduced in [6] into traced symmetric monoidal categories. Since our result tells us that every strong unique decomposition category is partially traced (Corollary 5.4), we can embed a strong unique decomposition category into a traced symmetric monoidal category by their result. On the other hand, our result also provides an embedding of a strong unique decomposition category into a traced symmetric monoidal category since a category with countable biproducts is traced (Theorem 3 in [16]). As we concentrate only on strong unique decomposition categories, our embedding tells us further information on strong unique decomposition categories: an explicit description of their trace operators, for example. However, there are some other partially traced symmetric monoidal categories that are not strong unique decomposition categories. At this point, we do not know clear comparison between our work and their work.

2 Preliminary

2.1 Kleene equality

For expressions e and e' that possibly include partial operations, we write $e \preceq e'$ if e is defined, then e' is defined, and they denote the same value. We use $\simeq$ for the *Kleene equality*: we write $e \simeq e'$ when we have $e \preceq e'$ and $e' \preceq e$.

For example, the following Kleene equality holds for all real numbers x and y.

$$\frac{x \cdot 3}{x^2} \cdot \frac{1}{y^2} \cdot y \simeq \frac{3}{x \cdot y}.$$

2.2 Biproducts

Definition 2.1 Let $\mathcal{C}$ be a category. For a set I, an *I-ary biproduct* of a family $\{X_i \in \mathcal{C}\}_{i \in I}$ consists of an object $\bigoplus_{i \in I} X_i$ and a family of $\mathcal{C}$-morphisms $\left\{\pi_i : \bigoplus_{i \in I} X_i \leftrightarrows X_i : \kappa_i\right\}_{i \in I}$ such that

- $\pi_i \circ \kappa_i = \mathrm{id}_{X_i}$ for every $i \in I$.
- $\bigoplus_{i \in I} X_i$ with $\{\pi_i\}_{i \in I}$ forms a product of $\{X_i\}_{i \in I}$.
- $\bigoplus_{i \in I} X_i$ with $\{\kappa_i\}_{i \in I}$ forms a coproduct of $\{X_i\}_{i \in I}$.
- For each $f_i : X_i \to Y_i$, the tupling $\langle f_i \circ \pi_i \rangle_{i \in I} : \bigoplus_{i \in I} X_i \to \bigoplus_{i \in I} Y_i$ coincides with the cotupling $[\kappa_i \circ f_i]_{i \in I} : \bigoplus_{i \in I} X_i \to \bigoplus_{i \in I} Y_i$.

A *zero object* 0 is a $\emptyset$-ary biproduct, and a *binary biproduct* of X_0 and X_1 is a $\{0,1\}$-ary biproduct of $\{X_i\}_{i \in \{0,1\}}$, for which we write $X_0 \oplus X_1$.

We use countable to mean at most countable. We say that $\mathcal{C}$ *has countable (finite) biproducts* when for every countable (finite) set I and every I-indexed family of $\mathcal{C}$-objects, there exists an I-ary biproduct of the family.

Definition 2.2 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories with finite biproducts. We say that F *preserves finite biproducts* when for any objects $X_0, X_1 \in \mathcal{C}$, the canonical morphisms $[F\kappa_0, F\kappa_1] : FX_0 \oplus FX_1 \leftrightarrows F(X_0 \oplus X_1) : \langle F\pi_0, F\pi_1 \rangle$ and $F0 \leftrightarrows 0$ form isomorphisms.

The definition of biproducts is from [9]. Definition 2.1 depends on neither abelian-group enrichment as in [13] nor existence of zero morphisms defined through a zero object as in [10]. The above definition of finite biproducts is equivalent to the definition of finite biproducts in [10].

2.3 Partial trace operators

Let $\mathcal{C}$ be a symmetric monoidal category (for the definition, see [13]). We recall the definition of partial trace operators in [6] that is a generalization of trace operators introduced in [11].

Definition 2.3 A *partial trace operator* of $\mathcal{C}$ is a family of partial maps

$$\{\mathrm{tr}^Z_{X,Y} : \mathcal{C}(X \otimes Z, Y \otimes Z) \rightharpoonup \mathcal{C}(X, Y)\}_{X,Y,Z \in \mathcal{C}}$$

subject to the following conditions:

- (Naturality) For $g : X' \to X$, $h : Y \to Y'$ and $f : X \otimes Z \to Y \otimes Z$,

$$h \circ \mathrm{tr}^{Z}_{X,Y}(f) \circ g \preceq \mathrm{tr}^{Z}_{X',Y'}((h \otimes \mathrm{id}_Z) \circ f \circ (g \otimes \mathrm{id}_Z)).$$

- (Dinaturality) For $f : X \otimes Z \to Y \otimes Z'$ and $g : Z' \to Z$,

$$\mathrm{tr}^{Z}_{X,Y}((\mathrm{id}_Y \otimes g) \circ f) \simeq \mathrm{tr}^{Z'}_{X,Y}(f \circ (\mathrm{id}_X \otimes g)).$$

- (Vanishing I) For $f : X \otimes \mathrm{I} \to Y \otimes \mathrm{I}$,

$$\mathrm{tr}^{\mathrm{I}}_{X,Y}(f) \simeq f.$$

- (Vanishing II) For $f : X \otimes Z \otimes W \to Y \otimes Z \otimes W$,

$$\mathrm{tr}^{W}_{X\otimes Z,Y\otimes Z}(f) \text{ is defined } \implies \mathrm{tr}^{Z\otimes W}_{X,Y}(f) \simeq \mathrm{tr}^{Z}_{X,Y}(\mathrm{tr}^{W}_{X\otimes Z,Y\otimes Z}(f)).$$

- (Superposing) For $f : X \otimes Z \to Y \otimes Z$,

$$\mathrm{id}_W \otimes \mathrm{tr}^{Z}_{X,Y}(f) \preceq \mathrm{tr}^{Z}_{W\otimes X,W\otimes Y}(\mathrm{id}_W \otimes f).$$

- (Yanking)

$$\mathrm{tr}^{X}_{X,X}(\sigma_{X,X}) \simeq \mathrm{id}_X.$$

Here we omit several coherence isomorphisms. Although our superposing rule is weaker than the original superposing rule in [6], we can derive the original superposing rule from the above axioms. A *trace operator* is a partial trace operator consisting of total maps. We say that a partial trace operator is *uniform* when for any $f : X \otimes Z \to Y \otimes Z$, $g : X \otimes Z' \to Y \otimes Z'$ and $h : Z \to Z'$, if $(\mathrm{id}_Y \otimes h) \circ f = g \circ (\mathrm{id}_X \otimes h)$, then $\mathrm{tr}^{Z}_{X,Y}(f) \simeq \mathrm{tr}^{Z'}_{X,Y}(g)$.

3 Σ-monoids

We recall the definition of Σ-monoids from [4]. For a set X, a *countable family on* X is a map $x : I \to X$ for a countable set I. We denote such a family x by $\{x_i\}_{i\in I}$. A *countable partition* of a set I is a countable family $\{I_j\}_{j\in J}$ consisting of pairwise disjoint subsets of I such that $\bigcup_{j\in J} I_j = I$. We define X^* to be the set of countable families on X whose indexing sets are subsets of the set of natural numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$. This restriction on indexing sets is to define $\mathit{\Sigma}$ in the following definition to be a set theoretic partial map.

Definition 3.1 A Σ-*monoid* is a non-empty set X with a partial map $\mathit{\Sigma} : X^* \rightharpoonup X$ subject to the following axioms:

- If I is a singleton $\{n\}$, then $\mathit{\Sigma}\{x_i\}_{i\in I} \simeq x_n$.
- If $\{I_j\}_{j\in J}$ is a countable partition of a countable subset $I \subset \mathbb{N}$, then for every countable family $\{x_i\}_{i\in I}$ on X, we have $\mathit{\Sigma}\{x_i\}_{i\in I} \simeq \mathit{\Sigma}\left\{\mathit{\Sigma}\{x_i\}_{i\in I_j}\right\}_{j\in J}$.

A countable family $\{x_i\}_{i \in I}$ is *summable* when $\mathit{\Sigma}\{x_i\}_{i \in I}$ is defined. We say that a Σ-monoid $(X, \mathit{\Sigma})$ is *total* when the operator $\mathit{\Sigma}$ is a total map.

In the following, we simply say that X is a Σ-monoid without mentioning its sum operator, and we write $\sum_{i \in I} x_i$ for $\mathit{\Sigma}\{x_i\}_{i \in I}$. We informally write $x_0 + x_1 + \cdots$ for $\sum_{i \in \mathbb{N}} x_i$ and $x_0 + x_1 + \cdots + x_n$ for $\sum_{i \in \{0,1,\cdots,n\}} x_i$. By the definition of Σ-monoids, every subfamily of a summable countable family is summable. Especially, the empty family $\emptyset$ is summable. The *zero element* $0 := \sum \emptyset$ behaves as a unit of the summation: $\sum_{i \in I} x_i \simeq \sum_{j \in \{i \in I \mid x_i \neq 0\}} x_j$. We note that $\mathit{\Sigma}\{x_i\}_{i \in I} \simeq \mathit{\Sigma}\{y_j\}_{j \in J}$ when there is a bijection $\theta : I \to J$ such that $x_i = y_{\theta(i)}$ for every $i \in I$. For a proof, see [7].

For every countable set S, we can define S-indexed summation $\sum_{s \in S} x_s$ by choosing a bijection $\theta : I \to S$ for some subset $I \subset \mathbb{N}$: we define $\sum_{s \in S} x_s$ to be $\sum_{i \in I} x_{\theta i}$. The definition does not depend on our choice of I and the bijection $\theta : I \to S$ since the summation is independent of renaming of indexing sets. Hence, the definition is well-defined. In the following, we implicitly extend summations in this way.

Example 3.2 Let M be a commutative monoid that does not have non-trivial subgroup. M forms a Σ-monoid by the following summation:

$$\sum_{i \in I} x_i := \begin{cases} \sum_{i \in I'} x_i & (I' := \{i \in I \mid x_i \neq 0\} \text{ is finite}) \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

Examples are the set of natural numbers and the set of non-negative reals associated with the addition. Another example is M/N where M is a commutative monoid, and N is the submonoid of M consisting of invertible elements in M. Generally, if an element of a Σ-monoid is invertible, then it is equal to the zero element:

$$x = x + 0 + 0 + \cdots = x + (-x) + x + (-x) + x + \cdots = 0.$$

Example 3.3 A bounded complete poset D forms a Σ-monoid:

$$\sum_{i \in I} x_i := \begin{cases} \bigvee_{i \in I} x_i & (\{x_i \in D \mid i \in I\} \text{ is bounded}) \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

3.1 The category of Σ-monoids

We define a category $\mathbf{M}$ of Σ-monoids: objects are Σ-monoids, and a morphism $f : X \to Y$ is a map $f : X \to Y$ such that for each summable countable family $\{x_i\}_{i \in I}$ on X, the summation $\sum_{i \in I} f x_i$ is defined to be $f(\sum_{i \in I} x_i)$. In this section, we show that $\mathbf{M}$ is a symmetric monoidal closed category. Due to lack of space, proofs of propositions in this section are in [8].

Definition 3.4 For a positive natural number n and Σ-monoids $X_1, \cdots, X_n$ and Y, we say that a map $f : X_1 \times \cdots \times X_n \to Y$ is *n-linear* when

$$f(x_1, \cdots, x_{k-1}, -, x_{k+1}, \cdots, x_n) : X_k \to Y$$

is an $\mathbf{M}$-morphism for all $k = 1, 2, \cdots, n$ and $x_1 \in X_1$, $\cdots$, $x_n \in X_n$. We write $\mathbf{M}(X_1, \cdots, X_n; Y)$ for the set of n-linear morphisms of the form $f : X_1 \times \cdots \times X_n \to Y$.

Proposition 3.5 *A functor* $\mathbf{M}(X, Y; -) : \mathbf{M} \to \mathbf{Set}$ *is representable, i.e., there is an object* $X \otimes Y$ *such that* $\mathbf{M}(X, Y; -) \cong \mathbf{M}(X \otimes Y, -)$.

We define I to be a Σ-monoid $\{0, 1\}$ associated with a summation

$$\sum_{i \in I} x_i := \begin{cases} 0 & (\{i \in I \mid x_i = 1\} \text{ is empty}) \\ 1 & (\{i \in I \mid x_i = 1\} \text{ is a singleton}) \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

For Σ-monoids X and Y, we define a Σ-monoid $[X, Y] := \mathbf{M}(X, Y)$ by

$$\sum_{i \in I} f_i := \begin{cases} \lambda x. \sum_{i \in I} f_i x & \left(\sum_{i \in I} f_i x \text{ is defined for all } x \in X\right) \\ \text{undefined} & (\text{otherwise}). \end{cases} \tag{1}$$

Proposition 3.6 $(\mathbf{M}, \mathrm{I}, \otimes, [-, -])$ *is a symmetric monoidal closed category.*

3.2 A reflective full subcategory $\mathbf{M}_{\mathrm{t}}$

We define $\mathbf{M}_{\mathrm{t}}$ to be the full subcategory of $\mathbf{M}$ consisting of total Σ-monoids.

Lemma 3.7 *The inclusion functor* $U : \mathbf{M}_{\mathrm{t}} \to \mathbf{M}$ *has a left adjoint functor.*

Proof. For $X \in \mathbf{M}$, let $\mathcal{S}$ be the set of total Σ-monoids whose underlying sets are quotients of X^*. We show that $\mathcal{S}$ satisfies the solution set condition: for each morphism $f : X \to Y$ whose codomain Y is in $\mathbf{M}_{\mathrm{t}}$, there exists a morphism $s : X \to A$ and a morphism $h : A \to Y$ for some $A \in \mathcal{S}$ such that $f = h \circ s$. We define a map $p : X^* \to Y$ by $p\{x_i\}_{i \in I} := \sum_{i \in I} f x_i$. Let A be the quotient of X^* by an equivalence relation on X^* given by $\{x_i\}_{i \in I} \approx \{x'_j\}_{j \in J} \overset{\text{def}}{\iff} p\{x_i\}_{i \in I} = p\{x'_j\}_{j \in J}$. Since the image of p is closed under the summation of Y, the total Σ-monoid structure of Y induces a total Σ-monoid structure of A, and we obtain a monomorphism $h : A \to Y$. Since the image of f is in the image of h, there exists a morphism $s : X \to A$ such that $f = h \circ s$. Hence, $\mathcal{S}$ satisfies the solution set condition. Since $\mathbf{M}_{\mathrm{t}}$ is small complete [8], and U preserves all limits, U has a left adjoint functor by the adjoint functor theorem [13]. □

For a category $\mathcal{C}$, a *reflective full subcategory* of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ such that the inclusion functor has a left adjoint functor. For a symmetric monoidal closed category $(\mathcal{C}, \mathrm{I}, \otimes, [-,-])$ and its full subcategory $\mathcal{B}$, we say that $\mathcal{B}$ is an *exponential ideal* of $\mathcal{C}$ when for any $X \in \mathcal{C}$ and $Y \in \mathcal{B}$, the exponential $[X, Y]$ is a $\mathcal{B}$-object.

Theorem 3.8 ([2]) *Let $\mathcal{B}$ be a reflective full subcategory of a symmetric monoidal closed category $\mathcal{C}$. If $\mathcal{B}$ is an exponential ideal of $\mathcal{C}$, then $\mathcal{B}$ has a symmetric monoidal closed structure, and the adjunction is symmetric monoidal.*

By the definition (1) of the exponential $[-,-]$ of $\mathbf{M}$, it is easy to check that $\mathbf{M}_\mathrm{t}$ is an exponential ideal of $\mathbf{M}$.

Corollary 3.9 *$\mathbf{M}_\mathrm{t}$ is a symmetric monoidal closed category, and the adjunction between $\mathbf{M}$ and $\mathbf{M}_\mathrm{t}$ is symmetric monoidal with respect to the structures.*

Let T be the symmetric monoidal monad on $\mathbf{M}$ induced by the symmetric monoidal adjunction. We show several properties of the unit $\eta_X : X \to TX$.

Definition 3.10 We say that an $\mathbf{M}$-morphism $f : X \to Y$ *reflects summability* when for every countable family $\{x_i\}_{i \in I}$ on X if $\sum_{i \in I} f x_i$ is summable and is in the image of f, then $\{x_i\}_{i \in I}$ is summable.

Lemma 3.11 *The unit $\eta_X : X \to TX$ is monic and reflects summability.*

Proof. We define a total Σ-monoid X' by $X' := X + \{\bot\}$ with a summation

$$\sum_{i \in I} y_i := \begin{cases} \mathrm{inl}\left(\sum_{i \in I} x_i\right) & \left(\begin{array}{l}\text{for each } i \in I,\ y_i \text{ is of the form } \mathrm{inl}(x_i), \\ \text{and } \{x_i\}_{i \in I} \text{ is summable}\end{array}\right) \\ \mathrm{inr}(\bot) & (\text{otherwise}) \end{cases}$$

where $\mathrm{inl}(-)$ is the left injection, and $\mathrm{inr}(-)$ is the right injection. We define an $\mathbf{M}$-morphism $h : X \to X'$ by $hx := \mathrm{inl}(x)$. Since an $\mathbf{M}$-morphism is monic if and only if its underlying map is injective, h is monic. Let $k : TX \to X'$ be the unique morphism such that $h = k \circ \eta_X$. Since $h : X \to X'$ is monic, the unit η_X is also monic. For a countable family $\{x_i\}_{i \in I}$ on X, if $\sum_{i \in I} \eta_X x_i$ is in the image of η_X, then we have

$$\sum_{i \in I} h x_i = \sum_{i \in I} k \eta_X x_i = k \left(\sum_{i \in I} \eta_X x_i \right) \in \mathrm{image}(k \circ \eta_X) = \mathrm{image}(h),$$

which means that $\{x_i\}_{i \in I}$ is summable. Hence, η_X reflects summability. $\square$

Although our construction of T is abstract, for some Σ-monoids X, we can concretely describe TX via the universality of T.

Example 3.12 For countable sets A and B, let $\mathbf{Pfn}(A,B)$ be the set of partial maps from A to B. The set $\mathbf{Pfn}(A,B)$ forms a Σ-monoid by the union of graph relations:

$$\sum_{i\in I} f_i := \begin{cases} \bigcup_{i\in I} f_i & (\bigcup_{i\in I} f_i \text{ represents a partial map}) \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

Let $\mathbf{Rel}(A,B)$ be the set of relations between A and B, which forms a total Σ-monoid by the union of graphs. There is an obvious inclusion $h : \mathbf{Pfn}(A,B) \to \mathbf{Rel}(A,B)$ between Σ-monoids. For a total Σ-monoid X and an $\mathbf{M}$-morphism $f : \mathbf{Pfn}(A,B) \to X$, there is an $\mathbf{M}$-morphism $g : \mathbf{Rel}(A,B) \to X$ given by $g(R) := \sum_{(a,b)\in R} f(\delta_{a,b})$ where $\delta_{a,b} := \{(a,b)\}$. Since every partial map in $\mathbf{Pfn}(A,B)$ is equal to a sum of partial maps of the form $\delta_{a,b}$, we obtain $g \circ h = f$. Such g is unique since g must satisfy the following equation.

$$g(R) = g\left(\sum_{(a,b)\in R} \delta_{a,b}\right) = \sum_{(a,b)\in R} g(\delta_{a,b}) = \sum_{(a,b)\in R} gh(\delta_{a,b}) = \sum_{(a,b)\in R} f(\delta_{a,b}).$$

By the universality of T, we see that $T\mathbf{Pfn}(A,B)$ is isomorphic to $\mathbf{Rel}(A,B)$.

Example 3.13 For a countable set A, we define sets $A^\star$ and $A^{\divideontimes}$ by

$$A^\star := \{x : A \to \mathbb{N} \mid \mathrm{dom}(x) \text{ is finite}\}, \qquad A^{\divideontimes} := \mathbf{Set}(A, \mathbb{N} \cup \{\infty\})$$

where $\mathrm{dom}(x) := \{a \in A \mid x(a) \neq 0\}$. The sets $A^\star$ and $A^{\divideontimes}$ are Σ-monoids with the pointwise summations. The Σ-monoid $A^{\divideontimes}$ is total. As in Example 3.12, we can show that $TA^\star$ is isomorphic to $A^{\divideontimes}$.

4 Unique decomposition categories

4.1 **M**-*categories*

With respect to the symmetric monoidal structure of $\mathbf{M}$, we consider $\mathbf{M}$-enrichment [12]. By Proposition 3.5, we can say that an $\mathbf{M}$-*enriched category* ($\mathbf{M}$-*category*) $\mathcal{C}$ is a category with a Σ-monoid structure on each hom-set $\mathcal{C}(X,Y)$ such that for any summable countable families $\{f_i : X \to Y\}_{i\in I}$ and $\{g_j : Y \to Z\}_{j\in J}$, the summation $\sum_{(i,j)\in I\times J} g_j \circ f_i$ is defined to be $(\sum_{j\in J} g_j) \circ (\sum_{i\in I} f_i)$, i.e., the composition distributes over the summations if they exist. We write $0_{X,Y} : X \to Y$ for the zero element in the Σ-monoid $\mathcal{C}(X,Y)$ and call $0_{X,Y}$ *a zero morphism*. By the definition of $\mathbf{M}$-categories, the composition of a morphism with a zero morphism is a zero morphism.

For $\mathbf{M}$-categories $\mathcal{C}$ and $\mathcal{D}$, an $\mathbf{M}$-*enriched functor* ($\mathbf{M}$-*functor*) $F : \mathcal{C} \to \mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$ such that for any $X, Y \in \mathcal{C}$, the map $F : \mathcal{C}(X,Y) \to$

$\mathcal{D}(FX, FY)$ is an $\mathbf{M}$-morphism. We say that $F : \mathcal{C} \to \mathcal{D}$ *reflects summability* when $F : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ reflects summability for all X and Y in $\mathcal{C}$.

By *symmetric monoidal* $\mathbf{M}$*-category*, we mean an $\mathbf{M}$-category with a symmetric monoidal structure on its underlying category. We do not assume that the symmetric monoidal structure is compatible with the $\mathbf{M}$-enrichment. For symmetric monoidal $\mathbf{M}$-category $\mathcal{C}$ and $\mathcal{D}$, a *symmetric monoidal* $\mathbf{M}$*-functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is an $\mathbf{M}$-functor from $\mathcal{C}$ to $\mathcal{D}$ that is symmetric monoidal.

4.2 Strong unique decomposition categories

We recall the definition of unique decomposition categories in [4], and we give a subclass of unique decomposition categories.

Definition 4.1 A *unique decomposition category* is a symmetric monoidal $\mathbf{M}$-category such that for all $i \in I$, there are morphisms called *quasi projections* $\rho_i : \bigotimes_{i\in I} X_i \to X_i$ and *quasi injections* $\iota_i : X_i \to \bigotimes_{i \in I} X_i$ subject to the following axioms:

$$\rho_i \circ \iota_j = \begin{cases} \mathrm{id}_{X_i} & (i = j) \\ 0_{X_j, X_i} & (\text{otherwise}), \end{cases} \qquad \sum_{i \in I} \iota_i \circ \rho_i \simeq \mathrm{id}_{\bigotimes_{i\in I} X_i}.$$

Definition 4.2 A *strong unique decomposition category* $\mathcal{C}$ is a symmetric monoidal $\mathbf{M}$-category $\mathcal{C}$ such that

- The identity on the unit I is equal to $0_{\mathrm{I},\mathrm{I}}$.
- $\mathrm{id}_X \otimes 0_{Y,Y} + 0_{X,X} \otimes \mathrm{id}_Y$ is defined to be $\mathrm{id}_{X \otimes Y}$.

We say that $\mathcal{C}$ is *total* when every hom-object is a total Σ-monoid.

The class of strong unique decomposition categories forms a subclass of unique decomposition categories: a strong unique decomposition category has binary quasi projections and binary quasi injections given as follows:

$$\rho_{X,Y} := X \otimes Y \xrightarrow{\mathrm{id}_X \otimes 0_{Y,\mathrm{I}}} X \otimes \mathrm{I} \xrightarrow{\cong} X \qquad \rho'_{X,Y} := X \otimes Y \xrightarrow{0_{X,\mathrm{I}} \otimes \mathrm{id}_Y} \mathrm{I} \otimes Y \xrightarrow{\cong} Y$$

$$\iota_{X,Y} := X \xrightarrow{\cong} X \otimes \mathrm{I} \xrightarrow{\mathrm{id}_X \otimes 0_{\mathrm{I},Y}} X \otimes Y \qquad \iota'_{X,Y} := Y \xrightarrow{\cong} \mathrm{I} \otimes Y \xrightarrow{0_{\mathrm{I},X} \otimes \mathrm{id}_Y} X \otimes Y.$$

We can similarly define quasi projections and quasi injections for general cases. It is easy to check that a strong unique decomposition category with the above morphisms forms a unique decomposition category.

Remark 4.3 As the main point of unique decomposition categories is their unique decomposition of morphisms into matrices of morphisms via quasi projections and quasi injections (Proposition 4.0.6 in [4]), it would be better to employ quasi projections and quasi injections as primal data for strong unique decomposition categories. We choose the above definition of strong unique decomposition categories because of its compactness. At this point, we do not know "equivalent" definition that employs quasi projections and quasi injections as primal data, which would consist of a series of equalities that require quasi projections and quasi injections to be natural and compatible with monoidal structural isomorphisms. In fact, the above quasi projections and quasi injections satisfy naturality and compatibility with monoidal structural isomorphisms; see Proposition 4.8 for the case of total unique decomposition categories.

Example 4.4 All the examples of unique decomposition categories in [5] are strong unique decomposition categories. For example, sets and partial injections, sets and partial maps, sets and relations are strong unique decomposition categories.

Example 4.5 The opposite category of a strong unique decomposition category is a strong unique decomposition category.

Example 4.6 A category $\mathcal{C}$ with countable biproducts is a total strong unique decomposition category, c.f. [4]. For a countable family $\{f_i\}_{i\in I}$ on $\mathcal{C}(X, Y)$, we define its summation by

$$\sum_{i\in I} f_i := X \xrightarrow{\delta_X} \bigoplus_{i\in I} X \xrightarrow{\bigoplus_{i\in I} f_i} \bigoplus_{i\in I} Y \xrightarrow{\gamma_X} Y$$

where δ_X and γ_X are the diagonal morphisms. Since the composition distributes over the summation, we obtain an $\mathbf{M}$-enrichment of $\mathcal{C}$. We take the finite biproducts as a symmetric monoidal structure of $\mathcal{C}$. By these data, $\mathcal{C}$ forms a strong unique decomposition category. Concrete examples are: sets and relations, sup-complete lattices and continuous maps, and $\mathbf{M}_{\mathrm{t}}$.

Example 4.7 Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor from a symmetric monoidal category $\mathcal{C}$ to a category $\mathcal{D}$ with countable biproducts. We say that $F : \mathcal{C} \to \mathcal{D}$ is *downward-closed* when for every countable family $\{f_i : X \to Y\}_{i\in I}$ on $\mathcal{C}$-morphisms, if the summation $\sum_{i\in I} Ff_i : FX \to FY$ is in the image of F, then for every subset $J \subset I$, the summation $\sum_{i\in J} Ff_i : FX \to FY$ is also in the image of F. If the faithful functor $F : \mathcal{C} \to \mathcal{D}$ is downward-closed, then $\mathcal{C}$ forms a strong unique decomposition category: for a countable family $\{f_i\}_{i\in I}$ on $\mathcal{C}(X, Y)$, we define $\sum_{i\in I} f_i$ to be f when $\sum_{i\in I} Ff_i$ is equal to Ff; when $\sum_{i\in I} Ff_i$ is not in the image of F, we do not define $\sum_{i\in I} f_i$.

Proposition 4.8 *If a strong unique decomposition category is total, then it has finite biproducts: the unit is a zero object, and $X \otimes Y$ with morphisms $(\rho_{X,Y}, \rho'_{X,Y}, \iota_{X,Y}, \iota'_{X,Y})$ forms a biproduct of X and Y. Furthermore, the symmetric monoidal structure coincides with the symmetric monoidal structure derived from the finite biproducts.*

Proof. In every strong unique decomposition category, the unit is a zero object since the identity on the unit is a zero morphism. When the strong unique decomposition category is total, $(X \otimes Y, \rho_{X,Y}, \rho'_{X,Y})$ forms a product of X and Y, and $(X \otimes Y, \iota_{X,Y}, \iota'_{X,Y})$ forms a coproduct of X and Y. For $f : X \to Y$ and $g : Z \to W$, the tupling $\langle f \circ \rho_{X,Z}, g \circ \rho'_{X,Z} \rangle$ is $\iota_{Y,W} \circ f \circ \rho_{X,Z} + \iota'_{Y,W} \circ g \circ \rho'_{X,Z}$, which is equal to the cotupling $[\iota_{Y,W} \circ f, \iota'_{Y,W} \circ g]$. Hence, $(X \otimes Y, \rho_{X,Y}, \rho'_{X,Y}, \iota_{X,Y}, \iota'_{X,Y})$ forms a biproduct of X and Y. By the universality of biproducts, we can check that coherence isomorphisms of the symmetric monoidal structure of the strong unique decomposition category coincide with the symmetric monoidal structure derived from the biproducts.□

5 A representation theorem

For a strong unique decomposition category $\mathcal{C}$, since T is a symmetric monoidal functor (Corollary 3.9), we can define a new $\mathbf{M}$-category $T_*\mathcal{C}$ by the action of T: objects are objects of $\mathcal{C}$, and $T_*\mathcal{C}(X, Y) := T(\mathcal{C}(X, Y))$. Furthermore, the unit $\eta_X : X \to TX$ induces an $\mathbf{M}$-functor $H : \mathcal{C} \to T_*\mathcal{C}$ given by $HX := X$ and $Hf := \eta_{\mathcal{C}(X,Y)}(f)$ for $f : X \to Y$.

Proposition 5.1 *$T_*\mathcal{C}$ is a total strong unique decomposition category, and H is a faithful strong symmetric monoidal $\mathbf{M}$-functor that* reflects summability.

Proof. We give a symmetric monoidal structure on the underlying category. For objects, we employ the symmetric monoidal structure of $\mathcal{C}$. For $f : X \to Y$ and $g : Z \to W$ in $T_*\mathcal{C}$, we define $f \otimes g : X \otimes Z \to Y \otimes W$ to be

$$H\iota_{Y,W} \circ f \circ H\rho_{X,Z} + H\iota'_{Y,W} \circ g \circ H\rho'_{X,Z}.$$

Functoriality of $\otimes$ follows from $\mathbf{M}$-enrichment of H. For example,

$$\mathrm{id}_X \otimes \mathrm{id}_Y = H(\iota_{X,Y} \circ \rho_{X,Y} + \iota'_{X,Y} \circ \rho'_{X,Y}) = H(\mathrm{id}_{X \otimes Y}) = \mathrm{id}_{X \otimes Y}.$$

We can similarly check that $\otimes$ is compatible with the composition of $\mathcal{C}$. By $\mathbf{M}$-enrichment of H again, we can check that $\otimes$ with $H\lambda_X, H\rho_X, H\alpha_{X,Y,Z}$ and $H\sigma_{X,Y}$ provide a symmetric monoidal structure on $T_*\mathcal{C}$ where $\lambda_X : X \otimes \mathrm{I} \to X$, $\rho_X : \mathrm{I} \otimes X \to X$, $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ and $\sigma_{X,Y} : X \otimes Y \to Y \otimes Z$ are the coherence isomorphisms of $\mathcal{C}$. The identity on the unit is the zero morphism. In fact, $H\mathrm{id}_\mathrm{I} = H0_{\mathrm{I},\mathrm{I}} = 0_{\mathrm{I},\mathrm{I}}$. We also have

$$\mathrm{id}_X \otimes 0_{Y,Y} + 0_{X,X} \otimes \mathrm{id}_Y = H\iota_{X,Y} \circ H\rho_{X,Y} + H\iota'_{X,Y} \circ H\rho'_{X,Y} = \mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}$$

in $T_*\mathcal{C}$. Therefore, we see that $T_*\mathcal{C}$ is a strong unique decomposition category. Since T constructs total Σ-monoids, $T_*\mathcal{C}$ is total. By the definition of symmetric monoidal structure of $T_*\mathcal{C}$, we see that H is strong symmetric monoidal. The $\mathbf{M}$-functor H is faithful and reflects summability by Lemma 3.11. □

Since $H : \mathcal{C} \to T_*\mathcal{C}$ is faithful and reflects summability, H completely characterizes the summation of $\mathcal{C}$-morphisms:

$$\sum_{i\in I} f_i \text{ is defined to be } f \iff Hf = \sum_{i\in I} Hf_i \quad \text{in } T_*\mathcal{C}(X,Y).$$

We go a bit farther so as to give an embedding into a category that is more familiar to us than total strong unique decomposition categories. For a total strong unique decomposition category $\mathcal{A}$, we define a category $\mathscr{B}(\mathcal{A})$ by:

- An object is a countable family on the set of $\mathcal{A}$-objects.
- A morphism $f : \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ is a family $\{f_{i,j} : X_i \to Y_j\}_{(i,j)\in I\times J}$.
- The identity $\mathrm{id}_{\{X_i\}_{i\in I}}$ on $\{X_i\}_{i\in I}$ and the composition $g \circ f$ are given by

$$\left(\mathrm{id}_{\{X_i\}_{i\in I}}\right)_{i,i'} := \begin{cases} \mathrm{id}_{X_i} & (i = i') \\ 0_{X_i,X_{i'}} & (i \neq i'), \end{cases} \qquad (g\circ f)_{i,k} := \sum_{j\in J} g_{j,k} \circ f_{i,j}.$$

$\mathscr{B}(\mathcal{A})$ has countable biproducts: a biproducts $\bigoplus_{i\in I}\{X_{ij}\}_{j\in J_i}$ of a countable family $\{\{X_{ij}\}_{j\in J_i}\}_{i\in I}$ is $\{X_{ij}\}_{(i,j)\in\coprod_{i\in I} J_i}$ whose i-th projection and i-th injection $\pi_i : \bigoplus_{i\in I}\{X_{ij}\}_{j\in J_i} \leftrightarrows \{X_{ij}\}_{j\in J_i} : \kappa_i$ for $i \in I$ are given as follows:

$$\pi_i((i',j'),j) := \begin{cases} \mathrm{id}_{X_{ij}} & ((i,j) = (i',j')) \\ 0_{X_{i'j'},X_{ij}} & (\text{otherwise}), \end{cases}$$

$$\kappa_i(j,(i',j')) := \begin{cases} \mathrm{id}_{X_{ij}} & ((i,j) = (i',j')) \\ 0_{X_{ij},X_{i'j'}} & (\text{otherwise}). \end{cases}$$

The induced summation of a countable family $\{f_k : \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}\}_{k\in K}$ is pointwise: the (i,j)-th entry of $\sum_{k\in K} f_k$ is $\sum_{k\in K}(f_k)_{i,j}$. By Example 4.6, $\mathscr{B}(\mathcal{A})$ is a total strong unique decomposition category. A similar construction appears in [13] called *matrix construction*.

We define a fully faithful functor $K : \mathcal{A} \to \mathscr{B}(\mathcal{A})$ by $KX := \{X\}$ and $Kf := \{f\}$ where we simply write $\{x\}$ for a family indexed by a singleton $\{\bullet\}$ such that $\{x\}\bullet = x$.

Lemma 5.2 *K is a fully faithful strong symmetric monoidal* $\mathbf{M}$*-functor.*

Proof. Since summations on hom-sets of $\mathscr{B}(\mathcal{A})$ are pointwise, the functor K preserves summations, i.e., K is an $\mathbf{M}$-functor. K is fully faithful by the definition. It remains to see that K is strong symmetric monoidal. Since the symmetric monoidal structure of $\mathcal{A}$ is given by the finite biproducts (Proposition 5.1 and Proposition 4.8), we show that K preserves finite biproducts. There are canonical morphisms $\varphi := \langle K\rho_{X,Y}, K\rho'_{X,Y}\rangle : K(X\otimes Y) \to KX \oplus KY$ and $\psi := [K\iota_{X,Y}, K\iota'_{X,Y}] : KX \oplus KY \to K(X \otimes Y)$. By the universality of biproducts and $\mathbf{M}$-enrichment of K, we see that $\varphi \circ \psi = \mathrm{id}_{KX\oplus KY}$

and $\psi \circ \varphi = K(\iota_{X,Y} \circ \rho_{X,Y}) + K(\iota'_{X,Y} \circ \rho'_{X,Y}) = \mathrm{id}_{K(X \otimes Y)}$. It is easy to check that KI is a zero object of $\mathscr{B}(\mathcal{A})$. $\square$

Now, we obtain a representation theorem for strong unique decomposition categories by composing two embeddings K and H.

Theorem 5.3 *For every strong unique decomposition category $\mathcal{C}$, there is a category $\mathcal{D}$ with countable biproducts and a faithful strong symmetric monoidal $\mathbf{M}$-functor $F : \mathcal{C} \to \mathcal{D}$ that is downward-closed and reflects summability.*

Proof. By Proposition 5.1 and Lemma 5.2, for every strong unique decomposition category $\mathcal{C}$, the category $\mathscr{B}(T_*\mathcal{C})$ has countable biproducts, and we have a faithful strong symmetric monoidal $\mathbf{M}$-functor $KH : \mathcal{C} \to \mathscr{B}(T_*\mathcal{C})$ that reflects summability. Downward-closedness of KH follows from the axioms of Σ-monoids and that KH reflects summability. $\square$

The faithful functor KH characterizes the Σ-monoid structure on $\mathcal{C}(X, Y)$:

$$\sum_{i \in I} f_i \text{ is defined to be } f \iff KHf = \sum_{i \in I} KHf_i.$$

So as to prove the fundamental property of strong unique decomposition categories, we construct a trace operator following the argument in [16]. Let $\mathcal{D}$ be a category with countable biproducts. For $f : X \oplus Z \to Y \oplus Z$ in $\mathcal{D}$, we define $f_{XY} : X \to Y$, $f_{XZ} : X \to Z$, $f_{ZY} : Z \to Y$ and $f_{ZZ} : Z \to Z$ by:

$$f_{XY} := \pi_0 \circ f \circ \kappa_0, \quad f_{XZ} := \pi_1 \circ f \circ \kappa_0, \quad f_{ZY} := \pi_0 \circ f \circ \kappa_1, \quad f_{ZZ} := \pi_1 \circ f \circ \kappa_1.$$

By Theorem 3 in [16] and the argument in the paper, $\mathcal{D}$ has a uniform trace operator given by

$$\mathrm{tr}^Z_{X,Y}(f) := X \xrightarrow{\langle X, \infty \rangle} X \oplus \textstyle\bigoplus_{i \in \mathbb{N}} X \xrightarrow{X \oplus u_f} X \oplus Z \xrightarrow{f} Y \oplus Z \xrightarrow{\pi_0} Y$$

where $\infty : X \to \bigoplus_{i \in \mathbb{N}} X$ is the diagonal morphism, and $u_f : \bigoplus_{i \in \mathbb{N}} X \to Z$ is the unique morphism such that $u_f \circ \kappa_i = f^i_{ZZ} \circ f_{XZ}$ for each $i \in \mathbb{N}$. By simple calculation, we see that the obtained trace operator is equal to the standard trace formula: $\mathrm{tr}^Z_{X,Y}(f) = f_{XY} + \sum_{i \in \mathbb{N}} f_{ZY} \circ f^i_{ZZ} \circ f_{XZ}$.

Corollary 5.4 *Every strong unique decomposition category $\mathcal{C}$ has a uniform partial trace operator. If the summation $\mathrm{Ex}^Z_{X,Y}(f) := f_{XY} + \sum_{i \in \mathbb{N}} f_{ZY} \circ f^i_{ZZ} \circ f_{XZ}$ is defined for all $X, Y, Z \in \mathcal{C}$ and $f : X \otimes Z \to Y \otimes Z$, then Ex is a uniform trace operator of $\mathcal{C}$.*

Proof. By the above argument, $\mathscr{B}(T_*\mathcal{C})$ has a uniform trace operator given by the standard trace formula. Since $KH : \mathcal{C} \to \mathscr{B}(T_*\mathcal{C})$ is strong monoidal and reflects summability, Ex provides a uniform partial trace operator of $\mathcal{C}$.

If $\mathrm{Ex}^{Z}_{X,Y}(f)$ is defined for all $X, Y, Z \in \mathcal{C}$ and $f : X \otimes Z \to Y \otimes Z$, then by the definition of trace operators, Ex is a trace operator of $\mathcal{C}$. $\square$

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A Proofs

Proposition A.1 **M** *and* $\mathbf{M}_{\mathrm{t}}$ *are small complete, and the inclusion* $U : \mathbf{M}_{\mathrm{t}} \to \mathbf{M}$ *preserves small limits.*

Proof. A Σ-monoid $\{0\}$ with the unique total summation is a terminal object. For Σ-monoids X and Y, we define a Σ-monoid $X \times Y$ by

$$\sum_{i\in I}(x_i, y_i) := \begin{cases} (\sum_{i\in I} x_i, \sum_{i\in I} y_i) & (\{x_i\}_{i\in I} \text{ and } \{y_i\}_{i\in I} \text{ are summable}) \\ \text{undefined} & (\text{otherwise}). \end{cases}$$

The Σ-monoid $X \times Y$ with the standard projections forms a product of X and Y. We can generalize this construction to small products. For parallel morphisms $f, g : X \rightrightarrows Y$ in **M**, since $E := \{x \in X \mid fx = gx\}$ is closed under the summation of X, the set E inherits the Σ-monoid structure of X. The obvious inclusion from E to X forms an equalizer of f and g. We can restrict these limits to $\mathbf{M}_{\mathrm{t}}$. Hence, $\mathbf{M}_{\mathrm{t}}$ is small complete, and the inclusion U preserves small limits. □

Proposition A.2 (Proposition 3.5) *A functor* $\mathbf{M}(X, Y; -) : \mathbf{M} \to \mathbf{Set}$ *is representable.*

Proof. $\mathbf{M}(X, Y; -)$ preserves small limits. We define $\mathcal{S}$ to be the set of Σ-monoids whose underlying sets are quotients of subsets of $(X \times Y)^*$. We show that $\mathcal{S}$ satisfies the solution set condition: for any 2-linear map $f : X \times Y \to W$, there exist a 2-linear map $s : X \times Y \to A$ and a morphism $h : A \to W$ for some $A \in \mathcal{S}$ such that $f = h \circ s$. We define a subset $Z \subset (X \times Y)^*$ by

$$Z := \left\{\{z_i\}_{i\in I} \in (X \times Y)^* \;\middle|\; \{fz_i\}_{i\in I} \text{ is summable}\right\},$$

and we define a map p from Z to W by $p\{z_i\}_{i\in I} := \sum_{i\in I} fz_i$. Let A be the quotient of Z by an equivalence relation $\{z_i\}_{i\in I} \approx \{z'_j\}_{j\in J} \overset{\text{def}}{\iff} p\{z_i\}_{i\in I} = p\{z'_j\}_{j\in J}$ on Z. Since the image of p is closed under the summation of W, the Σ-monoid structure of W induces a Σ-monoid structure of A, and we obtain a Σ-morphism $h : A \to W$ given by $h[\{z_i\}_{i\in I}] := \sum_{i\in I} fz_i$ where $[\{z_i\}_{i\in I}]$ is the equivalence class of $\{z_i\}_{i\in I}$. The morphism h is monic, and a countable family $\{a_i\}_{i\in I}$ on A is summable if and only if a countable family $\{ha_i\}_{i\in I}$ is summable. Since the image of $f : X \times Y \to W$ is in the image of h, there is a map $s : X \times Y \to A$ such that $f = h \circ s$. Since f is 2-linear, s is 2-linear. Therefore, $\mathcal{S}$ satisfies the solution set condition. Since **M** is small complete, $\mathbf{M}(X, Y; -)$ is representable by the adjoint functor theorem. □

Lemma A.3 *For Σ-monoids X and Y, the set of* **M***-morphisms* $[X, Y] := \mathbf{M}(X, Y)$ *with a summation on* $[X, Y]$ *given by (1) is a Σ-monoid.*

Proof. By the definition (1), $\sum_{i\in\{n\}} f_i$ is defined to be f_n. Let $\{I_j\}_{j\in J}$ be a countable partition of a countable set I. Then

$$\sum_{i\in I} f_i \simeq \lambda x. \sum_{j\in J}\sum_{i\in I_j} f_i x \simeq \lambda x. \sum_{j\in J}\left(\left(\sum_{i\in I_j} f_i\right) x\right) \simeq \sum_{j\in J}\left(\sum_{i\in I_j} f_i\right).$$

If $\{x_j\}_{j\in J}$ is a summable countable family on X, then

$$\begin{aligned}\left(\sum_{i\in I} f_i\right)\left(\sum_{j\in J} x_j\right) &= \sum_{i\in I} f_i \left(\sum_{j\in J} x_j\right)\\ &= \sum_{i\in I}\sum_{j\in J} f_i x_j = \sum_{j\in J}\sum_{i\in I} f_i x_j = \sum_{j\in J}\left(\sum_{i\in I} f_i\right) x_j.\end{aligned}$$

□

Lemma A.4 *We have the following bijections natural in* $X_1, \cdots, X_n$*:*

$$\begin{aligned}\mathbf{M}(X\,;Y) &\cong \mathbf{M}(\mathrm{I}\,;[X,Y])\\ \mathbf{M}(X_1,\cdots,X_n,X\,;Y) &\cong \mathbf{M}(X_1,\cdots,X_n\,;[X,Y])\\ \mathbf{M}(X_1,\cdots,X_n\,;Y) &\cong \mathbf{M}(X_{\sigma 1},\cdots,X_{\sigma n}\,;Y)\end{aligned}$$

where σ *is a permutation on* $\{1,2,\cdots,n\}$.

Proof. The first and the second bijection is given by currying and uncurrying. The third bijection is easy to check. □

By Proposition A.2, there exists a universal 2-linear map $p : X \times Y \to X \otimes Y$: for any 2-linear map $f : X \times Y \to Z$, there exists a unique morphism $h : X \otimes Y \to Z$ such that $h \circ p = f$. Generally, by Lemma A.4, we can inductively show that for every bracketing of $X_1 \otimes \cdots \otimes X_n$ and $X_1 \times \cdots \times X_n$, there exists a universal n-linear map $p_n : X_1 \times \cdots \times X_n \to X_1 \otimes \cdots \otimes X_n$: for any n-linear map $f : X_1 \times \cdots \times X_n \to Z$ there exists a unique morphism $h : X_1 \otimes \cdots \otimes X_n \to Z$ such that $h \circ p_n = f$.

We define a symmetric monoidal structure on $\mathbf{M}$. We extend $\otimes$ to a bifunctor as in the diagram (1), and we define coherence morphisms to be the unique morphisms in the following diagrams (2), (3), (4) and (5).

$$\begin{tikzcd}
X \otimes Y \arrow[r, "f\otimes g"] & X' \otimes Y' \\
X \times Y \arrow[u, "p"] \arrow[r, "f\times g"'] \arrow[ur, phantom, "(1)"] & X' \times Y' \arrow[u, "p"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes \mathrm{I} \arrow[r, "\lambda"] & X \\
X \times \mathrm{I} \arrow[u, "p"] \arrow[ur, "l"'] \arrow[ur, phantom, "(2)"', shift left=3]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I} \otimes X \arrow[r, "\rho"] & X \\
\mathrm{I} \times X \arrow[u, "p"] \arrow[ur, "r"'] \arrow[ur, phantom, "(3)"', shift left=3]
\end{tikzcd}$$

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] & (X \otimes Y) \otimes Z \\ X \times (Y \times Z) \arrow[u, "p_3"] \arrow[r, "\cong"'] \arrow[ur, phantom, "(4)"] & (X \times Y) \times Z \arrow[u, "p_3"'] \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "\sigma"] & Y \otimes X \\ X \times Y \arrow[u, "p"] \arrow[r, "\cong"'] \arrow[ur, phantom, "(5)"] & Y \times X \arrow[u, "p"'] \end{tikzcd}$$

where

$$l(x,y) := \begin{cases} x & (y=1) \\ 0 & (y=0), \end{cases} \qquad r(x,y) := \begin{cases} y & (x=1) \\ 0 & (x=0). \end{cases}$$

Proposition A.5 (Proposition 3.6) $(\mathbf{M}, \mathrm{I}, \otimes, [-,-])$ *with the above morphisms* λ, ρ, α, *and* σ *forms a symmetric monoidal closed category.*

Proof. By the universality of $\otimes$, we obtain a symmetric monoidal category $(\mathbf{M}, \mathrm{I}, \otimes)$. By Lemma A.4, we see that $[-,-]$ forms a closed structure. $\square$

B An example of a unique decomposition category

We give an example of a unique decomposition category that do not satisfy Proposition 1.1. Existence of such an example means that we need some assumptions on unique decomposition categories so as to prove Proposition 1.1.

Let G be a group. We define a category $\mathbf{Rel}(G)$ by: objects are sets, and a morphism $f : X \to Y$ is a map $f : X \times Y \to 2^G$. For $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{Rel}(G)$, the composition $g \circ f : X \to Z$ is given by

$$(g \circ f)(x,z) := \{ba \mid \exists y \in Y.\ a \in f(x,y) \wedge b \in g(y,z)\},$$

and we define the identity $\mathrm{id}_X : X \to X$ by

$$\mathrm{id}_X(x,x') := \begin{cases} \{e\} & (x = x') \\ \emptyset & (x \neq x'). \end{cases}$$

$\mathbf{Rel}(G)$ has a symmetric monoidal structure: the monoidal product $X \otimes Y$ is the set theoretic coproduct $X + Y$, and $f \otimes g : X \otimes Z \to Y \otimes W$ is given by

$$f \otimes g(u,v) = \begin{cases} f(x,y) & (u = \mathrm{inl}(x), v = \mathrm{inl}(y)) \\ g(z,w) & (u = \mathrm{inr}(z), v = \mathrm{inr}(w)) \\ \emptyset & (\text{otherwise}). \end{cases}$$

We define a total Σ-monoid structure on $\mathbf{Rel}(G)(X,Y)$ by the pointwise union: $\left(\sum_{i \in I} f_i\right)(x,y) := \bigcup_{i \in I} f_i(x,y)$. We fix an element $g \in G$ and define quasi projections $\rho_i : \bigotimes_{i \in I} X_i \to X_i$ and quasi injections $\iota_i : X_i \to \bigotimes_{i \in I} X_i$ by

$$\rho_i(w,x) := \begin{cases} \{g^{-1}\} & (w = \mathrm{inj}_i(x)) \\ \emptyset & (\text{otherwise}), \end{cases} \qquad \iota_i(x,w) := \begin{cases} \{g\} & (w = \mathrm{inj}_i(x)) \\ \emptyset & (\text{otherwise}), \end{cases}$$

where inj_i is the i-th injection of X_i into $\bigotimes_{i\in I} X_i$.

Proposition B.1 $\mathbf{Rel}(G)$ *is a unique decomposition category.*

Proposition B.2 *The execution formula defined with respect to the above quasi projections and quasi injections is not a trace operator.*

Proof. We suppose that the group G has an element a such that $gag^{-1} \neq a$. For example, the symmetric group S_n has such a and g. Let $f_a : \{*\} \otimes \emptyset \to \{*\} \otimes \emptyset$ be a $\mathbf{Rel}(G)$-morphism given by $(*, *) \mapsto \{a\}$. By the definition of the execution formula, $\left(\mathrm{Ex}^{\emptyset}_{\{*\},\{*\}}(f_a)\right)(*, *)$ is equal to $\{gag^{-1}\}$. However, if Ex is a trace operator, then $\left(\mathrm{Ex}^{\emptyset}_{\{*\},\{*\}}(f_a)\right)(*, *)$ must be equal to $\{a\}$. □

Remark B.3 The point in the above argument is naturality of quasi projections and quasi injections.

C Universality of the embedding $KH : \mathcal{C} \to \mathscr{B}(T_*\mathcal{C})$

Several reviewers pointed universality of our embedding $KH : \mathcal{C} \to \mathscr{B}(T_*\mathcal{C})$. In fact, KH has a universal property. In the following, we discuss universality of K and H separately. Universality of KH follows from universality of the two functors.

Definition C.1 Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbf{M}$-categories. For strong symmetric monoidal $\mathbf{M}$-functors $F, G : \mathcal{C} \to \mathcal{D}$, a *monoidal* $\mathbf{M}$*-natural transformation* $\alpha : F \to G$ is an $\mathbf{M}$-natural transformation $\alpha : F \to G$ that is monoidal natural with respect to the structures of the underlying symmetric monoidal functors F and G.

We introduce 2-categories $\mathfrak{B}$, $\mathfrak{U}$ and $\mathfrak{T}$:

$$\mathfrak{U} := \left[\begin{array}{c}\text{unique decomposition categories}\\ \text{strong symmetric monoidal } \mathbf{M}\text{-functors}\\ \text{monoidal } \mathbf{M}\text{-natural transformations}\end{array}\right],$$

$$\mathfrak{T} := \left[\begin{array}{c}\text{total unique decomposition categories}\\ \text{strong symmetric monoidal } \mathbf{M}\text{-functors}\\ \text{monoidal } \mathbf{M}\text{-natural transformations}\end{array}\right],$$

$$\mathfrak{B} := \left[\begin{array}{c}\text{categories with countable biproducts}\\ \text{countable biproducts preserving functors}\\ \text{natural transformations}\end{array}\right].$$

As we observed in Example 4.6, all objects in $\mathfrak{B}$ are objects in $\mathfrak{T}$. With respect to these structures of unique decomposition categories, we have an inclusion functor $\mathscr{I} : \mathfrak{T} \to \mathfrak{U}$. We write $\mathscr{J}$ for the inclusion functor from $\mathfrak{B}$ to $\mathfrak{T}$.

C.1 Universality of $H : \mathcal{C} \to T_*\mathcal{C}$

We show that the inclusion functor $\mathscr{I} : \mathfrak{T} \to \mathfrak{U}$ has a 2-left adjoint functor, and the unit of the 2-adjunction is $H : \mathcal{C} \to T_*\mathcal{C}$. So as to prove the statement, we define a functor

$$\overline{(-)} : \mathfrak{U}(\mathcal{C}, \mathcal{D}) \to \mathfrak{T}(T_*\mathcal{C}, \mathcal{D})$$

for $\mathcal{C} \in \mathfrak{U}$ and $\mathcal{D} \in \mathfrak{T}$.

First, for $F \in \mathfrak{U}(\mathcal{C}, \mathcal{D})$, we define an $\mathbf{M}$-functor $\overline{F} : T_*\mathcal{C} \to \mathcal{D}$ as follows:

- For $X \in T_*\mathcal{C}$, we define $\overline{F}X$ to be FX.
- We define $\overline{F} : T_*\mathcal{C}(X, Y) \to \mathcal{D}(\overline{F}X, \overline{F}Y)$ to be

$$\overline{F} : T_*\mathcal{C}(X, Y) \xrightarrow{TF} T_*\mathcal{D}(FX, FY) \xrightarrow{\cong} \mathcal{D}(FX, FY).$$

It is easy to check that $\overline{F} \circ H = F$. Let $m_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ and $n : \mathrm{I} \to F\mathrm{I}$ be the coherence isomorphisms for the monoidal functor F. To show that $\overline{F}$ with m and n forms a symmetric monoidal functor, it is enough to show that naturality of m. As we defined in the proof of Proposition 5.1, for $f \in T_*\mathcal{C}(X, X')$ and $g \in T_*\mathcal{C}(Y, Y')$, the monoidal product $f \otimes g$ is

$$H(\iota_{X',Y'}) \circ f \circ H(\rho_{X,Y}) + H(\iota'_{X',Y'}) \circ g \circ H(\rho'_{X,Y}).$$

Since $\overline{F}$ is $\mathbf{M}$-enriched and $\overline{F} \circ H = F$, we have

$$\begin{aligned}
m^{-1}_{X',Y'} \circ \overline{F}(f \otimes g) \circ m_{X,Y} &= m^{-1}_{X',Y'} \circ \overline{F}H\iota_{X',Y'} \circ \overline{F}f \circ \overline{F}H\rho_{X,Y} \circ m_{X,Y} \\
&\quad + m^{-1}_{X',Y'} \circ \overline{F}H\iota'_{X',Y'} \circ \overline{F}g \circ \overline{F}H\rho'_{X,Y} \circ m_{X,Y} \\
&= m^{-1}_{X',Y'} \circ F\iota_{X',Y'} \circ \overline{F}f \circ F\rho_{X,Y} \circ m_{X,Y} \\
&\quad + m^{-1}_{X',Y'} \circ F\iota'_{X',Y'} \circ \overline{F}g \circ F\rho'_{X,Y} \circ m_{X,Y} \\
&= \iota_{FX',FY'} \circ \overline{F}f \circ \rho_{FX,FY} + \iota'_{FX',FY'} \circ \overline{F}g \circ \rho'_{FX,FY} \\
&= \overline{F}f \otimes \overline{F}g.
\end{aligned}$$

Hence, m is natural.

Next, for $\alpha : F \to G$ in $\mathfrak{U}(\mathcal{C}, \mathcal{D})$, we define $\overline{\alpha}_X : \overline{F}X \to \overline{G}X$ to be $\alpha_X : FX \to GX$. Since the outer rectangle below commutes, the inner rectangle also commutes by the universality of H. Therefore, $\overline{\alpha}$ is natural.

$$\begin{tikzcd}
 & & \mathcal{D}(FX,FY) \arrow[dr, "(\overline{\alpha}_Y)_*"] & \\
\mathcal{C}(X,Y) \arrow[r, "H"] \arrow[urr, bend left, "F"] \arrow[drr, bend right, "G"'] & T_*\mathcal{C}(X,Y) \arrow[ur, "\overline{F}"] \arrow[dr, "\overline{G}"] & & \mathcal{D}(FX,GY) \\
 & & \mathcal{D}(GX,GY) \arrow[ur, "(\overline{\alpha}_X)^*"] &
\end{tikzcd}$$

Furthermore, $\overline{\alpha}$ is monoidal since α is monoidal. Now, we obtain a functor $\overline{(-)} : \mathfrak{U}(\mathcal{C},\mathcal{D}) \to \mathfrak{T}(T_*\mathcal{C},\mathcal{D})$.

Theorem C.2 *The inclusion functor* $\mathscr{I} : \mathfrak{T} \to \mathfrak{U}$ *is a 2-right adjoint functor, whose unit is* $H : \mathcal{C} \to T_*\mathcal{C}$*.*

Proof. By the universality of T, we see that $\overline{(-)}$ is the inverse of $(-) \circ H$ on 1-cells. On 2-cells, it is easy to check that $\overline{(-)}$ is the inverse of $(-) \circ H$ since both functors does almost nothing on data. □

C.2 Universality of $K : \mathcal{A} \to \mathscr{B}(\mathcal{A})$

Theorem C.3 *The inclusion functor* $\mathscr{J} : \mathfrak{B} \to \mathfrak{T}$ *has a left biadjoint, whose unit is* $K : \mathcal{A} \to \mathscr{B}(\mathcal{A})$*.*

Proof. For $\mathcal{A} \in \mathfrak{T}$ and $\mathcal{D} \in \mathfrak{B}$, we show that

$$(-) \circ K : \mathfrak{B}(\mathscr{B}(\mathcal{A}),\mathcal{D}) \to \mathfrak{T}(\mathcal{A},\mathcal{D})$$

is essentially surjective and fully faithful. Given $F : \mathcal{A} \to \mathcal{D}$ in $\mathfrak{T}$, we define $F' : \mathscr{B}(\mathcal{A}) \to \mathcal{D}$ in $\mathfrak{B}$ by

- $F'\{X_i\}_{i \in I} := \bigoplus_{i \in I} FX_i$
- $F'f : F'\{X_i\}_{i \in I} \to F'\{Y_j\}_{j \in J}$ is a unique morphism such that

  $$\pi_j \circ F'f \circ \iota_i = f_{ij}$$

  for $i \in I$ and $j \in J$. Recall that $f : \{X_i\}_{i \in I} \to \{Y_j\}_{j \in J}$ is a family of morphisms of the form $\{f_{ij}\}_{i \in I, j \in J}$.

By the definition of F', we have $F'K \cong F$, i.e., $(-) \circ K$ is essentially surjective.

Next, we show faithfulness on 2-cells. For $\alpha : F \to G : \mathscr{B}(\mathcal{C}) \to \mathcal{D}$, since F and G preserve biproducts, we have the following commutative diagram:

$$\begin{tikzcd}
FX_i \arrow[r, "\alpha_{X_i}"] \arrow[d, "\iota_i"'] \arrow[dr, "F\iota_i"] & GX_i \arrow[rr, "\delta_{ij}"] \arrow[dr, "G\iota_i"] & & GX_j \\
\bigoplus_{i \in I} FX_i \arrow[r, "\cong"'] & F\{X_i\}_{i \in I} \arrow[r, "\alpha"'] & G\{X_i\}_{i \in I} \arrow[r, "\cong"'] \arrow[ur, "G\pi_j"] & \bigoplus_{j \in I} GX_i \arrow[u, "\pi_j"']
\end{tikzcd}$$

where δ_{ij} is equal to 0 when $i \neq j$ and is equal to id when $i = j$. Therefore, α is completely determined by $\alpha * K$.

It remains to show fullness. Let $\alpha : F \circ K \to G \circ K$ be a 2-cell in $\mathfrak{T}$. We define $\alpha' : F \to G$ to be

$$\alpha'_{\{X_i\}_{i \in I}} := F\{X_i\}_{i \in I} \xrightarrow{\cong} \bigoplus_{i \in I} FX_i \xrightarrow{\bigoplus_{i \in I} \alpha_{X_i}} \bigoplus_{i \in I} GX_i \xrightarrow{\cong} G\{X_i\}_{i \in I} \, .$$

Since the outer rectangle below commutes for all i and j, the inner rectangle commutes.

$$\begin{tikzcd}
FX_i \arrow[r, "\alpha_{X_i}"] \arrow[d, "F\iota_i"'] \arrow[ddd, bend right=60, "Ff_{ij}"'] & GX_i \arrow[d, "G\iota_i"] \arrow[ddd, bend left=60, "Gf_{ij}"] \\
F\{X_i\}_{i \in I} \arrow[r, "\alpha'_X"] \arrow[d, "Ff"'] & G\{X_i\}_{i \in I} \arrow[d, "Gf"] \\
F\{Y_j\}_{j \in J} \arrow[r, "\alpha'_Y"] \arrow[d, "F\pi_j"'] & G\{Y_j\}_{j \in J} \arrow[d, "G\pi_j"] \\
F_{Y_j} \arrow[r, "\alpha_{Y_j}"'] & GY_j
\end{tikzcd}$$

Hence α' is natural, and we obtain a 2-cell $\alpha' : F \to G$. It is easy to see that $\alpha = \alpha' * H$. □

Corollary C.4 *The inclusion* $\mathscr{J} \circ \mathscr{I} : \mathfrak{B} \to \mathfrak{U}$ *has a left biadjoint, whose unit is* $KH : \mathcal{C} \to \mathscr{B}(T_*\mathcal{C})$.