THE UBIQUITOUS HYPERFINITE II_1 FACTOR

lectures 1-5

Kyoto U. & RIMS, April 2019

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Generalities on von Neumann algebras

A **von Neumann (vN) algebra** is a *-algebra of operators acting on a Hilbert space, $M \subset \mathcal{B}(\mathcal{H})$, that contains $1 = id_{\mathcal{H}}$ and satisfies any of the following equivalent conditions:

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- **Kaplansky Density Theorem** shows that if $M \subset \mathcal{B}(\mathcal{H})$ is a vN algebra and $M_0 \subset M$ is a *-sublgebra that's wo-dense in M, then $\overline{(M_0)_1}^{so} = (M)_1$.
- A vN algebra M is closed to polar decomposition and Borel functional calculus. Also, if $\{x_i\}_i \subset (M_+)_1$ is an increasing net, then $\sup_i x_i \in M$, and if $\{p_j\}_j \subset M$ are mutually orthogonal projections, then $\sum_j p_j \in M$.

Examples

- $\mathcal{B}(\mathcal{H})$ itself is a vN algebra.

- Let (X, μ) be a standard Borel probability measure space (pmp). Then the function algebra $L^\infty X = L^\infty(X, \mu)$ with its essential sup-norm $\| \ \|_\infty$, can be represented as a *-algebra of operators on the Hilbert space $L^2 X = L^2(X, \mu)$, as follows: for each $x \in L^\infty X$, let $\lambda(x) \in \mathcal{B}(L^2 X)$ denote the operator of (left) multiplication by x on $L^2 X$, i.e., $\lambda(x)(\xi) = x\xi$, $\forall \xi \in L^2 X$. Then $x \mapsto \lambda(x)$ is clearly a *-algebra morphism with $\|\lambda(x)\|_{\mathcal{B}(L^2 X)} = \|x\|_\infty$, $\forall x$. Its image $A \subset \mathcal{B}(L^2 X)$ satisfies $A' = A$, in other words A is a maximal abelian *-subalgebra (MASA) in $\mathcal{B}(L^2 X)$.

Indeed, if $T \in A'$ then let $\xi = T(1) \in L^2 X$. Denote by $\lambda(\xi) : L^2 X \to L^1 X$ the operator of (left) multiplication by ξ, which by Cauchy-Schwartz is bounded by $\|\xi\|_2$. But $T : L^2 X \to L^2 X \subset L^1 X$ is also bounded as an operator into $L^1 X$, and $\lambda(\xi)$, T coincide on the $\| \ \|_2$-dense subspace $L^\infty X \subset L^2 X$ (*Exercise!*) Thus, $\lambda(\xi) = T$ on all L^2, forcing $\xi \in L^\infty X$ (*Exercise!*).

This shows that A is a vN algebra (by vN's bicommutant thm).

A key example: the hyperfinite II_1 factor

A vN algebra M is called a **factor** if its center, $\mathcal{Z}(M) := M' \cap M$, is trivial, $\mathcal{Z}(M) = \mathbb{C}1$.

• Let R_0 be the algebraic infinite tensor product $\mathbb{M}_2(\mathbb{C})^{\otimes\infty}$, viewed as inductive limit of the increasing sequence of algebras $\mathbb{M}_{2^n}(\mathbb{C}) = \mathbb{M}_2(\mathbb{C})^{\otimes n}$, via the embeddings $x \mapsto x \otimes 1_{\mathbb{M}_2}$. Endow R_0 with the norm $\|x\| = \|x\|_{\mathbb{M}_{2^n}}$, if $x \in \mathbb{M}_{2^n} \subset R_0$, which is clearly a well defined operator norm, i.e., satisfies $\|x^*x\| = \|x\|_2$. One also endows R_0 with the functional $\tau(x) = Tr(x)/2^n$, for $x \in \mathbb{M}_{2^n}$, which is well defined, positive $(\tau(x^*x) \geq 0, \forall x)$ and satisfies $\tau(xy) = \tau(yx), \forall x, y \in R_0$, $\tau(1) = 1$, i.e., it is a **trace state**. Define the Hilbert space $L^2(R_0)$ as the completion of R_0 with respect to the Hilbert-norm $\|y\|_2 = \tau(y^*y)^{1/2}$, $y \in R_0$, and denote $\hat{R}_0$ the copy of R_0 as a subspace of $L^2(R_0)$.

For each $x \in R_0$ define the operator $\lambda(x)$ on $L^2(R_0)$ by $\lambda(x)(\hat{y}) = \hat{xy}$, $\forall y \in R_0$. Note that $R_0 \ni x \mapsto \lambda(x) \in \mathcal{B}(L^2)$ is a *-algebra morphism with $\|\lambda(x)\| = \|x\|$, $\forall x$. Moreover, $\langle \lambda(x)(\hat{1}), \hat{1} \rangle_{L^2} = \tau(x)$.

One similarly defines $\rho(x)$ to be the operator of right multiplication by x on $L^2(R_0)$, for which we have $[\lambda(y), \rho(x)] = 0$, $\forall x, y \in R_0$.

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One can easily see that the vN algebra $R := \overline{\lambda(R_0)}^{so} = \overline{\lambda(R_0)}^{wo}$ is a factor (*Exercise!*). It can alternatively be defined by $R = \rho(R_0)'$ (*Exercise!*). This is the **hyperfinite II_1 factor**.

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Yet another way to define R is as the completion of R_0 in the topology of convergence in the norm $\|x\|_2 = \tau(x^*x)^{1/2}$ of sequences that are bounded in the operator norm (*Exercise!*). Notice that, in both definitions, τ extends to a trace state on R. Note also that if one denotes by $D_0 \subset R_0$ the natural "diagonal subalgebra" (...), then $(D_0, \tau_{|D_0})$ coincides with the algebra of dyadic step functions on $[0, 1]$ with the Lebesgue integral. So its closure in R in the above topology, $(D, \tau_{|D})$, is just $(L^\infty([0, 1]), \int \mathrm{d}\mu)$.

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Note that (R_0, τ) (and thus R) is completely determined by the sequence of partial isometries $v_1 = e^1_{12}$, $v_n = (\Pi_{i=1}^{n-1} e^i_{22}) e^n_{12}$, $n \geq 2$, with $p_n = v_n v_n^*$ satisfying $\tau(p_n) = 2^{-n}$ and $p_n \sim 1 - \sum_{i=1}^{n} p_i$ (*Exercise!*)

Finite factors: some equivalent characterizations

Theorem A

Let M be a vN factor. The following are equivalent:

1° M is a **finite** vN algebra, i.e., if $p \in \mathcal{P}(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).

2° M has a **trace state** τ (i.e., a functional $\tau : M \to \mathbb{C}$ that's positive, $\tau(x^*x) \geq 0$, with $\tau(1) = 1$, and is tracial, $\tau(xy) = \tau(yx), \forall x, y \in M$).

3° M has a trace state τ that's **completely additive**, i.e., $\tau(\Sigma_i p_i) = \Sigma_i \tau(p_i)$, $\forall \{p_i\}_i \subset \mathcal{P}(M)$ mutually orthogonal projections.

4° M has a trace state τ that's **normal**, i.e., $\tau(\sup_i x_i) = \sup_i \tau(x_i)$, $\forall \{x_i\}_i \subset (M_+)_1$ increasing net.

Thus, a vN factor is finite iff it is tracial. Moreover, such a factor has a unique trace state τ, which is automatically normal and faithful, and satisfies $\overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap \mathbb{C}1 = \{\tau(x)1\}$, $\forall x \in M$.

Some preliminary lemmas

Lemma 1

If a vN factor M has a minimal projections, then $M = \mathcal{B}(\ell^2 I)$, for some I. Moreover, if $M = \mathcal{B}(\ell^2 I)$, then the following are eq.:

1° M has a trace.

2° $|I| < \infty$.

3° M is finite, i.e. $u \in M$, $u^* u = 1 \Rightarrow uu^* = 1$

Proof: Exercise.

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Proof: Exercise.

Lemma 2

If M is finite then:
(a) $p, q \in \mathcal{P}(M)$, $p \sim q \Rightarrow 1 - p \sim 1 - q$.
(b) pMp is finite $\forall p \in \mathcal{P}(M)$, i.e., $q \in \mathcal{P}(M)$, $q \leq p$, $q \sim p$, then $q = p$.

Proof: Use the comparison theorem (Exercise).

Lemma 3

If M vN factor with no atoms and $p \in \mathcal{P}(M)$ is so that $\dim(pMp) = \infty$, then $\exists P_0, P_1 \in \mathcal{P}(M)$, $P_0 \sim P_1$, $P_0 + P_1 = p$.

Proof: Consider the family $\mathcal{F} = \{(p_i^0, p_i^1)_i \mid$ with p_i^0, p_j^1 all mutually orthogonal $\leq p$ such that $p_i^0 \sim p_i^1$, $\forall i\}$, with its natural order. Clearly inductively ordered. If $(p_i^0, p_i^1)_{i \in I}$ is a maximal element, then $P_0 = \sum_i p_i^0, P_1 = \sum_i p_i^1$ will do (for if not, then the comparison Thm. gives a contradiction).

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Lemma 4

If M is a factor with no minimal projections, then $\exists \{p_n\}_n \subset \mathcal{P}(M)$ mutually orthogonal such that $p_n \sim 1 - \sum_{i=1}^{n} p_i$, $\forall n$.

Proof: Apply **L3** recursively.

Lemma 5

If M is a finite factor and $\{p_n\}_n \subset \mathcal{P}(M)$ are as in **L4**, then:

(a) If $p \prec p_n$, $\forall n$, then $p = 0$. Equivalently, if $p \neq 0$, then $\exists n$ such that $p_n \prec p$. Moreover, if n is the first integer such that $p_n \prec p$ and $p'_n \leq p$, $p'_n \sim p_n$, then $p - p'_n \prec p_n$.

(b) If $\{q_n\}_n \subset \mathcal{P}(M)$ increasing and $q_n \leq q \in \mathcal{P}(M)$ and $q - q_n \prec p_n$, $\forall n$, then $q_n \nearrow q$ (with so-convergence).

(c) $\sum_n p_n = 1$.

Proof: If $p \simeq p'_n \leq p_n$, $\forall n$, then $P = \sum_n p'_n$ and $P_0 = \sum_k p'_{2k+1}$ satisfy $P_0 < P$ and $P_0 \sim P$, contradicting the finiteness of M. Rest is *Exercise!*

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Lemma 6

Let M be a finite factor without atoms. If $p \in \mathcal{P}(M)$, $\neq 0$, then $\exists$ a unique infinite sequence $1 \leq n_1 < n_2 < \ldots$ such that p decomposes as $p = \sum_{k \geq 1} p'_{n_k}$, for some $\{p'_{n_k}\}_k \subset \mathcal{P}(M)$ with $p'_{n_k} \sim p_{n_k}$, $\forall k$.

Proof: Apply Part (a) of **L5** recursively (*Exercise!*).

If M is a finite factor without atoms, then we let $\dim : \mathcal{P}(M) \to [0,1]$ be defined by $\dim(p) = 0$ if $p = 0$ and $\dim(p) = \sum_{k=1}^{\infty} 2^{-n_k}$, if $p \neq 0$, where $n_1 < n_2 < \ldots$, are given by **L4**.

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Lemma 7

$\dim$ satisfies the conditions:

(a) $\dim(p_n) = 2^{-n}$

(b) If $p, q \in \mathcal{P}(M)$ then $p \sim q$ iff $\dim(p) \leq text\dim(q)$

(c) $\dim$ is completely additive: if $q_i \in \mathcal{P}(M)$ are mutually orthogonal, then $\dim(\Sigma_i q_i) = \Sigma_i \dim(q_i)$.

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Proof: *Exercise!*.

Lemma 8 (Radon-Nykodim trick)

Let $\varphi, \psi : \mathcal{P}(M) \to [0,1]$ be completely additive functions, $\varphi \neq 0$, and $\varepsilon > 0$. There exists $p \in \mathcal{P}(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$, and $\theta \geq 0$, such that $\theta\varphi(q) \leq \psi(q) \leq (1+\varepsilon)\theta\varphi(q)$, $\forall q \in \mathcal{P}(pMp)$.

Proof: Denote $\mathcal{F} = \{p \mid \exists n \text{ with } p \sim p_n\}$. Note first we may assume φ faithful: take a maximal family of mutually orthogonal non-zero projections $\{e_i\}_i$ with $\varphi(e_i) = 0$, $\forall i$, then let $f = 1 - \sum_i e_i \neq 0$ (because $\varphi(1) \neq 0$); it follows that φ is faithful on fMf, and by replacing with some $f_0 \leq f$ in $\mathcal{F}$, we may also assume $f \in \mathcal{F}$. Thus, proving the lemma for M is equivalent to proving it for fMf, which amounts to assuming φ faithful.

If $\psi = 0$, then take $\theta = 0$. If $\psi \neq 0$, then by replacing φ by $\varphi(1)^{-1}\varphi$ and ψ by $\psi(1)^{-1}\psi$, we may assume $\varphi(1) = \psi(1) = 1$. Let us show this implies:

(1) $\exists g \in \mathcal{F}$, s.t. $\forall g_0 \in \mathcal{F}$, $g_0 \leq g$, we have $\varphi(g_0) \leq \psi(g_0)$. For if not then

(2) $\forall g \in \mathcal{F}$, $\exists g_0 \in \mathcal{F}$, $g_0 \leq g$ s.t. $\varphi(g_0) > \psi(g_0)$.

Take a maximal family of mut. orth. projections $\{g_i\}_i \subset \mathcal{F}$, with $\varphi(g_i) > \psi(g_i)$, $\forall i$. If $1 - \sum_i g_i \neq 0$, then take $g \in \mathcal{F}$, $g \leq 1 - \sum_i g_i$ (cf. **L5**) and apply (2) to get $g_0 \leq g$, $g_0 \in \mathcal{F}$ with $\varphi(g_0) > \psi(g_0)$, contradicting the maximality. Thus,

$$1 = \varphi(\sum_i g_i) = \sum_i \varphi(g_i) > \sum_i \psi(g_i) = \psi(\sum_i g_i) = \psi(1) = 1,$$

a contradiction. Thus, (1) holds true.

Define $\theta = \sup\{\theta' \mid \theta'\varphi(g_0) \leq \psi(g_0), \forall g_0 \leq g, g_0 \in \mathcal{F}\}$.

Clearly $1 \leq \theta < \infty$ and $\theta\varphi(g_0) \leq \psi(g_0), \forall g_0 \leq g, g_0 \in \mathcal{F}$. Moreover, by def. of θ, there exists $g_0 \in \mathcal{F}$, $g_0 \leq g$, s.t., $\theta\varphi(g_0) > (1+\varepsilon)^{-1}\psi(g_0)$. We now repeat the argument for ψ and $\theta(1+\varepsilon)\varphi$ on $g_0 M g_0$, to prove that

(3) $\exists g' \in \mathcal{F}$, $g' \leq g_0$, such that for all $g_0' \in \mathcal{F}$, $g_0' \leq g_0$, we have $\psi(g_0') \leq \theta(1+\varepsilon)\varphi(g_0')$.

Indeed, for if not, then

(4) $\forall g' \in \mathcal{F}$, $g' \leq g_0$, $\exists g_0' \leq g'$ in $\mathcal{F}$ s.t. $\psi(g_0') > \theta(1+\varepsilon)\varphi(g_0')$.

But then we take a maximal family of mutually orthogonal $g_i' \leq g_0$ in $\mathcal{F}$, s.t. $\psi(g_i') \geq \theta(1+\varepsilon)\varphi(g_i')$, and using **L5** and (4) above we get $\sum_i g_i' = g_0$. This implies that $\psi(g_0) \geq \theta(1+\varepsilon)\varphi(g_0) > \psi(g_0)$, a contradiction. Thus, (3) above holds true for some $g' \leq g_0$ in $\mathcal{F}$. Taking $p = g'$, we get that any $q \in \mathcal{F}$ under p satisfies both $\theta\varphi(q) \leq \psi(q)$ and $\psi(q) \leq \theta(1+\varepsilon)\varphi(q)$. By complete additivity of φ, ψ and **L6**, we are done.

We now apply **L8** to $\psi = \dim$ and φ a vector state on $M \subset \mathcal{B}(\mathcal{H})$, to get:

Lemma 9

$\forall \varepsilon > 0$, $\exists p \in \mathcal{P}(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$, and a vector (thus normal) state φ_0 on pMp such that, $\forall q \in \mathcal{P}(pMp)$, we have $(1+\varepsilon)^{-1}\varphi_0(q) \leq \dim(q) \leq (1+\varepsilon)\varphi_0(q)$.

Proof: trivial by **L8**

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Proof: trivial by **L8**

Lemma 10

With p, φ_0 as in **L9**, let $v_1 = p, v_2, ..., v_{2^n} \in M$ such that $v_i v_i^* = p$, $\sum_i v_i^* v_i = 1$. Let $\varphi(x) := \sum_{i=1}^{2^n} \varphi_0(v_i x v_i^*)$, $x \in M$. Then φ is a normal state on M satisfying $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$, $\forall x \in M$.

Proof: Note first that $\varphi_0(x^*x) \leq (1+\varepsilon)\varphi_0(xx^*)$, $\forall x \in pMp$ (Hint: do it first for x partial isometry, then for x with x^*x having finite spectrum). To deduce the inequality for φ itself, note that $\sum_j v_i^* v_i = 1$ implies that for any $x \in M$ we have

$$\varphi(x^*x) = \sum_i \varphi_0(v_i x^* (\sum_j v_j^* v_j) x v_i^*) = \sum_{i,j} \varphi_0((v_i x^* v_j^*)(v_j x v_i))$$

$$\leq (1+\varepsilon)\sum_{i,j}\varphi_0((v_j x v_i)(v_i x^* v_j^*)) = ... = (1+\varepsilon)\varphi(xx^*).$$

$$\leq (1+\varepsilon)\sum_{i,j}\varphi_0((v_j x v_i)(v_i x^* v_j^*)) = \ldots = (1+\varepsilon)\varphi(xx^*).$$

Lemma 11

If φ is a state on M that satisfies $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$, $\forall x \in M$, then $(1+\varepsilon)^{-1}\varphi(p) \leq \dim(p) \leq (1+\varepsilon)\varphi(p)$, $\forall p \in \mathcal{P}(M)$.

Proof: By complete additivity, it is sufficient to prove it for $p \in \mathcal{F}$, for which we have for $v_1, \ldots, v_{2^n}$ as in **L10** $\varphi(p) = \varphi(v_j^* v_j) \leq (1+\varepsilon)\varphi(v_j v_j^*)$, $\forall j$, so that

$$2^n\varphi(p) \leq (1+\varepsilon)\sum_j \varphi(v_j v_j^*) = (1+\varepsilon)2^n\dim(p)$$

and similarly $2^n\dim(p) = 1 \leq (1+\varepsilon)2^n\varphi(p)$.

Proof of Thm A

Define $\tau : M \to \mathbb{C}$ as follows. First, if $x \in (M_+)_1$ then we let $\tau(x) = \tau(\Sigma_n 2^{-n} e_n) = \Sigma_n 2^{-n} \mathrm{dim}(e_n)$, where $x = \Sigma_n 2^{-n} e_n$ is the (unique) dyadic decomposition of $0 \leq x \leq 1$. Extend τ to M_+ by homothety, then further extend to M_h by $\tau(x) = \tau(x_+) - \tau(x_-)$, where for $x = x^* \in M_h$, $x = x_+ - x_-$ is the dec. of x into its positive and negative parts. Finally, extend τ to all M by $\tau(x) = \tau(\mathrm{Re} x) + i\tau(\mathrm{Im} x)$.

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By **L11**, $\forall \varepsilon > 0$, $\exists \varphi$ normal state on M such that $|\tau(p) - \varphi(p)| \leq \varepsilon$, $\forall p \in \mathcal{P}(M)$. By the way τ was defined and the linearity of φ, this implies $|\tau(x) - \varphi(x)| \leq \varepsilon$, $\forall x \in (M_+)_1$, and thus $|\tau(x) - \varphi(x)| \leq 4\varepsilon$, $\forall x \in (M)_1$. This implies $|\tau(x+y) - \tau(x) - \tau(y)| \leq 8\varepsilon$, $\forall x, y \in (M)_1$. Since $\varepsilon > 0$ was arbitrary, this shows that τ is a linear state on M.

By definition of τ, we also have $\tau(uxu^*) = \tau(x)$, $\forall x \in M$, $u \in \mathcal{U}(M)$, so τ is a trace state. From the above argument, it also follows that τ is a norm limit of normal states, which implies τ is normal as well.

Finite vN algebras

Theorem A'

Let M be a vN algebra that's countably decomposable (i.e., any family of mutually orthogonal projections is countable). The following are equivalent:

1° M is a **finite** vN algebra, i.e., if $p \in \mathcal{P}(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).

2° M has a faithful normal (equivalently completely additive) trace state τ.

Moreover, if M is finite, then there exists a unique normal faithful **central trace**, i.e., a linear positive map $ctr : M \to \mathcal{Z}(M)$ that satisfies $ctr(1) = 1$, $ctr(z_1 x z_2) = z_1 ctr(x) z_2$, $ctr(xy) = ctr(yx)$, $x, y \in M$, $z_i \in \mathcal{Z}$.

Any trace τ on M is of the form $\tau = \varphi_0 \circ ctr$, for some state φ_0 on $\mathcal{Z}$.

Also, $\overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap \mathcal{Z} = \{ctr(x)\}$, $\forall x \in M$.

Proof of $2^\circ \Rightarrow 1^\circ$: If τ is a faithful trace on M and $u^*u = 1$ for some $u \in M$, then $\tau(1 - uu^*) = 1 - \tau(uu^*) = 1 - \tau(u^*u) = 0$, thus $uu^* = 1$.

L^p-spaces from tracial algebras

• A *-operator algebra $M_0 \subset \mathcal{B}(\mathcal{H})$ that's closed in operator norm is called a **C*-algebra**. Can be described abstractly as a Banach algebra M_0 with a *-operation and the norm satisfying the axiom $\|x^*x\| = \|x\|^2$, $\forall x \in M_0$.

• If M_0 is a unital C*-algebra and τ is a faithful trace state on M_0, then for each $p \geq 1$, $\|x\|_p = \tau(|x|^p)^{1/p}$, $x \in M_0$, is a norm on M_0. We denote $L^p M_0$ the completion of $(M_0, \| \ \|_p)$. One has $\|x\|_p \leq \|x\|_q$, $\forall 1 \leq p \leq q \leq \infty$, thus $L^p M_0 \supset L^q M_0$.

Note that $L^2 M_0$ is a Hilbert space with scalar product $\langle x, y \rangle_\tau = \tau(y^* x)$. The map $M_0 \ni x \mapsto \lambda(x) \in \mathcal{B}(L^2)$ defined by $\lambda(x)(\hat{y}) = \hat{xy}$ is a *-algebra isometric representation of M_0 into $\mathcal{B}(L^2)$ with $\tau(x) = \langle \lambda(x)\hat{1}, \hat{1} \rangle_\varphi$. Similarly, $\rho(x)(\hat{y}) = \hat{yx}$ defines an isometric representation of $(M_0)^{op}$ on $L^2 M_0$. One has $[\lambda(x_1), \rho(x_2)] = 0$, $\forall x_i \in M_0$.

More generally, $\|x\| = \sup\{\|xy\|_p \mid \|y\|_p \leq 1\}$. Also, $\|y\|_1 = \sup\{|\tau(xy)| \mid x \in (M)_1\}$. In particular, τ extends to $L^1 M_0$.

Exercise!

Abstract characterizations of finite vN algebras

Theorem B

Let (M, τ) be a unital C*-algebra with a faithful trace state. The following are equivalent:

1° The image of $\lambda : M \to \mathcal{B}(L^2(M, \tau))$ is a vN algebra (i.e., is wo-closed).

2° $\lambda(M) = \rho(M)'$ (equivalently, $\rho(M) = \lambda(M)'$).

3° $(M)_1$ is complete in the norm $\|x\|_{2,\tau}$.

4° As Banach spaces, we have $M = (L^1(M, \tau))^*$, where the duality is given by $(M, L^1 M) \ni (x, Y) \mapsto \tau(xY)$.

Proof: One uses similar arguments as when we represented $L^\infty([0, 1])$ as a vN algebra and as in the construction of R (*Exercise!*).

II_1 factors: definition and basic properties

Definition

An ∞-dim finite factor M (so $M \neq \mathbb{M}_n(\mathbb{C}), \forall n$) is called a **$\mathrm{II}_1$ factor**.

• R is a factor, has a trace, and is ∞-dimensional, so it is a II_1 factor.

• The construction of the trace on a non-atomic factor satisfying the finiteness axiom in Thm A is based on splitting recursively 1 dyadically into equivalent projections, with the underlying partial isometries generating the hyperfinite II_1 factor R. Thus, R embeds into any II_1 factor.

• If $A \subset M$ is a maximal abelian *-subalgebra (MASA) in a II_1 factor M, then A is diffuse (i.e., it has no atoms).

• The (unique) trace τ on a II_1 factor M is a dimension function on $\mathcal{P}(M)$, i.e., $\tau(p) = \tau(q)$ iff $p \sim q$, with $\tau(\mathcal{P}(M)) = [0,1]$ (*continuous dimension*).

• If $B \subset M$ is vN alg, the orth. projection $e_B : L^2M \to \overline{\hat{B}}^{\| \ \|_2} = L^2B$ is positive on $\hat{M} = M$, so it takes M onto B, implementing a cond. expect. $E_B : M \to B$ that satisfies $\tau \circ E_B = \tau$. It is unique with this property.

Finite amplifications of II_1 factors

• If $n \geq 2$ then $\mathbb{M}_n(M) = \mathbb{M}_n(\mathbb{C}) \otimes M$ is a II_1 factor with trace state $\tau((x_{ij})_{i,j}) = \sum_i \tau(x_{ii})/n$, $\forall (x_{ij})_{i,j} \in \mathbb{M}_n(M)$.

• If $0 \neq p \in \mathcal{P}(M)$, then pMp is a II_1 factor with trace state $\tau(p)^{-1}\tau$, whose isomorphism class only depends on $\tau(p)$.

• Given any $t > 0$, let $n \geq t$ and $p \in \mathcal{P}(\mathbb{M}_n(M))$ be so that $\tau(p) = t/n$. We denote the isomorphism class of $p\mathbb{M}_n(M)p$ by M^t and call it the **amplification of** M **by** t (*Exercise*: show that this doesn't depend on the choice of n and p.)

• We have $(M^s)^t = M^{st}$, $\forall s, t > 0$ (*Exercise*). One denotes $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$. Clearly a multiplicative subgroup of $\mathbb{R}_+$, called the **fundamental group of** M. It is an isom. invariant of M.

∞-amplifications, II_∞ factors and semifinite vN alg

If $M_i \subset \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, are vN algebras, then $M_1 \overline{\otimes} M_2 \subset \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2)$ denotes the vN alg generated by alg tens product $M_1 \otimes M_2 \subset \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2)$.

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• If (M, τ) is tracial (finite) vN algebra, then $\mathcal{M} = M \overline{\otimes} \mathcal{B}(\ell^2 S) \subset \mathcal{B}(L^2 M \overline{\otimes} \ell^2 S)$ is a vN algebra with the property $\exists p_i \nearrow 1$ projections such that $p_i \mathcal{M} p_i$ is finite, $\forall i$. Such a vN algebra $\mathcal{M}$ is called **semifinite**. It has a normal faithful semifinite trace $\tau \otimes Tr$.

• If M is a type II_1 factor and $|S| = \infty$, then $\mathcal{M} = M \overline{\otimes} \mathcal{B}(\ell^2 S)$ is called a **II_∞ factor**. It can be viewed as the $|S|$-amplification of M.

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- *An important example*: If $B \subset M$ is a vN subalgebra and $e_B : L^2 M \to L^2 B$ as before, then: $e_B x e_B = E_B(x) e_B$, $\forall x \in \lambda(M) = M$, the vN algebra $\langle M, e_B \rangle$ generated by M and e_B in $\mathcal{B}(L^2 M)$ is equal to the wo-closure of the *-algebra $\mathrm{sp}\{x e_B y \mid x, y \in M\}$, and also equal to $\rho(B)' \cap \mathcal{B}(L^2 M)$. It has a normal semifinite faithful trace uniquely determined by $Tr(x e_B y) = \tau(xy)$. $(\langle M, e_B \rangle, Tr)$ is called the **basic construction** algebra for $B \subset M$.

vN representations and Hilbert M-modules

• If M is a vN algebra, then a *-rep $\pi : M \to \mathcal{B}(\mathcal{H})$ is a vN rep (i.e., $\pi(M)$ wo-closed) iff π is completely additive. We'll call such representations **normal representations** and $\mathcal{H}$ a (left) **Hilbert M-module**. Two Hilbert M-modules $\mathcal{H}, \mathcal{K}$ are equivalent if there exists a unitary $U : \mathcal{H} \simeq \mathcal{K}$ that intertwines the two M-module structures (reps).

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- If $M \subset \mathcal{B}(\mathcal{H})$ is a vN algebra and $p' \in M'$, then $M \ni x \mapsto xp' \in \mathcal{B}(p'(\mathcal{H}))$ is a vN representation of M. Also, if $\pi_i : M \to \mathcal{B}(\mathcal{H}_i)$ are vN representations of M, then $x \mapsto \oplus_i \pi_i(x) \in \mathcal{B}(\oplus_i \mathcal{H}_i)$ is a vN rep. of M.

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• If (M, τ) is a tracial vN algebra, then a *-rep $\pi : M \to \mathcal{B}(\mathcal{H})$ is a vN rep iff π is continuous from $(M)_1$ with the $\| \ \|_2$-topology to $\mathcal{B}(\mathcal{H})$ with the so-topology.

Classification of Hilbert modules of a II_1 factor

- If M is tracial vN algebra then any cyclic Hilbert M-module is of the form $\rho(p)(L^2 M) = L^2(Mp)$. Any Hilbert M-module $\mathcal{H}$ is of the form $\oplus_i L^2(Mp_i)$, for some projections $\{p_i\}_i \subset M$.

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• If $t = \dim({}_M\mathcal{H}) \geq 1$ and $p \in M^t$ has trace $1/t$ then ${}_M\mathcal{H} \simeq_M L^2(pM^t)$.

• If $t = \dim({}_M\mathcal{H}) < \infty$ then $\dim({}_{M'}\mathcal{H}) = 1/t$. Also, M' is naturally isomorphic to $(M^t)^{op}$, equivalently $\mathcal{H}$ has a natural Hilbert right M^t-module structure.

II_1 factors from groups and group actions

- Let Γ be a discrete group, $\mathbb{C}\Gamma$ its (complex) group algebra and $\mathbb{C}\Gamma \ni x \mapsto \lambda(x) \in \mathcal{B}(\ell^2\Gamma)$ the left regular representation. The wo-closure of $\lambda(\mathbb{C}\Gamma)$ in $\mathcal{B}(\mathcal{H})$ is called the **group von Neumann algebra** of Γ, denoted $L(\Gamma)$, or just $L\Gamma$. Denoting $u_g = \lambda(g)$ (the canonical unitaries), the algebra $L\Gamma$ can be identified with the set of ℓ^2-summable formal series $x = \sum_g c_g u_g$ with the property that $x \cdot \xi \in \ell^2$, $\forall \xi \in \ell^2\Gamma$. It has a normal faithful trace given by $\tau(\sum_g c_g u_g) = c_e$, implemented by the vector ξ_e, and is thus tracial (finite).

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- Similarly, if $\Gamma \curvearrowright^\sigma X$ is a pmp action, one associates to it the *group measure space vN algebra* $L^\infty(X) \rtimes \Gamma \subset \mathcal{B}(L^2(X) \otimes \ell^2\Gamma)$, as weak closure of the algebraic crossed product of $L^\infty(X)$ by Γ. Can be identified with the algebra of ℓ^2-summable formal series $\sum_g a_g u_g$, with $a_g \in L^\infty(X)$, with multiplication rule $a_g u_g a_h u_h = a_g \sigma_g(a_h) u_{gh}$. It is a II_1 factor if $\Gamma \curvearrowright X$ is *free ergodic*, in which case $A = L^\infty(X)$ is maximal abelian in $L^\infty(X) \rtimes \Gamma$ and its normalizer generates $L^\infty(X) \rtimes \Gamma$, i.e. A is a *Cartan subalgebra*.

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- *Ultraproduct* of finite factors $\Pi_\omega M_n$, notably the case $\Pi_\omega \mathbb{M}_{n\times n}(\mathbb{C})$ and the ultrapower R^ω of R (i.e., the case $M_n = R$, $\forall n$)

R is the unique AFD II_1 factor

• *Exercise*: Show that if (A, τ) is a diffuse (i.e., without atoms) countably generated abelian vN algebra, with faithful completely additive state τ, then $(A, \tau) \simeq (L^\infty([0,1], \mu), \int \, \mathrm{d}\mu)$. Hint: construct an increasing "dyadic" partitions by projections in A (of trace 2^{-n}) that "exhaust" it.

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Definition of AFD vN algebras

A tracial vN algebra (M, τ) is **approximately finite dimensional** (**AFD**) if $\forall F \subset M$ finite, $\forall \varepsilon > 0$, $\exists B \subset M$ fin dim s.t. $\|x - E_B(x)\|_2 \leq \varepsilon$, $\forall x \in F$.

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Proof: Exercise (Like in the abelian case above, construct increasing "non-commutative dyadic" approximations $\mathbb{M}_{2^{k_n}}(\mathbb{C}) \nearrow M$).

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A tracial vN algebra (M, τ) is **approximately finite dimensional** (**AFD**) if $\forall F \subset M$ finite, $\forall \varepsilon > 0$, $\exists B \subset M$ fin dim s.t. $\|x - E_B(x)\|_2 \leq \varepsilon$, $\forall x \in F$.

Theorem: Murray-von Neumann 1943

If M is an AFD II_1 factor that's countably generated ($\Leftrightarrow \| \; \|_2$-separable) then $M \simeq R$.

Proof: Exercise (Like in the abelian case above, construct increasing "non-commutative dyadic" approximations $\mathbb{M}_{2^{k_n}}(\mathbb{C}) \nearrow M$).

Corollary

$R^t \simeq R$, $\forall t > 0$, i.e., $\mathcal{F}(R) = \mathbb{R}_+$.

Amenability for groups and vN algebras

Definitions

- A group Γ is **amenable** if it has an **invariant mean**, i.e., a state φ on $\ell^\infty(\Gamma)$ such that $\varphi({}_g f) = \varphi(f)$, $\forall f \in \ell^\infty\Gamma$, $g \in \Gamma$, where ${}_g f(h) = f(g^{-1}h)$, $\forall h$.

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• A tracial vN algebra (M, τ) is **amenable** if it has a **hypertrace** (invariant mean), i.e., a state φ on $\mathcal{B}(L^2M)$ such that $\varphi(xT) = \varphi(Tx)$, $\forall x \in M$, $T \in \mathcal{B}$, and $\varphi_{|M} = \tau$ (Note: the 2nd condition is redundant if M is a II_1 factor).

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- $L\Gamma$ is amenable iff Γ is amenable

Proof. If φ is a state on $\mathcal{B}(\ell^2\Gamma)$ with $L\Gamma$ in its centralizer (a hypertrace on $L\Gamma$), then and $\mathcal{D} = \ell^\infty\Gamma$ is represented in $\mathcal{B}(\ell^2\Gamma)$ as diagonal operators, then $\varphi_{|D}$ is a state on $\mathcal{D}$ that satisfies $\varphi(u_g f u_g^*) = \varphi(f)$, $\forall f \in \mathcal{D} = \ell^\infty\Gamma$, where $u_g = \lambda(g)$. But $u_g f u_g^* =_g f$ (*Exercise*), so $\varphi_{|\mathcal{D}}$ is an invariant mean.

Conversely, if Γ is amenable and $\varphi \in (\ell^\infty\Gamma)^*$ is an invariant mean, then $\psi = \int \tau(u_g \cdot u_g^*)\mathrm{d}\varphi \in \mathcal{B}^*$ is a state on $\mathcal{B}$ which has $\{u_h\}_h$ in its centralizer and equals τ when restricted to $L\Gamma$. For any $x \in (L\Gamma)_1$ and $\varepsilon > 0$, let $x_0 \in \mathbb{C}\Gamma$ be so that $\|x - x_0\|_2 \leq \varepsilon$, $\|x_0\| \leq 1$ (Kaplansky). By Cauchy-Schwartz, if $T \in (\mathcal{B})_1$, then we have: $|\psi((x - x_0)T)| \leq \varepsilon$, $|\psi(T(x - x_0))| \leq \varepsilon$. Since $\psi(x_0 T) = \psi(T x_0)$ and ε arbitrary, this shows that $\psi(Tx) = \psi(xT)$.

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- Let (M, τ) be tracial vN algebra. The following are equiv:

1° M is amenable.

2° $M \subset \mathcal{B}(\mathcal{H})$ has a hypertrace for any normal rep. of M.

3° There exists a normal rep $M \subset \mathcal{B}(\mathcal{H})$ with a hypertrace.

Corollary

1° (M, τ) amenable and $B \subset M$ a vN subablegbra, then (B, τ) amenable.

2° Assume (M, τ) is tracial vN algebra, $B \subset M$ amenable vN subalgebra and $\pi : \Gamma \to \mathcal{U}(M)$ a representation of an amenable group Γ such that $\pi(g)(B) = B$, $\forall g$, and $B \vee \pi(\Gamma) = M$. Then (M, τ) is amenable.

Concrete examples of amenable II_1 factors

• We have already shown that if Γ amenable then $L\Gamma$ amenable. Some concrete examples of amenable group are: finite groups; more generally locally finite groups (e.g., S_∞); $\mathbb{Z}^n$, $n \geq 1$, in fact all abelian groups; $H \wr \Gamma_0$ with H, Γ_0 amenable; more generally if $1 \to H \to \Gamma \to \Gamma_0 \to 1$ is exact, then Γ amenable iff H, Γ_0 are amenable.

• If in addition Γ is ICC, then $L\Gamma$ is an amenable II_1 factor. Of the above amenable groups, S_∞ is ICC Also, $H \wr \Gamma_0$ are ICC whenever $|H| \geq 2$ and $|\Gamma_0| = \infty$, so groups like $(\mathbb{Z}/m\mathbb{Z}) \wr \mathbb{Z}^n$ with $m \geq 2, n \geq 1$ are all ICC amenable.

Concrete examples of amenable II_1 factors (continuation)

• Let $\mathcal{U}_0 \subset \mathcal{U}(R)$ be the subgroup of all unitaries in $R_0 = \mathbb{M}_2(\mathbb{C})^{\otimes\infty}$ that have only ± 1 and 0 as entries. Then $\mathcal{U}_0$ is locally finite so it is amenable and it clearly generates R.

Thus R is an amenable II_1 factor, and any vN subalgebra $B \subset M$ is amenable, in particular any II_1 subfactor of R is an amenable II_1 factor.

• By last Corollary, any abelian vN algebra is amenable (because it is generated by an abelian group of unitaries). Also, any group measure space vN algebra $L^\infty X \rtimes \Gamma$ with Γ an amenable group (e.g., like in the above examples), is an amenable vN algebra. Thus, if $\Gamma \curvearrowright X$ is free ergodic with Γ amenable then $L^\infty X \rtimes \Gamma$ is an amenable II_1 factor.

Følner condition for groups

Føloner's 1955 characterization of amenability for groups

A group Γ is amenable iff it satisfies the condition: $\forall F \subset \Gamma$ finite, $\varepsilon > 0$, $\exists K \subset \Gamma$ finite such that $|FK \setminus K| < \varepsilon |K|$.

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Proof: $\Leftarrow$ If $F_i \nearrow \Gamma$, $K_i \subset \Gamma$ are finite s.t. $|F_i K_i \setminus K_i| \leq |F_i|^{-1}$ then $f \mapsto \mathrm{Lim}_i |K_i|^{-1} \sum_{g \in K_i} f(g)$ is clearly an invariant mean for Γ (*Exercise!*).

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$\Rightarrow$ *Step 1: Day's trick*. $\exists \psi \in (\ell^1\Gamma_+)_1$ s.t. $\|\psi -_g \psi\|_1 \leq \varepsilon/|F|$, $\forall g \in F$.

Consider the Banach space $(\ell^1\Gamma)^{|F|}$ and its convex subspace $\mathcal{C} = \{(\psi -_g \psi)_{g \in F} \mid \psi \in (\ell^1_+)_1\}$. It is sufficient to show that 0 is in norm closure of $\mathcal{C}$. If $0 \notin \overline{\mathcal{C}}$, then $\exists f^g \in \ell^\infty\Gamma$ such that

$$\mathrm{Re} \sum_{g \in F} \langle \psi -_g \psi, f^g \rangle \geq c > 0, \forall \psi \in (\ell^1_+)_1$$

But the set of ψ as above is $\sigma((\ell^\infty)^*, \ell^\infty)$ dense in the state space of ℓ^∞, so the above holds true for all states on ℓ^∞, in particular for the invariant mean φ, which gives $0 > c$, a contradiction.

Step 2: Namioka's trick. If $b \in (\ell^1\Gamma)_+)_1$ satisfies $\sum_{g\in\Gamma} \|_g b - b\|_1 < \varepsilon$, then $\exists t > 0$ such that $e = e_t(b)$ (spectral projection of b, or "level set", corresponding to $[t, \infty)$) satisfies $\sum_{g\in\Gamma} \|_g e - e\|_1 < \varepsilon \|e\|_1$.

Note first that $\forall y_1, y_2 \in \mathbb{R}_+$ we have $\int_0^\infty |e_t(y_1) - e_t(y_2)| \mathrm{d}t = |y_1 - y_2|$. Thus, if $b_1, b_2 \in \ell^1\Gamma_+$, then $\int_0^\infty |e_t(b_1) - e_t(b_2)| \mathrm{d}t = |b_1 - b_2|$ (pointwise, as functions). Hence, $\int_0^\infty \|e_t(b_1) - e_t(b_2)\|_1 \mathrm{d}t = \|b_1 - b_2\|_1$. Applying this to $b_1 =_g b$, $b_2 = b$, we get:

$$\sum_{g\in F} \int_0^\infty \|_g e_t(b) - e_t(b)\|_1 \mathrm{d}t = \sum_{g\in F} \|_g b - b\|_1 < \varepsilon \|b\|_1 = \varepsilon \int_0^\infty \|e_t(b)\|_1 \mathrm{d}t$$

Thus, there must exist $t > 0$ such that $e = e_t(b)$ satisfies $\sum_{g\in F} \|_g e - e\|_1 < \varepsilon \|e\|_1$.

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Thus, there must exist $t > 0$ such that $e = e_t(b)$ satisfies $\sum_{g\in F} \|_g e - e\|_1 < \varepsilon\|e\|_1$.

Step 3: End of proof of Følner's Thm. But then the set $K \subset \Gamma$ with $\chi_K = e$ is finite and satisfies $|FK \setminus K| \leq \sum_{g\in F} |gK \setminus K| < \varepsilon|K|$.

Følner condition for II_1 factors

Connes' 1976 Følner-type characterization of amenable II_1 factors

Let $M \subset \mathcal{B}(L^2M)$ be a II_1 factor. Then M is amenable iff for any $F \subset \mathcal{U}(M)$ finite and $\varepsilon > 0$, there exists a finite rank projection $e \in \mathcal{B}(L^2M)$ such that $\|ueu^* - e\|_{2,Tr} < \varepsilon \|e\|_{2,Tr}$, $\forall u \in F$, where $\|X\|_{2,Tr} = Tr(X^*X)^{1/2}$ is the Hilbert-Schmidt norm on $\mathcal{B}(L^2M)$.

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$\Rightarrow$ *Step 1: Day-type trick*. $\exists b \in (L^1(\mathcal{B})_+)_1$ such that $\|ubu^* - b\|_{1,Tr} \leq \varepsilon$, $\forall u \in F$, where $\mathcal{B} = \mathcal{B}(L^2M)$, $\|X\|_{1,Tr} = Tr(|X|)$.

Proof of this part is same as proof of Step 1 of Følner's condition for amenable groups, using the fact that $L^1(\mathcal{B}, Tr)^* = \mathcal{B}(L^2M) = \mathcal{B}$.

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Switching to $\| \|_{2,Tr}$-estimate. With b as above, one has $\|ub^{1/2}u^* - b^{1/2}\|_{2,Tr} \leq 2\varepsilon^{1/2} = 2\varepsilon^{1/2}\|b^{1/2}\|_{2,Tr}$, $\forall u \in F$. This is due to the Powers-Stømer inequality, showing that if $b_1, b_2 \in L^1(\mathcal{B}, Tr)_+$ then

$$\|b_1^{1/2} - b_2^{1/2}\|_{2,Tr}^2 \leq \|b_1 - b_2\|_{1,Tr} \leq \|b_1^{1/2} - b_2^{1/2}\|_{2,Tr}\|b_1^{1/2} + b_2^{1/2}\|_{2,Tr}.$$

Step 2: "Connes' joint distribution trick" and "Namioka-type trick".

If $a \in L^2(\mathcal{B}, Tr)_+$ satisfies $Tr(a^2) = 1$ and $\sum_{g \in F} \|uau^* - a\|_{2,Tr}^2 < {\varepsilon'}^2$ then $\exists t > 0$ such that $\sum_{g \in F} \|ue_t(a)u^* - e_t(a)\|_{2,Tr}^2 < {\varepsilon'}^2 \|e_t(a)\|_{2,Tr}^2$.

This is because if $a_1, a_2 \in \mathcal{B}(L^2M)_+$ are finite rank positive operators then there exists a (discrete) measure m on $X = \mathbb{R}_+ \times \mathbb{R}_+$ such that for any Borel functs f_1, f_2 on $\mathbb{R}_+$ one has $\int_X f_1(t) f_2(s) \mathrm{d}m(t,s) = Tr(f_1(a_1) f_2(a_2))$. (this is Applying this to $a_1 = a$, $a_2 = uau^*$, one then gets:

$$\sum_{g \in F} \int_0^\infty \|ue_t(a)u^* - e_t(a)\|_{2,Tr}^2 \mathrm{d}t$$

$$= \sum_{g \in F} \|uau^* - a\|_{2,Tr}^2 < {\varepsilon'}^2 \|a\|_{2,Tr}^2 = {\varepsilon'}^2 \int_0^\infty \|e_t(a)\|_{2,Tr}^2 \mathrm{d}t$$

But then there must exist $t > 0$ such that $e = e_t(a)$ satisfies $\sum_{g \in F} \|ueu^* - e\|_{2,Tr}^2 < {\varepsilon'}^2 \|e\|_{2,Tr}^2$

$\Leftarrow$ *Exercise!*

Connes Thm: R is the unique amenable II_1 factor

C's 1976 Fundamental Thm: Any separable amenable II_1 factor is AFD and is thus isomorphic to the hyperfinite factor R.

From C's Følner-type condition to local AFD. Let $1 \in F \subset \mathcal{U}(M)$ finite and $\varepsilon > 0$. By the C's Følner condition, $\exists p = p_{\mathcal{H}_0}$ for some finite dim $\mathcal{H}_0 \subset L^2M$ s.t. $\|upu^* - p\|_{2,Tr} < \varepsilon \|p\|_{2,Tr}$, $\forall u \in F$. By density of M in L^2M, may assume $\mathcal{H}_0 \subset M$. Let $\{\eta_j\}_j$ be an orthonormal basis of $\mathcal{H}_0$, i.e., $\tau(\eta_i^* \eta_j) = \delta_{ij}$, $\sum_j \mathbb{C}\eta_j = \mathcal{H}_0$.

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Local quantization (LQ) lemma

$\forall F' \subset M$ finite, $\delta > 0$, $\exists q \in \mathcal{P}(M)$ s.t. $\|qxq - \tau(x)q\|_2 < \delta\|q\|_2$, $\forall x \in F'$.

We apply the LQ lemma to $F' := \{\eta_i^* u \eta_j \mid u \in F, i, j\}$. Note that, as $\delta \to 0$, the elements $\eta_i q \eta_j^*$ behave like matrix units e_{ij}, i.e., $e_{ij}e_{kl} \approx \delta_{jk}e_{il}$. Thus, the space $\mathcal{H}q\mathcal{H}^* = \Sigma_{i,j}\mathbb{C}\eta_i q\eta_j^*$ behaves as the algebra $B_0 = \Sigma_{i,j}\mathbb{C}e_{ij}$, with $1_{B_0} = \Sigma_j e_{jj} \approx \Sigma_j \eta_j q \eta_j^*$ satisfying $\|usu^* - s\|_2 < \varepsilon\|s\|_2$ and $\|sus - E_{B_0}(sus)\|_2 < \varepsilon\|s\|_2$, $\forall u \in F$.

Since any $y \in M$ is a combination of 4 unitaries in M, we have shown that the amenable II_1 factor M satisfies the following local AFD property:

$\forall F \subset M$ finite, $\varepsilon > 0$, $\exists B_0 \subset M$ non-zero fin dim *-subalgebra such that if $s = 1_{B_0}$ then $\|E_{B_0}(sys) - sys\|_2 \leq \varepsilon \|s\|_2$, $\|[s, y]\|_2 \leq \varepsilon \|s\|_2$, $\forall y \in F$.

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From local AFD to global AFD. One uses a *maximality argument* to get from this local AFD, a "global AFD". Let $\mathcal{F}$ be the set of families of subalgebras $(B_i)_i$ of M, with B_i finite dimensional, $s_i = 1_{B_i}$ mutually orthogonal, such that if $B = \oplus_i B_i \subset M$, $s = 1_B$, then $\|[s, y]\|_2 \leq \varepsilon \|s\|_2$, $\|E_B(sys) - sys\|_2 \leq \varepsilon \|s\|_2$, $\forall y \in F$. Clearly $\mathcal{F}$ with its natural order given by inclusion is inductively ordered. Let $(B_i)_i$ be a maximal family. Denote $p = 1 - 1_B$ and assume $p \neq 0$. Clearly pMp is amenable, so by local AFD $\exists B_0 \subset pMp$ fin dim *-subalgebra s.t. $s_0 = 1_{B_0}$ satisfies $\|[s_0, x]\|_2 \leq \varepsilon \|s_0\|_2$, $\|E_{B_0}(s_0 x s_0) - s_0 x s_0\|_2 \leq \varepsilon \|s_0\|_2$, $\forall x \in pFp$. By Pythagora, one gets that if $B_1 = B \oplus B_0$, $s_1 = 1_{B_1}$ then $\|E_{B_1}(s_1 y s_1) - s_1 y s_1\|_2 \leq \varepsilon \|s_1\|_2$, $\|[s_1, y]\|_2 \leq \varepsilon \|s_1\|_2$, $\forall y \in F$. So $(B_i)_i \cup \{B_1\}$ contradicts the maximality of $(B_i)_i$. Thus, $\sum_i s_i = 1$. But then for a large finite subfamily $(B_i)_{i \in I_0}$, we have that $B = \sum_{i \in I_0} B_i \oplus \mathbb{C}(1 - \Sigma s_i)$ is fin. dim. and satisfies $\|E_B(y) - y\|_2 \leq \varepsilon$, $\forall y \in F$. Thus, M follows AFD.

Some comments

- Connes' proof of "M amenable $\implies M \simeq R$" in Annals of Math 1976, which is different from the above, first shows that any amenable M embeds into R^ω and "splits off R". That original proof became a major source of inspiration in the effort to classify nuclear C^*-algebras (Elliott, Kirchberg, H. Lin, more recently Tikuisis-White-Winter, Schafhouser).

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• **Connes approximate embedding (CAE)** conjecture, stated in his Ann Math 1976 paper, predicts that in fact any (separable) II_1 factor M embeds into R^ω, equivalently into $\Pi_\omega \mathbb{M}_{n\times n}(\mathbb{C})$. For group algebras $M = L(\Gamma)$ this amounts to "simulating" Γ by unitary groups $U(n)$: $\forall F \subset \Gamma$, $m \geq 1$, $\varepsilon > 0$, $\exists n$ and $\{v_g\}_{g\in F} \subset U(n)$ such that for any word w of length $\leq m$ in the free group with generators in F, one has $|tr(w(\{v_g\}_g) - 1| \leq \varepsilon$ if $w(F) = e$ and $|tr(w(\{v_g\}_g))| \leq \varepsilon$ if $w(F) \neq e$.

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• An alternative characterization of R by K. Jung from 2007 shows that all embeddings of M in R^ω are unitary conjugate iff $M \simeq R$. A related open problem asks whether $(M' \cap M^\omega)' \cap M^\omega = M$ implies $M \simeq R$.

Some consequences to C's Fund Thm

• Connes theorem implies that for any countable ICC amenable group Γ we have $L\Gamma \simeq R$. Also, any group measure space II_1 factor $L^\infty X \rtimes \Gamma$ arising from a pmp action of countable amenable group Γ, is isomorphic to R.

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• More generally, if a II_1 factor M arises as a crossed product $B \rtimes \Gamma$ of a separable amenable tracial vN algebra (B, τ) by a countable amenable group Γ, then $M \simeq R$. In particular, if $\Gamma \curvearrowright R$, with Γ amenable and the action outer, then $R \rtimes \Gamma \simeq R$.

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• Since any vN subalgebra of R is amenable, it follows that any II_1 subfactor of R is isomorphic to R. In fact, one can easily deduce:

Classification of all vN subalgebras of R

If $B \subset R$ is a vN subalgebra, then $B \simeq \oplus_{n\geq 1}\mathbb{M}_n(A_n) \oplus R\overline{\otimes}A_0$, where $A_m, m \geq 0$ are abelian vN algebras.

Uniqueness of Cartan subalgebras of R

Regular and Cartan subalgebras: definition and examples

• (Dixmier 1954) If M is a II$_1$ factor and $B \subset M$ is a vN subalgebra, then $\mathcal{N}_M(B) = \{u \in \mathcal{U}(M) \mid uBu^* = B\}$ is the **normalizer** of B in M. B is **regular** (resp. **singular**) in M if $\mathcal{N}_M(B)'' = M$ (resp. $\mathcal{N}_M(B) = \mathcal{U}(B)$). A regular MASA $A \subset M$ called a **Cartan subalgebra** of M (Vershik, Feldman-Moore 1970s).

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• If $B \subset M$ is a regular vN subalgebra and $M \subset^{e_B} \langle M, e_B \rangle$ its basic construction, then its canonical normal faithful semifinite trace Tr (defined by $Tr(xe_By) = \tau(xy), \forall x, y \in M$) is semifinite on $B' \cap \langle M, e_B \rangle$.

Connes-Feldman-Weiss and Ornstein-Weiss Theorems 1980-1981

If M is a separable amenable II_1 factor and $A \subset M$ is Cartan, then $(A \subset M) \simeq (D \subset R)$. In particular, any two free ergodic pmp actions of countable amenable groups $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are orbit equivalent.

Proof. Note first that given any regular inclusion $B \subset M$, the trace Tr is semifinite on $\mathcal{M} := B' \cap \langle M, e_B \rangle$ (*Exercise!*). Also, if $u \in \mathcal{N}_M(B)$ then $\mathrm{Ad}(u)(\mathcal{M}) = \mathcal{M}$, $Tr \circ \mathrm{Ad}(u) = Tr$.

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Proof. Note first that given any regular inclusion $B \subset M$, the trace Tr is semifinite on $\mathcal{M} := B' \cap \langle M, e_B \rangle$ (*Exercise!*). Also, if $u \in \mathcal{N}_M(B)$ then $\mathrm{Ad}(u)(\mathcal{M}) = \mathcal{M}$, $Tr \circ \mathrm{Ad}(u) = Tr$.

Følner-type condition. If M is amenable and $B \subset M$ is regular, then $\forall F \subset \mathcal{N}_M(B)$ finite, $\varepsilon > 0$, $\exists p \in \mathcal{P}(\mathcal{M})$ with $Tr(p) < \infty$ such that $\|upu^* - p\|_{2,Tr} < \varepsilon \|p\|_{2,Tr}$, $\forall u \in F$.

Note first that the hypertrace for $M \subset \mathcal{B}(L^2 M)$ restricted to $\mathcal{M}$ gives a state φ on $\mathcal{M}$ such that $\varphi(uxu^*) = \varphi(x)$, $\forall u \in \mathcal{N}_M(B)$ and $x \in \mathcal{M}$. By using exactly as before Day's trick, one gets $b \in L^1(\mathcal{M}, Tr)_+$, $Tr(b) = 1$ such that $\|ubu^* - b\|_{1,Tr} < \varepsilon$, $\forall u \in F$. Using C's Joint Distribution trick and Namioka-type trick, one gets the desired p as $e_{[t,\infty)}(b)$ for some $t > 0$.

From the Følner-type condition to local AFD for $A \subset M$ Cartan. Any "finite" $p \in \mathcal{M}$ is of the form $\sum_j v_j e_A v_j^*$ for some finite set v_j of partial isometries normalizing A (*Exercise!*). By "local quantization" $\exists q \in \mathcal{P}(A)$ such that one approximately have $qv_i^* u v_j q \in \mathbb{C}q$, $\forall i, j$, $\forall u \in F$. This means $B_0 = \sum_{i,j} \mathbb{C} v_i q v_j$ is fin. dim. with diagonal $D_0 = \mathbb{C} v_i q v_i^* \subset A$ s.t. $s_0 = 1_{B_0}$ satisfies $\|[s, u]\|_2 \leq \varepsilon \|s\|_2$, $\|E_{B_0}(sus) - sus\|_2 \leq \varepsilon \|s\|_2$, $\forall u \in F$.
From local AFD to global AFD. Using a maximality argument, one shows that the local AFD implies: $\forall F \subset M$ finite, $\varepsilon > 0$, $\exists B_1 \subset M$ fin dim vN subalgebra, generated by matrix units $\{e_{ij}^k\}_{i,j,k}$ such that $e_{ii}^k \in A$ and e_{ij}^k normalize A. This shows that $A \subset M$ is AFD, which immediately implies $(A \subset M) \simeq (D \subset R)$ (*Exercise!*)

To see the last part of the CFW-OW theorems, about orbit equivalence of amenable group actions, we need some remarks/definitions.

Two remarks, by I.M. Singer 1955, Feldman-Moore 1977

(1) Let $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ be free ergodic pmp actions of countable groups. Then $(L^\infty X \subset L^\infty X \rtimes \Gamma) \simeq (L^\infty Y \rtimes L^\infty Y \rtimes \Lambda)$ iff $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are **orbit equivalent** (OE), i.e., $\exists \Delta : X \simeq Y$ such that $\Delta(\Gamma t) = \Lambda(\Delta(t))$, $\forall_{ae} t \in X$.

Thus, since any two free ergodic pmp actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ of countable amenable groups give rise to Cartan inclusions into R, the uniqueness of the Cartan in R shows that these two actions are OE. This is Ornstein-Weiss 1980 Thm.

(2) Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic pmp action of a countable group and $\mathcal{R}$ the corresponding orbit equivalence relation on X: $t \sim s$ if $\Gamma t = \Gamma s$.

One associates to it a II_1 factor $L(\mathcal{R})$ with a Cartan subalgebra $A = L^\infty X$, by taking the algebra of formal finite sums $\Sigma_\phi a_\phi \lambda(\phi)$, where $a_\phi \in A$, ϕ are local isomorphisms of X with graph in $\mathcal{R}$, endowed with its structure of multiplicative pseudo-group, endowed with the trace $\tau(av_\phi) = \int ai(\varphi)\mathrm{d}\mu$, where $i(\phi)$ is the characteristic function of the set $X_0 \subset X$ on which ϕ is the identity.

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Moreover, if $v : \mathcal{R} \times \mathcal{R} \to A$ is a 2-cocycle for $\mathcal{R}$, then one can form the v-twisted version $L(\mathcal{R}, v)$ of this algebra, where $\lambda(\phi)\lambda(\psi) = v_{\phi,\psi}\lambda(\phi\psi)$. Given any Cartan inclusion $A \subset M$, with M a countably generated II_1 factor, there exists $(\mathcal{R}, v)$ such that $(A \subset M) \simeq (L^\infty X \subset L(\mathcal{R}, v))$. Also, for Cartan inclusions we have $(A_1 \subset M_1) \simeq (A_2 \subset M_2)$ iff $(\mathcal{R}_1, v_1) \simeq (\mathcal{R}_2, v_2)$

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• Thus, by the uniqueness of the Cartan in R, we have that any two ergodic pmp actions of any two amenable group on non-atomic prob spaces are OE, and that any 2-cocycle v for such actions is co-boundary.

Next problem: classifying all regular inclusions $B \subset R$

• The CFW theorem shows that there exists just one Cartan subalgebra $A \subset R$, up to conjugacy by an automorphism of R. One would of course like to classify ALL regular inclusions $B \subset R$. A natural "homogeneity/irreducibility" condition to impose is that $B' \cap R = \mathcal{Z}(B)$. Besides the case $B = A$ abelian, a first case of interest is when $B = N$ is a subfactor. By Connes Thm, such N is necessarily isomorphic to R and the irreducibility condition amounts to $N' \cap R = \mathbb{C}$.

• It is an easy exercise to show that if $N \subset M$ is a regular irreducible inclusion of II_1 factors, then $\Gamma_{N \subset M} = \mathcal{N}_M(N)/\mathcal{U}(N)$ is a discrete group, which is countable if M is separable and it is amenable if $M \simeq R$ (all this will follow in a short while, from a more ample discussion).

The case $N \subset R$ is a regular subfactor

Ocneanu's Theorem 1985

Irreducible regular inclusions $N \subset R$ are completely classified (up to conjugacy by an automorphism of R) by the normalizing group, $\Gamma_{N \subset R} := \mathcal{N}_R(N)/\mathcal{U}(N)$.

More precisely, if $N_0 \subset R$ is another irreducible regular subfactor then there exists an automorphism θ of R s.t. $\theta(N_0) = N$ iff $\Gamma_{N_0 \subset R} \simeq \Gamma_{N \subset R}$.

Since any inclusion $N \subset M = N \rtimes \Gamma$ arising from a free action $\Gamma \curvearrowright N$ is irreducible and regular with $\Gamma_{N \subset M} = \Gamma$, the above is equivalent to saying that any irreducible regular inclusion of factors $(N \subset R)$ is isomorphic to $(N \subset N \rtimes \Gamma)$, where $\Gamma = \Gamma_{N \subset R}$ and $\Gamma \curvearrowright N = R = \mathbb{M}_2(\mathbb{C})^{\overline{\otimes}\Gamma}$ is the Bernoulli action.

Arbitrary cocycle actions

- A **cocycle action** of a group Γ on a tracial vN algebra (B, τ) is a map $\sigma : \Gamma \to \mathrm{Aut}(B, \tau)$ which is multiplicative modulo inner automorphisms of B,

$$\sigma_g \sigma_h = \mathrm{Ad}(v_{g,h})\sigma_{gh}, \forall g, h \in \Gamma,$$

with the unitary elements $v_{g,h} \in \mathcal{U}(B)$ satisfying the cocycle relation

$$v_{g,h} v_{gh,k} = \sigma_g(v_{h,k}) v_{g,hk}, \forall g, h, k \in \Gamma.$$

The cocycle action is **free** if σ_g properly outer $\forall g \neq e$ ($\theta \in \mathrm{Aut}(B, \tau)$ is properly outer if $b \in B$ with $\theta(x)b = bx$, $\forall x \in B$, implies $b = 0$; thus, if $B = N$ is a II_1 factor then this amounts to θ being outer).

- (σ, v) is a "genuine" action, if $v \equiv 1$.

Some examples

• **Connes-Jones cocycles** (1984): Let $\Gamma = \langle S \rangle$ infinite group and $\pi : \mathbb{F}_S \to \Gamma \to 1$ with kernel $ker(\pi) \simeq \mathbb{F}_\infty$. This gives rise to $N = L(\ker(\pi)) \subset L(\mathbb{F}_S) = M$ irreducible and regular, with $M = N \rtimes_{(\sigma,v)} \Gamma$ for some free cocycle action (σ, v) of Γ on $N = L(\mathbb{F}_\infty)$.

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• **Amplified cocycles**: Given any action $\Gamma \curvearrowright^\sigma N$ and $p \in \mathcal{P}(N)$, one has $p \sim \sigma_g(p)$ via some partial isometry $w_g \in N$. Then $\mathrm{Ad}(w_g) \circ \sigma_{g_{|pNp}}$ is a cocyle action of Γ on $N^t = pNp$, where $t = \tau(p)$. Denoted (σ^t, v^t), in which $v^t_{g,h} := w_g \sigma_g(w_h) w^*_{gh}, \forall g, h.$.

Crossed product vN algebras from cocycle actions

• Any cocycle action $\Gamma \curvearrowright^{(\sigma,v)} (B,\tau)$ gives rise to a crossed product inclusion $B \subset M = B \rtimes_{(\sigma,v)} \Gamma$, in a similar way we defined the usual crossed product for actions, where multiplication is given by $u_g u_h = v_{g,h} u_{gh}$ and $u_g b = \sigma_g(b) u_g$. Clearly B is regular in M.

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• One can easily show that the cocycle action is free iff $B' \cap M = \mathcal{Z}(B)$. In particular, if $B = N$ is a II_1 factor, then (σ, v) is free iff $N' \cap M = \mathbb{C}1$, i.e., N is **irreducible** in $M = N \rtimes \Gamma$.

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• Conversely, if $N \subset M$ is irreducible and regular and one denotes $\Gamma = \mathcal{N}_M(N)/\mathcal{U}(N)$, then choosing $U_g \in \mathcal{N}$ for each $g \in \Gamma$ and letting

$$\sigma_g = \mathrm{Ad}(U_g), \quad v_{g,h} = U_g U_h U_{gh}^*$$

shows that $M = N \rtimes_{(\sigma,v)} \Gamma$ (this is a remark by Jones, Sutherland 1980).

Equivalence of cocyle actions

• Two cocycle actions (σ_i, v_i) of Γ_i on (B_i, τ_i), $i = 1, 2$, are **cocycle conjugate** if $\exists\theta : (B_1, \tau_1) \simeq (B_2, \tau_2)$, $\gamma : \Gamma_1 \simeq \Gamma_2$ and $w_g \in \mathcal{U}(B_2)$ such that :

$$\theta\sigma_1(g)\theta^{-1} = \mathrm{Ad} \circ \sigma_2(\gamma(g)), \forall g,$$

$$\theta(v_1(g, h)) = w_g\sigma_2(g)(w_h)v_2(\gamma(g), \gamma(h))w_{gh}^*, \forall g, h.$$

• Two free cocycle actions $\Gamma_i \curvearrowright^{(\sigma_i, v_i)}$ on the II_1 factors N_i, $i = 1, 2$, are cocycle conjugate iff their associated crossed product inclusions are isomorphic, $(N_1 \subset N_1 \rtimes \Gamma_1) \simeq (N_2 \subset N_2 \rtimes \Gamma_2)$.

Untwisting cocycle actions

• The cocycle action (σ, v) of Γ on $(B\tau)$ **untwists** (or is **co-boundary**) if $\exists w_g \in \mathcal{U}(B)$ s.t. $v_{g,h} = w_g \sigma_g(w_h) w_{gh}^*$, $\forall g, h$. Thus, (σ, v) untwists iff it is cocycle conjugate to a genuine action.

Note this is a bit stronger than $\sigma'_g = \mathrm{Ad}(w_g) \circ \sigma_g$ being a "genuine" action. It is equivalent to: $\exists w_g \in \mathcal{U}(B)$ s.t. $U'_g = w_g U_g \in B \rtimes_{(\sigma,v)} \Gamma$ satisfy $U'_g U'_h = U'_{gh}$, $\forall g, h$.

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Example

- Clearly any cocycle action of $\Gamma = \mathbb{F}_n \in$ untwists.

Original formulation of Ocneanu's theorem

• O's original Thm is that any two free cocycle actions of a countable amenable group Γ on R are cocycle conjugate.

This result was already known in the case $\Gamma = \mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$ (Connes 1975) and in the case Γ finite (Jones 1980). In case Γ finite, Jones proved that any two free Γ-actions on R are in fact conjugate and that any 1-cocycle of a finite group action on any II_1 factor is co-boundary.

From the above discussion, we see that O's result implies that any cocycle action of a countable amenable group untwists.

If Γ is amenable, the crossed product $R \rtimes_{(\sigma,v)} \Gamma$ is amenable, so by C's Thm it is isomorphic to R. Thus, by the above remarks, the uniqueness (up to cocycle conjugacy) of free cocycle Γ-actions on R translates into the uniqueness (up to conjugacy by automorphisms of R) of irreducible regular subfactors $N \subset R$ with $\Gamma_{N \subset R} = \Gamma$. In particular, O's result shows that any such irreducible regular inclusion $N \subset R$ is a "true" (untwisted) crossed product construction, coming from a "genuine" Γ-action.

Sketch of proof of O's Thm (two approaches)....

Classifing regular inclusions $B \subset R$: remaining cases

• Let M be a II_1 factor and $B \subset M$ regular with $B' \cap M = \mathcal{Z}(B) = L^\infty(X, \mu)$. These assumptions imply B is "homogeneous", i.e., either $B = \mathbb{M}_n(\mathbb{C}) \overline{\otimes} L^\infty X$, for some $n \geq 1$, or $B = \int_X B_t \mathrm{d}\mu(t)$, where B_t are II_1 factors, $\forall_{ae} t \in X$. If in addition $M = R$, in this latter case we have $B_t \simeq R$ and $B \simeq R \overline{\otimes} L^\infty X$. The normalizer $\mathcal{N}_M(B)$ defines an amenable **discrete measured groupoid** $\mathcal{G} = \mathcal{G}_{B \subset M}$ together with a free cocycle action $(\alpha, v) = (\alpha_{B \subset M}, v_{B \subset M})$ of $\mathcal{G}$ on B. The iso class of the inclusion $B \subset M$ is completely encoded in the cocycle conjugacy class of $\mathcal{G} \curvearrowright^{(\alpha, v)} B$.

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• In the case $B \subset M = R$, the discrete groupoid $\mathcal{G}$ accounts for an amenable ergodic countable equivalence relation "along" the space $\mathcal{G}^{(0)} = X$ of units of $\mathcal{G}$, with amenable countable isotropy groups Γ_t at each $t \in X$ acting outerly on $B_t \simeq R$.

• When B is abelian, then $B \simeq L^\infty X$ and $\mathcal{G}$ is just a countable amenable equiv rel $\mathcal{R}$ on X, with α intrinsic to $\mathcal{R}$. The CFW Thm says that there is just one amenable countable equiv. rel. and it has vanishing coh v. This also implies that, for each $n \geq 1$, there is just one regular inclusion $B \subset R$ with $B' \cap R = \mathcal{Z}(B)$ and B of type I_n.

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• If B is a factor, then $B \simeq R$ and the groupoid $\mathcal{G}_{B \subset R}$ is the group $\Gamma = \mathcal{N}_R(B)/\mathcal{U}(B)$, which follows countable amenable, and (α, v) is the free cocycle action of Γ on B implemented by $\mathcal{N}_R(B)$. O's Thm then shows that $\mathcal{G}$ uniquely determines $B \subset R$. This clearly takes care of the case $\mathcal{Z}(B)$ atomic as well.

Solving the case $B \subset R$ with $B \simeq R\overline{\otimes}L^\infty X$

- So we are left with the case $B \subset R$ where $B = R\overline{\otimes}L^\infty X$, with X diffuse, i.e., to the problem of classifying $\mathcal{G} \curvearrowright^{(\alpha,v)} B = R\overline{\otimes}L^\infty X$ up to cocycle conjugacy, for all amenable groupoids $\mathcal{G}$ with $\mathcal{G}^{(0)} = X$. When $v \equiv 1$ (i.e., for "genuine" actions of $\mathcal{G}$) this was solved as follows:

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Sutherland-Takesaki Theorem 1985

Any two actions α_1, α_2 of the same amenable groupoid $\mathcal{G}$ on $R\overline{\otimes}L^\infty X$ are cocycle conjugate. Equivalently, any two regular inclusions of the form $B \subset R$ with $B' \cap R = \mathcal{Z}(B)$, with same $\mathcal{G}_{B\subset R}$ and with $v_{B\subset R} \equiv 1$, are conjugate by an automorphism of R.

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By the above result, it follows that we are left with proving that any 2-cocycle v for a cocycle action $\mathcal{G} \curvearrowright^{(\alpha,v)} R\overline{\otimes}L^\infty X$ of an amenable groupoid $\mathcal{G}$ is co-boundary. As it turns out, this is a rather difficult problem.

Untwisting cocycles on arbitrary II_1 factors

Theorem (P 2018)

Given any countable amenable group Γ, any free cocycle Γ-action $\Gamma \curvearrowright^{(\alpha,v)} N$ on an arbitrary II_1 factor N untwisis.

Untwisting cocycles on arbitrary II_1 factors

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Same actually holds true for $\Gamma = \Gamma_1 *_K \Gamma_2 *_K .... \in$, with Γ_n countable amenable and $K \subset \Gamma_n$ common finite subgroup, $\forall n$.

We prove this by building an embedding $R \hookrightarrow N$ that's $\alpha(\Gamma)$-equivariant, modulo an inner perturbation (α', v') of (α, v), and which is "large" in N, in the sense that $R' \cap N \rtimes \Gamma = \mathbb{C}$. This last condition forces v' to take values in R. By O's vanishing oh Thm, $(\alpha'_{|R}, v'$ can be perturbed to an actual action α'', with the untwisting of the cocycle v' in R, $v'_{g,h} = w_g \alpha'_g(w_h) w^*_{gh}$. But this means we have untwisted (α, v) as a cocycle action on N as well.

An amenable/non-amenable dichotomy

While the "universal vanishing cohomology" property for a group Γ holds true for $\Gamma = \mathbb{F}_n$ and more generally free products of amenable groups, the existence of Γ-equivariant embeddings of the hyperfinite factor characterizes amenability of Γ:

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(1) Any cocycle action σ of a countable amenable group Γ on an arbitrary II_1 factor N admits an inner perturbation σ' that normalizes a hyperfinite subfactor $R \subset N$ satisfying $R' \cap N \rtimes_\sigma \Gamma = \mathbb{C}$.

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PROOF of (1) uses subfactor techniques, constructing R as an inductive limit of relative commutants of a sequence of subfactors of finite index, coming from a "generalized tunnel" associated with a "diagonal subfactor" $(N \subset M_\sigma)$. Part (2) uses deformation-rigidity (Ozawa-Popa 2007).

Untwisting cocycle actions of amenable groupoids

Theorem: P-Shlyakhtenko-Vaes 2018

Let $\mathcal{G}$ be a discrete measured groupoid with $X = \mathcal{G}^{(0)}$ and $(B_t)_{t\in X}$ a measurable field of II_1 factors with separable predual. Assume that $\mathcal{G}$ is amenable and that (α, v) is a free cocycle action of $\mathcal{G}$ on $(B_t)_{t\in X}$. Then the cocycle v is a co-boundary: there exists a measurable field of unitaries $\mathcal{G} \ni g \mapsto w_g \in (B_t)_t$ s.t. $v(g,h) = \alpha_g(w_h^*)w_g^* w_{gh}$, $\forall (g,h) \in \mathcal{G}^{(2)}$.

Before discussing the proof, we mention that we have finally proved:

Complete classification of regular $B \subset R$ with $B' \cap R = \mathcal{Z}(B)$

Two regular vN subalgebras $B \subset R$ satisfying $B' \cap R = \mathcal{Z}(B)$ are conjugate by an automorphism of R iff they are of the same type and have isomorphic associated discrete measured groupoids $\mathcal{G}_{B\subset R}$.

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Any such B contains a Cartan subalgebra of R and if $A_1, A_2 \subset B$ are Cartan in R, there exists an automorphism θ of R satisfying $\theta(B) = B$ and $\theta(A_1) = A_2$.

About the proof

The proof of the vanishing 2-cohomology Thm uses the vanishing 2-coh for cocycle actions of amenable groups on II_1 factors (P 2018), the CFW vanishing of the con along $\mathcal{G}^{(0)} = X$, which we apply to the isotropy groups $\Gamma_t, t \in X$ of the amenable groupoid $\mathcal{G}$. To extend to the entire $\mathcal{G}$, we have to make equivariant choices of 2-cocycle vanishing, for the Γ_t, where the equivariance is w.r.t. to the isomorphisms $\Gamma_t \to \Gamma_s$ given by conjugation with an element $g \in \mathcal{G}$ with $s(g) = s$ and $t(g) = t$ (source and target of g).

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The proof of this latter part depends on two key points. The first one is a technical result showing that such an equivariant choice exists, provided that the 2-cocycle vanishing for Γ_t, can be done in an "approximately unique way". The fact that a 2-cocycle untwists in an "approximately unique way" amounts to the fact that 1-cocycles for actions are "approximately co-boundary". The second key point is to prove such approximate vanishing of 1-cocycles for arbitrary amenable groups, a result we discuss next because of its independent interest.

Approximate vanishing 1-cohomology

- A 1-cocycle for an action $\Gamma \curvearrowright^{\sigma} N$ is a map$w : \Gamma \to \mathcal{U}(N)$ s.t. $w_g \sigma_g(w_h) = w_{gh}$, $\forall g, h$. The cocycle w is co-boundary if $\exists u \in \mathcal{U}(N)$ such that $w_g = \sigma_g(u)u^*$, $\forall g$; it is approximate co-boundary if $\exists u_n \in \mathcal{U}(N)$ such that $\|w_g - \sigma_g(u_n)u_n^*\|_2 \to 0$, $\forall g$, equivalently w is co-boundary as a 1-cocycle for $\Gamma \curvearrowright^{\sigma^\omega} N^\omega$.

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Theorem (P-Shlyakhtenko-Vaes 2018)

Let Γ be a countable group. The following conditions are equivalent.

(i) Γ is amenable.

(ii) For any free action $\Gamma \curvearrowright^{\sigma} N$ the fixed point algebra of σ^ω on N^ω is a subfactor with trivial relative commutant in N^ω.

(iii) Any free action of Γ on any II_1 factor is non strongly ergodic.

(iv) Any 1-cocycle w for any $\Gamma \curvearrowright^{\sigma} N$ is approximate co-boundary.

About the proof of approx vanishing 1-coh

- Jones showed in 1980 that any 1-cocycle for a free action σ of a finite group Γ on a II_1 factor is co-boundary. The proof only uses that the fixed point algebra of any such action is an irreducible subfactor: let $\tilde{\sigma}$ be the action of Γ on $\tilde{N} = \mathbb{M}_2(N) = N \otimes \mathbb{M}_2(\mathbb{C})$ given by $\tilde{\sigma}_g = \sigma_g \otimes id$. If $\{e_{ij} \mid 1 \leq i, j \leq 2\} \subset \mathbb{M}_2 \subset \tilde{N}$ is a matrix unit, then $\tilde{w}_g = e_{11} + w_g e_{22}$ is a 1-cocycle for $\tilde{\sigma}$. If $Q \subset \tilde{N}$ denotes the fixed point algebra of the action $\tilde{\sigma}'_g = \mathrm{Ad}(\tilde{w}_g)\tilde{\sigma}$, then $e_{11}, e_{22} \in Q$. The existence of a unitary element $u \in N$ satisfying $w_g = u\sigma_g(u^*)$, $\forall g$, is equivalent to $e_{11} \sim e_{22}$ in Q. Since Q is a II_1 factor and e_{11}, e_{22} have equal trace $1/2$ in Q so indeed $e_{11} \sim e_{22}$ in Q, thus w is co-boundary.

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• Note that the above proof only uses that the fixed point algebra is a II_1 factor. This shows that $(ii) \Rightarrow (iv)$. To show that $(i) \Rightarrow (ii)$ we use the foll Lemma:

If $\Gamma \curvearrowright N$ is a free action of a countable group on a II_1 factor and $\mathcal{X} \subset N^\omega$ separable, then $\exists u \in \mathcal{U}(N^\omega)$ s.t. $\mathcal{X}, \{\sigma_g^\omega(uNu^*)\}_{g\in\Gamma}$ are all mutually free independent.

Sketch of proof of $(i) \Rightarrow (ii)$ in the Theorem

- With the notations in the previous lemma, let $Q = \vee_g \sigma_g(uNu^*) \simeq N^{*\Gamma}$. Note that Q is free independent to N and $\sigma^\omega(Q) = Q$, with $\rho = \sigma^\omega_{|Q}$ implementing on $Q \simeq N^{*\Gamma}$ the free Bernoulli Γ-action. Let $a = a^* \in N$ be a semi-circular element and denote by a_g its identical copies in the $(N)_g \simeq N$ components of $N^{*\Gamma}$, $g \in \Gamma$. Thus, ρ acts on the set $\{a_g\}_g$ by left translation, $\rho_h(a_g) = a_{hg}$. Let $K_n \subset \Gamma$ be a sequence of Folner sets and denote $b_n = |K_n|^{-1/2} \sum_{g \in K_n} a_g$. Then b_n is also a semicircular element and one has

$$\|\rho_h(b_n) - b_n\|_2^2 = |hF_n \Delta F_n|/|K_n| \to 0, \forall h \in \Gamma.$$

Thus, the element $\tilde{b} = (b_n)_n \in (N^{*\Gamma})^\omega$ is semicircular with $\rho_h(\tilde{b}) = \tilde{b}$, $\forall h \in \Gamma$, showing that ρ is not strongly ergodic.

This shows that there exist finite partitions $\{q_i\}_i \subset \mathcal{P}(Q)$ of arbitrary small mesh and which are almost σ^ω-invariant. So given any $x \in \mathcal{X}$, we have that $\|\sum_i q_i x q_i - \tau(x)1\|_2$ small, because Q is free independent to $x \in \mathcal{X}$. This readily implies $(N^\omega)^{\sigma^{\omega\prime}} \cap N^\omega = \mathbb{C}$.

From $\Gamma \curvearrowright^{\sigma} R$ to $\Gamma \curvearrowright^{(\tilde{\sigma}_\omega, v_\omega^\sigma)} R \vee R_\omega$

Proposition

1° $R_\omega = R' \cap R^\omega$ satisfies $R_\omega' \cap R^\omega = R$.

2° $\forall \theta \in \mathrm{Aut}(R)$, $\exists U_\theta \in \mathcal{N}_{R^\omega}(R)$ such that $\mathrm{Ad}(U_\theta)_{|R} = \theta$. If $U_\theta' \in \mathcal{N}_{R^\omega}(R)$ is another unitary satisfying $\mathrm{Ad}(U_\theta')_{|R} = \theta$, then $U_\theta' = vU_\theta = U_\theta v'$ for some $v, v' \in \mathcal{U}(R_\omega)$.

3° If θ, U_θ as in 2°, then $\mathrm{Ad}(U_\theta)_{|R_\omega}$ implements $\theta_\omega \in \mathrm{Out}(R_\omega)$ and $\tilde{\theta}_\omega = \mathrm{Ad}(U_\theta)_{|R \vee R_\omega} \in \mathrm{Out}(R \vee R_\omega)$, with $\theta \in \mathrm{Aut}(R)$ outer iff θ_ω outer and iff $\tilde{\theta}_\omega$ outer.

4° Any free action $\Gamma \curvearrowright^{\sigma} R$ gives rise to a free cocycle action $\tilde{\sigma}_\omega$ of Γ on $R \vee R_\omega$, by $\tilde{\sigma}_\omega(g) = \mathrm{Ad}(U_{\sigma(g)})_{|R \vee R_\omega}$, $g \in \Gamma$, with corresponding 2-cocycle $v_\omega^\sigma : \Gamma \times \Gamma \to \mathcal{U}(R_\omega)$.

Vanishing cohomology for $\tilde{\sigma}_\omega$ and the CE conjecture

Theorem

$\Gamma \curvearrowright^\sigma R$ free action of Γ on R. The II_1 factor $M = R \rtimes_\sigma \Gamma$ has the CAE property (i.e., is embeddable into R^ω) iff the $\mathcal{U}(R_\omega)$-valued 2-cocycle v_ω^σ vanishes, i.e., iff there exist unitary elements $\{U_g \mid g \in \Gamma\} \subset \mathcal{N}_{R^\omega}(R)$ that implement σ on R and satisfy $U_g U_h = U_{gh}$, $\forall g, h \in \Gamma$.

A related problem

We have seen that one has a group isomorphism

$$\mathrm{Out}(R) \ni \theta \mapsto \mathrm{Ad}(U_\theta) \in \mathrm{Out}(R \vee R_\omega)$$

which is also onto if on the right side we restrict to autom that leave R invariant. Lifting this map to a grp morphism into to $\mathcal{N}_{R^\omega}(R)$ when restricted to a countable subgroup $\Gamma \subset \mathrm{Out}(R)$ implementing a genuine action, is equiv. to CE conjecture for $R \rtimes \Gamma$. But even if CE holds true for these factors, it seems quite clear that such lifting is not possible for the entire $\Gamma = \mathrm{Out}(R)$. However, we do not have a proof for this.

A closer look at the two technical lemmas

In the proofs of C's Fund Thm, the CFW Thm, O's Thm, we used:

local quantization (LQ) lemma

$\forall F' \subset M$ finite, $\delta > 0$, $\exists q \in \mathcal{P}(M)$ s.t. $\|qxq - \tau(x)q\|_2 < \delta\|q\|_2$, $\forall x \in F'$.

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This result is in fact a consequence of the following more general:

Theorem (free independence in irreducible subfactors)

If $N \subset M$ is an irreducible inclusion of II_1 factors, then $\forall B \subset M^\omega$ separable vN algebra, $\exists A \subset N^\omega$ abelian diffuse such that $A \vee B \simeq A * B$.

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Indeed, taking $M = N \rtimes \Gamma$ we have $N' \cap M = \mathbb{C}$. Then apply the Thm to get $A \subset N^\omega$ free independent to the vN algebra $B = (\mathcal{X} \cup M)''$.

Ergodic embeddings of $L^\infty([0,1])$ and R into factors

The technical results above are in fact related,: the LQ lemma plays a key role in the proof of “free independence embeddings of $L^\infty([0,1])$”, while the free independence embeddings allow sharp quantitative versions of LQ lemma. To deduce them, we’ll go through several steps:

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(4) *blackboard comments* on the proof of "approximately free independent" embeddings of $L^\infty([0,1])$ and the incremental patching method.

Free random embeddings of $L^\infty([0,1])$ and R

The incremental patching method allows proving the following general

Theorem (approx. free independence with amalgamation)

Let M_n be a sequence of finite factors with $\dim M_n \to \infty$ and denote by $\mathbf{M}$ the ultraproduct II_1 factor $\Pi_\omega M_n$, over a free ultrafilter ω on $\mathbb{N}$. Let $\mathbf{Q} \subset \mathbf{M}$ be a vN subalgebra satisfying one of the following:

(a) $\mathbf{Q} = \Pi_\omega Q_n$, for some vN alg. $Q_n \subset M_n$ with $Q_n \not\prec_{M_n} Q_n' \cap M_n$, $\forall n$;

(b) $\mathbf{Q} = B' \cap \mathbf{M}$, for some separable amenable vN alg. $B \subset \mathbf{M}$.

Then given any separable subspace $X \subset \mathbf{M} \ominus (\mathbf{Q}' \cap \mathbf{M})$, there exists a diffuse abelian vN alg. $A \subset \mathbf{Q}$ such that A is free independent to X, relative to $\mathbf{Q}' \cap \mathbf{M}$, i.e. $E_{\mathbf{Q}'\cap\mathbf{M}}(x_0 \Pi_{i=1}^n a_i x_i) = 0$, for all $n \geq 1$, $x_0, x_k \in X \cup \{1\}$, $x_i \in X$, $1 \leq i \leq k-1$, $a_i \in A \ominus \mathbb{C}1$, $1 \leq i \leq n$.

• The above result led us to the discovery in 1990-1994 of the *reconstruction method* in subfactor theory, and the *axiomatisation* of the standard invariant of a subfactor.

Applications

• Existence of ergodic embeddings of AFD factors into arbitrary vN factors is crucial for establishing Stone-Weierstrass type theorems for inclusions of C^*-algebras (Kadison, Sakai, Glimm, J. Anderson, Bunce, etc). A complete solution to the "factor state" such result' was given using **(1)** above.

• Existence of ergodic embeddings of R into II_1 factors M were used to prove that $H^2(M, M) = 0$ (Kadison-Ringrose Hochshild-type 2nd coh) for a large class of II_1 factors M (Schmidt-Sinclair 95).

• Embeddings of $L^\infty([0,1])$ and R into a II_1 factor M that are asympt. free to M where key to establishing a variety of vanishing cohomology results:

(a) All derivations from a vN algebra M that take values in $\mathcal{K}(\mathcal{H})$ (more generally, all "smooth derivations") are inner, i.e., $H^1(M, \mathcal{K}) = 0$ (Popa 1984, Popa-Radulescu 1986, Galatan-Popa 2014).

(b) Vanishing of the Connes-Shlyakhtenko-Thom 1st L^2 cohomology, $H^1(M, \mathrm{Aff}(M \overline{\otimes} M^{op})) = 0$ (Popa-Vaes 2016).

(c) Approx. vanishing of 1-cohomology for any action of an amenable groups on any II_1 factor (Popa-Shlyakhtenko-Vaes 2018).

Coarse, mixing, and strongly malnormal embeddings

Coarse subalgebras and coarse pairs

A vN subalgebra $B \subset M$ is **coarse** if the vN algebra generated by left-right multiplication by elements in B on $L^2(M \ominus B)$ is $B\overline{\otimes}B^{op}$. The vN subalgebras $B, Q \subset M$ form a **coarse pair** if the vN algebra generated by left multiplication by B and right multiplication by Q on L^2M is $B\overline{\otimes}Q^{op}$.

Mixing subalgebras

A vN subalgebra $B \subset M$ is **mixing** if $\lim_{u\in\mathcal{U}(B)} \|E_B(xuy)\|_2 = 0$, $\forall x, y \in M \ominus B$, where the limit is over $u \in \mathcal{U}(B)$ tending wo to 0.

Strongly malnormal subalgebras

A vN subalgebra $B \subset M$ is **strongly malnormal** if its weak intertwining space $w\mathcal{I}_M(B, B)$ is equal to B, i.e., if $x \in M$ satisfies $\dim(L^2(A_0xB)_B) < \infty$, then $x \in B$.

Proposition

One has the implications "coarse $\Rightarrow$ mixing $\Rightarrow$ strongly malnormal".

Coarse embeddings of R and $L^\infty([0,1])$

Theorem (P 2018-19)

Any separable II_1 factor M contains a hyperfinite factor $R \subset M$ that's coarse in M (and thus also mixing and strongly malnormal in M). Moreover, given any irreducible subfactor $P \subset M$, any vN alg. $Q \subset M$ satisfying $P \not\prec_M Q$ and any $\varepsilon > 0$, the coarse subfactor $R \subset M$ can be constructed so that to be contained in P, make a coarse pair with Q and satisfy $R \perp_\varepsilon Q$.

Proof comments on blackboard.

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Corollary

Any separable II_1 factor M has a coarse MASA $A \subset M$, which in addition is strongly malnormal and mixing, with infinite multiplicity (Pukansky invariant equal to ∞). Moreover, given any irreducible subfactor $P \subset M$, any vN alg. $Q \subset M$ such that $P \not\prec_M Q$ and any $\varepsilon > 0$, the coarse MASA $A \subset M$ can be constructed inside P, coarse to Q, and satisfying $A \perp_\varepsilon Q$.

Coarseness and strong malnormality in $L\mathbb{F}_n$

Coarseness conjecture

Any maximal amenable (equivalently maximal AFD) von Neumann subalgebra B of $L(\mathbb{F}_t)$ is coarse, and thus also mixing and strongly malnormal, $\forall 1 < t \leq \infty$.

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- Note that if $B \subset M$ is strongly malnormal, then any weak intertwiner of B in M is contained in B, in particular if $u \in \mathcal{U}(M)$ is so that $uBu^* \cap B$ is diffuse, then $u \in B$. It also implies that if $B_0 \subset M$ amenable and $B_0 \cap B$ diffuse, then $B_0 \subset B$. Thus, the above coarseness conjecture implies the *Peterson-Thom conjecture*, which predicts that any $B_0 \subset L\mathbb{F}_n$ amenable diffuse is contained in a unique maximal amenable subalgebra of $L\mathbb{F}_n$.

More on R-embeddings

• *Connes Approximate Embedding* (CAE) conjecture asks whether any countably generated tracial vN algebra has an "approximate embeding" into R, i.e., M embeds into R^ω, equivalently into $\Pi_\omega \mathbb{M}_n(\mathbb{C})$. (Can any tracial vN algebra be "simulated" by matrix algebras?).

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Ergodic embeddings of R (work in progress: to be checked)

Any vN factor $\mathcal{M}$ that's not of type I and has separable predual, contains an ergodic copy of R, i.e., a hyperfinite subfactor $R \subset \mathcal{M}$ with trivial relative commutant, $R' \cap \mathcal{M} = \mathbb{C}1$.