On extension groups of Cuntz–Krieger algebras and K-theoretic duality

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Abstract

Let us denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space H. An extension of a C^*-algebra $\mathcal{A}$ by $\mathcal{K}(H)$, an extension of $\mathcal{A}$ for brevity, is a short exact sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0,$$

that is, $\mathcal{K}(H)$ is an ideal of a C^*-algebra $\mathcal{E}$ and its quotient is $\mathcal{A}$. It is well-known that the exact sequence bijectively corresponds to a $*$-homomorphism $\tau : \mathcal{A} \longrightarrow \mathcal{Q}(H)$ from $\mathcal{A}$ to the Calkin algebra $\mathcal{Q}(H)$, called the Busby invariant. To classify extensions of C^*-algebras, Brown–Douglas–Fillmore (Ann Math. 1977) introduced the extension (semi) groups $Ext_*(\mathcal{A})$, which were motivated by classification of essentially normal operators on a Hilbert space and homotopy theory in algebraic topology. As an algebraic invariant of a C^*-algebra, Cuntz-Krieger (Invent. Math. 1980) computed the extension group written $Ext(\mathcal{O}_A)$ such that

$$Ext(\mathcal{O}_A) = \mathbb{Z}^N/(I - A)\mathbb{Z}^N.$$

The group was nothing but the Bowen-Franks group $BF(A)$ which is an "almost" complete invariant of flow equivalence of the topological Markov shift defined by the underlying matrix A. The group $Ext(\mathcal{O}_A)$ which Cuntz-Krieger computed was the weak extension group $Ext_w(\mathcal{O}_A)$, which is defined by weak equivalence classes of extensions of $\mathcal{O}_A$. As is well-known that there are several kinds of equivalence relations in extensions of a C^*-algebra. The weak equivalence is one of them. If we restrict our interest to the class of a separable unital nuclear C^*-algebras $\mathcal{A}$, the several kinds of the equivalence relations in extensions are essentially reduced to the two equivalence relations, weak equivalence relation and strong equivalence relation. The groups defined by the equivalence relations are written $Ext_w(\mathcal{A})$ and $Ext_s(\mathcal{A})$, respectively.

In this talk, we study the strong extension groups $Ext_s(\mathcal{O}_A)$ of Cuntz–Krieger algebras $\mathcal{O}_A$, and present a formula to compute the groups. We also detect the position of the Toeplitz extension of a Cuntz–Krieger algebra in the strong extension group and in the weak extension group to see that the weak extension group with the position of the Toeplitz extension is a complete invariant of the isomorphism class of the Cuntz–Krieger algebra associated with its transposed matrix. We also introduce the notion of K-theoretic duality for extensions of unital nuclear C^*-algebras by using K-homology long exact sequence. We then prove that the Toeplitz extension of a Cuntz-Krieger algebra is the K-theoretic dual of the Toeplitz extension of the Cuntz-Krieger algebra for the transposed matrix.