Recall the definition of an amalgam of groups (NB: not the amalgamated free product). It is a family $\{A_i\}$ of groups and subgroups $A_{i,j} \leq A_i$ with $A_{i,j} = A_{j,i}$ (i.e., not only that $A_{i,j}$ and $A_{j,i}$ are abstractly isomorphic, but that $A_{i,j}$ and $A_{j,i}$ are identified via a distinguished isomorphism). The amalgam is said to be realizable (or embeddable) in a group G if there are injective homomorphisms $A_i \hookrightarrow G$ such that $A_i \cap A_j = A_{i,j}$ in G for all (i,j). It is simply said to be realizable if realizable in some group. It is asked in [B. H. Neumann and H. Neumann; PLMS (1953)] whether every realizable amalgam of finitely many finite groups is realizable in a finite group. K. S. Brown (1992) has disproved it by showing that Thompson's infinite simple group is the universal group generated by a certain amalgam of three finite groups (i.e., a triangle of groups).

Theorem (W. Slofstra; `arXiv:1606.03140, arXiv:1703.08618`). *There is an amalgam $\{A_i\}_i$ of finitely many finite abelian groups which is realizable in a sofic group but not realizable in finite groups.*

Proof following `arXiv:1703.08618`. First we note that soficity is closed under the amalgamated free product over a common amenable group and hence under the HNN extension over an amemable group. We use many times the following easy fact: for any involutions a and b, the relation $(ab)^2 = e$ is equivalent to $[a,b] = e$. Consider

$$K_0 := \langle a, b, c, y : a^2 = b^2 = c^2 = e,\ a = bc,\ yby^{-1} = c,\ ycy^{-1} = b \rangle \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z},$$

$$K := \langle K_0, x : xyx^{-1} = y^2 \rangle \cong \mathrm{HNN}((\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z};\ y \mapsto y^2).$$

The group K is sofic and the element a is non-trivial in K, but a is trivial in any finite quotient. Indeed in any finite quotient y has odd order and hence $b = c$ there. The map $\sigma \colon x \mapsto x,\ y \mapsto y^{-1}$ defines an involutive automorphism on $\langle x, y \rangle$ (which extends on K).

$$K_1 := \langle K, t : t^2 = [t,x] = e,\ tyt = y^{-1} \rangle \cong (K *_{g=\sigma(g)'} K') \rtimes \mathbb{Z}/2.$$

Here $K' := \{g' : g \in K\}$ is a copy of K and $K *_{g=\sigma(g)'} K'$ is the amalgamated free product over the solvable subgroups $\langle x, y \rangle$ and $\langle x', y' \rangle$ via σ. By putting $s := yt$ and replacing y with st, one has

$$K_1 = \langle b, c, s, t, x : b^2 = c^2 = s^2 = t^2 = e,\ [b,c] = e,\ stbts = c,\ stcts = b,$$
$$[t,x] = e,\ xsx^{-1} = xstsx^{-1} \rangle$$

The map $x \leftrightarrow x^{-1}$ defines an involutive automorphism on $\langle x \rangle$.

$$K_2 := \langle K_1, u : u^2 = e,\ uxu = x^{-1} \rangle \cong (K_1 *_{x=(x')^{-1}} K_1') \rtimes (\mathbb{Z}/2)$$

By putting $v := ux$ and replacing x with uv, one has

$$K_2 = \langle b, c, s, t, u, v : b^2 = c^2 = s^2 = t^2 = u^2 = v^2 = e,\ [b,c] = e,$$
$$stbts = c,\ stcts = b,\ [t,x] = e,\ uvsvu = uvstsvu \rangle$$

By adding slack variables and expanding the relations, e.g. $stbts = c$ as $tbt = b'$ and $sb's = c$, one arrives at the presentation

$$K_2 = \langle x_1, \ldots, x_n : x_i^2 = e \text{ for all } i \text{ and } x_i x_j x_i = x_k \text{ for } (i,j,k) \in C \rangle$$

for some n and $C \subset [n]^3$. We need some construction which makes $x_i x_j$ involution-like. In $(K_2 \times K_2) \rtimes_\sigma \mathbb{Z}/2$, the element $\sigma \cdot (x_i x_j, x_j x_i)$ is an involution. We need an extra-room to produce more commuting relations. Thus we look at the sofic group

$$K_4 := K_2 \wr (\mathbb{Z}/2 \times \mathbb{Z}/2) = (\bigoplus_{\mathbb{Z}/2\times\mathbb{Z}/2} K_2) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$$

and the (possibly non-sofic??) group

$$\begin{aligned}\tilde{K}_4 := \langle x_{l,i}, y_i^{(m)}, y_{I3}^{(m)}, \dots, y_{I7}^{(m)}, f_1, \dots, f_3 \text{ for } l = 0,1,2,\ i \in [n],\ m \in \mathbb{Z}/2,\ I \in C :\\ \text{generators are all involutive}\\ f_3 = f_1 f_2,\ x_{0,i} = y_i^{(0)} y_i^{(1)},\ f_l x_{0,i} = x_{l,i} \text{ for all } i \text{ and } l = 1,2;\\ f_3 y_j^{(m)} = y_{I3}^{(m)},\ y_{I3}^{(m)} y_{I4}^{(m)} = y_{I5}^{(m)},\ y_{I5}^{(m)} y_{I6}^{(m)} = x_{1,i}\\ y_{I4}^{(m)} y_{I7}^{(m)} = x_{2,i},\ y_{I6}^{(m)} y_{I7}^{(m)} = y_k^{(m+1)} \text{ for } m \in \mathbb{Z}/2 \text{ and } I = (i,j,k) \in C\rangle.\end{aligned}$$

We claim that $x_i \mapsto x_{0,i}$ defines a homomorphism α from K_2 into $\tilde{K}_4$ and there is a homomorphism $\pi\colon \tilde{K}_4 \to K_4$ such that $\pi \circ \alpha$ is the diagonal embedding of K_2 into K_4. For the first claim, observe that for every $(i,j,k) \in I$ one has

$$x_{0,i} y_j^{(m)} x_{0,i} = x_{0,i} f_3 y_{I3}^{(m)} y_{I5}^{(m)} y_{I6}^{(m)} f_1 = f_3 x_{0,i} y_{I4}^{(m)} y_{I7}^{(m)} y_k^{(m+1)} f_1 = f_1 y_k^{(m+1)} f_1$$

and hence

$$x_{0,i} x_{0,j} x_{0,i} = x_{0,i} y_j^{(0)} y_j^{(1)} x_{0,i} = f_1 y_k^{(1)} y_k^{(0)} f_1 = f_1 x_{0,k} f_1 = x_{0,k}.$$

We view $(\mathbb{Z}/2\times\mathbb{Z}/2) \curvearrowright (\mathbb{Z}/2\times\mathbb{Z}/2)$ as $S := \{(), (12)(34), (14)(23), (13)(24)\} \curvearrowright \{1,2,3,4\}$ and $K_2 \wr (\mathbb{Z}/2 \times \mathbb{Z}/2) = (\bigoplus_{n=1}^4 K_2) \rtimes_\sigma S$. Then the hom $\pi\colon \tilde{K}_4 \to K_4$ defined by

$$\begin{aligned}&\pi(x_{0,i}) := (x_i, x_i, x_i, x_i) \in \bigoplus K_2,\ \pi(y_i^{(0)}) := (x_i, x_i, e, e),\ \pi(y_i^{(1)}) := (e, e, x_i, x_i),\\ &\pi(f_1) := \sigma_{(13)(24)},\ \pi(f_2) := \sigma_{(14)(23)},\ \pi(f_3) = \pi(f_1 f_2) = \sigma_{(12)(34)},\\ &\pi(y_{I3}^{(0)}) = \pi(f_3 y_j^{(m)}) = \sigma_{(12)(34)}(x_j, x_j, e, e),\\ &\pi(y_{I4}^{(0)}) := \sigma_{(14)(23)}(x_i x_j, x_i, x_i, x_j x_i),\ \text{check involutiveness}\\ &\pi(y_{I5}^{(0)}) = \pi(y_{I3}^{(0)} y_{I4}^{(0)}) = \sigma_{(13)(24)}(x_i x_j, x_i, x_j x_i, x_i),\ \text{check involutiveness}\\ &\pi(y_{I6}^{(0)}) = \pi(y_{I5}^{(0)} f_1 x_{0,i}) = (x_j, e, x_i x_j x_i, e) = (x_j, e, x_k, e),\\ &\pi(y_{I7}^{(0)}) = \pi(y_{I4}^{(0)} f_2 x_{0,i}) = (x_j, e, e, x_i x_j x_i) = (x_j, e, e, x_k),\ \text{check } \pi(y_{I6}^{(0)} y_{I7}^{(0)}) = y_k^{(1)}\end{aligned}$$

is indeed a well-defined homomorphism such that $\pi \circ \alpha$ is the diagonal embedding. The presentation of $\tilde{K}_4$ gives rise to an amalgam of several copies of $\mathbb{Z}/2 \times \mathbb{Z}/2$. It is realizable in the sofic group K_4 but not in any finite quotient because $a = bc \in K$ (which belongs to one of $\mathbb{Z}/2 \times \mathbb{Z}/2$'s) is trivial in any finite quotient. □

Remark: For any G and a nontrivial involution $a \in G$, one has (G' a copy of G)

$$G \hookrightarrow \langle G, G' : J := aa' \text{ is a nontrivial central involution}\rangle =: G_J$$

Indeed, consider $G \times \langle J \rangle$ and an involutive automorphism σ on $\langle a, J \rangle$, given by $a \mapsto aJ$, $J \mapsto J$. Then, there is $G_J \to ((G \times \langle J \rangle) * (G \times \langle J \rangle)) \rtimes \mathbb{Z}/2$ given by $g \mapsto g$ and $g' \mapsto sgs$.

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