GABORIAU'S THEOREM AFTER LÜCK, SAUER AND THOM

1. Background in homological algebra

Throughout this section, R is a unital ring and V is a left R-module.

Definition. A *complex* $\mathbb{V}$ consists of sequences of modules and morphisms

$$\mathbb{V}: \qquad \cdots \longrightarrow V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \longrightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n. The n-th homology module of $\mathbb{V}$ is defined to be $H_n(\mathbb{V}) = \ker \partial_n / \operatorname{ran} \partial_{n+1}$. The complex $\mathbb{V}$ is *exact* if $H_n(\mathbb{V}) = 0$ for all n.

A *morphism* $\boldsymbol{\varphi}\colon \mathbb{V} \to \mathbb{W}$ consists of a sequence of morphisms $\varphi_n\colon V_n \to W_n$ such that $\varphi_n \circ \partial_{n+1} = \partial'_{n+1} \circ \varphi_{n+1}$ for all n. Since $\varphi_n(\operatorname{ran} \partial_{n+1}) \subset \operatorname{ran} \partial'_{n+1}$ and $\varphi_n(\ker \partial_{n+1}) \subset \ker \partial'_{n+1}$, the morphism $\boldsymbol{\varphi}$ induces morphisms $\varphi_{*,n}\colon H_n(\mathbb{V}) \to H_n(\mathbb{W})$.

A morphism $\boldsymbol{\varphi}\colon \mathbb{V} \to \mathbb{W}$ is *null-homotopic* if there is a sequence of morphisms $h_n\colon V_n \to W_{n+1}$ such that $\varphi_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V_{n+1} & \xrightarrow{\partial_{n+1}} & V_n & \xrightarrow{\partial_n} & V_{n-1} & \longrightarrow & \cdots \\
 & \swarrow & \downarrow{\scriptstyle \varphi_{n+1}} & \swarrow{\scriptstyle h_n} & \downarrow{\scriptstyle \varphi_n} & \swarrow{\scriptstyle h_{n-1}} & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
\cdots & \longrightarrow & W_{n+1} & \xrightarrow{\partial'_{n+1}} & W_n & \xrightarrow{\partial'_n} & W_{n-1} & \longrightarrow & \cdots
\end{array}$$

Morphisms $\boldsymbol{\varphi}$, $\boldsymbol{\psi}\colon \mathbb{V} \to \mathbb{W}$ are *homotopic* if $\boldsymbol{\varphi} - \boldsymbol{\psi}$ is null-homotopic.

Lemma 1. *If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are homotopic, then $\varphi_{*,n} = \psi_{*,n}$ for all n.*

Proof. If $\boldsymbol{\varphi}$ is null-homotopic, then $\varphi_n(\ker \partial_n) = (\partial'_{n+1} \circ h_n)(\ker \partial_n) \subset \operatorname{ran} \partial'_{n+1}$ and hence $\varphi_{*,n} = 0$. The general case follows from this. □

Theorem 2. *Let complexes $\mathbb{V}$, $\mathbb{W}$ and a morphism $\varphi\colon V \to W$ be given*

$$\begin{array}{lccccccccc}
\mathbb{V}: & \cdots \longrightarrow & V_n & \xrightarrow{\partial_n} & V_{n-1} & \longrightarrow & \cdots & \longrightarrow V_0 & \twoheadrightarrow & V \\
 & & & & & & & & & \downarrow{\scriptstyle \varphi} \\
\mathbb{W}: & \cdots \longrightarrow & W_n & \xrightarrow{\partial'_n} & W_{n-1} & \longrightarrow & \cdots & \longrightarrow W_0 & \twoheadrightarrow & W
\end{array}$$

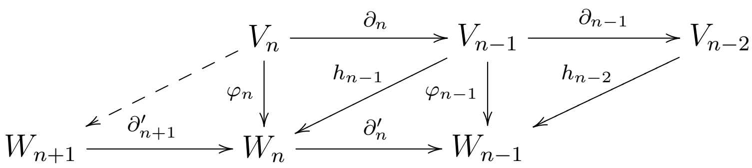
such that every V_n ($n \geq 0$) is projective and $\mathbb{W}$ is exact. Then, there exists a morphism $\boldsymbol{\varphi}\colon \mathbb{V} \to \mathbb{W}$ which extends φ. Moreover, the extension $\boldsymbol{\varphi}$ is unique up to homotopy.

Proof. (Existence.) We proceed by induction. Let $\varphi_{-1} = \varphi$ and $\varphi_{-2} = 0$, and suppose we have constructed $\varphi_{-2}, \cdots, \varphi_{n-1}$ satisfying $\varphi_{m-2} \circ \partial_{m-1} = \partial'_{m-1} \circ \varphi_{m-1}$ for $m \le n$:

$$\begin{CD} V_n @>\partial_n>> V_{n-1} @>\partial_{n-1}>> V_{n-2} \\ @VVV @V\varphi_{n-1}VV @V\varphi_{n-2}VV \\ W_n @>\partial'_n>> W_{n-1} @>\partial'_{n-1}>> W_{n-2} \end{CD}$$

Since $\partial'_{n-1} \circ \varphi_{n-1} \circ \partial_n = \varphi_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$, one has $\operatorname{ran} \varphi_{n-1} \circ \partial_n \subset \operatorname{ran} \partial'_n$ by exactness. Since V_n is projective, there is a morphism $\varphi_n \colon V_n \to W_n$ which lifts $\varphi_{n-1} \circ \partial_n$ through ∂'_n, i.e., $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n$.

(Uniqueness.) It suffices to show that any extension $\boldsymbol{\varphi}$ of $\varphi = 0$ is null-homotopic. Let $h_{-1} = 0$ and $h_{-2} = 0$, and suppose we have constructed $h_{-2}, \dots, h_{n-1}$ satisfying $\varphi_{m-1} = \partial'_m \circ h_{m-1} + h_{m-2} \circ \partial_{m-1}$ for $m \le n$:



Since $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n = (\partial'_n \circ h_{n-1} + h_{n-2} \circ \partial_{n-1}) \circ \partial_n = \partial'_n \circ h_{n-1} \circ \partial_n$, one has $\operatorname{ran}(\varphi_n - h_{n-1} \circ \partial_n) \subset \operatorname{ran} \partial'_{n+1}$ by exactness. Since V_n is projective, there is a morphism $h_n \colon V_n \to W_{n+1}$ such that $\partial'_{n+1} \circ h_n = \varphi_n - h_{n-1} \circ \partial_n$. $\square$

Definition. For a module V, a *projective resolution* of V is an exact complex

$$\mathbb{V}: \qquad \cdots \longrightarrow V_n \longrightarrow \ \cdots \ \longrightarrow V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} V \longrightarrow 0$$

with all V_n $(n \ge 0)$ projective.

Definition. For a right R-module M and a left R-module V, define

$$\operatorname{Tor}_n^R(M, V) = H_n(M \otimes_R \mathbb{V}_{\ge 0}),$$

where $\mathbb{V}$ is any projective resolution of V and $M \otimes_R \mathbb{V}_{\ge 0}$ is the complex

$$M \otimes_R \mathbb{V}_{\ge 0}: \qquad \cdots \longrightarrow M \otimes_R V_n \longrightarrow \ \cdots \ \longrightarrow M \otimes_R V_1 \xrightarrow{\partial_1} M \otimes_R V_0 \longrightarrow 0.$$

Note that $M \otimes_R \mathbb{V}_{\ge 0}$ is given by omitting the term $M \otimes_R V$ from $M \otimes_R \mathbb{V}$.

Remark. Every module V has a projective (or even free) resolution, and the projective resolution is unique up to homotopy. It follows that the complex $M \otimes_R \mathbb{V}_{\ge 0}$ used to define $\operatorname{Tor}_\bullet^R(M, V)$ is also unique up to homotopy and hence $\operatorname{Tor}_\bullet^R(M, V)$ does not depends on the choice of a projective resolution of V.

We recall that the relative tensor product $M \otimes_R V$ is defined to be the $\mathbb{Z}$-module generated by $\{a \otimes \xi : a \in M,\ \xi \in V\}$ and factored out by the relations $a\otimes\xi + b\otimes\xi - (a+b)\otimes\xi$, $a\otimes\xi + a\otimes\eta - a\otimes(\xi+\eta)$, and $ar\otimes\xi - a\otimes r\xi$. If M is an S-R-module, then $M \otimes_R V$ is naturally a left S-module. We note that the relative tensor product operation $\otimes_R$ is associative and distributive w.r.t. a direct sum.

Examples. $M \otimes_R R = M$ and $R \otimes_R V = V$.

The module $\operatorname{Tor}_n^R(M,V)$ can be non-zero because $M \otimes_R \cdot$ needs not be a short exact functor. Namely, $V_2 \rightarrowtail V_1$ does not imply $M \otimes_R V_2 \rightarrowtail M \otimes_R V_1$. (The symbol $\rightarrowtail$ is used for injection.) However the functor $M \otimes_R \cdot$ is always right exact.

Lemma 3 (Right exactness). *Let M be arbitrary. If $V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \to 0$ is exact, then $M \otimes_R V_2 \xrightarrow{\mathrm{id}\otimes\partial_2} M \otimes_R V_1 \xrightarrow{\mathrm{id}\otimes\partial_1} M \otimes_R V_0 \to 0$ is exact.*

Proof. Exactness at $M \otimes_R V_0$ is clear. Since $(\mathrm{id}\otimes\partial_1)\circ(\mathrm{id}\otimes\partial_2) = \mathrm{id}\otimes(\partial_1\circ\partial_2) = 0$, the morphism $\mathrm{id}\otimes\partial_1$ induces a morphism $\tilde{\partial}_1 \colon M \otimes_R V_1/\operatorname{ran}(\mathrm{id}\otimes\partial_2) \twoheadrightarrow M \otimes_R V_0$. It is left to show that $\tilde{\partial}_1$ is injective. For this, it suffices to construct the left inverse σ of $\tilde{\partial}_1$: For $\sum a_i \otimes \xi_i \in M \otimes_R V_0$, define $\sigma(\sum a_i\otimes\xi_i) = \sum a_i \otimes \tilde{\xi}_i + \operatorname{ran}(\mathrm{id}\otimes\partial_2)$, where $\tilde{\xi}_i \in V_1$ is any lift of ξ_i. Then, σ is a well-defined morphism with $\sigma\circ\tilde{\partial}_1 = \mathrm{id}$. □

Definition. A right S-module N is *flat* if $N \otimes_S \cdot$ is an exact functor.

Note that free modules and projective modules are flat.

Lemma 4. *For a right S-module N, the following are equivalent.*

(1) N is flat.
(2) $\ker(\mathrm{id}\otimes\varphi) = N \otimes_S \ker\varphi$ for any morphism $\varphi\colon W \to V$.
(3) $H_\bullet(N \otimes_S \mathbb{V}) = N \otimes_S H_\bullet(\mathbb{V})$ for any complex $\mathbb{V}$ of S-modules.
(4) $N \otimes_S V \rightarrowtail N \otimes_S F$ for every f.g. modules $V \subset F$ with F free.

In particular, if N is flat, then for any S-R-module M and any left R-module V,

$$N \otimes_S \operatorname{Tor}_\bullet^R(M,V) = \operatorname{Tor}_\bullet^R(N \otimes_S M, V).$$

Proof. It is routine to check the equivalence of the conditions (1)–(3). (Use right exactness.) We only prove the implication (4)⇒(1). We first observe that the f.g. assumption on V and F can be dropped by continuity of a tensor product w.r.t. inductive limits. Let $\iota\colon W_1 \hookrightarrow W_2$ be given. We will show $N \otimes_S W_1 \rightarrowtail N \otimes_S W_2$. Take a free S-module F and a surjection $\pi\colon F \twoheadrightarrow W_2$, and set $V = \ker\pi$. Then, we have a commuting diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & \ker(\mathrm{id}\otimes\iota) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & N\otimes_S V & \longrightarrow & N\otimes_S \pi^{-1}(W_1) & \longrightarrow & N\otimes_S W_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N\otimes_S V & \longrightarrow & N\otimes_S F & \longrightarrow & N\otimes_S W_2 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

which is exact everywhere. By Snake Lemma, one has $\ker(\mathrm{id}\otimes\iota) = 0$. □

For the later purpose, we need the following. A (full) subcategory $\mathcal{D}$ of modules is a *Serre subcategory* if for every short exact sequence $0 \to V_2 \to V_1 \to V_0 \to 0$, one has $V_1 \in \mathcal{D} \Leftrightarrow V_0, V_2 \in \mathcal{D}$. A morphism $\varphi\colon V \to W$ is an *isomorphism modulo* $\mathcal{D}$ if both $\ker\varphi$ and $\operatorname{coker}\varphi = V/\operatorname{ran}\varphi$ are in $\mathcal{D}$.

Lemma 5. *Let $\mathcal{D}$ be a Serre subcategory. Let $\mathbb{V}$ and $\mathbb{W}$ be complexes of modules and $\boldsymbol{\varphi}\colon \mathbb{V} \to \mathbb{W}$ be a morphism consisting of isomorphisms modulo $\mathcal{D}$. Then all $\varphi_{*,\bullet}\colon H_\bullet(\mathbb{V}) \to H_\bullet(\mathbb{W})$ are also isomorphisms modulo $\mathcal{D}$.*

Proof. Consider the following commuting exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\partial_n & \hookrightarrow & V_n & \xrightarrow{\partial_n} & \operatorname{ran}\partial_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n-1}} & & \\
0 & \longrightarrow & \ker\partial'_n & \hookrightarrow & W_n & \xrightarrow{\partial'_n} & \operatorname{ran}\partial'_n & \longrightarrow & 0
\end{array}$$

Since φ_n is an isomorphism modulo $\mathcal{D}$ and $\ker\varphi_{n-1} \cap \operatorname{ran}\partial_n$ is in $\mathcal{D}$, Snake Lemma implies that other two column morphisms are also isomorphisms modulo $\mathcal{D}$. Now, applying Snake Lemma again to the following commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{ran}\partial_{n+1} & \hookrightarrow & \ker\partial_n & \longrightarrow & H_n(\mathbb{V}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{*,n}} & & \\
0 & \longrightarrow & \operatorname{ran}\partial_{n+1} & \hookrightarrow & \ker\partial_n & \longrightarrow & H_n(\mathbb{W}) & \longrightarrow & 0
\end{array}$$

one sees that $\varphi_{*,n}$ is an isomorphism modulo $\mathcal{D}$. □

2. Dimension function (after Lück)

Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and recall that $\mathrm{Proj}(\mathcal{M})$ is a lattice such that $\tau(p)+\tau(q)=\tau(p\vee q)+\tau(p\wedge q)$ for every $p,q\in\mathrm{Proj}(\mathcal{M})$. Throughout this section, a module means a left $\mathcal{M}$-module. Note that

$$\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})=\mathbb{M}_{m,n}(\mathcal{M}) \text{ by the right multiplication.}$$

Definition. A module V is *finitely generated and projective* (abbreviated as f.g.p.) if $V\cong\mathcal{M}^{\oplus m}P$ for some $m\in\mathbb{N}$ and some idempotent $P\in\mathbb{M}_m(\mathcal{M})$.

Remark. In the original definition, a module V is *projective* if every surjection onto it splits. We note that a concrete realization $\mathcal{M}^{\oplus m}P$ of V is *not* among the structures of V. We can take P to be self-adjoint, because if we set $P_0=l(P)$, then $P=SP_0S^{-1}$ for $S=I+P_0-P$. For the following, we generally assume that P is self-adjoint.

A ring R is said to be "semi-hereditary" if every f.g. R-submodule of a free R-module is projective. Every von Neumann algebra has this property.

Lemma 6. *(1) Every weakly closed submodule V of $\mathcal{M}^{\oplus m}$ is of the form $\mathcal{M}^{\oplus m}P$.*
(2) For every $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})$, both $\ker\varphi$ and $\mathrm{ran}\,\varphi$ are f.g.p.
(3) Every f.g. submodule V of $\mathcal{M}^{\oplus m}$ is projective.

Proof. Ad(1): One observes that $V=\mathcal{M}^{\oplus m}P$ for the orthogonal projection P in $\mathbb{M}_m(\mathcal{M})$ from $L^2\mathcal{M}^{\oplus m}$ onto the L^2-norm closure of V.
Ad(2): $\ker\varphi=\mathcal{M}^{\oplus m}P$ by (1) and $\mathrm{ran}\,\varphi\cong\mathcal{M}^{\oplus m}P^{\perp}$ by Isomorphism Theorem.
Ad(3): If V is f.g., then $V=\mathrm{ran}\,\varphi$ for some $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus n},\mathcal{M}^{\oplus m})$. □

Definition. For a f.g.p. module $V\cong\mathcal{M}^{\oplus m}P$, define $\dim_{\mathcal{M}}V=(\mathrm{Tr}\otimes\tau)(P)$.

Remark. The $\mathcal{M}$-dimension $\dim_{\mathcal{M}}V$ is well-defined: If $\mathcal{M}^{\oplus m}P\cong\mathcal{M}^{\oplus n}Q$, then $(\mathrm{Tr}\otimes\tau)(P)=(\mathrm{Tr}\otimes\tau)(Q)$. In particular, if $W\cong V$ (resp. $W\subset V$) are f.g.p. modules, then $\dim_{\mathcal{M}}W=\dim_{\mathcal{M}}V$ (resp. $\dim_{\mathcal{M}}W\le\dim_{\mathcal{M}}V$).

Definition. For every module V, we define the $\mathcal{M}$-dimension of V by

$$\dim_{\mathcal{M}}V=\sup\{\dim_{\mathcal{M}}W : W\subset V \text{ f.g.p. submodule}\}\in[0,\infty].$$

Note that the definitions are consistent for f.g.p. modules. The dimension function is *continuous* in the following sense: if $V=\bigcup V_i$ is a directed union of modules, then one has $\dim_{\mathcal{M}}V=\lim\dim_{\mathcal{M}}V_i$.

For $V\subset\mathcal{M}^{\oplus m}$, we denote by $\overline{V}$ the weak closure of V. Although there is a way defining $\overline{V}$ purely algebraically for arbitrary module V, we do not elaborate it.

Proposition 7. *Let $V\subset\mathcal{M}^{\oplus m}$ be a submodule with $\overline{V}=\mathcal{M}^{\oplus m}P$. Then, there exists a net of projections $P_i\in\mathbb{M}_m(\mathcal{M})$ such that $\mathcal{M}^{\oplus m}P_i\subset V$ and $P_i\to P$. In particular, one has $\dim_{\mathcal{M}}V=\dim_{\mathcal{M}}\overline{V}$.*

Proof. Let $V \subset \mathcal{M}^{\oplus m}$ be given. Let $i = (W, \varepsilon)$ be a pair of f.g. submodule $W \subset V$ and $\varepsilon > 0$. We choose $n \in \mathbb{N}$ and $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $W = \mathcal{M}^{\oplus n}T$, and $\delta > 0$ such that $P_i = \chi_{[\delta,1]}(T^*T) \in \mathbb{M}_m(\mathcal{M})$ satisfies $\tau(r(T) - P_i) < \varepsilon$. Since $P_i = ST$ for $S = \chi_{[\delta,1]}(T^*T)(T^*T)^{-1}T^* \in \mathbb{M}_{m,n}(\mathcal{M})$, we have $\mathcal{M}^{\oplus m}P_i \subset \mathcal{M}^{\oplus n}T \subset V$. It is not hard to see $P_i \nearrow P$. This implies that $\dim_{\mathcal{M}} V \geq \sup \dim_{\mathcal{M}} \mathcal{M}^{\oplus m}P_i = \dim_{\mathcal{M}} \overline{V}$. The converse inequality is trivial. □

Theorem 8 (Lück)**.** *For every short exact sequence* $0 \to V_2 \xrightarrow{\iota} V_1 \xrightarrow{\pi} V_0 \to 0$, *one has* $\dim_{\mathcal{M}} V_1 = \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$.

Proof. Let $W \subset V_0$ be any f.g.p. submodule. Then, one has $\pi^{-1}(W) \cong W \oplus \iota(V_2)$ by the projectivity of W. Hence,

$$\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} \pi^{-1}(W) \geq \dim_{\mathcal{M}} W + \dim_{\mathcal{M}} \iota(V_2).$$

Taking the supremum over all $W \subset V_0$, one gets $\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$. In particular, we have proved that $\dim_{\mathcal{M}}$ decreases under a surjection. To prove the converse inequality, let $W \subset V_1$ be any f.g.p. submodule. We realize W as $\mathcal{M}^{\oplus m}P$. Then, one has $\overline{\iota(V_2) \cap W} = \mathcal{M}^{\oplus m}Q$ for some projection $Q \in \mathbb{M}_m(\mathcal{M})$ with $Q \leq P$. This implies that $W/\overline{\iota(V_2) \cap W} \cong \mathcal{M}^{\oplus m}(P - Q)$. It follows by Proposition 7 that

$$\begin{aligned}\dim_{\mathcal{M}} W &= \dim_{\mathcal{M}} W/\overline{\iota(V_2) \cap W} + \dim_{\mathcal{M}} \overline{\iota(V_2) \cap W}\\ &\leq \dim_{\mathcal{M}} W/(\iota(V_2) \cap W) + \dim_{\mathcal{M}} \iota(V_2) \cap W\\ &\leq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} \iota(V_2),\end{aligned}$$

where we have applied the first part to $W/(\iota(V_2) \cap W) \twoheadrightarrow W/\overline{\iota(V_2) \cap W}$. □

We call V a *torsion module* if $\dim_{\mathcal{M}} V = 0$. Torsion modules form a Serre subcategory and every module V has the unique largest torsion submodule $V_T \subset V$.

Corollary 9. *For every f.g. module V, one has $V \cong V_P \oplus V_T$, where V_P is f.g.p. with* $\dim_{\mathcal{M}} V_P = \dim_{\mathcal{M}} V$.

Proof. We prove that the f.g. module $V_P = V/V_T$ is projective (and hence there is a splitting $V_P \hookrightarrow V$). Take a surjection $\varphi\colon \mathcal{M}^{\oplus m} \twoheadrightarrow V_P$. Since $\overline{\ker\varphi}/\ker\varphi$ is a torsion submodule of $\mathcal{M}^{\oplus m}/\ker\varphi \cong V_P$, it is zero. It follows that $\ker\varphi$ is closed and $V_P \cong \mathcal{M}^{\oplus m}/\ker\varphi$ is projective. □

Although we do not use it explicitly, this corollary, in combination with continuity, is useful to reduce the proof of dimensional equations to those for f.g.p. modules.

Definition. A morphism $\varphi\colon V \to W$ is a $\dim_{\mathcal{M}}$-*isomorphism* if it is an isomorphism modulo torsion modules, i.e., $\dim_{\mathcal{M}} \ker\varphi = 0 = \dim_{\mathcal{M}} \operatorname{coker}\varphi$.

Lemma 10. *The morphism $\mathcal{M} \hookrightarrow L^2\mathcal{M}$ is a $\dim_{\mathcal{M}}$-isomorphism.*

Proof. Let $\xi \in L^2\mathcal{M}$ be given. We view it as a closed square-integrable operator affiliated with $\mathcal{M}$. Then, for $p_n = \chi_{[0,n]}(\xi\xi^*) \in \mathcal{M}$, one has $p_n \to 1$ and $p_n\xi \in \mathcal{M}$. We note that $p_n\xi \in \mathcal{M}$ means that $p_n\xi = 0$ in $L^2\mathcal{M}/\mathcal{M}$. □

Remark. From this lemma, one observes that $\dim_{\mathcal{M}}$ agrees with the von Neumann dimension function for normal Hilbert $\mathcal{M}$-modules.

3. Definition of the ℓ_2-Betti numbers (after Lück)

Definition. For a discrete group Γ, we define the n-th ℓ_2-Betti number of Γ by

$$\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(\mathcal{L}\Gamma, \mathbb{C}),$$

where $\mathbb{C}$ is the trivial $\mathbb{C}\Gamma$-module: $f \cdot z = \sum_{s\in\Gamma} f(s)z$.

Exercise. Prove that $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C})$. (Hint: You have to show that the functor $\ell_2\Gamma \otimes_{\mathcal{L}\Gamma} \cdot$ is exact and $\dim_{\mathcal{L}\Gamma}$-preserving.)

Example. For $d = 1, 2, \ldots$, one has

$$\beta_n^{(2)}(\mathbb{F}_d) = \begin{cases} d-1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $g_1, \ldots, g_d$ be the canonical generators of $\mathbb{F}_d$. We consider the complex

$$\mathbb{V}: \qquad 0 \longrightarrow (\mathbb{C}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathbb{C}\mathbb{F}_d \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0,$$

where $\partial_0(\xi) = \sum_{s\in\mathbb{F}_d} \xi(s)$ and $\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i$. (We define $(\xi \cdot s)(t) = \xi(ts)$.) We will show that the complex $\mathbb{V}$ is exact. We check $\ker \partial_1 = 0$. Let $\chi_j \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words starting at g_j. It is not hard to see that $\chi_j \cdot g_i^{-1} = \chi_j + \delta_{i,j}\delta_e$ for every i, j. If $(\xi_i)_{i=1}^d \in \ker \partial_1$, then for every $s \in \Gamma$ and j, one has

$$0 = \langle \sum_{i=1}^d \xi_i \cdot g_i - \xi_i,\, s \cdot \chi_j \rangle = \sum_{i=1}^d \langle \xi_i,\, s \cdot (\chi_j \cdot g_i^{-1} - \chi_j) \rangle = \xi_j(s)$$

and $(\xi_i)_{i=1}^d = 0$. We next check $\operatorname{ran} \partial_1 = \ker \partial_0$. It is easy to see $\partial_0 \circ \partial_1 = 0$. Let $\chi_i^\vee \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words ending at g_i^{-1}. We observe that $\chi_i - s \cdot \chi_i^\vee$ is finitely supported for every $s \in \mathbb{F}_d$. (Indeed, it suffices to check this for $g_1, \ldots, g_d$.) Moreover, since $\chi_i^\vee \cdot g_i - \chi_i^\vee$ is the characteristic function of the reduced words ending at other than $g_i^{\pm 1}$, one has $\sum_{i=1}^d \chi_i^\vee \cdot g_i - \chi_i^\vee = (d-1)\mathbb{1} + \delta_e$. Now, suppose $\xi \in \ker \partial_0$. Then, since

$$\xi = -(\sum_{s\neq e} \xi(s)\delta_e) + \sum_{s\neq e} \xi(s)\delta_s = \sum_{s\neq e} \xi(s)(\delta_e - \delta_s),$$

$\xi_i = \xi * \chi_i^\vee \in \mathbb{C}\Gamma$ by the above observation, and since $\xi * \mathbb{1} = 0$, one has

$$\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i = \sum_{i=1}^d \xi * (\chi_i^\vee \cdot g_i - \chi_i^\vee) = \xi.$$

We have proved that $\mathbb{V}$ is a projective resolution of $\mathbb{C}$. Since

$$\mathcal{L}\mathbb{F}_d \otimes_{\mathbb{C}\mathbb{F}_d} \mathbb{V}_{\geq 0} : \qquad 0 \longrightarrow (\mathcal{L}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathcal{L}\mathbb{F}_d \longrightarrow 0,$$

one has

$$\operatorname{Tor}_n^{\mathbb{C}\mathbb{F}_d}(\mathcal{L}\mathbb{F}_d, \mathbb{C}) = \begin{cases} \mathcal{L}\mathbb{F}_d / \operatorname{ran}\partial_1 & \text{if } n = 0 \\ \ker \partial_1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Since $\lambda(s) - \lambda(t) \in \operatorname{ran}\partial_1$ and $\lambda(t) \to 0$ weakly as $t \to \infty$, one has $\lambda(s) \in \overline{\operatorname{ran}\partial_1}$ for every $s \in \mathbb{F}_d$ and hence $\overline{\operatorname{ran}\partial_1} = \mathcal{L}\mathbb{F}_d$. It follows that $\beta_0^{(2)}(\mathbb{F}_d) = 0$ and $\beta_1^{(2)}(\mathbb{F}_d) = \dim_{\mathcal{L}\mathbb{F}_d} \ker \partial_1 = d - \dim_{\mathcal{L}\mathbb{F}_d} \operatorname{ran}\partial_1 = d - 1$. □

Below, we sketch an argument showing that the above definition of ℓ_2-Betti numbers is consistent with another(?). We denote by $\mathcal{F}(\Gamma, X)$ the set of functions from a set Γ into X. Now Γ be a discrete group and consider ℓ_Γ as a right Γ-module. The is a natural complex

$$0 \longrightarrow \ell_2\Gamma \xrightarrow{\partial_0} \mathcal{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial_1} \mathcal{F}(\Gamma^2, \ell_2\Gamma) \longrightarrow \cdots,$$

where $(\partial_0 f)(s) = f - f \cdot s$ and $(\partial_1 b)(s,t) = b(t) - b(st) + b(s) \cdot t$, etc. We then define the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ by $H_n(\Gamma, \ell_2\Gamma) = \ker \partial_n / \operatorname{ran}\partial_{n+1}$. Since ∂_n commutes with the $\mathcal{L}\Gamma$-action on $\ell_2\Gamma$, the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ is naturally an $\mathcal{L}\Gamma$-module. We define $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} H_n(\Gamma, \ell_2\Gamma)$. Let us calculate $\beta_n^{(2)}(\Gamma)$ for $n = 0, 1$. Since $H_0 \subset \ell_2\Gamma$ is the subspace of constant functions, one has $\beta_0^{(2)}(\Gamma) = |\Gamma|^{-1}$. We note that $D(\Gamma) = \ker \partial_1$ is the space of derivations and $D_0(\Gamma) = \operatorname{ran}\partial_0$ is the space of inner derivations. To see what $\beta_1^{(2)}(\Gamma)$ is, we assume that Γ is generated by a finite subset $\{s_1, \ldots, s_d\}$. Then, there is an $\mathcal{L}\Gamma$-module map

$$D(\Gamma) \ni b \longmapsto (b(s_i))_{i=1}^d \in \bigoplus_{i=1}^d \ell_2\Gamma,$$

which is an isomorphism onto a closed subspace. We note that $D_0(\Gamma)$ is closed in $\bigoplus_{i=1}^d \ell_2\Gamma$ iff Γ is finite or non-amenable, and that $\dim_{\mathcal{L}\Gamma} \overline{D_0(\Gamma)} = \dim_{\mathcal{L}\Gamma}(\ker \partial_0)^\perp = 1 - |\Gamma|^{-1}$. Hence, one has $\dim_{\mathcal{L}\Gamma} D(\Gamma) = \beta_1^{(2)}(\Gamma) + \dim_{\mathcal{L}\Gamma} D_0(\Gamma) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$. We view ∂_0 as a map from $\ell_2\Gamma$ into $\bigoplus_{i=1}^d \ell_2\Gamma$ and consider

$$\partial_0^* \colon \bigoplus_{i=1}^d \ell_2\Gamma \ni (\xi_i) \longmapsto \sum_i \xi_i - \xi_i \cdot s_i^{-1} \in \ell_2\Gamma.$$

Lemma 11. *One has* $(\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp = D(\Gamma)$.

Proof. We note that the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ is defined consistently on $\mathbb{C}\Gamma \times \mathcal{F}(\Gamma,\mathbb{C})$ and on $\ell_2\Gamma \times \ell_2\Gamma$. Moreover, $\mathcal{F}(\Gamma,\mathbb{C})$ is the algebraic dual of $\mathbb{C}\Gamma$ w.r.t. this scalar product. Suppose that $b \in D(\Gamma)$. It is not hard to show that $b = f - f\cdot s$ for some $f \in \mathcal{F}(\Gamma,\mathbb{C})$. It follows that for every $\xi \in \ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma$, one has

$$\langle \xi, b\rangle = \sum \langle \xi_i, b(s_i)\rangle = \sum_i \langle \xi_i - \xi_i \cdot s_i^{-1}, f\rangle = 0.$$

Conversely, if $b \in \ell_2\Gamma$ is such that $b \perp (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)$, then the linear functional $\langle\,\cdot\,,b\rangle$ on $\bigoplus \mathbb{C}\Gamma$ factors through ∂_0^* and there is $f \in \mathcal{F}(\Gamma,\mathbb{C})$ such that $\langle \xi, b\rangle = \langle \partial_0^*(\xi), f\rangle$ for every $\xi \in \bigoplus \mathbb{C}\Gamma$. It follows that $b(s) = f - f\cdot s$ and $b \in D(\Gamma)$. □

Since $(\ker \partial_0^*)^\perp = \overline{\mathrm{ran}}\partial_0 = \overline{D_0(\Gamma)}$, one has

$$\begin{aligned} D(\Gamma)/\overline{D_0(\Gamma)} &\cong (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \ominus (\ker \partial_0^*)^\perp \\ &= \ker \partial_0^* \cap (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \quad \cong \mathrm{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}). \end{aligned}$$

The last isomorphism follows from the following observation:

$$\mathbb{V}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^{d} \mathbb{C}\Gamma \xrightarrow{\partial_0^*} \mathbb{C}\Gamma \longrightarrow \mathbb{C}$$

is a free resolution of the trivial left $\mathbb{C}\Gamma$-module $\mathbb{C}$ and

$$\ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \mathbb{V}_{\geq 0}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^{d} \ell_2\Gamma \xrightarrow{\partial_0^*} \ell_2\Gamma \longrightarrow 0$$

with $\mathrm{ran}\, \partial_1^* = \ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma}) \subset \overline{\mathbb{C}\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})} = \overline{\ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})}$.

4. Rank metric (after Thom)

Definition. Let V be a left $\mathcal{M}$-module. For $\xi \in V$, we define its *rank norm* by

$$[\xi] = \inf\{\tau(p) : p \in \mathrm{Proj}(A),\ p\xi = \xi\} \in [0,1].$$

We record several basic properties of the rank norm.

Lemma 12. *For a left $\mathcal{M}$-module V, the following are true.*

1. *Triangle inequality:* $[\xi + \eta] \leq [\xi] + [\eta]$ *for every* $\xi, \eta \in V$.
2. $[x\xi] \leq \min\{[x], [\xi]\}$ *for every* $x \in \mathcal{M}$ *and* $\xi \in V$.
3. $V_T = \{\xi \in V : [\xi] = 0\}$.
4. *A submodule* $W \subset V$ *is dense in rank norm if and only if* $\dim_{\mathcal{M}} V/W = 0$.
5. *Every* $\varphi \in \mathrm{Mor}(V,W)$ *is a rank contraction:* $[\varphi(\xi)] \leq [\xi]$.

(6) *For every* $\varphi \in \mathrm{Mor}(V, W)$, $\eta \in \operatorname{ran} \varphi$ *and* $\varepsilon > 0$, *there exists* $\xi \in \varphi^{-1}(\eta)$ *such that* $[\eta] \leq [\xi] + \varepsilon$.

Proof. The triangle inequality follows from the fact that $\tau(p \vee q) \leq \tau(p) + \tau(q)$. The second assertion follows from the fact that $p\xi = \xi$ implies $xp\xi = \xi$ and $[x\xi] \leq \tau(l(xp)) \leq \tau(p)$. For the third assertion, we observe that $[\xi] = 0$ iff $\mathcal{M}\xi$ is a torsion submodule. Indeed, the "if" part is rather easy and the "only if" part follows by considering the morphism $\varphi\colon M \ni x \mapsto x\xi \in V$. Since $\ker \varphi$ is a left ideal with $\dim_{\mathcal{M}} \ker \varphi = 1$, i.e., $\overline{L} = \mathcal{M}$, Proposition 7 implies that there is a net $p_i \in L$ of projections such that $p_i \to 1$. This means $[\xi] = 0$. The rest are trivial. □

We recall that the *completion* of a metric space (X, d) is the metric space of all equivalence classes of Cauchy sequences in X. Here, two Cauchy sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if $d((x_n), (y_n)) := \lim_n d(x_n, y_n) = 0$.

Definition. The *rank completion* of a left $\mathcal{M}$-module V is the completion $C(V)$ of V w.r.t. the *rank metric* d, where $d(\xi, \eta) = [\xi - \eta]$ for $\xi, \eta \in V$. We observe that

$$C(V) = \{\text{ Cauchy sequences in V }\}/\{\text{ Null sequences }\}$$

and that $C(V)$ is naturally a left $\mathcal{M}$-module (thanks to Lemma 12).

The rank metric is actually a pseudo-metric. More precisely, it is a metric on V/V_T. The constant "embedding" $c\colon V \to C(V)$ is a $\dim_{\mathcal{M}}$-isomorphism and it induces a canonical inclusion $V/V_T \hookrightarrow C(V)$. Moreover, $C(V)$ is the unique torsion-free complete $\mathcal{M}$-module containing V/V_T as a dense submodule. Indeed, one has:

Lemma 13. *Let* V, W *be* $\mathcal{M}$*-modules with* W *torsion-free and complete. Then, every* $\varphi \in \mathrm{Mor}(V, W)$ *extends to* $\tilde{\varphi} \in \mathrm{Mor}(C(V), W)$, *i.e.,* $\tilde{\varphi} \circ c = \varphi$.

Proposition 14. *The rank completion* c *is an exact functor.*

Proof. Let a short exact sequence $0 \longrightarrow V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \longrightarrow 0$ be given.

Exactness at $C(V_0)$**.** Let $\xi \in C(V_0)$ and choose a representing Cauchy sequence $(\xi_n)_n$ in V_0 such that $d(\xi_n, \xi_{n+1}) < 2^{-(n+1)}$. We will construct $\eta_1, \eta_2, \ldots$ such that $\partial_1(\eta_n) = \xi_n$ and $d(\eta_n, \eta_{n+1}) < 2^{-n}$. Suppose $\eta_1, \ldots, \eta_n$ have been chosen. Lift $\xi_{n+1} - \xi_n \in V_0$ to $\zeta_{n+1} \in V_1$ with $[\zeta_{n+1}] \leq [\xi_{n+1} - \xi_n] + 2^{-(n+1)}$. Set $\eta_{n+1} = \eta_n + \zeta_{n+1}$ and we are done. Now the sequence $(\eta_n)_n$ is Cauchy in V_1 and hence converges to an element η in $C(V_1)$ such that $\partial_1(\eta) = \xi$.

Exactness at $C(V_1)$**.** It is clear that $C(\partial_1) \circ C(\partial_2) = 0$ by continuity. Let $\xi \in \ker C(\partial_1)$ be given and choose $(\xi_n)_n$ in V_1 such that $\xi_n \to \xi$. Since $\partial_1(\xi_n) \to C(\partial_1)(\xi) = 0$, the sequence $(\partial_1(\xi_n))_n$ is null. Hence, one can lift $(\partial_1(\xi_n))_n$ to a null sequence $(\eta_n)_n$ in V_1. It follows that $(\xi_n - \eta_n)_n$ is a Cauchy sequence in $\ker \partial_1 = \operatorname{ran} \partial_2$. Therefore,

$$\xi = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} (\xi_n - \eta_n) \in \overline{\operatorname{ran} \partial_2} = \operatorname{ran} C(\partial_2),$$

where we used the result of the previous paragraph for the last equality.

Exactness at $C(V_2)$. Since ∂_2 is an isometry, $C(\partial_2)$ is an isometry as well. Since $C(V_2)$ does not have a non-zero torsion element, $C(\partial_2)$ is injective. □

5. Gaboriau's theorem (after Sauer and Thom)

Proposition 15. *Let $\mathcal{M} \subset \mathcal{N}$ be finite von Neumann algebras with $\tau_{\mathcal{M}} = \tau_{\mathcal{N}}|_{\mathcal{M}}$. Then, $\mathcal{N}$ is a flat $\mathcal{M}$-module and $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for any $\mathcal{M}$-module V.*

Proof. We use Lemma 4 to prove flatness. Let $V \subset \mathcal{M}^{\oplus m}$ be a f.g. submodule. It follows that there is $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $V = \mathcal{M}^{\oplus n}T$. Let P be the left support of T and observe that $V = \mathcal{M}^{\oplus n}T \ni \xi T \mapsto \xi P \in \mathcal{M}^{\oplus n}P$ is an isomorphism. Since $\mathcal{M}^{\oplus n}P$ is a direct summand of $\mathcal{M}^{\oplus n}$, one has the following kosher identifications

$$\mathcal{N} \otimes_{\mathcal{M}} V \cong \mathcal{N} \otimes_{\mathcal{M}} (\mathcal{M}^{\oplus n}P) \cong \mathcal{N}^{\oplus n}P \cong \mathcal{N}^{\oplus n}T \subset \mathcal{N}^{\oplus m} \cong \mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}^{\oplus m}.$$

It follows from Lemma 4 that $\mathcal{N}$ is flat.

Since the dimension function is continuous w.r.t. inductive limits, it suffices to check the identity $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for a f.g. V. Since $\mathcal{M}^{\oplus m}P \hookrightarrow V$ implies $\mathcal{N}^{\oplus m}P \hookrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$, one has $\dim_{\mathcal{M}} V \leq \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$. To prove the converse inequality, take a surjection $\pi\colon \mathcal{M}^{\oplus n} \twoheadrightarrow V$. Then, $\mathrm{id} \otimes \pi\colon \mathcal{N}^{\oplus n} \twoheadrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$ is also a surjection such that $\ker(\mathrm{id} \otimes \pi) = \mathcal{N} \otimes_{\mathcal{M}} \ker \pi$ by flatness. It follows that

$$\dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V = n - \dim_{\mathcal{N}} \ker(\mathrm{id} \otimes \pi) \leq n - \dim_{\mathcal{M}} \ker \pi = \dim_{\mathcal{M}} V$$

by the previous inequality. □

Let $\Gamma \curvearrowright (X, \mu)$ be an essentially-free probability-measure-preserving action. Let $\mathcal{A} = L^\infty(X, \mu)$, $\mathcal{M} = \mathcal{L}\Gamma$ and $\mathcal{N} = \mathcal{A} \rtimes \Gamma$. Let $R_0 \subset \mathcal{N}$ (resp. $R \subset \mathcal{N}$) be the $\mathbb{C}$-algebra generated by $\mathcal{A}$ and Γ (resp. by $\mathcal{A}$ and the full group $[\Gamma]$). Then,

$$\mathcal{A} \subset R_0 \subset R \subset \mathcal{N}$$

and $\mathcal{A}$ is a left R-module: $a\varphi \cdot f = a\varphi_*(f)$ for $a, f \in \mathcal{A}$ and $\varphi \in [\Gamma]$. Now, Gaboriau's theorem that $\beta^{(2)}_\bullet(\Gamma)$ is an invariant of $[\Gamma]$ follows from the following equalities:

$$\begin{aligned}
\beta^{(2)}_\bullet(\Gamma) &= \dim_{\mathcal{M}} \mathrm{Tor}^{\mathbb{C}\Gamma}_\bullet(\mathcal{M}, \mathbb{C}) \\
&= \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} \mathrm{Tor}^{\mathbb{C}\Gamma}_\bullet(\mathcal{M}, \mathbb{C}) && \text{by Proposition 15} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}^{\mathbb{C}\Gamma}_\bullet(\mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}, \mathbb{C}) && \text{since } \mathcal{N} \text{ is flat over } \mathcal{M} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}^{R_0}_\bullet(\mathcal{N}, \mathcal{A}) && (\spadesuit) \\
&= \dim_{\mathcal{N}} \mathrm{Tor}^{R}_\bullet(\mathcal{N}, \mathcal{A}) && (\heartsuit)
\end{aligned}$$

The proof of ($\spadesuit$) is rather routine: Since $\mathcal{N}$ is also a right R_0-module and R_0 is a free left $\mathbb{C}\Gamma$-module (Consider the conditional expectation onto $\mathcal{A}$), one has

$$\mathrm{Tor}^{\mathbb{C}\Gamma}_\bullet(\mathcal{N}, V) = \mathrm{Tor}^{R_0}_\bullet(\mathcal{N}, R_0 \otimes_{\mathbb{C}\Gamma} V)$$

for any $\mathbb{C}\Gamma$-module V. Indeed, if $\mathbb{V}$ is a projective resolution of V, then $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}$ is a projective resolution of $R_0 \otimes_{\mathbb{C}\Gamma} V$ with $\mathcal{N} \otimes_{R_0} (R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}) \cong \mathcal{N} \otimes_{\mathbb{C}\Gamma} \mathbb{V}$. We then observe that $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \cong \mathcal{A}$ as an R_0-module. The proof of ($\heartsuit$) is more involved, but reduces to the fact that $R_0 \subset R$ is dense in an appropriate sense.

We write $[\xi]_{\mathcal{A}}$ (resp. $[\xi]_{\mathcal{N}}$) for the rank norm w.r.t. $\mathcal{A}$ (resp. $\mathcal{N}$) and note that $[\xi]_{\mathcal{N}} \leq [\xi]_{\mathcal{A}}$. In particular, one has $[x]_{\mathcal{A}} = \inf\{\,\tau(p) : p \in \mathrm{Proj}(\mathcal{A}),\ px = x\}$ for $x \in \mathcal{N}$. For $x \in \mathcal{N}$, we define

$$|[x]|_{\mathcal{A}} = \sup\{\,[xp]_{\mathcal{A}}/[p]_{\mathcal{A}} : p \in \mathrm{Proj}(\mathcal{A})\} \in [0,\infty].$$

We record several basic properties of this norm.

Lemma 16. *(1) $|[\alpha x]|_{\mathcal{A}} = |[x]|_{\mathcal{A}}$ for every $\alpha \in \mathbb{C} \setminus \{0\}$ and $x \in \mathcal{N}$.*

(2) $|[v]|_{\mathcal{A}} = 1$ for every non-zero pseudo-normalizer v of $\mathcal{A}$ in $\mathcal{N}$.

(3) $|[x+y]|_{\mathcal{A}} \leq |[x]|_{\mathcal{A}} + |[y]|_{\mathcal{A}}$ and $|[xy]|_{\mathcal{A}} \leq |[x]|_{\mathcal{A}}|[y]|_{\mathcal{A}}$ for every $x, y \in \mathcal{N}$.

(4) $|[x]|_{\mathcal{A}} < \infty$ for every $x \in R$.

(5) For every $x \in R$, there is a sequence $(x_n)_n$ in R_0 such that $[x_n - x]_{\mathcal{A}} \to 0$ and $\sup |[x_n]|_{\mathcal{A}} < \infty$.

(6) If V is an R_0-module, then $[x\xi]_{\mathcal{A}} \leq |[x]|_{\mathcal{A}}[\xi]_{\mathcal{A}}$ for every $x \in R_0$ and $\xi \in V$. The same thing holds for R.

Lemma 17. *Let V be a left R_0-module. Then, the rank completion $C(V)$ w.r.t. $\mathcal{A}$ is naturally a left R-module. Moreover, C is a natural functor from the category of R_0-modules into the category of complete R-modules.*

Proof. By the previous lemma, one knows that $C(V)$ is naturally an R_0-module. Let $x \in R$ and $\xi \in C(V)$ be given. Choose a sequence $(x_n)_n$ in R_0 such that $[x - x_n]_{\mathcal{A}} \to 0$. Then, $(x_n\xi)_n$ is a Cauchy sequence in $C(V)$ and has a limit $x\xi$ in $C(V)$. We note that the limit is independent of the choice of $(x_n)_n$. Moreover, if $|[x_n]|_{\mathcal{A}}$ is bounded and $[y_m - y]_{\mathcal{A}} \to 0$, then $[x_n y_m - xy]_{\mathcal{A}} \to 0$. This shows $(xy)\xi = x(y\xi)$. ☐

Lemma 18. *Let V be a left R_0-module. Then the constant embedding*

$$\mathrm{id} \otimes c \colon \mathcal{N} \otimes_{R_0} V \to \mathcal{N} \otimes_{R_0} C(V)$$

is a $\dim_{\mathcal{N}}$-isomorphism. The same thing holds for R.

Proof. Suppose that $\sum_{i=1}^n x_i \otimes \xi_i \in \ker(\mathrm{id} \otimes c)$ and $\varepsilon > 0$ be given. Then, one has

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_j b_j r_j \otimes \eta_j - b_j \otimes r_j \eta_j \ \text{ in } \mathcal{N} \otimes_{\mathbb{C}} C(V).$$

Choose $p_j \in \mathrm{Proj}(\mathcal{A})$ such that $p_j^{\perp}\eta_j \in V$ and $n \sum (1 + |[r_j]|_{\mathcal{A}})\tau(p_j) < \varepsilon$. It follows that there is $p \in \mathrm{Proj}(\mathcal{A})$ such that $pp_j = p_j$, $pr_jp_j = r_jp_j$ and $\tau(p) < \varepsilon/n$. Since

$\sum_j b_j r_j \otimes p_j^{\perp} \eta_j - b_j \otimes r_j p_j^{\perp} \eta_j$ is zero in $\mathcal{N} \otimes_{R_0} V$, subtracting it from the both sides of the above equation, we may assume that

$$\sum_{i=1}^{n} x_i \otimes \xi_i = \sum_j b_j r_j \otimes p_j \eta_j - b_j \otimes r_j p_j \eta_j \quad \text{in } \mathcal{N} \otimes_{\mathbb{C}} C(V).$$

It follows that $\sum x_i \otimes \xi_i = \sum x_i \otimes p\xi_i$ in $\mathcal{N} \otimes_{\mathbb{C}} C(V)$, and *a fortiori* in $\mathcal{N} \otimes_{\mathbb{C}} V$ since $\mathcal{N} \otimes_{\mathbb{C}} V \subset \mathcal{N} \otimes_{\mathbb{C}} C(V)$ (recall any module over a field is free). Hence, one has

$$\sum_{i=1}^{n} x_i \otimes \xi_i = \sum_{i=1}^{n} x_i \otimes p\xi_i = \sum_{i=1}^{n} x_i p \otimes \xi_i \quad \text{in } \mathcal{N} \otimes_{R_0} V.$$

This implies that $[\sum_{i=1}^{n} x_i \otimes \xi_i]_{\mathcal{N}} \leq \sum [x_i p]_{\mathcal{N}} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, one sees that $\ker(\mathrm{id} \otimes c)$ is a torsion submodule. That $\mathrm{ran}(\mathrm{id} \otimes c)$ is dense in $\mathcal{N} \otimes_{R_0} C(V)$ follows from the fact that $[x \otimes \xi]_{\mathcal{N}} \leq [\xi]_{\mathcal{A}}$ for every $x \in \mathcal{N}$ and $\xi \in C(V)$. □

We omit the proof of the next lemma, which is similar to that of the previous one.

Lemma 19. *Let V be a left R-module, then the surjection*

$$N \otimes_{R_0} V \twoheadrightarrow N \otimes_R V$$

is a $\dim_{\mathcal{N}}$*-isomorphism.*

We are now in position to complete the proof of Gaboriau's theorem.

Proof of (♡). Let $\mathbb{V}$ (resp. $\mathbb{W}$) be a projective resolution of $\mathcal{A}$ as an R_0-module (resp. as an R-module). Then, by Theorem 2 (and Proposition 14), the identity morphism $\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ (resp. the constant embedding $c \colon \mathcal{A} \to C(\mathcal{A})$) extends to a morphism $\boldsymbol{\varphi} \colon \mathbb{V} \to \mathbb{W}$ (resp. a morphism $\boldsymbol{\psi} \colon \mathbb{W} \to C(\mathbb{V})$):

$$\begin{array}{llcccccc}
\mathbb{V}: & \cdots \longrightarrow & V_n & \longrightarrow \cdots \longrightarrow & V_0 & \longrightarrow\!\!\!\!\rightarrow & \mathcal{A} \\
& & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_0} & & \|{\scriptstyle \mathrm{id}_A} \\
\mathbb{W}: & \cdots \longrightarrow & W_n & \longrightarrow \cdots \longrightarrow & W_0 & \longrightarrow\!\!\!\!\rightarrow & \mathcal{A} \\
& & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle c} \\
C(\mathbb{V}): & \cdots \longrightarrow & C(V_n) & \longrightarrow \cdots \longrightarrow & C(V_0) & \longrightarrow\!\!\!\!\rightarrow & C(\mathcal{A}) \\
& & \downarrow{\scriptstyle \tilde{\varphi}_n} & & \downarrow{\scriptstyle \tilde{\varphi}_0} & & \|{\scriptstyle \mathrm{id}_{C(A)}} \\
C(\mathbb{W}): & \cdots \longrightarrow & C(W_n) & \longrightarrow \cdots \longrightarrow & C(W_0) & \longrightarrow\!\!\!\!\rightarrow & C(\mathcal{A})
\end{array}$$

By the uniqueness part of Theorem 2, the compositions $\boldsymbol{\psi}\circ\boldsymbol{\varphi}$ and $\tilde{\boldsymbol{\varphi}}\circ\boldsymbol{\psi}$ are homotopic to the morphisms of the constant embeddings. Taking tensor products, one has

$$\begin{array}{lccccc}
\mathcal{N}\otimes_{R_0}\mathbb{V}_{\geq 0}: & \cdots \longrightarrow & \mathcal{N}\otimes_{R_0} V_n & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_n} & \\
\mathcal{N}\otimes_{R}\mathbb{W}_{\geq 0}: & \cdots \longrightarrow & \mathcal{N}\otimes_{R} W_n & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\psi_n} & \\
\mathcal{N}\otimes_{R} C(\mathbb{V})_{\geq 0}: & \cdots \longrightarrow & \mathcal{N}\otimes_{R} C(V_n) & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\tilde{\varphi}_n} & \\
\mathcal{N}\otimes_{R} C(\mathbb{W})_{\geq 0}: & \cdots \longrightarrow & \mathcal{N}\otimes_{R} C(W_n) & \longrightarrow \cdots
\end{array}$$

The morphism from $\mathcal{N}\otimes_{R_0}\mathbb{V}_{\geq 0}$ to $\mathcal{N}\otimes_R C(\mathbb{V})_{\geq 0}$ and the morphism from $\mathcal{N}\otimes_R\mathbb{W}_{\geq 0}$ to $\mathcal{N}\otimes_R C(\mathbb{W})_{\geq 0}$ are homotopic to the morphisms of constant embeddings. Since constant embeddings are $\dim_{\mathcal{N}}$-isomorphisms by Lemmas 18 and 19, the induced morphisms on the homology modules are all $\dim_{\mathcal{N}}$-isomorphisms by Lemma 5. It follows that $\varphi_{*,\bullet}\colon \operatorname{Tor}^{R_0}_{\bullet}(\mathcal{N},\mathcal{A}) \to \operatorname{Tor}^{R}_{\bullet}(\mathcal{N},\mathcal{A})$ are all $\dim_{\mathcal{N}}$-isomorphisms. □

Let $p\in\mathcal{N}$ be a projection and V be an $\mathcal{N}$-module. It is not hard to check that $p\mathcal{N}\otimes_{\mathcal{N}}V\cong pV$ and $\dim_{p\mathcal{N}p} p\mathcal{N}\otimes_{\mathcal{N}}V=\tau(p)^{-1}\dim_{\mathcal{N}}V$, where one uses the normalized trace $\tau(p)^{-1}\tau(\,\cdot\,)$ for $p\mathcal{N}p$. If $p\in\operatorname{Proj}(\mathcal{A})$ is a projection such that $\sum_i v_ipv_i^*=1$ for some pseudo-normalizers $v_1,\ldots,v_n$, then $\mathcal{N}\otimes_R V\cong\mathcal{N}p\otimes_{pRp}pV$ for every R-module V. whose central support in $\mathcal{N}$ is 1. It follows that

$$\begin{aligned}
\dim_{\mathcal{N}}\operatorname{Tor}^{R}_{\bullet}(\mathcal{N},\mathcal{A}) &= \tau(p)\dim_{p\mathcal{N}p} p\mathcal{N}\otimes_{\mathcal{N}}\operatorname{Tor}^{R}_{\bullet}(\mathcal{N},\mathcal{A})\\
&= \tau(p)\dim_{p\mathcal{N}p}\operatorname{Tor}^{pRp}_{\bullet}(p\mathcal{N}p,p\mathcal{A}).
\end{aligned}$$

With little more analysis, one can show the above equation for every $p\in\operatorname{Proj}(\mathcal{A})$ with full central support.

References

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