ORBIT EQUIVALENCE AND OPERATOR ALGEBRAS

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ABSTRACT. This treatise is based on the lecture given by Narutaka Ozawa at the University of Tokyo during the winter semester 2006-2007. It includes an elementary theory of orbit equivalence via type II_1 von Neumann algebras, Lück's dimension theory [6] and its application to L^2 Betti numbers [5], convergence of the semigroup associated to a derivation and a theorem of Popa on embeddability of subalgebras.

CONTENTS

1. INTRODUCTION

1.1. Orbit equivalence.

Definition 1.1. Let Y be a topological space, B_Y the σ-algebra of the Borel sets of Y. When Y is a separable complete metric space, (Y, B_Y) (or, by abuse of language, Y) is said to be a standard Borel space (standard σ-algebra).

Remark 1.2. When X is a standard Borel space, X is either (at most) countable or isomorphic to the closed interval $[0, 1]$.

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Definition 1.3. A standard Borel space with a Borel probability measure is said to be a (standard) probability space. A point x of a probability space (X, μ) is said to be an atom of (X, μ) when $\mu(x) > 0$. A probability space (X, μ) is said to be diffuse when it has no atom.

Example 1.4. (Examples of standard probability spaces)

(1) The infinite product $(\prod_{n\in\mathbb{N}} \{0,1\}, \otimes_n \mu_n)$, where μ_n is a probability measure on $\{0,1\}$ for each $n \in \mathbb{N}$ is standard.
(2) When G is a separable compact group, the normalized Haar measure on G makes G into a standard probability space.

When (X, μ) is a probability space, we obtain a (w*-) separable von Neumann algebra $L^\infty X$ and a normal state (also denoted by μ) on it. To each isomorphism $\phi\colon (X, \mu) \to (Y, \nu)$ of probability spaces, we obtain an isomorphism $\phi_*\colon L^\infty Y \to L^\infty X$, $f \mapsto f \circ \phi$ satisfying $\mu \circ \phi^* = \nu$.

Theorem 1.5. *(von Neumann)*

(1) *When (X, μ) and (Y, ν) are diffuse probability spaces, there is an isomorphism $(L^\infty(X, \mu), \mu) \simeq (L^\infty(Y, \nu), \nu)$.*
(2) *For each isomorphism $\sigma\colon L^\infty Y \to L^\infty X$ with $\mu\sigma = \nu$, there exists a Borel isomorphism $\phi\colon X \to Y$ such that $\phi^*\mu = \nu$ and $\phi_* = \sigma$.*

Proof. (Outline): (1) We may assume that $Y = \prod_{\mathbb{N}} \{0,1\}, \mu = \otimes_{\mathbb{N}}(\frac{1}{2}, \frac{1}{2})$. Since X is diffuse, we have a decomposition $X = X_0 \coprod X_1$ by Borel sets with $\mu(X_0) = \frac{1}{2}$. We can continue this procedure as $X_0 = X_{00} \coprod X_{01}, \mu(X_{00}) = \frac{1}{4}$, so on. The partition by $X_{**\cdots}$ can be made fine enough because there is a separating family $(B_n)_{n\in\mathbb{N}}$ in B_X, which will imply the desired isomorphism between $L^\infty X$ and $L^\infty Y$ compatible with the normal states.

(2) Let λ denote the Lebesgue measure on the closed interveal $[0,1]$. Since there exists an isomorphism $(L^\infty Y, \nu) \simeq (L^\infty[0,1], \lambda)$, we may assume that $Y = [0,1]$ and $\nu = \lambda$ here. For each $r \in \mathbb{Q} \cap [0,1]$, put $E_r = \sigma(\chi_{[0,r)})$. Define a mapping $\phi\colon X \to [0,1]$ by $\phi(x) = \inf \{r : x \in E_r\}$. The inverse image of $[0,t)$ under ϕ is equal to $\cup_{r<t} E_r$. The latter is obviously Borel, which means that ϕ is a Borel map. By $\sigma(\chi_{[0,r)}) = \phi^*(\chi_{[0,r)})$ for $r \in \mathbb{Q} \cap [0,1]$, we have $\sigma = \phi^*$ and $\phi_*\mu =$ Lebesgue measure.

It remains to replace ϕ by a Borel isomorphism. Let $(B_n)_{n\in\mathbb{N}}$ be a separating family of X. For each n, there exists $F_n \in B_Y$ such that $\phi_*\chi_{F_n} = \chi_{B_n}$. Thus $N = \cup_n B_n \triangle \phi^{-1} F_n$ is a null set. On $X \setminus N$, the condition $x \in B_n$ is equivalent to $\phi(x) \in F_n$. If x and y are distinct points of $X \setminus N$, there exists an integer n such that $x \in B_n$ while $y \notin B_n$. Thus $\phi(x) \neq \phi(y)$ and ϕ is injective on $X \setminus N$. We may assume that N and $Y \setminus \phi(X \setminus N)$ are uncountable so that there is an isomorphism of N to $Y \setminus \phi(X \setminus N)$. □

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action by a discrete countable group. (We may assume that it acts by Borel isomorphisms.) Let s be an element of Γ. When f is a complex Borel function defined on X, put $\alpha_s(f)\colon x \mapsto f(s^{-1}x)$. This induces a μ-preserving $*$-automorphism on $L^\infty X$. This way we obtain an action $\alpha\colon \Gamma \curvearrowright L^\infty(X, \mu)$ preserving the state μ.

Definition 1.6. Two actions $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are said to be conjugate when there exists an probability space isomorphism $\phi\colon (X, \mu) \to (Y, \nu)$ witch is a.e.

Γ-equivariant. This is equivalent to the existence of a Γ-equivariant state preserving isomorphism $\sigma\colon L^\infty(Y,\nu) \to L^\infty(X,\mu)$.

Definition 1.7. Let $\Gamma \curvearrowright (X,\mu)$ be an action by measure preserving Borel isomorphisms. The subset $\mathscr{R}_{\Gamma\curvearrowright(X,\mu)} = \{(sx,x) : s \in \Gamma\}$ of $X \times X$ is called the orbit equivalence relation of the action.

Definition 1.8. Two actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are said to be orbit equivalent when there exists a measure preserving Borel isomorphism $\phi\colon Y \to X$ satisfying $\Gamma\phi(y) = \phi(\Lambda y)$ for a.e. $y \in Y$.

Definition 1.9. A partial Borel isomorphism on X is a triple (ϕ, A, B) consisting of $A, B \in B_X$ and a Borel isomorphism ϕ of A onto B.

Definition 1.10. A measure preserving standard orbit equivalence is a subset $\mathscr{R}$ of $X \times X$ satisfying the following conditions:

(1) $\mathscr{R}$ is a Borel subset with respect to the product space structure.
(2) $\mathscr{R}$ is an equivalence relation on X.
(3) For each $x \in X$, the $\mathscr{R}$-equivalence class of x is at most countable.
(4) Any partial Borel isomorphism ϕ whose graph is contained in $\mathscr{R}$, ϕ preserves measure.

Theorem 1.11. *(Lusin) Let X, Y be standard spaces.*

(1) *When $\phi\colon X \to Y$ is a countable-to-one Borel map, $\phi(X)$ is Borel. Moreover there exists a Borel partition $X = \coprod X_n$ such that $\phi|_{X_n}$ is a Borel isomorphism onto $\phi(X_n)$.*
(2) *When $\mathscr{R}$ is a standard orbit equivalence, $\mathscr{R} = \cup_n \mathscr{G}(\phi_n)$ where ϕ_n is a partial Borel isomorphism for each n.*

Lemma 1.12. *Let A be a subset of a standard space X, ϕ a mapping of A into X. ϕ and A are Borel if and only if the graph $\mathscr{G}(\phi) = \{(\phi x, x) : x \in A\}$ of ϕ is Borel in $X \times X$.*

Proof. $\Leftarrow$ is an immediate consequence of Theorem 1.11.

$\Rightarrow$: Let $(B_n)_{n\in\mathbb{N}}$ be a separating family of X. The condition $y \neq \phi(x)$ is equivalent to $(y,x) \in \cup_n (\complement B_n) \times \phi^{-1}(B_n)$. Thus $\mathscr{G}(\phi) = \complement(\cup(\complement B_n) \times \phi^{-1}(B_n))$. □

1.2. **Preliminaries on von Neumann algebras.** Let H be a Hilbert space, $B(H)$ the involutive Banach algebra of the continuous endomorphisms of H, A a $*$-subalgebra of $B(H)$. (typically A generates a von Neumann algebra M of our interest.) In the following A is often assumed to admit a cyclic tracial vector $\xi_\tau \in H$, i.e. $\|\xi_\tau\| = 1$, $A\xi_\tau$ is dense in H, and that the vector state $\tau(a) = \langle a\xi_\tau, \xi_\tau\rangle$ is tracial.

Remark 1.13. A state τ is tracial means that by definition the two sesquilinear forms $\tau(ab^*)$ and $\tau(b^*a)$ in (a,b) are same. To check this property, by polarization it is enough to show $\tau(aa^*) = \tau(a^*a)$. Under the assumption above ξ_τ becomes a separating vector for A''. Indeed, $a\xi_\tau = 0$ implies $\tau(bc^*a) = 0$ for $b, c \in A$, which means $\tau(c^*ab) = 0$ and in turn $\langle aH, H\rangle = 0$.

Notation. Let $\hat{a}$ denote $a\xi_\tau$. (Hence we have $\langle \hat{a}, \hat{b}\rangle = \tau(ab^*)$.)

Remark 1.14. We have a conjugate linear map $J\colon H \to H$ determined by $\hat{a} \mapsto \widehat{a^*}$. Then we have $JaJ\hat{b} = \widehat{ba^*}$ which implies $JAJ \subset A'$ and $JA''J \subset A'$. On the other hand, for any $x \in A'$ and $a \in A$

$$\langle Jx\xi_\tau, a\xi_\tau\rangle = \langle Ja\xi_\tau, x\xi_\tau\rangle = \langle a^*\xi_\tau, x\xi_\tau\rangle = \langle x^*\xi_\tau, a\xi_\tau\rangle.$$

Thus $Jx\xi_\tau = x^*\xi_\tau$, thence ξ_τ is a cyclic tracial vector for A'. The J-operator for (A', ξ_τ) is exactly equal to the original J. Doing the same argument as above, we obtain $JA'J \subset A''$.

Remark 1.15. The map $A'' \to A', a \mapsto JaJ$ is a conjugate linear $*$-algebra isomorphism.

1.3. **Crossed products.** Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a discrete group on a standard probability space X. Recall that we have an action $\Gamma \curvearrowright L^\infty X$ induced by $\alpha_s(f) = f(s^{-1}-)$ for $s \in \Gamma$.

On the other hand, we get a unitary representation $\pi\colon \Gamma \curvearrowright L^2(X, \mu)$ given by the same formula $\pi_s f = \alpha_s f$ as the one on $L^\infty X$. Note that $\pi_s f \pi_s^* = \alpha_s(f)$ for $s \in \Gamma$ and $f \in L^\infty X$.

Definition 1.16. Let $\lambda\colon \Gamma \curvearrowright B(\ell_2\Gamma)$ denote the regular representation. The von Neumann algebra $L^\infty X \rtimes \Gamma$ on $L^2(X) \otimes \ell_2\Gamma$ is generated by the operators $\pi \otimes \lambda(s)$ for $s \in \Gamma$ and $f \otimes 1$ for $f \in L^\infty X$ is called the crossed product of $L^\infty X$ by α.

Let A denote $\{\sum_{\text{finite}} f_s \otimes 1 \cdot \pi \otimes \lambda(s)\} \subset L^\infty X \rtimes \Gamma$. By abuse of notation, in the following f stands for $f \otimes 1$ and $\lambda(s)$ for $\pi \otimes \lambda(s)$. Now $\xi_\tau = \mathbf{1} \otimes \delta_e \in L^2 X \otimes \ell_2\Gamma$ is a cyclic tracial vector for A. Indeed, it is obviously cyclic, while $\tau(f\lambda(s)) = \delta_{e,s}\mu(f)$ implies the tracial property:

$$\tau(f\lambda(s)g\lambda(t)) = \delta_{st,e} f\alpha_s(g) = \delta_{ts,e}\alpha_t(f)g = \tau(g\lambda(t)f\lambda(s)).$$

Note that the above expressions are nonzero only if $s = t^{-1}$.

Let V denote the isometry $L^2(X) \to L^2(X) \otimes \ell_2\Gamma$, $f \mapsto f \otimes \delta_e$. Then the contraction $E\colon L^\infty X \rtimes \Gamma \to B(L^2(X))$, $a \mapsto V^*aV$ has image $L^\infty X$, i.e. E is a conditional expectation (see Definition 2.6) of $L^\infty X \rtimes \Gamma$ onto $L^\infty X$. Note that $\tau = \mu \circ E$.

1.4. **von Neumann algebras of orbit equivalence.** Let $\mathscr{R}$ be a standard orbit equivalence on X. Hence it is a countable disjoint union $\coprod_n \mathscr{G}(\phi_n)$ of the graphs of partial isometries. We may assume that $\phi_0 = \mathrm{Id}_X$. We will define a "Borel probability measure" on $\mathscr{R}$.

Observe that when $f\colon \mathscr{R} \to \mathbb{C}$ is a Borel function, $X \to \mathbb{C}$, $x \mapsto \sum_y f(y, x) = \sum_n f(\phi_n x, x)$ is also Borel. Define a measure ν on $\mathscr{R}$ by putting

$$\int_{\mathscr{R}} \xi d\nu = \int_X \sum_{y\mathscr{R}x} \xi(y, x) d\mu(x)$$

for each Borel function ξ on $\mathscr{R}$. Thus when B is a Borel subset of $\mathscr{R}$, $\nu(B) = \int |\pi_r^{-1}(x) \cap B| d\mu(x)$ for the second projection $\pi_r\colon \mathscr{R} \to X$, $(y, x) \mapsto x$.

We get a pseudogroup $[\![\mathscr{R}]\!]$ whose underlying set is

$$\{\phi : \text{partial Borel isomorphism}, \mathscr{G}(\phi) \subset \mathscr{R}\}.$$

The composition $\phi \circ \psi$ of ϕ and ψ is defined as the composition of the maps on $\psi^{-1}\operatorname{dom}(\phi)$. In particular, the identity maps of the Borel sets are the units of $[\![\mathscr{R}]\!]$, and $\phi \in [\![\mathscr{R}]\!]$ implies $\phi^{-1} \in [\![\mathscr{R}]\!]$.

For each $\phi \in [\![\mathscr{R}]\!]$, define a partial isometry $v_\phi \in B(L^2(\mathscr{R}, \nu))$ by $v_\phi \xi(y, x) = \xi(\phi^{-1}y, x)$. Thus $v_\phi v_\psi = v_{\phi\circ\psi}$. On the other hand, the set $\left\{\chi_{\mathscr{G}(\phi)} : \phi \in [\![\mathscr{R}]\!]\right\}$ is total in $L^2(\mathscr{R}, \nu)$ and $v_\phi \chi_{\mathscr{G}\psi} = \chi_{\mathscr{G}\phi\circ\psi}$. Moreover, we have

$$\langle v_\phi \chi_{\mathscr{G}\psi}, \chi_{\mathscr{G}\theta}\rangle = \int \mathscr{G}(\phi\psi) \cap \mathscr{G}(\theta) d\nu = \mu\left\{x : \phi\psi x = \theta x\right\} = \langle \chi_{\mathscr{G}\psi}, v_{\phi^{-1}}\chi_{\mathscr{G}\theta}\rangle,$$

which implies $v_\phi^* = v_{\phi^{-1}}$.

Definition 1.17. The von Neumann algebra $\mathsf{vN}\mathscr{R}$ on $L^2(\mathscr{R}, \nu)$ generated by $\{v_\phi : \phi \in [\![\mathscr{R}]\!]\}$ is called the von Neumann algebra of $\mathscr{R}$.

$\xi_\tau = \chi_{\mathscr{G}(\mathrm{Id}_X)}$ is a cyclic tracial vector for $\mathsf{vN}\mathscr{R}$: in fact,

$$\begin{aligned}\tau(v_{\phi\psi}) &= \mu(\{x : \phi \circ \psi(x) = x\}) \\ &= \mu(\{y : \psi\phi y = y\}) \quad (y = \phi^{-1}x) \\ &= \tau(v_{\psi\phi}).\end{aligned}$$

Note that $L^\infty X$ is contained "in the diagonal" of $\mathsf{vN}\mathscr{R}$, subject to the relation $v_\phi f = (f\circ\phi^{-1})v_\phi$. We have a conditional expectation $E\colon \mathsf{vN}\mathscr{R} \to L^\infty X$, $a \mapsto V^*aV$ implemented by the "diagonal inclusion" isometry $V\colon L^2X \to L^2\mathscr{R}$. We have $E(v_\phi) = \chi_{\{x:\phi x=x\}}$.

2. Elementary theory of orbit equivalence

2.1. **Essentially free action of countable discrete groups.** Suppose we are given a measure preserving action $\Gamma \curvearrowright (X, \mu)$ by a discrete group on a standard probability space. As in the last section we get two inclusions of von Neumann algebras:

(1) $L^\infty X \subset L^\infty X \rtimes \Gamma$ in $B(L^2X \otimes \ell_2\Gamma)$.
(2) $L^\infty X \subset \mathsf{vN}(\mathscr{R}_{\Gamma\curvearrowright(X,\mu)})$ in $B(L^2\mathscr{R})$.

In general these are different, e.g. when the action is trivial.

Definition 2.1. An action $\Gamma \curvearrowright (X, \mu)$ is said to be essentially free when the fixed point set of s has measure 0 for any $s \in G \setminus \{e\}$.

Theorem 2.2. *When the action $\Gamma \curvearrowright (X, \mu)$ is essentially free, the above two inclusions of von Neumann algebras are equal.*

Remark 2.3. $J\hat{v_\phi} = \hat{v_{\phi^{-1}}}$ implies $J\xi(x, y) = \overline{\xi(y, x)}$.

Proof of the theorem. Identification of the representation Hilbert spaces is given by $U\colon L^2X\otimes\ell_2\Gamma \to L^2\mathscr{R}$, $g\otimes\delta_t \mapsto g\cdot\chi_{\mathscr{G}(t)}$. When we have an equality $f\chi_{\mathscr{G}(s)} = g\chi_{\mathscr{G}(t)}$ of nonzero vectors in $L^2\mathscr{R}$, s must be equal to t by the essential freeness assumption. Now,

$$U^*v_sU(g \otimes \delta_t) = U^*\alpha_s(g)v_s\chi_{\mathscr{G}(t)} = U^*\alpha_s(g)\chi_{\mathscr{G}(st)} = \alpha_s(g) \otimes \delta_{st}.$$

This shows $U^*v_sU = \pi \otimes \lambda(s)$. On the other hand, $U^*fU = f \otimes 1$ is trivial. Thus, via U, $L^2X \rtimes \Gamma$ is identified to $L^2\mathscr{R}$. ∎

Definition 2.4. Let M be a finite von Neumann algebra, A a von Neumann subalgebra (in the following A is often assumed to be commutative). The subset $\mathcal{N}A = \{u \in \mathcal{U}M : uAu^* = A\}$ of $\mathcal{U}M$ is called the normalizer of A. Likewise $\mathcal{N}^p A = \{v \in M : \text{partial isometry}, v^*v, vv^* \in A, vAv^* = Avv^*\}$ is called the partial normalizer of A.

Lemma 2.5. *For any $v \in \mathcal{N}^p A$, there exist $u \in \mathcal{N}A$ and $e \in \mathrm{Proj}(A)$ such that $v = ue$. For any $\phi \in [\![\mathscr{R}]\!]$, there exists a Borel isomorphism $\tilde{\phi}$ whose graph is contained in $\mathscr{R}$ and $\tilde{\phi}|_{\mathrm{dom}\,\phi} = \phi$.*

Proof. We prove the second assertion as the demonstration of the first one is an algebraic translation of it. Put $E = \mathrm{dom}\,\phi$ and $F = \mathrm{ran}\,\phi$. When $\mu(E \triangle F) = 0$, there is nothing to do. When $\mu(E \triangle F) \neq 0$, $\exists k > 0$ such that $\phi^k(E \setminus F) \cap (F \setminus E)$ is non-null. If not, $\phi^k(E \setminus F) \subset F \cap \complement(F \setminus E) = F \cap E \subset E$ up to a null set and ϕ^{k+1} can be defined a.e. on $E \setminus F$. Thus we would get a sequence $(\phi^k(E \setminus F))_{k\in\mathbb{N}}$ of subsets with nonzero measure. For any pair $m < n$, $\phi^m(E \setminus F) \cap \phi^n(E \setminus F)$ is equal to $\phi^m(\phi^{n-m}(E \setminus F) \cap (E \setminus F))$ which is null. This contradicts to $\mu(X) = 1$.

Now, given such k, put $\phi_1 = \phi \coprod (\phi^{-k}|_{\phi^k(E\setminus F)\cap(F\setminus E)})$. Then we can use the maximality argument (Zorn's lemma) to obtain a globally defined Borel isomorphism. □

2.2. **Inclusion of von Neumann algebras.**

Definition 2.6. Let $M \subset N$ be an inclusion of von Neumann algebras. A unital completely positive map $E : N \to M$ is said to be a conditional expectation when it satisfies $E(axb) = aE(x)b$ for $a, b \in M$ and $x \in N$.

Fact. When N is finite with a faithful tracial state τ, there exists a unique conditional expectation E that preserves τ. Then we obtain an orthogonal projection $e_M : L^2N \to \overline{M\xi_\tau} \simeq L^2M$ extending E.

Remark 2.7. (Martingale) If we are given $N_1 \subset N_2 \subset \cdots \subset M$ with $N = \vee_i N_i$ or $M \supset N_1 \supset N_2 \supset \cdots$ with $N = \cap_i N_i$, together with conditional expectations $E_n \colon M \to N_n$ and $E \colon M \to N$, $e_n \to e$ in the strong operator topology implies $\|E(x) - E_n(x)\|_2 \to 0$.

For example, let $A \subset M$ be a finite dimensional commutative subalgebra, e_i $(1 \leq i \leq n)$ the minimal projections of A. Then $E_{A'\cap M}(x) = \sum_{i=1}^n e_i x e_i$. If we have a sequence $A_1 \subset A_2 \subset \cdots \subset M$ of finite dimensional commutative subalgebras and $A = \vee A_i$, we have $E_{A_n'\cap M} \to E_{A'\cap M}$. The latter is equal to E_A if and only if A is a maximal abelian subalgebra.

Definition 2.8. A von Neumann subalgebra $A \subset M$ is said to be a Cartan subalgebra of M when it is a maximal abelian subalgebra in M and $\mathcal{N}(A)'' = M$. (Then we also have $M = \mathcal{N}^p(A)''$.)

Theorem 2.9. *$L^\infty X \subset \mathsf{vN}\mathscr{R}$ is a Cartan subalgebra.*

Proof. Since the generators v_ϕ are in $\mathcal{N}A$, it is enough to show that $L^\infty X$ is maximal abelian in $\mathsf{vN}\mathscr{R}$. Recall that $\mathscr{R} = \coprod \mathscr{G}(\phi_n)$ with $\phi_0 = \mathrm{Id}_X$. Then let a be an element of the relative commutant of $L^\infty X$. $\hat{a}$ can be written as $\sum_n f_n \chi_{\mathscr{G}(\phi_n)}$. By assumption $fa = af$ for any $f \in L^\infty X$. Thus,

$$\widehat{fa} = \sum f f_n \chi_{\mathscr{G}(\phi_n)}, \qquad \widehat{af} = J\bar{f}J\hat{a} = \sum f \circ \phi_n^{-1} \cdot f_n \chi_{\mathscr{G}(\phi_n)}.$$

Hence $ff_n = f \circ \phi_n f_n$ for any n and any f, which implies $f_n = 0$ except for $n = 0$. □

Definition 2.10. $\mathscr{R}$ is said to be ergodic when any $\mathscr{R}$-invariant Borel subset of X is of measure either 0 or 1. An action $\Gamma \curvearrowright (X, \mu)$ is said to be ergodic when $\mathscr{R}_{\Gamma \curvearrowright X}$ is ergodic.

Corollary 2.11. $\mathsf{vN}\mathscr{R}$ *is a factor if and only if* $\mathscr{R}$ *is ergodic.*

Proof. The Cartan subalgebra $L^\infty X$ contains the center of $\mathsf{vN}\mathscr{R}$. The central projections are the characteristic functions of the $\mathscr{R}$-invariant Borel subsets. □

Let $v \in \mathcal{N}^p L^\infty$, $E, F \in B_X$ the Borel sets (up to null sets) respectively representing the projections v^*v and vv^* in A. The map $L^\infty E \to L^\infty F$, $f \mapsto vfv^*$e is a $*$-isomorphism. Thus there exists a Borel isomorphism $\phi_v \colon E \to F$ such that $vfv^* = f \circ \phi_v^{-1}$. ($v = \sigma v_{\phi_v}$ for some $\sigma \in \mathcal{U}L^\infty F$.)

Theorem 2.12. *In the notation as above,* $v\xi v^* = \xi(\phi_v^{-1}(y), x)$ ν*-a.e. for any* $v \in \mathcal{N}^p L^\infty$ *and any* $\xi \in L^\infty \mathscr{R}$*. In particular,* $\phi_v \in [\![\mathscr{R}]\!]$ *up to a null set. Moreover, we have* $L^\infty \vee JL^\infty J = L^\infty \mathscr{R}$*.*

Proof. Put $A = L^\infty X$. First, $fJgJ \in L^\infty$ for $f, g \in A$: indeed, $fJgJ$ is the multiplication by the function $f(y)\overline{g(x)}$ on $\mathscr{R}$.

$$vfJgJv^* = vfv^*JgJ = f \circ \phi_v^{-1} JgJ \quad (JMJ = M').$$

Hence $v\xi v^*(y, x) = v(\phi_v^{-1}y, x)$ for $\xi \in A \vee JAJ$. It remains to show $\chi_{\mathscr{G}(\mathrm{Id}_X)} \in A \vee JAJ$. Because, if this is satisfied, we will have $\chi_{\mathscr{G}(\phi_v)} = v\chi_{\mathscr{G}(\mathrm{Id})}v^* \in L^\infty \mathscr{R}$.

Take an increasing sequence $A_1 \subset A_2 \subset \cdots$ of finite dimensional algebras with $A = \vee A_k$. The conditional expectation $E_n \colon \mathsf{vN}\mathscr{R} \to A_n$ is equal to $\sum_k e_k^{(n)} J e_k^{(n)} J$ (as an operator on $L^2\mathscr{R}$) for the minimal projections $(e_k^{(n)})_k$ of A_n. Now, $(E_n)_n$ converges to the conditional expectation E_A onto A which is equal to the multiplication by $\chi_{\mathscr{G}(\mathrm{Id}_X)}$ in the strong operator topology. Hence $\chi_{\mathscr{G}(\mathrm{Id})} \in A \vee JAJ$. □

Remark 2.13. (2-cocycle [4]) Suppose we are given a map $\sigma_{\phi,\psi} \colon \mathrm{ran}(\phi\psi) \to \mathbb{T}$ for each pair $\phi, \psi \in [\![\mathscr{R}]\!]$, satisfying $\sigma_{\phi,\psi}\sigma_{\phi\psi,\theta} = (\sigma_{\psi,\theta} \circ \phi^{-1})\sigma_{\phi,\psi\theta}$. Then $v_\phi^\sigma v_\psi^\sigma = \sigma_{\phi,\psi} v_{\phi\psi}^\sigma$ determines an associative product on $\mathbb{C}[\![\mathscr{R}]\!]$ with a trace τ. The GNS representation gives an inclusion $L^\infty X \subset \mathsf{vN}(\mathscr{R}, \sigma) \subset B(L^2\mathscr{R})$ of von Neumann algebras.

Fact. Any Cartan subalgebra of $\mathsf{vN}(\mathscr{R}, \sigma)$ is isomorphic to $L^\infty X$.

Theorem 2.14. *Let* $\mathscr{R}$ *(resp.* $\mathscr{S}$*) be an orbit equivalence on* X *(resp.* Y*),* $F \colon X \to Y$ *a measure preserving Borel isomorphism. The induced isomorphism* $F_* \colon L^\infty X \to L^\infty Y$ *can be extended to a normal* $*$*-homomorphism* $\mathsf{vN}\mathscr{R} \to \mathsf{vN}\mathscr{S}$ *if and only if* $F\mathscr{R} \subset \mathscr{S}$ *up to a* ν*-null set.*

Proof. For simplicity we identify Y with X by means of F. If $[\![\mathscr{R}]\!] \subset [\![\mathscr{S}]\!]$, the required homomorphism is induced by the isometry $L^2\mathscr{R} \to L^2\mathscr{S}$. Conversely, if $\pi\colon \mathsf{vN}\mathscr{R} \to \mathsf{vN}\mathscr{S}$ is an extension of F_*, for any $\phi \in [\![\mathscr{R}]\!]$ we have

$$\pi(v_\phi)\pi(f)\pi(v_\phi)^* = \pi(f \circ \phi^{-1}) = f \circ \phi^{-1},$$

which implies $\pi(v_\phi) = \sigma_\phi v_\phi$ for some $\sigma_\phi \in L^\infty X$. □

Let M be a finite von Neumann algebra with trace τ, identified to a subalgebra of $B(L^2M)$. Suppose A is a von Neumann subalgebra of M. Let e_A be the projection onto the span of $A\xi_\tau$ and put $\langle M, A\rangle = (M \cup \{e_A\})''$.

For any $x \in M$ and $\hat{a} \in L^2A$,

$$e_A x\hat{a} = e_A\widehat{xa} = \widehat{E_A(xa)} = \widehat{E_A(x)a}$$

which implies $e_A x e_A = E_A(x)e_A$. In particular, we have

$$\langle M, A\rangle = \overline{\left\{\sum x_j e_A y_j + z : x_j, y_j, z \in M\right\}}^{\mathrm{wop}}.$$

Now,

$$e_A JxJe_A\hat{a} = e_A\widehat{ax^*} = \widehat{E_A(ax^*)} = a\widehat{E_A(x^*)} = JE_A(x)J\hat{a}$$

implies $\langle M, A\rangle' = M' \cap \{e_A\}' = JAJ$, consequently $\langle M, A\rangle = (JAJ)'$. Note that when A is commutative $e_A JaJ = a^* e_A$ for $a \in A$.

We have the "canonical trace" Tr on $\langle M, A\rangle$ which is a priori unbounded defined by $\sum_i x_i e_A y_i \mapsto \tau(\sum_i x_i y_i)$. Still, Tr is normal semifinite, and its tracial property is verified as follows:

$$\begin{aligned}\Big\|\sum x_i e_A y_i\Big\|_{2,\mathrm{Tr}}^2 &= \mathrm{Tr}(\sum y_i^* e_A x_i^* x_j e_A y_j) = \sum \tau(y_i^* E_A(x_i^* x_j) y_j) \\ &= \sum \tau(E_A(y_j y_i^*) E_A(x_i^* x_j)) = \|y_i e_A x_i^*\|_{2,\mathrm{Tr}}^2 .\end{aligned}$$

Suppose $A \subset M$ is Cartan. Put $\tilde{A} = \{A, JAJ\}'' \subset \langle M, A\rangle$.

Example 2.15. When $A = L^\infty X$, $M = \mathsf{vN}\mathscr{R}$, we have $\tilde{A} = L^\infty\mathscr{R}$, $e_A = \chi_\Delta$ and $\mathrm{Tr}|_{\tilde{A}} = \int d\nu$ on $L^\infty\mathscr{R}$. Indeed,

$$\mathrm{Tr}(fe_A) = \tau(f) = \int_\Delta f d\mu = \int f d\nu \quad (f \in L^\infty X)$$

implies

$$\mathrm{Tr}(ufe_Au^*) = \mathrm{Tr}(fe_A) = \int_\Delta f d\mu = \int ufe_Au^* d\nu \quad (f \in L^\infty X, u \in \mathcal{N}A).$$

Remark 2.16. When $A \subset M$ is Cartan and $p \in \mathrm{Proj}(A)$, $A_p \subset pMp$ is also Cartan since $\mathcal{N}^p_{pMp}(A_p) = p\mathcal{N}^p_M(A)p$.

Example 2.17. When $Y \subset X$, the restricted equivalence $\mathscr{R}|_Y = Y \times Y \cap \mathscr{R}$ gives $\mathsf{vN}(\mathscr{R}|_Y) = p_Y(\mathsf{vN}\mathscr{R})p_Y$.

Exercise 2.18. Show that when A is a Cartan subalgebra of a factor M, $\tau p_1 = \tau p_2$ for $p_1, p_2 \in \mathrm{Proj}(A)$ implies the existence of $v \in \mathcal{N}^p A$ such that $p_1 \sim p_2$ via v. This implies that given an ergodic relation $\mathscr{R}$ on X, subsets Y_1 and Y_2 of X with the same measure, one would obtain $(A_{p_{Y_1}} \subset M_{p_{Y_1}}) \simeq (A_{p_{Y_2}} \subset M_{p_{Y_2}})$ via v.

2.3. **Theorem of Connes-Feldman-Weiss.**

Definition 2.19. A discrete group Γ is said to be amenable when $\ell_\infty\Gamma$ has a left Γ invariant state.

Example 2.20. Commutative groups, or more generally solvable groups are amenable. The union of an countable increasing sequence of amenable groups are again amenable.

Definition 2.21. A cartan subalgebra $A \subset M$ is said to be amenable when there exists a state $m\colon \tilde{A} \to \mathbb{C}$ invariant under the adjoint action of $\mathcal{N}A$. An orbit equivalence $\mathscr{R}$ on X is said to be amenable when $L^\infty X \subset \mathsf{vN}\mathscr{R}$ is amenable.

Remark 2.22. Let $\Gamma \curvearrowright X$ be a measure preserving essentially free action. Since Γ is assumed to be discrete, $\mathscr{R}$ can be identified to $\Gamma \times X$ as a measurable space and an invariant measure on $\mathscr{R}$ is nothing but a product measure on $\Gamma \times X$ of an invariant measure on Γ times an arbitrary measure on X. Thus, $\mathscr{R}$ is amenable if and only if Γ is amenable.

Definition 2.23. A von Neumann algebra M on H is said to be injective when there exists a conditional expectation $\Phi\colon B(H) \to M$.

Fact. The above condition is independent of the choice of a faithfull representation $M \hookrightarrow B(H)$. Moreover, M is injective if and only if it is AFD [2].

Theorem 2.24. *(Connes-Feldman-Weiss [3]) Let M be a factor with separable predual, A a Cartan subalgebra of M. The following conditions are equivalent:*

1. *The pair $A \subset M$ is amenable.*
2. *This pair is AFD in the sense that for any finite subset $\mathscr{F}$ of $\mathcal{N}A$ and a positive real number $\epsilon > 0$, there exists a finite dimensional subalgebra B of M such that*
   - *B has a matrix unit consisting of elements of $\mathcal{N}^p A$.*
   - *$\|v - E_B(v)\| < \epsilon$ for any $v \in \mathscr{F}$.*
3. *(A, M) is isomorphic to $(D, \bar{\otimes} M_2\mathbb{C}$ where $D = \bar{\otimes} D_2$ for the diagonal subalgebra $D_2 \subset M_2$. (Note that $\mathcal{N}^p D$ is generated by the “matrix units” of $M_{2^\infty} = \otimes M_2$.)*
4. *M is injective.*

Lemma 2.25. *In the assertion of (2), B may be assumed to be isomorphic to M_{2^N} for some N.*

Proof of the lemma. Perturbing a bit, we may assume that $\tau(e_{ij}^{(d)}) \in 2^{-N}\mathbb{N}$ for large enough N where $(e_{ij}^{(d)})_{d,1\le i,j\le n_d}$ is a matrix unit of $B = \oplus_d M_{n_d}$. By taking a partition if necessary, we may assume that $\tau(e_{ii}^{(d)} = 2^{-N}$ for any d and i. Then, since M is a factor, we have $e_{ii}^{(d)} \sim e_{jj}^{(f)}$ in M for any d, f, i and j. This means that B is contained in a subalgebra of M which is isomorphic to M_{2^N}. □

Proof of (2) $\Rightarrow$ (3): Note that there is a total (with respect to the 2-norm) sequence $(v_k)_{k\in\mathbb{N}} \subset \mathcal{N}^p A$. We are going to construct an increasing sequence of subalgebras $(B_k)_k$ in M with compatible matrix units $(e_{i,j}^{(k)})_{i,j}$ satisfying $B_k \simeq M_{2^{N_k}}$ and $\|E_{B_k}(v_l) - v_l\|^2 < \frac{1}{k}$ for $l \le k$.

Suppose we have constructed $B_1, \dots, B_k$. Applying the assertion of (2) to the finite set $\mathscr{F}' = \left\{e^{(k)}_{i,r} v_l e^{(k)}_{r,1}\right\}$, we obtain a matrix units $(f_{ij})_{i,j}$ in $\mathcal{N}^p A$ such that $\sum f_{ii} = e^{(k)}_{11}$ and

$$\left\|E_{\operatorname{span} f_{ij}}(x) - x\right\| < \frac{1}{n(k)^2(k+1)}$$

where $n(k)$ denotes the size of B_k. By the assumption that A is a maximal abelian subalgebra in M, the projections of $\mathcal{N}^p A$ are actually contained in A. Thus we obtain an inclusion $D \subset A$ (hence the equality between them) under the identification $M \simeq \otimes_{\mathbb{N}} M_2 = (\cup B_k)''$.

Proof of (3) $\Rightarrow$ (4): By assumption $M = (\cup B_n)''$ where B_n are finite dimensional subalgebras of M, $M' = (\cup J B_n J)''$. Let Φ_n denote the conditional expectation of $B(H)$ onto $(JB_nJ)'$: $\Phi_n(x) = \int_{\mathcal{U}(JB_nJ)} uxu^* du$ where du denotes the normalized Haar measure on the compact group $\mathcal{U}(JB_nJ)$. For each x, the sequence $(\|\Phi_n(x)\|)_n$ is bounded above by $\|x\|$. Thus we can take a Banach limit $\Phi(x)$ of $(\Phi_n(x))_n$, which defines a conditional expectation of $B(H)$ onto $\cap_n (JB_nJ)' = (\cup JB_nJ)' = M$.

Proof of (4) $\Rightarrow$ (1): Put $H = L^2 M$ and let Φ be a conditional expectation of $B(H)$ onto M. Then $\tau\Phi$ is an $\mathrm{Ad}\,\mathcal{U}M$-invariant state on $B(H)$. $\mathcal{N}A$ is obviously contained in $\mathcal{U}M$ and so is $\tilde{A}$ in $B(H)$.

Remark 2.26. When $A \subset M$ is an amenable Cartan subalgebra and e is a projection in A, the Cartan subalgebra $A_e \subset M_e$ is also amenable.

We are going to complete the proof of Theorem 2.24 by showing (1) $\Rightarrow$ (2).

Lemma 2.27. *Let ϕ be a measure preserving partial Borel isomorphism on a standard probability space (X, μ). Let E_0 denote the fixed point set $X^\phi = \{x \in \operatorname{dom}\phi : \phi x = x\}$. There exist Borel sets B_1, B_2, B_3 of X satisfying $X = \coprod_{0 \le i \le 3} E_i$ and $\phi E_i \cap E_i$ is null for $i > 0$.*

Proof. Take E_1 to be a Borel set with a maximal measure which satisfies $\phi E_1 \cap E_1 = \emptyset$. Put $E_2 = \phi E_1$. Then $\phi E_2 \cap E_2 = \emptyset$ by the injectivity of ϕ. Finally, put $E_3 = \complement(\cup_{0 \le i \le 2} E_i)$. Then $\phi E_3 \cap E_3$ is null by the maximality of E_1. □

Corollary 2.28. *For any finite set $\mathscr{F}$ of $\mathcal{N}^p A$, there exist projections $q_1, \dots, q_m$ of A ($m = 4^{|\mathscr{F}|}$) satisfying $\sum q_k = 1$ and that $q_k v q_k$ is either 0 or in $\mathcal{U}A_{q_k}$ for any $v \in \mathscr{F}$.*

Lemma 2.29. *(Dye) For any finite subset $\mathscr{F} \subset \mathcal{N}A$ and $\epsilon > 0$, there exists $a \in \tilde{A}_+$ with $\mathrm{Tr}(a) = 1$ and $\sum_{u \in \mathscr{F}} \|uau^* - a\|_{1,\mathrm{Tr}} < \epsilon$. (Here, $\|x\|_{1,\mathrm{Tr}} = \mathrm{Tr}(|x|)$.)*

Proof. Let $m\colon \tilde{A} \to \mathbb{C}$ be an $\mathrm{Ad}\,\mathcal{N}A$-invariant state. Since L^1 is w*-dense in $(L^\infty)^*$, there exists a net $a_i \in \tilde{A}_+$ satisfying $\mathrm{Tr}(a_i) = 1$ and $\mathrm{Tr}(a_i x) \to m(x)$ for any $x \in \tilde{A}$. Then, for any $u \in \mathcal{N}A$ and $x \in \tilde{A}$

$$\mathrm{Tr}((ua_iu^* - a_i)x) = \mathrm{Tr}(a_i u^* x u) - \mathrm{Tr}(a_i x) \to m(u^* x u) - m(x) = 0.$$

Thus $ua_iu^* - a_i$ is weakly convergent to 0. By Hahn-Banach's theorem, by taking the convex closure of the sets $\{ua_iu^* - a_i : k < i\}$, we find a sequence $(b_i)_i$ as convex combinations of the a_i satisfying $\|ub_iu^* - b_i\|_{1,\mathrm{Tr}} \to 0$ uniformly for $u \in \mathscr{F}$. □

Lemma 2.30. *(Namioka) Let $\mathscr{F}$, ϵ be as above. There exists a projection p of $\tilde{A}$ satisfying* $\mathrm{Tr}(p) < \infty$ *and* $\sum_{u\in\mathscr{F}} \|upu^* - p\|_{2,\mathrm{Tr}}^2 < \epsilon \|p\|_{2,\mathrm{Tr}}^2$.

Proof. Let $a \in \tilde{A}_+$ be an element given by Lemma 2.29. For each $r > 0$ put $P_r = \chi_{(r,\infty)}(a)$. We have

$$\|uau^* - a\|_{1,\mathrm{Tr}} = \int_0^\infty \|uP_ru^* - P_r\|_{1,\mathrm{Tr}}\,dr \qquad 1 = \|a\|_{1,\mathrm{Tr}} = \int_0^\infty \|P_r\|_{1,\mathrm{Tr}}\,dr.$$

Hence

$$\int_0^\infty \sum_{u\in\mathscr{F}} \|uP_ru^* - P_r\|_{1,\mathrm{Tr}}\,dr < \epsilon \int_0^\infty \|P_r\|_{1,\mathrm{Tr}}\,dr.$$

Thus there exists r such that $p = P_r$ satisfies $\sum \|upu^* - p\|_{1,\mathrm{Tr}} < \epsilon \|p\|_{1,\mathrm{Tr}}$. Since the summands are differences of projections, $\|-\|_{1,\mathrm{Tr}}$ is approximately equal to $\|-\|_{2,\mathrm{Tr}}^2$. □

Lemma 2.31. *(Local AFD approximation by Popa) Let $\mathscr{F}$, ϵ be as above. There exists a finite dimensional subalgebra $B \subset M$ with matrix units in $\mathcal{N}^pA$, satisfying $\left\|E_B(eue) - (u - e^\perp ue^\perp)\right\|_2^2 < \epsilon \|e\|_2^2$ for every $u \in \mathscr{F}$, where e denotes the multiplicative unit of B and E_B the conditional expectation $eMe \to B$.*

Proof. We may assume $1 \in \mathscr{F}$. Take $p \in \tilde{A}_+$ as in Lemma 2.30. Since $\mathrm{Tr}\,p < \infty$, we may assume that p can be written as $\sum_{i=1}^n v_ie_Av_i^*$ for $v_i \in \mathcal{N}^pA$. By Corollary 2.28, there exist projections $(q_k)_k$ in A with $\sum q_k = 1$ and each $q_kv_i^*uv_jq_k$ is either 0 or is in $\mathcal{U}(Aq_k)$ for $1 \le i,j \le n, u \in \mathscr{F}$. Taking finer partition if necessary, we deduce that $\mathrm{dist}(q_kv_i^*uv_jq_k, \mathbb{C}q_k) < \sqrt{\epsilon/n}$.

On the other hand,

$$\sum_{u\in\mathscr{F},k} \|(upu^* - p)Jq_kJ\|_{2,\mathrm{Tr}}^2 = \sum_{u\in\mathscr{F}} \|upu^* - p\|_{2,\mathrm{Tr}}^2 < \epsilon \|p\|_{2,\mathrm{Tr}}^2 = \epsilon \sum_k \|pJq_kJ\|_{2,\mathrm{Tr}}^2.$$

Hence for some k, $q = q_k$ satisfies $\sum \|(upu^* - p)JqJ\|^2 \le \epsilon \|pJqJ\|^2$. By $pJqJ = \sum v_ie_AJqJv_i^* = \sum v_iqe_Av_i^*$ since A is commutative, replacing v_i by v_iq, we may assume $v_i^*v_j = \delta_{i,j}q$ and $pJqJ = p$. (Note that $p = \sum v_ie_Av_i^*$ is a projection, which means that the ranges of v_i are mutually orthogonal.)

This way we obtain $\sum \|upu^* - p\|^2 \le \epsilon \|p\|^2$, each $v_iuv_j^* \in A_q$ is close to a constant z_{ij} by $\sqrt{\epsilon/n}$, and $(v_i)_i$ is a matrix unit in A_q. Put $e = \sum v_iv_i^*$. Thus,

$$\|p\|_{2,\mathrm{Tr}}^2 = \mathrm{Tr}(\sum v_ie_Av_i^*) = \tau(\sum v_iv_i^*) = \|e\|_\tau^2.$$

Consequently,

$$\begin{aligned}\|upu^* - p\|_{2,\mathrm{Tr}}^2 &= 2\,\mathrm{Tr}\,p - 2\,\mathrm{Tr}(upu^*p) = 2\tau(e) - 2\,\mathrm{Tr}(\sum uv_ie_Av_i^*u^*v_je_Av_j) \\ &= 2\tau(e) - 2\tau(\sum uv_iv_i^*u^*v_jv_j^*) = 2\tau(e) - 2\tau(ueu^*e) \\ &= \|ueu^* - e\|_{2,\tau}^2.\end{aligned}$$

Hence $\sum_{u\in\mathscr{F}} \|ue - eu\|_2^2 < \epsilon \|e\|_2^2$. Now $eue = \sum v_iv_i^*uv_jv_j^* \approx \sum z_{ij}v_iv_j^* \approx \epsilon \|e\|^2$ in $\|-\|_{2,\tau}^2$. Hence

$$\|eue - E_B(eue)\|_{2,\tau}^2 < \epsilon \|e\|_{2,\tau}^2 \qquad \left\|E_B(eue) - (u - e^\perp ue^\perp)\right\|_{2,\tau}^2 < 2\epsilon \|e\|_{2,\tau}^2.$$

When we have a family (B_i) of mutually orthogonal finite dimensional algebras satisfying the assertion of the lemma, $e = \sum 1_{B_i}$ satisfies

$$\left\|E_{\oplus B_i}(eue) - (u - e^{\perp}ue^{\perp})\right\|_{2,\tau}^2 < 2\epsilon \left\|e\right\|_{2,\tau}^2 . \qquad \square$$

Lemma 2.32. *In the notation of Lemma 2.31, $e = 1$.*

Proof. Otherwise we can apply Lemma 2.31 to $A_{e^\perp} \subset M_{e^\perp}$ and $\mathscr{F}' = e^\perp \mathscr{F} e^\perp$, to obtain a finite dimensional algebra $B_0 \subset M_{e^\perp}$ satisfying the assertion of Lemma 2.31. Use the Pythagorean equality. $\square$

Proof of (1) $\Rightarrow$ (2): Take $B_1, \ldots, B_m$ satisfying $\|\sum_m 1_{B_i}\|_2^2 > 1 - \epsilon$. Put $B = \oplus_i B_i \oplus \mathbb{C}(\sum 1_{B_i})^\perp$. Then we have $\|E_B(u) - u\|_2^2 < 3\epsilon$ for $u \in \mathscr{F}$. $\square$

3. L^2-Betti numbers

3.1. **Introduction.** Let $\mathfrak{F}(\Omega, X)$ denote the set of the mappings of a set Ω into another set X. Let Γ be a discrete group, λ the left regular representation of Γ on $\ell_2\Gamma$. We have the "standard complex" of right Γ modules

$$0 \longrightarrow \ell_2\Gamma \xrightarrow{\partial} \mathfrak{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial} \mathfrak{F}(\Gamma^2, \ell_2\Gamma) \longrightarrow \cdots$$

given by

$$\partial(f)(s_1, \ldots, s_{n+1}) = \lambda(s_1)f(s_2, \ldots, s_{n+1}) + \sum_{1 \le j \le n} (-1)^j f(s_1, \ldots, s_j s_{j+1}, \ldots, s_{n+1}) + (-1)^{n+1} f(s_1, \ldots, s_n).$$

Conceptually, the above complex can be regarded as $\mathrm{Hom}_{\mathbb{C}\Gamma}(P_*, {}_{\mathbb{C}\Gamma}\ell_2\Gamma)$ where P_* denotes the standard free resolution of the trivial left Γ-module $\mathbb{C}$. For each $n \in \mathbb{N}$, P_n is the vector space with basis Γ^{n+1} as a vector space over $\mathbb{C}$. Since Γ^{n+1} is a left Γ-set by $s.(s_0, \ldots, s_n) = (s.s_0, s_1, \ldots, s_n)$, P_n has the canonically induced left action of Γ.

Let $H_i(\Gamma, \ell_2\Gamma)$ denote the i-th (co)homology group of this complex. Note that this complex consists of $R\Gamma$ modules given by the action on $\ell_2\Gamma$, with boundary maps being $R\Gamma$-homomorphisms. The space of 1-cocycles

$$Z_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : b(st) = b(s) + \lambda(s)b(t)\}$$

is identified with the space of the derivations from Γ to $\ell_2\Gamma$ with respect to the trivial right action. When $b \in Z_1$ the map

$$s \mapsto \begin{pmatrix} \lambda(s) & b(s) \\ 0 & 1 \end{pmatrix}$$

of Γ into $B(\ell_2\Gamma \oplus \mathbb{C})$ becomes multiplicative. On the other hand the space of 1-coboundaries

$$B_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : \exists f \in \ell_2\Gamma, b(s) = \lambda(s)f - f\}$$

is identified with the space of the inner derivations. Note that for any $b \in Z_1$, there is a function $f \in \mathfrak{F}(\Gamma, \mathbb{C})$ satisfying $b(s) = \lambda(s)f - f$ if we do not require the square summability of f. Indeed, a vector system $(b(s))_{s\in\Gamma}$ is a derivation if and only if we have $\langle b(s), \delta_t\rangle = \langle b(st) - b(t), \delta_e\rangle$ for any $s, t \in \Gamma$, and in such a case we may put $f(s) = \langle b(s), \delta_s\rangle$ to obtain $b(s) = \lambda(s)f - f$.

Remark 3.1. The 0-th homology grouop $H_0 = Z_0$ is the space of the Γ-invariant vectors in $\ell_2\Gamma$. Thus this becomes the 0-module if and only if Γ is infinite.

In the following we assume that Γ admits a finite generating set $\mathscr{S}$. Let $D\Gamma$ denote the space Z_1 of the derivations, $\operatorname{Inn} D\Gamma$ the space B_1 of the inner derivations. Let $\mathscr{O}_{\mathscr{S}}$ denote the mapping $b \mapsto (b(s))_{s\in\mathscr{S}}$ of $D\Gamma$ into $\oplus_{\mathscr{S}}\ell_2\Gamma$. This is an injective $R\Gamma$-module map. Note that the range of $\mathscr{O}_{\mathscr{S}}$ is closed. Indeed, $(f(s))_{s\in\mathscr{S}}$ is in $\operatorname{ran}\mathscr{O}_{\mathscr{S}}$ if and only if

$$f(s_1) + \lambda(s_1)f(s_2) + \cdots + \lambda(s_1\cdots s_{n-1})f(s_n) = 0$$

holds for each relation $s_1\cdots s_n = e$ among elements of $\mathscr{S}$.

A sequence $(f_n)_{n\in\mathbb{N}}$ of unit vectors is said to be an approximate kernel of the restriction $\mathscr{O}_{\mathscr{S}}|_{\operatorname{Inn} D\Gamma}$ when $\lambda(s)f_n - f_n$ tends to zero (in norm) for any $s \in \mathscr{S}$. $\mathscr{O}_{\mathscr{S}}|_{\operatorname{Inn} D\Gamma}$ has an approximate kernel if and only if Γ is amenable. Thus $\mathscr{O}_{\mathscr{S}}(\operatorname{Inn} D\Gamma)$ is closed if and only if Γ is finite or non-amenable.

Let P, Q denote the orthogonal projections onto $\mathscr{O}_{\mathscr{S}}(D\Gamma)$ and $\mathscr{O}_{\mathscr{S}}(\operatorname{Inn} D\Gamma)$. These commute with the diagonal action of $R\Gamma$ on $\oplus_{\mathscr{S}}\ell_2\Gamma$, i.e. $P, Q \in M_{\mathscr{S}}L\Gamma$. We can measure them by the trace $\tilde{\tau} = \operatorname{Tr}\otimes\tau$. The first Betti number $\beta_1^{(2)} = \dim_{L\Gamma} H_1(\Gamma, \ell_2)$ is equal to the difference $\tilde{\tau}(P) - \tilde{\tau}(Q)$.

Example 3.2. When Γ is a finite group, $\beta_0^{(2)} = \frac{1}{|\Gamma|}$ while $\beta_i^{(2)} = 0$ for $0 < i$ because any $\mathbb{C}\Gamma$ module is projective. On the other hand when Γ is equal to the free group $\mathbb{F}_n$ generated by a set $\mathscr{S}$ consisting of n elements, $\operatorname{ran}\mathscr{O}_{\mathscr{S}} = \oplus_{\mathscr{S}}\ell_2\Gamma$ and $\beta_1^{(2)} = n - 1$.

We omit the injection $\mathscr{O}_{\mathscr{S}}$ and identify $D\Gamma$ with a subspace of $\oplus_{\mathscr{S}}\ell_2\Gamma$. Thus $\partial^0\colon \ell_2\Gamma \to \mathfrak{F}(\Gamma, \ell_2\Gamma)$ factors through $\oplus_{\mathscr{S}}\ell_2\Gamma$ and $\partial^0\colon \ell_2\Gamma \to \oplus_{\mathscr{S}}\ell_2\Gamma$ is written as $f \mapsto (\lambda(s)f - f)_{s\in\mathscr{S}}$.

Let $\epsilon_1^{(2)}\colon \oplus_{\mathscr{S}} \ell_2\Gamma \to \ell_2\Gamma$ denote the adjoint of ∂. Thus $\epsilon_1^{(2)}$ is expressed as $(\xi_s)_{s\in\mathscr{S}} \mapsto \sum_{s\in\mathscr{S}}(\lambda(s^{-1}) - 1)\xi_s$ and the orthogonal complement of $\ker\epsilon_1^{(2)}$ is equal to the closure of $\operatorname{ran}\partial = \operatorname{Inn} D\Gamma$.

Proposition 3.3. *When we identify $\mathbb{C}\Gamma$ with the space of vectors with finite support in $\ell_2\Gamma$, we have $D\Gamma = (\ker\epsilon_1^{(2)} \cap \oplus_{\mathscr{S}}\mathbb{C}\Gamma)^\perp$.*

Proof. The space $\mathbb{C}\Gamma$ has $\mathfrak{F}(\Gamma, \mathbb{C})$ as its algebraic dual. A vector system $b \in \oplus_{\mathscr{S}}\ell_2$ is in $D\Gamma$ if and only if there is an $f \in \mathfrak{F}(\Gamma, \mathbb{C})$ such that $b(s) = \lambda(s)f - f$. The latter implies

$$\forall\xi \in \ker\epsilon_1^{(2)} \cap \oplus_{\mathscr{S}}\mathbb{C}\Gamma, \langle\xi, b\rangle = \sum_s \langle\xi(s), b(s)\rangle = \sum_s \langle(\lambda(s^{-1}) - 1)\xi(s), f\rangle = 0.$$

Conversely, when $(b(s))_{s\in\mathscr{S}}$ is orthogonal to $\ker\epsilon_1^{(2)} \cap \oplus_{\mathscr{S}}\mathbb{C}\Gamma$, the functional $\langle b, -\rangle$ on $\oplus_{\mathscr{S}}\mathbb{C}\Gamma$ is induced by a functional f on the kernel of the map $\mathbb{C}\Gamma \to \mathbb{C}$. This f can be extended to a linear map on the whole $\mathbb{C}\Gamma$, and we have $b(s) = \lambda(s)f - f$, i.e. $b \in D\Gamma$. □

Remark 3.4. The i-th cohomology group $H^i(\Gamma, \ell_2\Gamma)$ is dimension isomorphic to $\operatorname{Tor}_i^{\mathbb{C}\Gamma}(\mathbb{C}, \ell_2\Gamma)$. This is seen by considering the exact functors $E \to E^*$ on the category of $L\Gamma$-modules and that of $L\Gamma$-bimodules, where E^* denotes the dual module of the weak closure of E. We have functors $(A, B) \to A \otimes_{\mathbb{C}\Gamma} B$ and $(A, B) \to$

$\mathrm{Hom}_{\mathbb{C}\Gamma}(A,B)$ of $\mathbb{C}\Gamma$-mod $\times$ $L\Gamma$-bimod into $L\Gamma$-mod. Then the functor equivalence $(A \otimes_{\mathbb{C}\Gamma} B)^* \simeq \mathrm{Hom}_{\mathbb{C}\Gamma}(A, B^*)$ up to dimension implies the dimension equivalence between the derived functors $\mathrm{Tor}_p(A,B)^* \simeq \mathrm{Ext}^p(A,B^*)$. The case $A = \mathbb{C}$ and $B = \ell_2\Gamma$ describes the desired isomorphism.

For example, we have a flat resolution $P.$ of the trivial Γ-module $\mathbb{C}$ with $P_0 = \mathbb{C}\Gamma$ and $P_1 = \mathbb{C}\Gamma \otimes_{\mathbb{C}} \mathbb{C}\mathscr{S}$, with $d_1(a \otimes b) = ab - a$. The first torsion group $\mathrm{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C})$ is by definition the quotient $\ker(id_{\ell_2\Gamma} \otimes d_1)/\ell_2\Gamma \otimes \ker d_1$. $\underline{\text{Now } id_{\ell_2\Gamma} \otimes d_1} = \epsilon_1^{(2)}$ implies $\ker(id_{\ell_2\Gamma} \otimes d_1) = \mathrm{Inn}\, D\Gamma^{\perp}$ while $\ell_2\Gamma \otimes \ker d_1 = \overline{\ker \epsilon_1^{(2)} \cap \oplus_{\mathscr{S}} \mathbb{C}\Gamma}$ implies $\ell_2\Gamma \otimes \ker d_1 = D\Gamma^{\perp}$.

3.2. **Operators affiliated to a finite von Neumann algebra.** Let (M, τ) be a finite von Neumann algebra with a faithful normal tracial state (τ is unique if M is a factor), L^2M the induced Hilbert M-M module. For each $n \in \mathbb{N}$ put $\tilde{\tau} = \tau \otimes \mathrm{Tr}$ on $M \otimes M_n\mathbb{C} \simeq M_nM$.

Definition 3.5. Let H be a left Hilbert module over M. A densely defined closed operator T on H is said to be affiliated to M, written as $T \sim M$, when we have $uT = Tu$ for any $u \in \mathcal{U}(M')$. Here the equality entails the agreement of the domains, i.e. $u\,\mathrm{dom}\, T = \mathrm{dom}\, T$.

Remark 3.6. An operator T is affiliated to M if and only if for the polar decomposition $T = v|T|$ the partial isometry v and the spectral projections of $|T|$ are in M. Note that in such cases τ takes the same value on the left support $l(T) = vv^*$ of T and the right support $r(T) = v^*v$.

We consider the case $H = L^2M$. Suppose $T \sim M$. It is said to be square integrable when $\hat{1} \in \mathrm{dom}\, T$. This condition is equivalent to

$$\tau(|T|^2) = \|T\hat{1}\|^2 = \int t^2 d\tau(E) < \infty$$

for the spectral measure $T = \int t dE$ of T. For each $\xi \in L^2M$ let L_ξ° denote the unbounded operator defined by $\mathrm{dom}\, L_\xi^\circ = \hat{M} \subset L^2M$ and $L_\xi^\circ x = \xi x$.

Proposition 3.7. *The operator $L_\xi^\circ x$ is closable and its closure L_ξ is affiliated to M. Moreover we have $L_\xi^* = L_{J\xi}$. If T is affiliated to M and square integrable, $T = L_{T\hat{1}}$.*

Proof. We show the inclusion $L_{J\xi}^\circ \subset (L_\xi^\circ)^*$. For any elements $x, y \in M$,

$$\langle L_\xi^\circ \hat{x}, \hat{y}\rangle = \langle \xi x, y\rangle = \langle J\hat{y}, J(\xi x)\rangle = \langle \hat{1}y^*, x^*J\xi\rangle = \langle \hat{x}, (J\xi)y\rangle.$$

On the other hand, when $u \in \mathcal{U}\mathscr{R}_M$, $uL_\xi^\circ = L_\xi^\circ u$ implies $uL_\xi = L_\xi u$.

Next we show the inclusion $(L_\xi)^* \subset L_{J\xi}$. Let $\eta \in \mathrm{dom}(L_\xi^\circ)^*$. Consider the polar decomposition $L_{J\xi} = v|L_{J\xi}|$ and the spectral decomposition $|L_{J\xi}| = \int_0^\infty \lambda de_\lambda$. Then $e_\lambda v^* L_{J\xi} = e_\lambda |L_{J\xi}|$ is bounded (i.e. is in M_+) for any λ. By definition, $L_{J\xi}(y\hat{1}) = (J\xi)y$ for $y \in M$. Hence $e_\lambda v^* L_{J\xi}(y\hat{1}) = e_\lambda v^*((J\xi)y) = (e_\lambda v^* J\xi)y$. Putting $y = 1$, we obtain $e_\lambda v^* L_{J\xi}\hat{1} = e_\lambda v^* J\xi \in M.\hat{1}$ for any $\lambda > 0$.

Thus, by definition of $(L_\xi)^*$, we have

$$\begin{aligned}\langle (L_\xi)^*\eta, (e_\lambda v^*)^* y\hat{1}\rangle &= \langle \eta, L_\xi (e_\lambda v^*)^* y\hat{1}\rangle = \langle \eta, \xi (e_\lambda v^*)^* y\rangle = \langle \eta, Jy^*(e_\lambda v^*)J\xi\rangle \\ &= \langle \eta, Jy^* e_\lambda v^* L_{J\xi}\hat{1}\rangle \quad \text{(by using above)} \\ &= \langle \eta, (e_\lambda v^* L_{J\xi})^* y\hat{1}\rangle.\end{aligned}$$

Hence $e_\lambda v^*(L_\xi)^*\eta = e_\lambda v^* L_{J\xi}\eta = |L_{J\xi}|e_\lambda\eta$ for any $\lambda > 0$. By letting $\lambda \to \infty$, $e_\lambda\eta \to \eta$ and $|L_{J\xi}|e_\lambda\eta \to v^*(L_\xi)^*\eta$. Since $|L_{J\xi}|$ is a closed operator, $\eta \in \mathrm{dom}(|L_{J\xi}|) = \mathrm{dom}(L_{J\xi})$. Hence $(L_\xi)^* \subset L_{J\xi}$ and $|L_{J\xi}| = v^*(L_\xi)^*$.

Finally, let us prove the last part. Let $T \sim M$ with the polar decomposition $v|T| = T$. Note that $\hat{v^*} = \hat{1}v^* \in \mathrm{dom}\, T$, $\hat{1} \in \mathrm{dom}\, T^*$, $T^*\hat{1} = |T|\hat{v^*}$. Put $\xi = T\hat{1}, \eta = T^*\hat{1}$. Since $T \sim M$, $L^\circ_\xi \subset T$, $L^\circ_\eta \subset T^*$ and we obtain $L_\xi \subset T \subset L_{J\eta}$. □

3.3. **Projective modules over a finite von Neumann algebra.** Let $m, n \in \mathbb{N}$. We have an isomorphism $\mathrm{Mor}(M^{\oplus m}, M^{\oplus n}) = M_{m,n}(M)$ by multiplication of matrices on column vectors.

Definition 3.8. An left M-module V is said to be finitely generated projective module when it is a projective object in the category of the M-modules and has a finite set generating itself.

Remark 3.9. Any finitely projective M module is isomorphic to some $M^{\oplus m}.P$ for a natural number m and an idempotent matrix P in $M_m M$.

Lemma 3.10. *In the above we may replace P with an orthogonal projection $P^* = P$ without changing the value of $\tilde{\tau}(P)$.*

Proof. Let P_0 be the right support of P. $P(P - P_0) = 0$ implies $P_0(P - P_0) = 0$. Thus $S = \mathrm{Id} + (P - P_0)$ is invertible. With respect to the orthogonal decomposition $\mathrm{Id} = P_0 \oplus P_0^\perp$, these operators are expressed as

$$P_0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} \mathrm{Id} & 0 \\ ? & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} \mathrm{Id} & 0 \\ ? & \mathrm{Id} \end{pmatrix}.$$

The operator $SP_0 = SP_0S^{-1}$ is self adjoint. □

Remark 3.11. When $M^{\oplus m}P$ and $M^{\oplus n}Q$ are isomorphic, $\tilde{\tau}(P) = \tilde{\tau}(Q)$.

Definition 3.12. For each finitely projective M-module V isomorphic to $M^{\oplus m}P$ where P is a orthogonal projection in $M_m M$, $\dim_M V - \tilde{\tau}(P)$ is called the τ-dimension

Lemma 3.13. *Let V be a submodule of $M^{\oplus n}$. When V is closed $M^{\oplus n}$ with respect to the L^2-norm (V is weakly closed), V is finitely generated and projective.*

Proof. The L^2 completion $\bar{V}^{\|\cdot\|_2} \subset L^2 M^{\oplus n}$ is written as $L^2 M^{\oplus} P$ for an orthogonal projection P. Then V is equal to $M^{\oplus n}P$. □

Lemma 3.14. *For each $T \in \mathrm{Mor}(M^{\oplus m}, M^{\oplus n})$, its kernel and range are finitely generated projective modules.*

Proof. Obviously the kernel of T is weakly closed in $M^{\oplus m}$. On the other hand for the projection P such that $\ker T = M^{\oplus m}P$, T induces an isomorphism $MP^{\perp} \to \operatorname{ran} T$. □

Remark 3.15. When a submodule $V \subset M^{\oplus m}$ is finitely generated, V is projective. In fact, $V = M^{\oplus m}A$ for some $A \in M_{m,n}(M)$. Thus we have

$$V \simeq M^{\oplus n}l(A) \simeq M^{\oplus m}r(A) \simeq \bar{V}.$$

Hence $\dim_M V = \dim_M \bar{V}$.

Remark 3.16. If $W \subset V$ are finitely generated projective modules, $\dim_M W \leq \dim_M V$.

Definition 3.17. Let V be an M-module. Put

$$\dim_M V = \sup \{\dim_M W : W \subset V, W \text{ is projective}\} \in [0, \infty].$$

Remark 3.18. Note that the above definition of $\dim_M$ is compatible with the previous one for finitely generated projective modules. In general, $W \subset V$ implies $\dim_M W \leq \dim_M V$ and $(V_i)_{i \in I} \uparrow V$ ($V = \cup_{i \in I} V_i$) implies $\dim_M V = \lim_i \dim_M V_i$.

Theorem 3.19. *(Lück [6]) When*

$$0 \longrightarrow V_0 \xrightarrow{\iota} V_1 \xrightarrow{\pi} V_2 \longrightarrow 0$$

is exact, we have $\dim_M V_1 = \dim_M V_0 + \dim_M V_2$.

Proof. When $W \subset V_2$ is finitely generated and projective, $\pi^{-1}W$ is identified to $W \oplus \iota V_0$. Hence $\dim V_1 \geq \dim V_0 + \dim V_2$. Conversely, let $W \subset V_1$ be finitely generated projective. The weak closure $\overline{\iota V_0 \cap W}$ is closed in a finite free module, hence is projective. From the sequence $\overline{\iota V_0 \cap W} \to W \to W/\overline{\iota V_0 \cap W}$, we have $\dim W = \dim \overline{\iota V_0 \cap W} + \dim W/\overline{\iota V_0 \cap W}$. Note that there is a natural surjection $W/\iota V_0 \cap W \to W/\overline{\iota V_0 \cap W}$. By the first part of the argument this implies the dimension inequality $\dim \overline{\iota V_0 \cap W} \leq \dim \iota V_0 \cap W$. On the other hand $W/\iota V_0 \cap W$ is identified to a submodule of V_2. □

Corollary 3.20. *Let V be a finitely generated M-module. We have a decomposition $V = V_p \oplus V_t$ where V_p is projective and* $\dim V = \dim V_p$. *(Hence* $\dim V_t = 0$.*)*

Proof. We have a surjection $T \colon M^{\oplus m} \to V$. Note that $\ker T$ may not be closed since we have no matrix presentation of T. Nonetheless, $V \simeq M^{\oplus m}/\ker T$ and the next lemma imply that $V_p = M^{\oplus m}/\ker T$ satisfies

$$\dim V = m - \dim \ker T = m - \dim \overline{\ker T} = \dim V_p.$$ □

Lemma 3.21. *Let W be a subset of a finite free module $M^{\oplus m}$. We have* $\dim W = \dim \bar{W}$.

Proof. Put $L = \{A \in M_m M : M^{\oplus}.A \subset W\}$. This is a left ideal of $M_m M$. We get a right approximate identity A_i of L. For the orthogonal projection P such that $\bar{W} = M^{\oplus m} P$, the right support $r(A_i$ converges to P in strong operator topology (for any normal representation, thus, in the ultrastrong topology). Thus for any $\epsilon > 0$, $P_{\epsilon,i} - \chi_{[\epsilon,1]}(A_i^* A_i)$ is in L and converges to P in the ultrastrong operator topology. □

Proposition 3.22. *For any $L\Gamma$-module V, $\dim V = 0$ is equivalent to*

$$\forall \xi \in V, \epsilon > 0, \exists P \in \operatorname{Proj} M : \tau P > 1 - \epsilon \text{ and } P\xi = 0.$$

Proof. $\Rightarrow$: Let $\xi \in V$. Consider the exact sequence $0 \to L \to M \to M.\xi \to 0$ where L is the annihilator of ξ. $\dim L = \dim M$ implies the existence of projections P_i convergent to 1 in the ultrastrong topology.

$\Leftarrow$: If $V \supset M.Q$, P satisfies $\tau P > 1 - \tau Q$ and $PQ \neq 0$. □

Definition 3.23. A homomorphism $\phi\colon V \to W$ of L-modules is said to be a dimension isomorphism when $\dim_M \ker \phi = \dim_M \operatorname{cok} \phi = 0$.

Remark 3.24. The torsion N-modules $\mathcal{T} = \{V : \dim_N V = 0\}$ form a Serre subcategory of N-mod. Analyzing N-modules up to dimension isomorphisms amounts to considering the localization N-mod$/\mathcal{T}$ of N-mod by $\mathcal{T}$. Thus, in general, when a morphism $V_* \to W_*$ of complexes is a dimension isomorphism at each degree, the induced homomorphism between the cohomology groups is also an dimension isomorphism because it factors through an isomorphism in the localization category $\mathcal{C}^*(N\text{-mod}/\mathcal{T})$ of the N-module complex category over the torsion module category.

Lemma 3.25. *The standard inclusion $M \to L^2(M)$ is a dimension isomorphism.*

Proof. Let $\xi \in L^2 M$. We get the corresponding square integrable operator affiliated with M. Put $P_n = \chi_{[0,n]}(\xi\xi^*) \in \operatorname{Proj} M$. Then $P_n \xi \in M$ and $P_n \to 1$, thus $P_n[\xi] - 0$ in the quotient $L^2 M/M$. □

When H is a Hilbert M-module, i.e. a normal representation of M on H, $H \simeq L^2 M^{\oplus n}.P$ for some cardinal n and an idempotent P in $M_n M$.

Lemma 3.26. *In the above notation,* $\dim_M H = \tilde{\tau}(P)$.

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} M^{\oplus n}.P & \hookrightarrow & L^2 M^{\oplus n}.P & \longrightarrow & \operatorname{cok} \\ \big\downarrow & p.b. & \big\downarrow & \circlearrowleft & \big\downarrow \\ M^{\oplus n} & \longrightarrow & L^2 M^{\oplus n} & \longrightarrow & \operatorname{cok}. \end{array}$$

The cokernel in the lower row has dimension 0, thus so does the one in the upper row. □

Definition 3.27. $\beta_n^{(2)}(\Gamma) = \dim_{L\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(L\Gamma, \mathbb{C}_{\text{triv}})$ is called the n-th L^2-Betti number of Γ.

Remark 3.28. $\beta_n^{(2)}(\Gamma)$ is equal to $\dim_{L\Gamma} \mathrm{Tor}_n^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}_{\mathrm{triv}})$.

Example 3.29. $\beta_n^{(2)}(\mathbb{F}_r) = r-1$ when $n=2$, 0 otherwise. This is seen as follows: let $g_1, \ldots, g_r$ be the standard generators of $\mathbb{F}_r$. A free resolution of the trivial $\mathbb{C}[\mathbb{F}_r]$-module $\mathbb{C}$ is given by

$$0 \longrightarrow (\mathbb{C}[\mathbb{F}_r])^r \underset{d_1}{\longrightarrow} \mathbb{C}[\mathbb{F}_r] \underset{\alpha}{\longrightarrow} \mathbb{C}$$

where $d_1 \colon (\xi_k)_{k=1}^r \mapsto \sum(\lambda_{g_k}^* - 1)\xi_k$ and α is the augmentation map. Now, d_1 is injective: let $\chi_j \in \ell_\infty \mathbb{F}_r$ be the characteristic function of $\mathbb{F}_r g_j$. Then $(\lambda_{g_k} - 1)\chi_j = \delta_{j,k}\delta_e$ and $(\xi_k)_k \in \ker d_1$ implies

$$0 = \langle \sum_k (\lambda_{g_k}^* - 1)\xi_k, \chi_j \rangle = \sum_k \langle \xi_k, (\lambda_{g_k} - 1)\chi_j \rangle = \xi_j(e).$$

Replacing χ_j by $\chi_j^t = \chi_j(-t^{-1})$ for $t \in \Gamma$, we have $\xi_j(t) = 0$ for any j and t. Thus, the torsion group is the cohomology of the complex

$$0 \longrightarrow (L^2\mathbb{F}_r)^r \underset{d_1}{\longrightarrow} L^2\mathbb{F}_r \longrightarrow 0.$$

Let R be a ring. Recall that a right R-module N is flat if and only if the tensor product functor $N \otimes_R -$ preserves injections $V \hookrightarrow F$ where F is a finitely generated free module. The latter holds if and only if $N \otimes_R -$ preserves the injectivity of inclusion $I \hookrightarrow R$ of the left ideals.

Theorem 3.30. *Let $M \hookrightarrow N$ be a trace preserving inclusion of finite von Neumann algebras. Then N is flat over M and* $\dim_N N \otimes_M V = \dim_M V$ *for any M-module V.*

Proof. Recall that any finitely generated submodule of a free M-module is projective. (That is, M is semihereditary.) To see this, let V be a finitely generated submodule of a finitely generated free module $M^{\oplus m}$. $V \simeq M^{\oplus n}A$ for some (m,n)-matrix A. Then V is projective, being isomorphic to $M^{\oplus}.l(A)$. Now,

$$N \otimes V \simeq N^{\oplus n}.l(A) \simeq N^{\oplus m}A \hookrightarrow N^{\oplus} \simeq N \otimes M^{\oplus m}.$$

Thence N is flat over M.

Let V be a finitely generated M-module. Suppose we had an inclusion $\Phi \colon M^{\oplus m}.P \hookrightarrow V$ of a projective module. Then $N^{\oplus m}.P \hookrightarrow N \otimes V$ by the flatness of N. This shows that $\dim_N N \otimes_M V \leq \dim_M V$. On the other hand, for any surjection $\pi M^{\oplus n} \Rightarrow V$, the induced homomorphism $\pi_* : N^{\oplus n} \to N \otimes V$ is surjective and $\dim N \otimes V = n - \dim \pi_*$, thus $\dim N \otimes V \leq \dim V$. □

3.4. Application to orbit equivalence.

Notation. Let $\alpha \colon \Gamma \curvearrowright (X,\mu)$ be a probability measure preserving essentially free action. Put $A = L^\infty(X,\mu), M = L\Gamma$, $N = L^\infty(X,\mu) \rtimes \Gamma = \mathrm{vN}(\mathscr{R}_{\Gamma \curvearrowright X})$. Let R_0 denote the linear span $alg(L^\infty(X,\mu),\Gamma)$ of $f\lambda(s)$ for the $f \in A, s \in \Gamma$. Let R denote the linear span $alg(N(A))$ of fv_ϕ for the $f \in A, \phi \in [[\mathscr{R}]]$.

Remark 3.31. R_0 is free over $\mathbb{C}\Gamma$ and $\mathbb{R}_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \simeq L^\infty(X)$. The induced left R_0-structure on $L^\infty(X)$ is given by $\sum f_s\lambda_s).g = \sum f_s\alpha_s(g)$ thus $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \simeq A$ and we have $\mathrm{Tor}_*^{R_0}(N, A) \simeq \mathrm{Tor}_*^{\mathbb{C}\Gamma}(N, \mathbb{C})$. The latter is isomorphic to $N \otimes_M \mathrm{Tor}_*^{\mathbb{C}\Gamma}(M, \mathbb{C})$ by the flatness of N. Note that $\dim_N N \otimes_M \mathrm{Tor}_n^{\mathbb{C}\Gamma}(M, \mathbb{C}) = \dim_M \mathrm{Tor}_n^{\mathbb{C}\Gamma}(M, \mathbb{C}) = \beta_n^{(2)}(\Gamma)$.

Our goal is to show the equality $\dim_N \mathrm{Tor}_n^{R_0}(N, A) = \dim_N \mathrm{Tor}_n^{R}(N, A)$. Note that the latter only depends on the orbit equivalence $\mathscr{R}_{\Gamma\curvearrowright}$.

Lemma 3.32. *For any $x \in R$ and $\epsilon > 0$, there is a projection p in A such that $\tau p > \epsilon$ and $xp^\perp \in R_0$.*

Proof. When x is of the form $v_\phi f$, the assertion is trivial by the expression $v_\phi = \sum \lambda(g_k)e_k$. The general case reduces o the above by $\tau(p \vee q) \leq \tau p + \tau q$. □

For the time being let A denote an arbitrary finite von Neumann algebra.

Definition 3.33. Let V be a left A-module. For $\xi \in V$,

$$[\xi] = \inf \{\tau p : p \in \mathrm{Proj}\, A, p\xi - \xi\}$$

is called the rank norm of ξ.

Remark 3.34. $[\xi]$ is subadditive and scalar invariant. $V_t = \{\xi : [\xi] = 0\}$ is the largest submodule with $\dim_A V_t = 0$. Any A-module homomorphism $\phi\colon V \to W$ contracts $[\xi]$. Moreover for any $\eta \in \ker\phi$ and $\epsilon > 0$, there is an element $\xi\eta \in \phi^{-1}\eta$ such that $[\xi] \leq [\eta] + \epsilon$.

Definition 3.35. Let V be an A-module. Consider a metric on V defined by $d(\xi, \eta) = [\xi - \eta]$. Let $C(V)$ denote the completion of V with respect to d.

Remark 3.36. $C(V)$ admits an left action of A: the continuity with respect to d follows from $[a\xi] \leq \min[a], [\xi]$: $p\xi = \xi$ implies

$$ap\xi = l(ap)ap\xi - l(ap)a\xi \Rightarrow [a\xi] \leq \tau(l(ap)) = \tau(r(ap))$$

C(V) contains V/V_t as a dense subspace.

Remark 3.37. $V \subset W$ is dense if and only if for any $\xi \in W$ and $\epsilon > 0$, there exists $p \in A$ such that $\tau p < \epsilon$ such that $p^\perp\xi \in V$, which, in turn, happens if and only if $\dim W/V = 0$.

Lemma 3.38. *The functor $V \mapsto CV$ is exact.*

Proof. Right exactness: consider an exact sequence $V_1 \to V_0 \to 0$. Let $\xi \in CV_0$, $(\xi_n)_{n\in\mathbb{N}} \subset V_0$ a sequence convergent to ξ. We may assume that $d(\xi, \xi_n) \leq 2^{-(n+1)}$. We can inductively lift (ξ_n) to (η_n) in V_1 such that $d(\eta_n, \eta_{n+1} \leq 2^{-n}$.

General exactness: let

$$V_2 \xrightarrow[g]{} V_1 \xrightarrow[f]{} V_0$$

be an exact sequence, ξ an element of $\ker C(f)$. Choose a sequence $(\xi_n)_n$ convergent to ξ. Then $f(\xi_n) \to C(f)(\xi) = 0$. This implies the existence of a sequence $(\eta_n)_n$, convergent to 0 and $f\eta_n = f\xi_n$. $\xi = \lim \xi_n - \eta n$ is in the closure of the image of g, which, by the right exactness of C, is equal to the image of $C(g)$. □

Now we turn to the orbit equivalence situation: $A \subset R_0 \subset R \subset N$. We consider A-rank metric on R_0-modules.

Lemma 3.39. *When V is an R_0 (resp. R) module, CV admits an R_0 (resp. R) module structure.*

Proof. If $x - \sum_{n=1}^{N} v_{\phi_n} f_n$, for any $\xi \in V$ we have the estimate $[x\xi] \leq n[\xi]$. □

Lemma 3.40. *When V is an R_0 module, CV admits an R-module structure.*

Proof. Let $x \in R$, $(x_n)_n$ be a sequence in R_0 convergent to x. For any $\xi \in V$, $x_n\xi$ is A-rank convergent to $x\xi$. □

Lemma 3.41. *When V is a left R_0-module. $N \otimes_{R_0} V \to N \otimes_{R_0} CV$ is a dimension isomorphism.*

Proof. Suppose $x = \sum a_i \otimes \xi_i$ ($a_i \in N, \xi_i \in V$) represents 0 in $N \otimes_{R_0} CV$. In the tensor product over $\mathbb{C}$,

$$\sum a_i \otimes \xi_i = \sum (b_j v_j \otimes \eta_j - b_j \otimes v_j \eta_j)$$

for $b_j \in N, v_j \in R_0, \eta_j \in CV$. For each j, there is a projection p_j such that $\tau(p_j) \sim 0$ and $p_j^\perp \eta_j \in V$. Thus we get a representative of x given by

$$\sum (b_j v_j \otimes p_j \eta_j - b_j \otimes v_j p_j \eta_j) + \sum (b_j v_j \otimes p_j^\perp \eta_j - b_j \otimes v_j p_j^\perp \eta_j)$$

The second summand becomes 0 in $N \otimes_{R_0} V$. Now, choose the smallest projection p in N such that $p v_j p_j = v_j p_j, p_j \leq p$. Then $x = (1 \otimes p)x$ and $[x]_N \sim 0$. Hence $N \otimes V \to N \otimes CV$ is an isometry. When $\xi_n \in V$ converges to $\xi \in CV$, $a \otimes \xi_n$ converges to $a \otimes \xi$ in $[-]_N$. □

Remark 3.42. For any R-module W, $N \otimes_{R_0} W \to N \otimes_R W$ is an $\dim_N$-isomorphism.

Theorem 3.43. $\dim_N \operatorname{Tor}_n^{R_0}(N, A) = \dim_N \operatorname{Tor}_n^{R}(N, A)$.

Proof. Consider projective resolutions of A: $P_* \to A$ as an R_0-module, $Q_* \to A$ as an R-module. We have morphisms $\phi_* \colon P_* \to Q_*$ and $\psi_* \colon Q_* \to CP_*$. Thus we get a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A \\
 & & \big\downarrow{\scriptstyle \phi_n} & & & & \big\downarrow{\scriptstyle \phi_0} & & \big\| \\
\cdots & \longrightarrow & Q_n & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & A \\
 & & \big\downarrow{\scriptstyle \psi_n} & & & & \big\downarrow{\scriptstyle \psi_0} & & \big\downarrow \\
\cdots & \longrightarrow & CP_n & \longrightarrow & \cdots & \longrightarrow & CP_0 & \longrightarrow & CA \\
 & & \big\downarrow{\scriptstyle C\phi_n} & & & & \big\downarrow{\scriptstyle C\phi_0} & & \big\| \\
\cdots & \longrightarrow & CQ_n & \longrightarrow & \cdots & \longrightarrow & CQ_0 & \longrightarrow & CA.
\end{array}$$

By the uniqueness of projective resolution up to homotopy, compositions of two homomorphisms $\psi_n\phi_n$ and $C\phi_n\psi_n$ are homotope to the the standard inclusion isomorphisms.

Now, the standard inclusion $P_* \to CP_*$ induces a $\dim_N$-isomorphism after applying the $N \otimes_{R_0} -$ functor by Lemma 3.41. Thus, $\mathrm{Id}_N \otimes \phi_*$: and $\mathrm{Id}_N \otimes \psi_*$ are inverse to each other via the identification of $N \otimes P_* \simeq N \otimes CP_*$ and $N \otimes Q_* \simeq N \otimes CQ_*$. Hence $\mathrm{Id}_N \otimes \phi_*$ induces a dimension isomorphism on cohomology groups. □

Corollary 3.44. *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be essentially free probability measure preserving actions. If $\mathscr{R}_{\Gamma \curvearrowright X} \simeq \mathscr{R}_{\Lambda \curvearrowright Y}$, $\beta_n^{(2)}(\Gamma) = \beta_n^{(2)}(\Lambda)$.*

Remark 3.45. Put $\beta^2_*(A, N) = \dim_N \mathrm{Tor}^R_*(N, A)$. For any nonzero projection p in A, $\beta^2_*(A, N) = \tau(p)\beta^2_*(pA, pNp)$.

4. Derivations on von Neumann algebras

In the following we only consider normal Hilbert (bi)modules over von Neumann algebras. Examples of such modules include the identity bimodule L^2N and the coarse (M, N)-module $L^2M \otimes L^2N$.

Let Γ be a countable discrete group, (π, H_0) a unitary representation of Γ. A map $b\colon \Gamma \to H_0$ is said to be a derivation when it satisfies $b(st) = b(s) + \pi(s)b(t)$ i. e. a derivation for the (π, triv)-bimodule structure. A derivation b is said to be inner when there exists $\xi \in H_0$ such that $b(s) = \pi(s)\xi - \xi$. Put

$$H^1(\Gamma, \pi) = \{\text{derivations}\} / \{\text{inner derivations}\}.$$

When b is a derivation, $\phi_r(s) = e^{-r\|b(s)\|^2}$ for $r \geq 0$ determines a positive semidefinite semigroup. Our goal is to show that it extends to a semigroup $\tilde{\phi}_r\colon L\Gamma \to L\Gamma$ of τ preserving completely positive maps.

4.1. Densely defined derivations.

Let M denote $L^2\Gamma$. Consider $H = M \otimes H_0$. A left action $M \to B(H)$ is defined by $\lambda(f) \mapsto \lambda \otimes \pi(f)$ (this is possible by the Fell absorption.) On the other hand we have a right action $M^o \to B(H)$ is defined by $\rho(g) \mapsto \rho(g) \otimes id$. Put $\delta(s) = \delta_s \otimes b(s) \in \ell_2\Gamma\bar{\otimes}H_0$. By

$$\delta(st) = \delta_{st} \otimes (b(s) + \pi(s)b(t)) = \rho \otimes 1(t^{-1})\delta(s) + \lambda \otimes \pi(s)\delta(t),$$

δ extends to a (possibly unbounded) derivation $\mathbb{C}\Gamma \to H$ satisfying $\delta(xy) = x\delta(y) + \delta(x)y$.

Notation. Let (M, τ) be a finite von Neumann algebra with a faithful normal tracial state, $\mathscr{D}$ a weak*-dense $*$-subalgebra of M. Let H be a Hilbert bimodule over M, $\delta\colon M \to H$ a derivation defined on $\mathscr{D}$ which is closable as a densely defined operator $L^2M \to H$. Let $\bar{\delta}$ denote its closure.

We are going to show that the domain of $\bar{\delta}$ is a $*$-subalgebra of $\mathscr{L}(H)$ and that $\bar{\delta}$ is a derivation.

Notation. Let $\|{-}\|_{\mathrm{Lip}}$ denote the 1-Lipschitz norm:

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Let Lip_0 denote the space of 1-Lipschitz continuous functions which map 0 to 0.

For any $x \in L^2M_{sa}$, regarded as a self adjoint unbounded operator on L^2M, we can consider its functional calculus $f(x)$.

Proposition 4.1. *When $x, y \in L^2M_{sa}$ and $f \in \mathrm{Lip}_0$, the functional calculus $f(x), f(y)$ is in L^2M and*

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\mathrm{Lip}} \|x - y\|_2 .$$

Proof. For the spectral measure $E(t)$ of x, $x = \int t dE(t)$ and $\|x\|_2^2 = \int |x|^2 d\tau E(T)$. Thus $\|f(x)\|_2^2 = \int |f(t)|^2 d\tau E(t) \leq \|f\|_{\mathrm{Lip}} \int |t|^2 d\tau E(t)$ and $f(x)$ is in L^2M. For the second assertion, consider the bilinear map

$$C_0(\mathbb{R})^2 \ni (f, g) \mapsto \tau(f(x)g(x)) = \langle f(x)\hat{1}f(y), \hat{1}\rangle.$$

This defines a linear form $C_0(\mathbb{R} \times \mathbb{R}) \to \mathbb{C}$, i.e. $\tau(f(x)g(y)) = \int f g d\mu$ for some measure μ on $\mathbb{R} \times \mathbb{R}$. Thus, $\tau(|f(x) - f(y)|^2)$ is equal to

$$\int |f(s) - f(t)|^2 d\mu(s,t) \leq \|f\|_{\mathrm{Lip}}^2 \int |s - t|^2 d\mu(s,t) = \|f\|_{\mathrm{Lip}}^2 \|x - y\|_2^2 . \qquad \square$$

Definition 4.2. Let I be a bounded closed interval in $\mathbb{R}$, $f \in C^1(I)$. The function

$$\tilde{f}(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y) \\ f'(x) & (x = y) \end{cases}$$

is called the difference quotient of f.

Note that $\left\|\tilde{f}\right\|_\infty = \|f'\|_\infty$. When $a \in M_{sa}$ and $[-\|a\|, \|a\|] \subset I$, we have $\pi_a \colon C(I \times I) \to B(H)$ by $\pi_a(f \otimes g)\xi = f(a)\xi g(a)$.

Lemma 4.3. *For any $a \in \mathscr{D}$ and $f \in C^1(I)$, the operator $f(a)$ is in $\mathrm{dom}\,\bar{\delta}$ and $\bar{\delta}(f(a)) = \pi_a(\tilde{f})\delta(a)$.*

Proof. The assertion is obvious for polynomial functions. The equality for the general C^1-functions follows from it because it is compatible with the C^1-norm. $\square$

Remark 4.4. When T is a closed operator on H, $x_n \to x$ $(n \to \infty)$ in H and $\sup_n \|Tx_n\| < \infty$ imply that $x \in \mathrm{dom}\, T$ and that $Tx \in \bigcap_{m=0}^\infty \overline{\mathrm{conv}}\, \{Tx_n : n \geq m\}$, where $\overline{\mathrm{conv}}$ denotes the closed convex span. This is because, taking a suitable subsequence if necessary, we may assume that the bounded sequence Tx_n is weakly convergent to some y. Taking the convex closure, we can find a sequence $(z_n)_{n\in\mathbb{N}}$ such that $Tz_n \to y$ in norm and that z_n is in the algebraic convex closure of $\{x_k : k \geq n\}$. By construction, $(z_n)_{n\in\mathbb{N}}$ converges to x.

Lemma 4.5. *Let x be an unbounded self adjoint operator on L^2M which is in $\mathrm{dom}\,\bar{\delta}$, $f \in \mathrm{Lip}_0$. Then $f(x) \in \mathrm{dom}\,\bar{\delta}$ and $\left\|\bar{\delta}(f(x))\right\| \leq \|f\|_{\mathrm{Lip}} \left\|\bar{\delta}(x)\right\|$.*

Proof. Choose a mollifier $(\phi_n)_n$ and set $f_n = f * \phi_n$. Thus f_n is of C^1 class and $f_n \to f$ uniformly on I. By

$$|f_n(y) - f_n(z)| = \int |f(y-r) - f(z-r)|\phi_n(r)dr \leq \|f\|_{\mathrm{Lip}} |y - z|,$$

we have $\|f_n\|_{\mathrm{Lip}} \leq \|f\|_{\mathrm{Lip}}$. Now take a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{D}_{sa}$ which is convergent to x in $\|-\|_2$-norm. Then

$$\left\|\bar{\delta}(f_n(a))\right\| = \left\|\pi_a(\tilde{f_n})\delta(a)\right\| \leq \|f_n\|_{\mathrm{Lip}} \|\delta a\| .$$

This shows $f(x) \in \operatorname{dom}\bar{\delta}$. □

Definition 4.6. A derivation $\delta \colon M \to H$ is said to be real when we have

$$\langle \delta(x), \delta(y)z\rangle = \langle z^* \delta(y^*, x^*\rangle$$

for any $x, y, z \in M$.

Remark 4.7. We summarize a few properties of real derivations.

- When M is the group von Neumann algebra $L\Gamma$ of a group Γ, the above condition is equivalent to $\langle \delta(s), \delta(t)\rangle \in \mathbb{R}$.
- In general, when we have a J-operator, δ is real if and only if $Jx\delta(y)z = z^*\delta(y^*)x^*$, since, by definition, $\langle \delta(x), \delta(y)z\rangle$ is equal to $\langle z^* J\delta(y), J\delta(x)\rangle$.
- When δ is real, $\operatorname{dom}\bar{\delta}$ is self adjoint.

Let $\overline{\mathscr{D}}$ denote $\operatorname{dom}\bar{\delta}$.

Lemma 4.8. *Let δ be a real derivation. When $x \in \overline{\mathscr{D}}$, $|x|$ is also in $\overline{\mathscr{D}}$ and $M \cap \overline{\mathscr{D}}$ is a *-subalgebra of M.*

Proof. Consider the linear map $\delta^{(2)} \colon M_2\mathscr{D} \to M_2 H \simeq H^{\oplus 4}$. Then $\delta^{\bar{(2)}} = \bar{\delta}^{(2)}$ and for any $z \in \overline{\mathscr{D}}$,

$$w = \begin{bmatrix} 0 & z^* \\ z & 0 \end{bmatrix} \in \operatorname{dom}\delta^{\bar{(2)}} \Rightarrow w^2 = \begin{bmatrix} |z|^2 & 0 \\ 0 & |z^*|^2 \end{bmatrix} \in \operatorname{dom}\delta^{\bar{(2)}}.$$

Thus $|z|^2$ is in $\mathscr{D}$.

Let $x, y \in \overline{\mathscr{D}}$. The polarization

$$x^* y = \frac{1}{4}\sum i^k |x + i^k y|$$

shows $x^* y \in \overline{\mathscr{D}}$, and in particular $x^* \in \overline{\mathscr{D}}$ follows from $1 \in \mathscr{D}$. □

Lemma 4.9. *For any $x \in \overline{\mathscr{D}} \cap M_{sa}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}_{sa}$ such that*

$$\|x_n - x\|_2 \to 0, \left\|\delta(x_n) - \bar{\delta}(x)\right\| \to 0 and \|x\|_\infty \leq \|x\|_\infty .$$

In particular, $x_n \to x$ in the ultrastrong topology.

Proof. The only nontrivial part is the last inequality. This is achieved by the functional calculus with respect to the function

$$f(t) = \begin{cases} \|x\|_\infty & (\|x\|_\infty < t) \\ t & (|t| \leq \|x\|_\infty) \\ -\|x\|_\infty & (t < -\|x\|_\infty). \end{cases} \quad \square$$

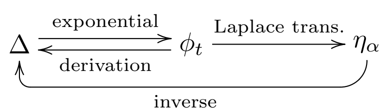
Theorem 4.10. *The restriction of $\bar{\bar{\delta}}$ to $\overline{\mathscr{D}} \cap M$ is a derivation.*

Proof. Let $x \in \overline{\mathscr{D}} \cap M$. Choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}$ weakly convergent to x and $\delta(x_n) \to \bar{\delta}(x)$. For each $y \in \mathscr{D} \cap M$, we have $x_n y \to xy$ in the $\|-\|_2$-norm. Since y is bounded, we have $\delta(x_n)y \to \bar{\delta}y$. On the other hand, the representation of M on H is normal, which implies $x_n\delta(y) \to x\delta(y)$. Thus we have $\bar{\delta}(xy) = x\delta(y) + \bar{\delta}(x)y$. Similar approximation in y shows that $\bar{\delta}(xy) = x\bar{\delta}(y) + \bar{\delta}(x)y$. for any $y \in \overline{\mathscr{D}} \cap M$. □

4.2. **Semigroup associated to a derivation.** In the following we assume $M \cap \overline{\mathscr{D}} = \mathscr{D}$. Put $\Delta = \delta^*\bar{\delta}$. This is a positive self adjoint operator on L^2M satisfying $\Delta\hat{1} = \hat{1}$ and commutes with the J operator so that we have "$\Delta(x^*) = (\Delta x)^*$." Put $\phi_t = e^{-t\Delta}$. This is a semigroup of positive contractions satisfying $\phi_t\hat{1} = \hat{1}$ and $\phi_t \nearrow \mathrm{Id}$ as $t \searrow 0$. The normalized resolvents

$$\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$$

for $\alpha > 0$ are again positive contractions on L^2M satisfying $\eta_\alpha \nearrow \mathrm{Id}$ as $\alpha \nearrow \infty$. These operators are related to each other as follows:



where the Laplace transform is given by

$$\eta_\alpha = \alpha \int_0^\infty e^{-\alpha t}\phi_t dt = \int_0^\infty e^{-t}\phi_{\frac{t}{\alpha}} dt.$$

Recall that any unital completely positive map $\phi\colon M \to M$ is expressed as $V^*\pi(x)V$ for some representation $\pi\colon M \to B(K)$ and an isometry $V\colon L^2M \to K$ (Steinespring's theorem). When ϕ is normal, π can be taken as a normal representation (we may take the normal part of a possibly non-normal π given by Steinespring's theorem). Thus,

(1) For any $x \in M$, $\phi(x^*x) - \phi(x^*)\phi(x) = V^*\pi x^*(1 - VV^*)\pi x V \geq 0$. When ϕ preserves τ, $\|\phi(x)\|_2 \leq \|x\|$.
(2) When ϕ preserves τ, $\|\phi(x^*y) - \phi(x^*)\phi(y)\|_2 = \left\|V^*\pi x(1 - VV^*)\pi y V\hat{1}\right\|$ is bounded from above by

$$\|\phi(x^*x) - \phi x^*\phi x\|_\infty^{\frac{1}{2}} \left(\tau(\phi(y^*y) - \phi y^*\phi y)\right)^{\frac{1}{2}} \leq 2\,\|x\|_\infty \|y - \phi(y)\|_2$$

by $\tau(\phi(y^*y) - \phi y^*\phi y) = \|y\|_2^2 - \|\phi y\|_2^2$, etc.

Fact. Consider the 1-norm $\|x\|_1 = \sup\{|\tau(xy)| : \|y\|_\infty \leq 1\}$ for $x \in M$. $x \in L^2M$ is in M if and only if $\sup\{|\tau(xy)| : \|y\|_1 \leq 1, xy \in M\}$ is finite.

Theorem 4.11. *(Sauvageot, [1]?) The contractions ϕ_t and η_α map M into M, are unital completely positive and τ-symmetric, i. e. $\tau(\phi_t(x)y) = \tau(x\phi_t(y))$ etc.*

Proof. Observe that $\phi_t^{(n)} = e^{-t\Delta^{(n)}}$ where $\Delta^{(n)} = \delta^{(n)*}\bar{\delta}^{(n)}$ for $\delta^{(n)}\colon M_n\mathscr{D} \to M_nH$. Thus, it is enough to show that the maps are positive to conclude that they are actually completely positive. Put

$$\Delta_\alpha = \frac{\alpha\Delta}{\alpha + \Delta} = \alpha(1 - \eta_\alpha).$$

Then

$$\phi_t = e^{-t\Delta}e = \lim_{\alpha\nearrow\infty} e^{-t\Delta_\alpha} = \lim_{\alpha\nearrow\infty} e^{-t\alpha}\sum_{n=0}^{\infty}\frac{t\alpha\eta_\alpha}{n!}$$

where the limit is taken in the strong operator topology (note: this might be the norm topology, as we are using c_0 functions converging from below). The last expression is compatible with the $x \mapsto \tau(xy)$ ($\|y\|_1 \le 1$) functionals. Thus it reduces to show that η_α restricts to a positive map on M.

By scaling δ, we may assume that $\alpha = 1$. Let $x \in M_+$ and put $y = (1+\Delta)^{-1}x \in \operatorname{dom}\Delta$. We have

$$\|\delta y\|^2 + \|y\|_2^2 = \langle y, \Delta y\rangle + \langle y, y\rangle = \langle y, x\rangle$$

Then the function $\Phi(z) = \|\bar\delta(z)\|^2 + \|z-x\|_2^2$ for $z \in \overline{\mathscr{D}}_{sa}$ satisfies

$$\begin{aligned}\|\bar\delta(z-y)\|^2 + \|z-y\|_2^2 &= \|\bar\delta(z)\|^2 - 2\langle z, \Delta y\rangle + \|\bar\delta(y)\|^2 + \|z\|^2 - 2\langle z, y\rangle + \|y\|^2\\ &= \|\bar\delta(z)\|^2 + \|z\|^2 - 2\langle z, x\rangle + \|x\|^2\\ &\quad - (\|\bar\delta(y)\|^2 + \|y\|^2 - 2\langle y, x\rangle + \|x\|^2)\\ &= \Psi(z) - \Psi(y).\end{aligned}$$

Consider a function

$$f(t) = \begin{cases}\|x\|_\infty & (\|x\|_\infty < t)\\ t & (0 \le t \le \|x\|_\infty)\\ 0 & (t < 0).\end{cases}$$

of Lip_0 class with $\|f\|_{\mathrm{Lip}} = 1$. Then

$$\Psi(f(z)) = \|\bar\delta(f(z))\|^2 + \|f(z) - f(x)\| \le \Psi(z).$$

Take a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathscr{D}_{sa}$ with $\|z_n - y\|_2 \to 0$ and $\|\delta z_n - \bar\delta y\| \to 0$. Then we have

$$\|fz_n - y\|_2^2 \le \Psi(fz_n) - \Psi(y) \le \Psi(z_n) - \Psi(y) \to 0.$$

Thus $y = \lim fz_n$ and $0 \le y \le \|x\|$ and η_1 is shown to be unital positive. □

Let B be a von Neumann subalgebra of M. Then we are interested in "when ϕ_t converges uniformly on B_1?" Roughly, this means "δ is inner on B."

Lemma 4.12. *Let $\Omega \subset M_1$. Then $\phi_t \to id$ uniformly on Ω as $t \to 0$ if and only if $\eta_\alpha \to id$ uniformly on Ω as $\alpha \to \infty$.*

Proof. $\Rightarrow$: We have

$$\|x - \eta_\alpha x\|_2 \le \int_0^\infty e^{-s}\left\|x - \phi_{\frac{s}{a}}(x)\right\|_2 ds,$$

but $\left\|x - \phi_{\frac{s}{a}}(x)\right\|_2$ does not exceed 2.

$\Leftarrow$: Suppose ϕ_s did not converge uniformly on Ω. Then there is a constant c such that for any t there exists an element x_t of Ω satisfying $\langle x_t - \phi_t x_t, x_t\rangle \ge c$.

Then

$$\begin{aligned}\langle x_t - \eta_{\frac{1}{t}} x_t, x_t\rangle &= \int_0^\infty e^{-s}\langle x_t - \phi_{st}x_t, x_t\rangle ds \\ &\geq \int_0^1 e^{-s}\langle x_t - \phi_t(x_t), x_t\rangle ds \\ &\geq c(1-e^{-1})\end{aligned}$$

and η_α is not uniformly convergent on Ω. □

Lemma 4.13. *For the latter convenience we record the following equalities:*

(1) *In* $B(L^2M)$,

$$\eta_\alpha^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t}\eta_{\frac{\alpha(1+t)}{t}}\,dt.$$

(2) *In* $B(L^2M)$,

$$(\mathrm{Id}-\eta_\alpha)^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t}(1-\eta_{\frac{\alpha(1+t)}{t}})dt = \mathrm{Id}-\theta_\alpha$$

where θ_α *restricts to a unital completely positive map on* M.

(3) $\psi_t = e^{-t\Delta^{\frac{1}{2}}}$ *is* τ*-symmetric and unital completely positive on* M.

Proof. (1): we have

$$s^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \frac{s}{s+t}t^{-\frac{1}{2}}dt \Rightarrow \eta_\alpha^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \frac{\eta_\alpha}{t+\eta_\alpha}t^{-\frac{1}{2}}dt.$$

On the other hand,

$$\frac{\eta_\alpha}{t+\eta_\alpha} = \frac{\alpha}{\alpha(1+t)+t\Delta} = \frac{1}{1+t}\eta_{\frac{\alpha(1+t)}{t}}.$$

(3): We have $\Delta_\alpha^{\frac{1}{2}} = \alpha^{\frac{1}{2}}(\mathrm{Id}-\eta_\alpha)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}(\mathrm{Id}-\theta_\alpha)$. Thus ψ_t can be written as

$$\lim_{\alpha\to\infty} e^{-t\Delta_\alpha^{\frac{1}{2}}} = \lim_{\alpha\to\infty} e^{-\alpha^{\frac{1}{2}}t}e^{t\alpha^{\frac{1}{2}}\theta_\alpha}.$$ □

Lemma 4.14. *For* $x, y \in \mathscr{D}$*, put* $\Gamma(x^*, y) = \Delta^{\frac{1}{2}}(x^*)y + x^*\Delta^{\frac{1}{2}}(y) - \delta^{\frac{1}{2}}(x^*y)$*. Then we have*

$$\|\Gamma(x^*, y)\|_2 \leq 4\,\|\delta(x)\|\,\|x\|_\infty\,\|\delta(y)\|\,\|y\|_\infty\,.$$

Proof. First we have

$$\Gamma(x^*, y) = \frac{d}{dt}(\psi_t(x^*y) - \psi_t(x^*)\psi_t(y))\Big|_{t=0}.$$

Note that $\|\psi_t x\| \leq \|x\|$. Define a sesquilinear form on $\mathscr{D}\otimes M$ by $\langle y\otimes b, x\otimes a\rangle = \tau(a^*\Gamma(x^*, y)b)$. This is positive semidefinite by

$$\langle \sum x_i\otimes a_i, \sum x_i\otimes a_i\rangle = \lim_{t\to 0}\tau(\sum a_i\frac{\psi_t(x_i^*x_j) - \psi_t x_i^*\psi_t(x_j)}{t}a_j) \leq 0.$$

For $z = v|z| \in M$, we have

$$|\tau(\Gamma(x^*, y)z)| = |\langle y\otimes v|z|^{\frac{1}{2}}, x\otimes |z|^{\frac{1}{2}}\rangle| \leq \langle y\otimes v|z|^{\frac{1}{2}}, y\otimes v|z|^{\frac{1}{2}}\rangle^{\frac{1}{2}}\langle x\otimes |z|^{\frac{1}{2}}, x\otimes |z|^{\frac{1}{2}}\rangle^{\frac{1}{2}}.$$

Here, $\langle x\otimes a, x\otimes a\rangle \leq \|aa^*\|_2 \|\Gamma(x^*,x)\|_2$ and

$$\begin{aligned}\|\Gamma(x^*,x)\| &\leq \left\|\Delta^{\frac{1}{2}}x^*\right\|_2 \|x\|_\infty + \|x^*\|_\infty \left\|\Delta^{\frac{1}{2}}x\right\|_2 + \left\|\Delta^{\frac{1}{2}}(x^*x)\right\|_2 \\ &\leq 4\|\delta(x)\| \|x\|_\infty .\end{aligned}$$

(Here we used the fact that $\left\|\Delta^{\frac{1}{2}}(x^*x)\right\|_2 = \|\delta(x^*)x + x^*\delta(x)\|$.) Hence we arrive at

$$|\tau(\Gamma(x^*,y)z)|^2 \leq \|\Gamma(x^*,x)\|_2 \|z\|_2 \|(y^*,y)\| \|z\|_2 ,$$

thus $\|\Gamma(x^*,y)\|^2 \leq \|\Gamma(x^*,x)\|_2 2\|(y^*,y)\|$. □

Put $\zeta_\alpha = \eta_\alpha^{\frac{1}{2}}$. $\Delta^{\frac{1}{2}}\zeta_\alpha = \Delta_\alpha^{\frac{1}{2}} = (\mathrm{Id} - \eta_\alpha)^{\frac{1}{2}}$ (hence bounded) and $\left\|\Delta_\alpha^{\frac{1}{2}}x\right\|_2^2 = \alpha\langle(\mathrm{Id}-\eta_\alpha)x,x\rangle$. Put $\tilde{\delta}_\alpha = \alpha^{\frac{1}{2}}\delta\zeta_\alpha$. Thus $\left\|\tilde{\delta}_\alpha(x)\right\| = \langle(\mathrm{Id}-\eta_\alpha)x,x\rangle$ and $\|\delta\|_\alpha(x) \to 0$ if and only if $\|x-\eta_\alpha x\|_2 \to 0$.

Theorem 4.15. *(Peterson?) Let $\Omega \subset M_1$ and suppose $\eta_\alpha \to \mathrm{Id}$ uniformly on Ω. Then we have $\left\|\tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x)\right\| \to 0$ ($\alpha\to\infty$) uniformly for $a\in\Omega$ and $x \in M_1$.*

Proof. By assumption ζ_α and θ_α converge uniformly to Id on Ω, by e.g. .

$$\theta_\alpha = \frac{1}{\pi}\int_0^\infty \frac{t^{\frac{1}{2}}}{1+t}\eta_{\frac{\alpha t}{1+t}}\,dt.$$

In particular, $\theta_\alpha(ax) \approx \theta_\alpha(a)\theta_\alpha(x) \approx a\theta_\alpha(x)$ whre $\approx$ means the 2-norm convergence under $\alpha\to\infty$. Now,

$$\begin{aligned}\alpha^{-\frac{1}{2}}\Delta^{\frac{1}{2}}\zeta_\alpha(ax) &= \alpha^{-\frac{1}{2}}(\mathrm{Id}-\theta_\alpha)(ax) \approx \alpha^{-\frac{1}{2}}a(\mathrm{Id}-\theta_\alpha)(x) \approx \alpha^{-\frac{1}{2}}\zeta_\alpha(a)(\mathrm{Id}-\theta_\alpha)(x) \\ &= \alpha^{-\frac{1}{2}}\zeta_\alpha(a)\Delta^{\frac{1}{2}}\zeta_\alpha(x) \approx \alpha^{-\frac{1}{2}}\Delta^{\frac{1}{2}}(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x)\end{aligned}$$

where the last approximation is given by applying Lemma 4.14 to get the error estimate

$$4\sqrt{\alpha^{-\frac{1}{2}}\left\|\delta^{\frac{1}{2}}(\zeta_\alpha(a)\right\|\left\|\alpha^{\frac{1}{2}}\delta\zeta_\alpha x\right\|}.$$

Here, $\alpha^{-\frac{1}{2}}\left\|\delta^{\frac{1}{2}}(\zeta_\alpha(a)\right\| \sim 0$ and $\left\|\alpha^{\frac{1}{2}}\delta\zeta_\alpha x\right\|$ is bounded by 1.

Finally we arrive at

$$\tilde{\delta}_\alpha(ax) \approx \alpha^{-\frac{1}{2}}\delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) = \zeta_\alpha(a)\tilde{\delta}_\alpha(x).$$ □

Theorem 4.16. *(Haagerup) Let M be a von Neumann algebra. M is finite injective if and only if for any nonzero central projection p of M, there exist $n\in\mathbb{N}$ and $u_1,\ldots,u_n \in \mathcal{U}(pM)$ such that $\|\sum_{i=1}^n u_i\otimes u_i\|_\infty = n$.*

Proof. (Outline) $\Rightarrow$: By Connes' theorem, $M\otimes_{\min}\bar{M} \to B(L^2M)$ can be defined by $(a\otimes b).\hat{x} = \widehat{axb^*}$. Now, $(\sum_{i=1}^n u_i\otimes\bar{u}_i).\hat{1} = n\hat{1}$ when $u_i\in\mathcal{U}M$.

$\Leftarrow$: The minimal tensor product $M\otimes_{\min}\bar{M}$ acts on $H\hat{\otimes}\bar{H}$ i.e. the Hilbert-Schmidt space of H. For any finite set $F\subset\mathcal{U}M$ containing 1 and $\left\|\sum_{u\in F}u\otimes\bar{u}\right\| = |F|$, there exists $T\in HS(H)$ of 2-norm 1, $\left\|\sum_{u\in F}uTu^*\right\| \approx |F|$. Then $uTu^*\approx T$. Now, define $\phi_F(x) = \mathrm{Tr}(T^*xT)$. Then $\phi_F(uxu^*) \approx \phi_\alpha(x)$ for $u\in F$. We obtain an ultrafilter convergence $\phi_F \to \phi \in S(B(H))$ such that $\phi(uxu^*) = \phi(x)$ for any $u \in \mathcal{U}M$. This holds under any central projection, which means M is injective. $\square$

Recall that we are investigating closable real derivations on M. Thus, H is an M-bimodule with a J-operator: $J(a\delta(x)b) = b^*\delta(x^*)a^*$. We have the operators

$$\eta_\alpha = \frac{\alpha}{\alpha + \delta^*\bar{\delta}}, \zeta_\alpha = \eta_\alpha^{\frac{1}{2}}, \tilde{\delta}_\alpha = \alpha^{-\frac{1}{2}}\delta\zeta_\alpha \colon M \to H.$$

As $\alpha \to \infty$, we have $\left\|\tilde{\delta}_\alpha(a)\right\|^2 = \left\|(\mathrm{Id} - \eta_\alpha)^{\frac{1}{2}} a\right\|_2^2 = \tau((a - \eta_\alpha a)a^*) \searrow 0$.

Theorem 4.17. *Let $(M;\tau)$ be a finite von Neumann algebra, $H = (L^2M \otimes L^2M)^{\oplus\mathbb{N}}$. Suppose $Q \subset M$ is a von Neumann subalgebra without injective summand. Then $\phi_t \to \mathrm{Id}$ uniformly on $(Q' \cap M)_1$.*

Proof. It is enough to show that for any nonzero central projection $p \in Q$, there exists a central projection $q \le p$ in Q such that $\phi_t \to \mathrm{Id}$ on $q(Q' \cap M)_1$. In fact, then by the maximal argument we would get a family $(p_i)_{i \in I}$ of nonzero central projections such that $\sum_{i \in I} p_i = 1$ and $\phi_t \to \mathrm{Id}$ on $p_i(Q' \cap M)_1$ for each i. Taking a finite subset $I_0 \subset I$ such that $\tau(\sum_{\complement I_0} p_i) < \frac{\epsilon}{3}$, we find t_0 such that $t > t_0$ implies $\|\phi_t(a) - a\|_2 < \frac{\epsilon}{3}$ for $a \in p_{I_0}(Q' \cap M)_1$. On the other hand, for any $a \in p_{I_0}(Q' \cap M)_1$ $\tau(a - p_{I_0}a) < \frac{\epsilon}{3}$.

Thus we are going to prove the negation of the above claim leads to that pQ is injective. Let $q \le p$ be a nonzero central projection in Q, $u_1, \ldots, u_n \in \mathcal{U}(qQ)$. As ϕ_t does not converge uniformly on $q(Q' \cap M)_1$, there exists $x_\alpha \in q(Q' \cap M)_1$ for any α such that $\liminf \left\|\tilde{\delta}_\alpha(x_\alpha)\right\| > 0$.

Applying Theorem 4.15 to the finite subset $\Omega = \{u_1, \ldots, u_n\}$ on which ϕ_t is uniformly convergent, for any $x \in q(Q' \cap M)$, as $\alpha \to \infty$,

$$\sum_i \zeta_\alpha(u_i)\tilde{\delta}_\alpha(x)\zeta_\alpha(u_i^*) \approx \sum_i \tilde{\delta}_\alpha(u_i x u_i^*) = n\tilde{\delta}_\alpha(x).$$

Thus, $\left\|\sum_i \zeta_\alpha(u_i) \otimes \overline{\zeta_\alpha(u_i)}\right\|_{\min} \to n$ as $\alpha \to \infty$. On the other hand, since ζ_α is a normal unital completely positive map, $\left\|\sum_i \zeta_\alpha(u_i) \otimes \overline{\zeta_\alpha(u_i)}\right\|_{\min}$ is always bounded by $\|\sum u_i \otimes \bar{u}_i\|$, which shows that $\|\sum u_i \otimes \bar{u}_i\| = n$. Thus we have the injectivity of pQ by Theorem 4.16. $\square$

Remark 4.18. If a 1-cocycle $b \colon \mathbb{F}_r \to \ell_2\mathbb{F}_r^{\oplus n}$ satisfies $\|b(s)\|_2^2 = |s|$, we obtain a derivation δ on $\ell_2\mathbb{F}_r \otimes \ell_2\mathbb{F}_r^{\oplus n}$ given by $\delta(s) = \delta_\Delta \otimes b$ where δ_Δ is the "diagonal" operator on $\ell_2\mathbb{F}_r$ which multiplies the standard base δ_s by $|s|$. The semigroup ϕ_t associated to this derivation is written as $\phi_t(\lambda(s)) = e^{-t|s|}\lambda(s)$, thus it is in $\mathbb{K}(L^2M)$.

When B is a von Neumann subalgebra of $L\mathbb{F}_r$, $\phi_t \to \mathrm{Id}$ uniformly on B_1 if and only if B is a direct sum $\oplus M_{n_i}$ of finite dimensional algebras.

Corollary 4.19. *Let Q be a von Neumann subalgebra of $L\mathbb{F}_r$ without injective summand. Then the relative commutant $Q' \cap L\mathbb{F}_r$ is completely atomic. In particular, $Q \otimes L^\infty[0,1] \not\subset L\mathbb{F}_r$.*

Theorem 4.20. *Let $(M;\tau)$ be a finite von Neumann algebra, $H=(L^2M\otimes L^2M)^{\mathbb{N}}$, δ a closable real derivation. If $B\subset M$ is diffuse (i.e. without minimal projection) von Neumann subalgebra such that ϕ_t converges to* Id *uniformly on B_1, one has $\phi_t\to \mathrm{Id}$ uniformly on $N(B)''_1$.*

Proof. Since B is diffuse, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{U}B$ ultraweakly convergent to 0 (e.g. $e^{2\pi int}\in L^\infty[0,1]$ for $n\in\mathbb{N}$). For any $u\in\mathcal{N}(B)$,

$$\begin{aligned}\left\|\tilde{\delta}_\alpha(u)\right\| &\leq \liminf\left\|\tilde{\delta}_\alpha(u)-\zeta_\alpha(v_n)\tilde{\delta}_\alpha(u)\zeta_\alpha(u^*v_n^*u)\right\|\\ &\to \left\|\tilde{\delta}_\alpha(u)-\tilde{\delta}_\alpha(v_nuu^*v_n^*u)\right\|=0 \quad (n\to\infty)\end{aligned}$$

The convergence holds uniformly for u. It remains to apply the following lemma to $N(B)=G$. □

Lemma 4.21. *When $\phi_t\to\mathrm{Id}$ uniformly on $G\subset\mathcal{U}M$, we have the uniform convergence $\phi_t\to\mathrm{Id}$ on G''_1.*

Proof of the lemma. Let $\phi\colon M\to M$ be a τ-symmetric unital completely positive map (hence a contraction). Consider the Stinespring construction on $M\otimes_{\mathrm{alg}}L^2M$ by $\langle a\otimes x,b\otimes y\rangle=\langle\phi(b^*a)x,y\rangle$. This is positive semi definite by the unital completely positivity. The M-M-action $a.(c\otimes x).b=ac\otimes xb$ is bounded and induces an M-bimodule structure on the completion.

Now, for $\xi_0=1\otimes\hat{1}\in M\otimes L^2M$,

$$\|a\xi_0-\xi_0a\|^2=\tau(\phi(aa^*))+\tau(aa^*)-2\Re\tau(\phi(aa^*))=2\tau((a-\phi(a))a^*).$$

On the other hand,

$$\frac{1}{2}\|a-\phi(a)\|_2^2\leq\|a\xi_0-\xi_0a\|\leq 2\|a-\phi(a)\|_2\cdot\|a\|_2\,.$$

Thus, if $\|u-\phi(u)\|\leq\epsilon$, we have $\|\xi_0-u\xi_0u^*\|\leq\sqrt{2\epsilon}$. By taking the circumcenter of $\{u\xi_0u^*:u\in G\}$, we get a G-invariant vector η_0 satisfying $\|\xi_0-\eta_0\|\leq\sqrt{2\epsilon}$ (this is possible by the Ryll-Nardzewski's fixed point theorem). Thus we obtain $\|a\xi_0-\xi_0a\|\leq 2\sqrt{2\epsilon}$ for $a\in(G'')_1$. □

Appendix A. Embeddability of subalgebras

Let $A\subset M$ be an inclusion of finite von Neumann algebras with a trace τ on M. Recall that we have the associated Jones projection $e_A\in B(L^2M)$, the orthogonal projection onto $L^2A=\overline{A\hat{1}}$ and the basic extension $\langle M,A\rangle$ of M:

$$\langle M,A\rangle=\mathsf{vN}\{M,e_A\}=\left\{\sum_{\text{finite}}x_ie_Ay_i:x_i,y_i\in M\right\}''$$

and the semifinite trace $\mathrm{Tr}(\sum x_ie_Ay_i)=\sum\tau(x_iy_i)$ on $\langle M,A\rangle$.

Theorem A.1. *(Popa) Let $A\subset M$ be an inclusion of separable finite von Neumann algebras, p a nonzero projection in M, $B\subset pMp$ a von Neumann subalgebra. The the followings are equivalent:*

1. *There are no sequence $(w_n)_n$ in $\mathcal{U}B$ such that $\|E_A(y^*w_nx)\|_2\to 0$ for any $x,y\in M$.*

(2) *There exists a nonzero positive element* $d \in \langle M, A\rangle$ *of finite trace such that* $0 \notin \overline{\mathrm{conv}}^w \{wdw^* : w \in \mathcal{U}B\}$
(3) *There exists a closed nonzero* B-A *submodule* H *of* pL^2M *such that* $\dim_A H_A$ *is finite.*
(4) *There exists a projection* e *in* A*, another* $0 \neq f$ *in* B *and a normal* $*$*-homomorphism* $\theta\colon fBf \to eAe$ *such that there exists a nonzero partial isometry* $v \in M$ *satisfying* $xv = v\theta(x)$ *for any* $x \in fBf$*, and* $vv^* \in (fBf)' \cap fMf$*,* $v^*v \in \theta(fBf)' \cap eMe$*.*

Proof. (1) ⇒ (2): By assumption there exits a finite set $\mathscr{F} \subset M$ and $\epsilon > 0$ such that

$$\inf_{w\in\mathcal{U}B} \sum_{x,y\in\mathscr{F}} \|E_A(y^*wx)\|_2^2 \geq \epsilon.$$

Now, put $d = \sum_{y\in\mathscr{F}} ye_Ay^* \in \langle M, A\rangle_+$. By definition $\mathrm{Tr}(d) < \infty$ and we have

$$\sum_{x\in\mathscr{F}} \langle w^*dw\hat{x}, \hat{x}\rangle = \sum_{x,y\in\mathscr{F}} \langle e_A\widehat{y^*wx}, \widehat{y^*wx}\rangle = \sum_{x,y\in\mathscr{F}} \|E_A(y^*wx)\|_2^2 \geq \epsilon$$

for any $w \in \mathcal{U}B$.

(2) ⇒ (3): Let $\mathcal{C}$ denote the closed convex hull of $\{wdw^* : w \in \mathcal{U}B\}$ in $L^2\langle M, A\rangle$. We can take the circumcenter d_0 of $\mathcal{C}$ which is not equal to zero by (2). Then d_0 is in $B' \cap p\langle M, A\rangle p$ and $\mathrm{Tr}(d_0) \leq \mathrm{Tr}(d) < \infty$. Thus we can take a nonzero spectral projection q of d_0 such that $\mathrm{Tr}(q) < \infty$. Now, $H = qL^2M$ is a B-A submodule with $\dim_A H_A = \mathrm{Tr}(q)$.

(3) ⇒ (4): Fact. When H is a B-A module with $\dim_A H_A < \infty$, there exists a nonzero projection$f$ of B, an fBf-A module $K \subset fH$ such that $K_A \hookrightarrow L^2A_A$ as a right A-module.

Thus, let V denote such an injection $K_A \to L^2A_A$. When $x \in fBf$, $VxV^* \in \mathrm{End}_A(L^2A_A) = A$. Thus $\theta(x) = VxV^*$ defines a normal $*$-homomorphism (since V is injective) θ of fBf into eAe for $e = VV^*$. Put $\xi = V^*\hat{1} \in K$. Since $V\xi = VV^*\hat{1} = \hat{e}$, $\xi \neq 0$. On the other hand, for any $x \in fBf$,

$$\begin{aligned} x\xi &= V^*VxV^*\hat{1} = V^*\theta(x)\hat{1} \\ &= V^*\hat{1}\theta(x) \quad (\theta(x) \in eAe) \\ &= \xi\theta(x). \end{aligned}$$

Now we are going to investigate

$$\xi \in K \subset fH \subset pL^2M \subset L^2M$$

as a square integrable operator affiliated with M. By above we have $xL^\xi = L_\xi\theta(x)$ for any $x \in \mathcal{U}(fBf)$. Let $v|L_\xi|$ be the polar decomposition of L_ξ. Then

$$|L_\xi|^2 = (xL_\xi)^*(xL_\xi) = (L_\xi\theta(x))^*L_\xi\theta(x) = \theta(x)^*|L_\xi|^2\theta(x)$$

for $x \in \mathcal{U}(fBf)$. Thus $|L_\xi|$ commutes with $\theta(fBf)$. In particular $v^*v = s(|L_\xi|) \in \theta(fBf)' \cap eMe$. Finally,

$$xv|L_\xi| = xL_\xi = L_\xi\theta(x) = v|L_\xi|\theta(x) = v\theta(x)|L_\xi|,$$

which implies $xvv^*v = v\theta(x)v^*v$, i.e. $xv = v\theta(x)$ for any $x \in fBf$.

(4) ⇒ (1): Take e, f, v as in (4). Let E_θ denote the conditional expectation $eMe \to \theta(fBf)$. Then $0 \neq E_\theta(v^*v) \in Z(\theta(fBf))$, $vE_\theta(v^*v)^2v^* \in (fBf)' \cap fMf$.

Let $(f_i)_{i\in I}$ be a maximal family of mutually orthogonal nonzero projections satisfying $f_0 = f$ and $f_i \precsim f$ in B. Thus, $\sum f_i$ is equal to the central support $z_B(f)$ of f in B. Put $u_0 = f$. For each i, take a partial isometry u_i satisfyikng $u_i u_i^* = f_i$ and $u_i^* u_i \le f$. Put $v_i = u_i v$. Now we have, for $w \in \mathcal{U}B$,

$$\sum_i \|E_A(v_i^* w v_0)\|_2^2 \ge \sum_i \|vv^* E_\theta(v_i^* w v_0)\|_2^2 = \cdots = \tau(E_\theta(v^*v)^3) > 0.$$

Since $\sum \|v_i^*\|_2^2 \le 1$ and $\|E_A(v_i^* w v_0)\|_2 \le \|v_i^*\|_2$, there exists a finite subset $\mathscr{F}$ of $\{v_i : i \in I\}$ containing v_0 and $\sum_{v_i \notin \mathscr{F}} \|v_i^*\|_2^2 < \tau(E_\theta(v^*v)^3)/2$. $\square$

Definition A.2. Let A and B be von Neumann subalgebras of M. B is said to embed into A inside M when the equivalent conditions of Theorem A.1 hold for B and A.

Corollary A.3. *If B does not embed into A inside M, there exists a commutative von Neumann subalgebra B_0 of B which does not embed into A inside M. Equivalently, if any commutative subalgebra of B embeds into A, B also embeds into A.*

Remark A.4. The above theorem is useful when we have τ-symmetric unital completely positive maps $\phi_i \colon M \to M$ which restrict to the identity map on A, giving $\hat{\phi}_i \in \langle M, A\rangle \cap A'$. Often one has $\hat{\phi}_i \in \mathbb{K}\langle M, A\rangle = C^*(xe_A y : x, y \in M)$.

$B \subset M$ is said to be rigid when $\phi_i \to \mathrm{Id}$ uniformly on the unit ball of B_1. Then, taking $\phi = \phi_{i_0}$ that satisfies

$$\|\phi(b) - b\|_2 < \frac{1}{3} \quad (\forall b \in B_1),$$

$d = \chi_{[\frac{1}{2},1]}(\hat{\phi})$ satisfies $\mathrm{Tr}(d) < \infty$ and

$$\left\|wdw^*\hat{1} - \hat{1}\right\| \le \frac{1}{2} + \left\|w\hat{\phi}w^*\hat{1} - \hat{1}\right\| = \frac{1}{2} + \|\phi(w^*) - w^*\|_2 \le \frac{5}{6}.$$

Hence $\overline{\mathrm{conv}}^2\{wdw^*\}$ does not contain 0 and B embeds into A inside M.

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