LECTURE ON THE FURSTENBERG BOUNDARY AND C*-SIMPLICITY

NARUTAKA OZAWA

Abstract. This is a handout for the lecture at the domestic "Annual Meeting of Operator Theory and Operator Algebras" at Toyo university, 24–26 December 2014. In this note, we first review the theory of the Furstenberg boundary for locally compact groups and prove Kalantar and Kennedy's theorem which identifies the Furstenberg boundary with the Hamana boundary. We then deal with applications of the boundary theory to the study of simplicity of C*-algebras of discrete groups and their actions.

1. The Furstenberg boundary

Let G be a locally compact group. For a compact space X, we denote by $\mathcal{M}(X)$ the space of Radon probability measures, equipped with the weak-topology. There is a natural continuous embedding of X into $\mathcal{M}(X)$ as the point masses. We often identify $\mathcal{M}(X)$ with the state space of $C(X)$ with the weak*-topology. When X is a *compact G-space* (i.e., X is a compact topological space with a distinguished continuous G-action $G \times X \ni (s,x) \mapsto sx \in X$), the space $\mathcal{M}(X)$ is also a compact G-space which contains X as a G-invariant closed subspace. Also, $C(X)$ becomes a *G-C*-algebra*. Namely, it is a C*-algebra on which G acts continuously by $*$-automorphisms: $(sf)(x) = f(s^{-1}x)$ for $s \in G$, $f \in C(X)$, and $x \in X$. A compact G-space is called a *G-boundary* in the sense of Furstenberg ([F1, F2]) if X is minimal and strongly proximal, or equivalently if X is the unique minimal G-invariant closed subspace of $\mathcal{M}(X)$. (Here and there the term "minimal" means "minimal nonempty.") Recall that a compact G-space is said to be *minimal* if there is no nontrivial G-invariant closed subset; and it is said to be *strongly proximal* if for every $\mu \in \mathcal{M}(X)$ one has $\overline{G\mu} \cap X \neq \emptyset$.

Example 1 ([F1, F2])**.** Let G be a connected simple Lie group and H be a maximal closed amenable subgroup (e.g., $G = \mathrm{SL}(n,\mathbb{R})$ and H upper triangular matrices). Then, G/H is compact and is a G-boundary. See Proposition 10.

Example 2. Let X be a compact G-space. An element s is said to be "hyperbolic" if there are points $x_s^{\pm}$ in X such that $\lim_{n\to\infty} s^n x \to x_s^+$ for all $x \in \partial G \setminus \{x_s^-\}$. When the set $\{x_s^+ : s \text{ hyperbolic}\}$ of attracting points has more than two elements (i.e., when the action is "non-elementary"), its closure X_∞ is a G-boundary. Indeed, let $\mu \in \mathcal{M}(X)$ be given. Then for any hyperbolic element t, one has $t^n\mu \to \mu(\{x_t^-\})\delta_{x_t^-} + (1-\mu(\{x_t^-\}))\delta_{x_t^+}$ by the bounded convergence theorem. Take another hyperbolic element s such that

$x_s^- \notin \{x_t^\pm\}$. Now, one has $\lim_m \lim_n s^m t^n \mu = \delta_{x_s^+} \in X$. This proves that X_∞ is strongly proximal. The proof of minimality is similar.

When G is a discrete non-elementary hyperbolic group and ∂G is its Gromov boundary, every infinite order element acts hyperbolically and the set of attracting points is dense in ∂G, and so ∂G is a G-boundary.

Lemma 3. *If $\{X_i\}_{i\in I}$ is a family of compact strongly proximal G-spaces, then $\prod_{i\in I} X_i$ with the diagonal G-action is also strongly proximal.*

Proof. By the definition of the product topology, it suffices to prove this when I is finite, which in turn reduces to prove that $X \times Y$ is strongly proximal when X and Y are so. Let $\mu \in \mathcal{M}(X \times Y)$ be given and let $Q_*\colon \mathcal{M}(X\times Y) \to \mathcal{M}(X)$ denote the pushforward map. Since $Q_*(\overline{G\mu}) = \overline{GQ_*(\mu)}$ contains δ_x for some $x \in X$, there is $\nu \in \mathcal{M}(Y)$ such that $\delta_x \otimes \nu \in \overline{G\mu}$. Then, there is a net (s_n) in G and $y \in Y$ such that $s_n\nu \to \delta_y$. By compactness, we may assume that $s_n x \to x'$ in X. It follows that $\delta_{x'} \otimes \delta_y \in \overline{G\mu}$. $\square$

A map between G-spaces is said to be *G-equivariant* or a *G-map* if it intertwines the G-actions. Unital (completely) positive maps between unital commutative C*-algebras are simply referred to as *morphisms*. There is a one-to-one correspondences between G-morphisms $\phi\colon C(X) \to C(Y)$ and continuous G-maps $\phi_*\colon Y \to \mathcal{M}(X)$, given by

$$\phi(f)(y) = \langle \phi_*(y), f\rangle \text{ for } f \in C(X) \text{ and } y \in Y.$$

The following lemma is the most fundamental observation of the boundary theory.

Lemma 4 (Furstenberg)**.** *Let X be a G-boundary and Y be a minimal compact G-space. Then, every continuous G-map from Y into $\mathcal{M}(X)$ has X as its range. Equivalently, every G-morphism from $C(X)$ into $C(Y)$ is an isometric $*$-homomorphism. Moreover there is at most one such map.*

Proof. Since X is a boundary, the G-invariant closed subset $\phi_*(Y)$ of $\mathcal{M}(X)$ contains X. Since Y is minimal, the nonempty G-invariant closed subset $\phi_*^{-1}(X)$ coincides with Y. If there are two such maps ϕ_* and ψ_*, then $(\phi_* + \psi_*)/2$ is also a continuous G-map and hence it ranges in point masses, which implies that $\phi_* = \psi_*$. $\square$

Every G-equivariant quotient of a G-boundary is again a G-boundary. The (*maximal*) *Furstenberg boundary* $\partial_{\mathrm{F}} G$ is a G-boundary which is universal in the sense that it has every G-boundary as a G-quotient ([F1, F2]). Such a maximal G-boundary exists: Take the set $\{X_i\}$ of all G-boundaries (up to G-homeomorphisms) and define $\partial_{\mathrm{F}} G$ to be a minimal G-invariant closed subset of $\prod X_i$. By Lemma 3, it is a G-boundary and by Lemma 4 such a maximal G-boundary is unique.

The following result says G-boundary is ubiquitous. A *compact convex G-space* is a compact convex subset K of a locally convex topological vector space, equipped with a continuous G-action on K by affine homeomorphisms.

Proposition 5 ([Gl, Theorem III.2.3]). *Let K be a compact convex G-space. Then, K contains a G-boundary. In fact, if K is a minimal compact convex G-space, then the closed extreme boundary $\overline{\mathrm{ex}}(K)$ is a G-boundary.*

Proof. First, we observe that $\overline{\mathrm{ex}}(K)$ is a compact G-space. Since every compact convex G-space contains a minimal compact convex G-space (which is not a minimal compact G-space), we may assume K is minimal. We recall that there is a barycenter map $\beta\colon \mathcal{M}(K) \to K$ such that $\int f\,d\mu = f(\beta(\mu))$ for every continuous affine function f on K. The map β is continuous, affine, and G-equivariant. Moreover, for any extreme point x in K, one has $\beta(\mu) = \delta_x$ if and only if $\mu = \delta_x$. (See III.2 in [Gl] for the proof of these facts.) It follows that for any $\mu \in \mathcal{M}(K)$, one has $\beta(\overline{\mathrm{conv}}(G\mu)) = \overline{\mathrm{conv}}(G\beta(\mu)) = K$ by minimality. Hence, $\mathrm{ex}(K) \subset \overline{\mathrm{conv}}(G\mu)$. This proves that $\overline{\mathrm{ex}}(K)$ is a G-boundary. □

2. The Hamana boundary

Let $C_{\mathrm{b}}^{\mathrm{lu}}(G) = \{f \in L^\infty(G) : G \ni s \mapsto sf \in L^\infty(G) \text{ is norm continuous}\}$ be the C*-algebra of bounded left uniformly continuous functions G. Here $(sf)(x) = f(s^{-1}x)$ for $s \in G$, $f \in L^\infty(G)$, and $x \in G$. Let V be a Banach G-space (i.e., a Banach space on which G acts continuously by isometries). Then there is a bijective correspondence between $v^* \in V^*$ and bounded linear G-maps $\theta_{v^*}\colon V \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$, given by

$$\theta_{v^*}(v)(x) = \langle x^{-1}v, v^*\rangle = \langle v, xv^*\rangle.$$

This implies that $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ is G-injective in the category of Banach G-spaces: for any Banach G-spaces $V \subset W$ and any bounded linear G-map $\theta\colon V \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$, there is a norm-preserving G-equivariant extension $\tilde{\theta}\colon W \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$. We will work with the category of G-operator systems: a *G-operator system* is a unital $*$-closed subspace V of a unital C*-algebra, equipped with a continuous G-action on V by unital completely isometric isomorphisms. (Actually, we only deal with G-C*-algebras.) A *G-morphism* will mean a unital completely positive G-map. Since unital linear functionals are positive if and only if contractive, $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ is also G-injective in the category of G-operator systems.

Hamana ([H1, H2]) has proved that every G-operator system has a unique minimal G-injective extension, called a *G-injective envelope*. The G-injective envelope of the trivial G-C*-algebra $\mathbb{C}$ is a commutative G-C*-algebra and we call its Gelfand spectrum $\partial_{\mathrm{H}}G$ the *Hamana boundary*.

Theorem 6 (Kalantar–Kennedy [KK]). *$\partial_{\mathrm{F}}G = \partial_{\mathrm{H}}G$.*

In particular, for every G-operator system V, there is a G-morphism from V into $C(\partial_{\mathrm{F}}G)$.

Proof. Theorem means that $C(\partial_{\mathrm{F}}G)$ is G-injective. Once this is proven, one sees that for any G-injective G-operator system V, there are G-morphisms $\phi\colon V \to C(\partial_{\mathrm{F}}G)$ and $\psi\colon C(\partial_{\mathrm{F}}G) \to V$ (that extend the trivial G-morphism $\mathbb{C}1_V \leftrightarrow \mathbb{C}1_{C(\partial_{\mathrm{F}}G)}$). By Lemma 4, they satisfy $\phi \circ \psi = \mathrm{id}_{C(\partial_{\mathrm{F}}G)}$. Now let us prove $C(\partial_{\mathrm{F}}G)$ is G-injective. Note that we will not use Hamana's theorem which assures existence of the Hamana boundary. Take $\mu \in \mathcal{M}(\partial_{\mathrm{F}} G)$ and consider the G-morphism $\theta_\mu \colon C(\partial_{\mathrm{F}} G) \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$. Let X denote the Gelfand spectrum of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$, i.e., X is the compact G-space such that $C_{\mathrm{b}}^{\mathrm{lu}}(G) = C(X)$. By Proposition 5 and the maximality of the Furstenberg boundary, there is a continuous G-map $\phi_* \colon \partial_{\mathrm{F}} G \to \mathcal{M}(X)$. Let $\phi \colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to C(\partial_{\mathrm{F}} G)$ be the corresponding G-morphism. Then, one has $\phi \circ \theta_\mu = \mathrm{id}_{C(\partial_{\mathrm{F}} G)}$ by Lemma 4. Thus G-injectivity of $C(\partial_{\mathrm{F}} G)$ follows from that of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$. The second statement is a consequence of G-injectivity applied to the trivial G-morphism $\mathbb{C}1_V \to \mathbb{C}1_{C(\partial_{\mathrm{F}} G)}$. □

This theorem will be used in combination with the following fact about the *multiplicative domain*. See around Definition 1.5.8 in [BO] for a proof and more information.

Lemma 7. *For a morphism $\phi \colon A \to B$ between* C*-*algebras, one has*

$$\begin{aligned}\mathrm{mult}(\phi) &:= \{a \in A : \phi(ax) = \phi(a)\phi(x) \text{ and } \phi(xa) = \phi(x)\phi(a) \text{ for all } x \in A\}\\ &= \{a \in A : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}\\ &= \overline{\mathrm{span}}\{u \in A : \|u\| = 1 \text{ and } \phi(u) \text{ is unitary in } B\}.\end{aligned}$$

In particular, the multiplicative domain $\mathrm{mult}(\phi)$ *of ϕ is the largest* C*-*subalgebra of A to which the restriction of ϕ is multiplicative.*

Proposition 8. *If G is a discrete group, then $C(\partial_{\mathrm{F}} G)$ is an injective* C*-*algebra, or equivalently $\partial_{\mathrm{F}} G$ is a Stonean space. In particular, $\partial_{\mathrm{F}} G$ is either a one-point space or a non-second countable space.*

Proof. This is because $C_{\mathrm{b}}^{\mathrm{lu}}(G) = \ell_\infty(G)$ is an injective C*-algebra. Since $\partial_{\mathrm{F}} G$ is a G-boundary, it does not admit a G-invariant probability measure, unless it consists of a point. Every non-finite Stonean space is non-second countable. □

It will be shown (Corollary 12) that $\partial_{\mathrm{F}} G$ consists of a point if and only if G is amenable. When G is not discrete, $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ need not be an injective C*-algebra (although it is G-injective). In particular, $\partial_{\mathrm{F}} G$ can be "small," e.g., for a connected group (which has a cocompact closed amenable subgroup). See Proposition 10.

3. Kernel of the boundary action

For every G-space X and $x \in X$, we denote by $G_x = \{g \in G : gx = x\}$ the stabilizer subgroup of x. Recall that a subgroup $H \leq G$ is said to be *relatively amenable* in G if there is an H-invariant state on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$. Since $C(\partial_{\mathrm{F}} G)$ is G-injective, this is equivalent to the existence of an H-invariant probability measure on $\partial_{\mathrm{F}} G$. When G is a discrete group, there is an H-morphism from $C_{\mathrm{b}}^{\mathrm{lu}}(H) = \ell_\infty(H)$ into $C_{\mathrm{b}}^{\mathrm{lu}}(G) = \ell_\infty(G)$ and so the notions of relatively amenability and amenability coincide, but it is not known (!) whether they coincide in general. See [CM] for more information. In particular, if G has a cocompact amenable subgroup, then G is amenable at infinity and so amenability and relative amenability coincide for subgroups of G ([CM]).

Lemma 9. *For every $x \in \partial_{\mathrm{F}} G$, the subgroup G_x is relatively amenable in G. In particular, G_x is amenable when G is a discrete group.*

Proposition 10. *Suppose that G has a cocompact closed (relatively) amenable subgroup. Then, a maximal closed (relatively) amenable subgroup H is unique up to conjugacy in G and $\partial_{\mathrm{F}} G \cong G/H$ as a compact G-space.*

Proof. Let P be a cocompact closed relatively amenable subgroup, and take a compact subset K of G such that $G = KP$ and a P-invariant probability measure μ on $\partial_{\mathrm{F}} G$. Then, $G\mu = K\mu$ is a G-invariant compact subset of $\mathcal{M}(\partial_{\mathrm{F}} G)$ and hence it contains $\partial_{\mathrm{F}} G$. But this implies that μ is a point mass and for the stabilizer subgroup $H = G_\mu$ one has $\partial_{\mathrm{F}} G = G\mu \cong G/H$. It follows that H is relatively amenable and contains P. In particular, $P = H$ when P was a maximal relatively amenable subgroup. Since G acts transitively on $\partial_{\mathrm{F}} G$, all stabilizer subgroups are conjugate to each other. $\square$

The *amenable radical* $R(G)$ of G is the largest closed amenable normal subgroup of G that contains all amenable normal subgroups of G. Existence of $R(G)$ follows from the fact that the class of amenable groups is closed under directed unions and extensions.

Theorem 11 ([Fu, Proposition 7])**.** $\ker(G \curvearrowright \partial_{\mathrm{F}} G) = R(G)$.
Moreover, $\partial_{\mathrm{F}} G \cong \partial_{\mathrm{F}}(G/R(G))$ as a compact G-space.

Proof. We first observe that $N := \ker(G \curvearrowright \partial_{\mathrm{F}} G) = \bigcap_x G_x$ is a closed relatively amenable normal subgroup of G. We claim that N is amenable (Proposition 3 in [CM]). Let μ be an N-invariant state of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ and consider the G-morphism $\theta'_\mu \colon L^\infty(G) \to L^\infty(G) = L^1(G)^*$ defined by $\langle \theta'_\mu(f), \xi \rangle = \langle \check{\xi} * f, \mu \rangle$ for $f \in L^\infty(G)$ and $\xi \in L^1(G)$. Here $(\check{\xi} * f)(x) = \int_G \xi(t) f(tx)\,dt$ for the left Haar measure dt on G and note that it belongs to $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ and that θ'_μ is indeed a G-map because $(\check{s\xi}) * (sf) = \check{\xi} * f$ for every $s \in G$. Moreover, θ'_μ maps $L^\infty(G)$ into the subspace $L^\infty(G/N)$ of right N-invariant functions. Indeed, for every $f \in L^\infty(G)$ and every $a \in N$, denoting by $(\xi a)(x) = \Delta_G(a)\xi(xa)$, one has

$$((\check{\xi a}) * f)(x) = \int_G \Delta_G(a)\xi(ta) f(tx)\,dt = \int_G \xi(t) f(ta^{-1}x)\,dt = (a(\check{\xi} * f))(x)$$

and so

$$\langle \theta'_\mu(f), \xi a \rangle = \langle (\check{\xi a}) * f, \mu \rangle = \langle a(\check{\xi} * f), \mu \rangle = \langle \check{\xi} * f, \mu \rangle = \langle \theta'_\mu(f), \xi \rangle$$

for all $\xi \in L^1(G)$, which implies that $\theta'_\mu(f)$ is right N-invariant. But since N is normal, the left N-action on $L^\infty(G/N)$ is trivial. Thus composing θ'_μ with any state on $L^\infty(G/N)$, one obtains an N-invariant state on $L^\infty(G)$. Since there is an N-morphism from $L^\infty(N)$ into $L^\infty(G)$ by Kehlet's cross section theorem, this implies that N is amenable. We have shown that $N \subset R(G)$.

Since $\partial_{\mathrm{F}}(G/R(G))$ is a G-boundary, there is a (unique) G-quotient map Q from $\partial_{\mathrm{F}} G$ onto $\partial_{\mathrm{F}}(G/R(G))$. We will show Q is a homeomorphism. For this, it suffices to show there is a continuous G-map from $\partial_{\mathrm{F}}(G/R(G))$ into $\partial_{\mathrm{F}} G$. Take an $R(G)$-invariant state

μ on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ and consider the G-morphism $\theta_\mu\colon C_{\mathrm{b}}^{\mathrm{lu}}(G)\to C_{\mathrm{b}}^{\mathrm{lu}}(G)$ (recall that it is defined by $\theta_\mu(f)(x)=\langle f,x\mu\rangle$). Since μ is N-invariant, the map θ_μ ranges in $C_{\mathrm{b}}^{\mathrm{lu}}(G/R(G))$. Thus there is a G-morphism $\psi\colon C(\partial_{\mathrm{F}}G)\to C(\partial_{\mathrm{F}}(G/R(G)))$, which, in view of Lemma 4, gives rise to a continuous G-map from $\partial_{\mathrm{F}}(G/R(G))$ into $\partial_{\mathrm{F}}G$. □

Corollary 12. *G is amenable if and only if $\partial_{\mathrm{F}}G$ is a one-point space.*

4. Tracial States

Let G be a discrete group. For a subgroup H, we denote by E_H the canonical conditional expectation from the reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ onto $\mathrm{C}^*_{\mathrm{r}}(H)\subset\mathrm{C}^*_{\mathrm{r}}(G)$, defined by $E_H(\lambda_s)=\lambda_s$ if $s\in H$ and $E_H(\lambda_s)=0$ otherwise. When $H=\mathbf{1}$, it coincides with the canonical tracial state τ_λ on $\mathrm{C}^*_{\mathrm{r}}(G)$, given by $\tau_\lambda(\lambda_s)=1$ if $s=1$ and else 0.

Theorem 13 ([B+]). *Let G be a discrete group and τ be a tracial state τ on $\mathrm{C}^*_{\mathrm{r}}(G)$. Then, $\tau=\tau\circ E_{R(G)}$. In particular, if $R(G)=\mathbf{1}$, then $\tau=\tau_\lambda$.*

Proof. [1] We view τ as a G-morphism from $\mathrm{C}^*_{\mathrm{r}}(G)$ to $\mathbb{C}1\subset C(\partial_{\mathrm{F}}G)$ and extend it to a G-morphism ϕ from the reduced crossed product $C(\partial_{\mathrm{F}}G)\rtimes_{\mathrm{r}}G$ into $C(\partial_{\mathrm{F}}G)$. Since $\phi|_{C(\partial_{\mathrm{F}}G)}=\mathrm{id}_{C(\partial_{\mathrm{F}}G)}$ by Lemma 4, the map ϕ is a conditional expectation. For every $s\in G\setminus R(G)$, there is nonzero $f\in C(X)$ such that $\operatorname{supp}f\cap s\operatorname{supp}f=\emptyset$ by Theorem 11. It follows that $f\lambda_s f=f(sf)\lambda_s=0$ in $C(\partial_{\mathrm{F}}G)\rtimes_{\mathrm{r}}G$ and so $\tau(\lambda_s)f^2=\phi(f\lambda_s f)=0$, which implies $\tau=\tau\circ E_{R(G)}$. □

5. Simplicity of reduced crossed products

Let G be a discrete group. A *G-C*-algebra* is a C*-algebra A equipped with a G-action on it. The canonical tracial state τ_λ on $\mathrm{C}^*_{\mathrm{r}}(G)$ extends to the canonical conditional expectation E from the reduced crossed product $A\rtimes_{\mathrm{r}}G$ onto A, which is given by $E(a\lambda_s)=a$ if $s=1$ and $E(a\lambda_s)=0$ otherwise. We note that E is G-equivariant and faithful. On the other hand, if ϕ is a G-invariant state on A, then it extends to a canonical conditional expectation E_ϕ from $A\rtimes_{\mathrm{r}}G$ onto $\mathrm{C}^*_{\mathrm{r}}(G)$, which is given by $E_\phi(a\lambda_s)=\phi(a)\lambda_s$.

A G-C*-algebra is called *G-simple* if there is no nontrivial G-invariant closed ideal. When X is a compact G-space, the G-C*-algebra $C(X)$ is G-simple if and only if X is minimal. For a compact G-space X and $x\in X$, let G_x° denote the subgroup consisting of elements of G that act as identity on some neighborhood of x. It is a normal subgroup of the stabilizer subgroup $G_x=\{g\in G: gx=x\}$. The compact G-space X is said to be *free* (resp. *topologically free*) if $G_x=\mathbf{1}$ (resp. $G_x^\circ=\mathbf{1}$) for all $x\in X$.

Let X be a compact G-space and $C(X)\rtimes_{\mathrm{r}}G$ be the reduced crossed product. Then for every $x\in X$ the conditional expectation E_{G_x} from $\mathrm{C}^*_{\mathrm{r}}(G)$ onto $\mathrm{C}^*_{\mathrm{r}}(G_x)$ extends to a canonical conditional expectation E_x from $C(X)\rtimes_{\mathrm{r}}G$ onto $\mathrm{C}^*_{\mathrm{r}}(G_x)$, which is given by $E_x(f\lambda_s)=f(x)E_{G_x}(\lambda_s)$.

[1]Perhaps, it is surprising that the proof is only 5-line modulo Hamana's theorem [H1] in 1985.

A discrete group G is said to be C*-*simple* if the reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ is simple. We note that if G is C*-simple, then the amenable radical $R(G)$ of G is trivial, because for any amenable normal subgroup N the quotient map from G onto G/N extends to a $*$-homomorphism Q from $\mathrm{C}^*_{\mathrm{r}}(G)$ onto $\mathrm{C}^*_{\mathrm{r}}(G/N)$. (For this, observe that the state $\tau_\lambda \circ Q$ is continuous on $\mathrm{C}^*_{\mathrm{r}}(G)$ if and only if the unit character τ_0 is continuous on $\mathrm{C}^*_{\mathrm{r}}(N)$ if and only if N is amenable.) In particular, if G is an amenable C*-simple group, then $G = \mathbf{1}$. Whether $R(G) = \mathbf{1}$ implies C*-simplicity or not is a major open problem. While it is likely that the answer will be negative, we can prove a weaker assertion in Corollary 18.[2] See [dlH] for a recent survey on this topic.

It would be interesting to find a characterization, in terms of stabilizer subgroups, of a minimal compact G-space X for which $C(X) \rtimes_{\mathrm{r}} G$ is simple.[3] The following result is inspired from Kawamura–Tomiyama [KT] and Archbold–Spielberg [AS].

Theorem 14. *Let G be a discrete group and X be a minimal compact G-space.*

(1) If G_x is C-simple for some $x \in X$, then $C(X) \rtimes_{\mathrm{r}} G$ is simple. In particular, if X is topologically free, then $C(X) \rtimes_{\mathrm{r}} G$ is simple.*

(2) If $C(X) \rtimes_{\mathrm{r}} G$ is simple and G_x° is amenable for some $x \in X$, then $G_x^\circ = \mathbf{1}$, i.e., X is topologically free.

Proof. Suppose there is a nontrivial closed ideal I in $C(X) \rtimes_{\mathrm{r}} G$ and let $x \in X$ be given. We will prove that G_x is not C*-simple. We observe that $E(I)$ is a nonzero (possibly non-closed) G-invariant ideal of $C(X)$ and hence it is dense in $C(X)$ because of the minimality assumption. It follows that $E_x(I)$ is also a nonzero (possibly non-closed) ideal of $\mathrm{C}^*_{\mathrm{r}}(G_x)$, since $\tau_\lambda \circ E_x = \delta_x \circ E$ is nonzero on I. Thus it remains to prove that $E_x(I)$ is not dense in $\mathrm{C}^*_{\mathrm{r}}(G_x)$. Since X is minimal, $C(X) \cap I = \mathbf{0}$. Hence, the state

$$C(X) + I \to (C(X) + I)/I \cong C(X)/(C(X) \cap I) = C(X) \overset{\delta_x}{\to} \mathbb{C}$$

is well-defined and extends to a state ϕ_x on $C(X) \rtimes_{\mathrm{r}} G$ such that $\phi_x(I) = \mathbf{0}$. We claim that $\phi_x = \phi_x \circ E_x$. Indeed, ϕ_x is multiplicative on $C(X)$ by Lemma 7, and for every $s \in G \setminus G_x$ one has $\phi_x(\lambda_s) = 0$, because there is $h \in C(X)$ such that $h(x) = 1$ and $\operatorname{supp} h \cap s \operatorname{supp} h = \emptyset$ and hence $\phi_x(\lambda_s) = \phi_x(h\lambda_s h) = 0$. Since $\phi_x(E_x(I)) = \phi_x(I) = \mathbf{0}$, the ideal $E_x(I)$ is not dense in $\mathrm{C}^*_{\mathrm{r}}(G_x)$.

Now, let us stick to the notation of the previous paragraph and assume that X is topologically free. We will prove that for every $c \in I$, there is $x \in X$ such that $\delta_x(E(c)) = 0$. This would contradict the fact that $E(I)$ is dense in $C(X)$. Given $c \in I$, take a countable subgroup H of G such that $c \in C(X) \rtimes_{\mathrm{r}} H$. Since X is topologically free, by Baire's category theorem one can find $x \in X$ such that $G_x \cap H = \mathbf{1}$. It follows that

[2] Exercise: Recall that G is said to be ICC if the conjugacy class of any non-neutral element is infinite; and G is ICC if and only if $\mathbb{C}[G]$ has a trivial center. Prove that if $R(G) = \mathbf{1}$, then G is ICC.

[3] I think it should have something to do with the C*-simplicity of G_x and/or G_x°. For example, does simplicity of $C(X) \rtimes_{\mathrm{r}} G$ imply C*-simplicity of G_x° ?

$E_x(c) = \delta_x(E(c))1$. Since the above state ϕ_x satisfies $\phi_x(E_x(c)) = \phi_x(c) = 0$, one has $\delta_x(E(c)) = 0$.

Next, we assume that $C(X) \rtimes_{\mathrm{r}} G$ is simple and $x \in X$ is such that G_x° is amenable. Since X is minimal, topological freeness is equivalent to that $G_x^\circ = \mathbf{1}$. Let $s \in G_x^\circ$ be arbitrary and take a nonempty open subset U of X on which s acts as identity. We consider the representation π of $C(X) \rtimes_{\mathrm{r}} G$ on $\ell_2(G/G_x^\circ)$ given by $\pi(f\lambda_s)\delta_p = f(spx)\delta_{sp}$ for $f \in C(X)$, $s \in G$, and $p \in G/G_x^\circ$. That π is continuous follows from the fact that the state $\langle \pi(\,\cdot\,)\delta_1, \delta_1\rangle = \tau_0 \circ E_{G_x^\circ} \circ E_x$ is continuous, where τ_0 is the unit character on $\mathrm{C}^*_{\mathrm{r}}(G_x^\circ)$. Take $f \in C(X) \setminus \{0\}$ such that $\operatorname{supp} f \subset U$. Then, $f(px) \neq 0$ implies $px \in U$ and hence $sp = p$ in G/G_x°. Thus $\pi((1-\lambda_s)f) = 0$. Since π is injective, one has $s = 1$. This implies $G_x^\circ = \mathbf{1}$. □

Theorem 15 ([B+, KK])**.** *For a discrete group G, the following are equivalent.*

1. *G is C^*-simple.*
2. *$C(\partial_{\mathrm{F}} G) \rtimes_{\mathrm{r}} G$ is simple or equivalently $\partial_{\mathrm{F}} G$ is (topologically) free.*
3. *There is a topologically free G-boundary.*
4. *Every minimal compact G-space X for which G_x is amenable for some $x \in X$ is topologically free.*
5. *$A \rtimes_{\mathrm{r}} G$ is simple for every unital G-simple G-C^*-algebra A. In particular, $C(X) \rtimes_{\mathrm{r}} G$ is simple for every minimal compact G-space X.*

For the proof, we need a few lemmas.

Lemma 16 (cf. Theorem 6.2 in [KK])**.** *Let A be a unital G-C^*-algebra and X be a G-boundary. Then for any nontrivial closed ideal I of $A \rtimes_{\mathrm{r}} G$, the ideal J of $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$ generated by I is nontrivial.*

Proof. Let $\pi\colon A \rtimes_{\mathrm{r}} G \to \mathbb{B}(\mathcal{H})$ be a $*$-representation such that $\ker \pi = I$. We extend it to a morphism $\bar\pi$ from $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$ into $\mathbb{B}(\mathcal{H})$. We note that $A \rtimes_{\mathrm{r}} G \subset \operatorname{mult}(\bar\pi)$ by Lemma 7. In particular, $\pi(A)$ and $\bar\pi(C(X))$ commute. Take a G-morphism $\psi\colon \mathrm{C}^*(\bar\pi(C(X))) \to C(\partial_{\mathrm{F}} G)$ (Theorem 6). Then, $\psi \circ \bar\pi$ is the inclusion of $C(X)$ into $C(\partial_{\mathrm{F}} G)$ by Lemma 4. It follows that $\bar\pi(C(X)) \subset \operatorname{mult}(\psi)$ by Lemma 7 and ψ is a $*$-homomorphism from $\mathrm{C}^*(\bar\pi(C(X)))$ onto $C(X)$. The C^*-algebra $\mathrm{C}^*(\bar\pi(C(X)))$ is a G-C^*-algebra with the conjugation G-action through π and the ideal $K = \ker\psi$ is G-invariant. We consider

$$D = \mathrm{C}^*\Big(\bar\pi\big((A \otimes C(X)) \rtimes_{\mathrm{r}} G\big)\Big) = \operatorname{closure}\Big(\mathrm{C}^*(\bar\pi(C(X))) \cdot \pi(A \rtimes_{\mathrm{r}} G)\Big)$$

and its ideal

$$L = \operatorname{closure}\big(K \cdot \pi(A \rtimes_{\mathrm{r}} G)\big).$$

An element $d \in D$ belongs to L if and only if $e_i d \to d$ for an approximate unit (e_i) of K. This implies that $L \cap \mathrm{C}^*(\bar\pi(C(X))) = K$. Let ψ still denote the quotient map from D onto D/L. Then, $\psi \circ \bar\pi$ is a $*$-homomorphism, since it a morphism which is multiplicative and covariant on $A \otimes C(X)$ and G. The ideal $\ker(\psi \circ \bar\pi)$ is proper and contains I. □

Let B be a G-C*-algebra and K be a G-invariant closed ideal of B. Then,

$$K \mathbin{\bar{\rtimes}_{\mathrm{r}}} G := \ker\big(B \rtimes_{\mathrm{r}} G \to (B/K) \rtimes_{\mathrm{r}} G\big) = \{b \in B \rtimes_{\mathrm{r}} G : E(b\lambda_s^*) \in K \,\forall s \in G\}$$

is a closed ideal in $B \rtimes_{\mathrm{r}} G$ which contains $K \rtimes_{\mathrm{r}} G$ (these two ideals coincide whenever G is exact). The following is inspired from [AS].

Lemma 17. *Let A be a unital G-C*-algebra, X be a free compact G-space, and J be a closed ideal in $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$. Then, for $J_A = J \cap (A \otimes C(X))$, one has $J_A \rtimes_{\mathrm{r}} G \subset J \subset J_A \mathbin{\bar{\rtimes}_{\mathrm{r}}} G$.*

Proof. For $x \in X$, let $J_A^x = (\mathrm{id}_A \otimes \delta_x)(J_A)$ be the ideal of A (which may not be proper). Here $\mathrm{id}_A \otimes \delta_x$ is the homomorphism from $A \otimes C(X)$ onto A given by evaluation at $x \in X$. Let π_x denote the induced homomorphism from $(A \otimes C(X))/J_A$ onto A/J_A^x. We note that any irreducible representation of $(A \otimes C(X))/J_A$ factors through some π_x and hence $\{\pi_x\}$ is a faithful family.

Let $x \in X$ be such that $J_A^x \neq A$. Fix a faithful representation $A/J_A^x \subset \mathbb{B}(\mathcal{H})$ and consider the representation

$$J + A \otimes C(X) \xrightarrow{Q} (J + A \otimes C(X))/J \cong (A \otimes C(X))/J_A \xrightarrow{\pi_x} A/J_A^x \subset \mathbb{B}(\mathcal{H}).$$

By Arveson's extension theorem, it extends to a morphism Φ_x from $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$ into $\mathbb{B}(\mathcal{H})$. We claim that $\Phi_x = \Phi_x \circ E$, where E is the canonical conditional expectation onto $A \otimes C(X)$. Indeed, $A \otimes C(X) \subset \mathrm{mult}(\Phi_x)$ by Lemma 7, and for every $s \in G \setminus \{1\}$ one has $\Phi_x(\lambda_s) = 0$, because there is $h \in C(X)$ such that $h(x) = 1$ and $\mathrm{supp}(h) \cap s\,\mathrm{supp}(h) = \emptyset$ and hence $\Phi_x(\lambda_s) = \Phi_x(h\lambda_s h) = 0$. This proves the claim. Thus, we see that $\pi_x(Q(E(J))) = \Phi_x(E(J)) = \Phi_x(J) = 0$ for all $x \in X$. This implies $E(J) \subset J_A$, or equivalently $J \subset J_A \mathbin{\bar{\rtimes}_{\mathrm{r}}} G$. The other inclusion $J_A \rtimes_{\mathrm{r}} G \subset J$ is obvious. □

Proof of Theorem 15. **Ad** (1) ⇒ (2): Let G be a C*-simple group. We first prove that $C(X) \rtimes_{\mathrm{r}} G$ is simple for every G-boundary X. It suffices to show every quotient map $\pi\colon C(X) \rtimes_{\mathrm{r}} G \to B$ is injective. Since $\mathrm{C}^*_{\mathrm{r}}(G)$ is simple, the canonical trace τ_λ is continuous on $\pi(\mathrm{C}^*_{\mathrm{r}}(G))$. We view it as a G-morphism from $\pi(\mathrm{C}^*_{\mathrm{r}}(G))$ into $C(\partial_{\mathrm{F}} G)$ and extend it to a G-morphism ϕ on B. By Lemma 4, $\phi \circ \pi|_{C(X)}$ is the identity inclusion of $C(X)$ into $C(\partial_{\mathrm{F}} G)$. It follows that $C(X) \subset \mathrm{mult}(\phi \circ \pi)$ by Lemma 7, and so $\phi \circ \pi = E$, the canonical conditional expectation from $C(X) \rtimes_{\mathrm{r}} G$ onto $C(X)$. Since E is faithful, so is π. This proves simplicity of $C(X) \rtimes_{\mathrm{r}} G$. By Lemma 9 and Theorem 14, the maximal Furstenberg boundary $\partial_{\mathrm{F}} G$ is topologically free. Since $\partial_{\mathrm{F}} G$ is a Stonean space, the fixed point set of any homeomorphism on it is clopen by Frolík's theorem. Hence $\partial_{\mathrm{F}} G$ is free.

Ad (2) ⇒ (5): Let A be a unital G-C*-algebra and I be a closed proper ideal in $A \rtimes_{\mathrm{r}} G$. By Lemma 16, the ideal J of $(A \otimes C(\partial_{\mathrm{F}} G)) \rtimes_{\mathrm{r}} G$ generated by I is proper. By Lemma 17 for $J_A = J \cap (A \otimes C(\partial_{\mathrm{F}} G))$ one has $J \subset J_A \mathbin{\bar{\rtimes}_{\mathrm{r}}} G$. It follows that $I_A = J \cap A$ is a proper ideal such that $I \subset I_A \mathbin{\bar{\rtimes}_{\mathrm{r}}} G$. By assumption that A has no nontrivial G-invariant closed ideal, $I_A = \mathbf{0}$ and so $I = \mathbf{0}$.

Ad (5) $\Rightarrow$ (1): Take $A = \mathbb{C}1$.

Ad (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2): This follows from Theorem 14. Note that if there is a topologically free G-boundary, then $\partial_{\mathrm{F}}G$ is topologically free. $\square$

6. C*-simple groups

Theorem 18 ([B+]). *If G is not* C*-*simple, then G has an amenable subgroup H such that $\bigcap_{t\in F} tHt^{-1} \neq \mathbf{1}$ for every finite subset $F \subset G$.*

Proof. By Theorem 15, if G is not C*-simple, then $\partial_{\mathrm{F}}G$ is not topologically free. Thus $G_x \neq \mathbf{1}$ for every $x \in \partial_{\mathrm{F}}G$. Moreover, G_x is amenable by Lemma 9. Let $x \in \partial_{\mathrm{F}}G$ be arbitrary and we claim that $H = G_x$ satisfies the above property. Take $s \in G \setminus \{1\}$ which acts as identity on a non-empty open subset U. Then by strong proximality, for every finite subset $F \subset G$, one can find $r \in G$ such that $rFx \subset U$. It follows that $srtx = rtx$ for every $t \in F$. This means that $r^{-1}sr \in \bigcap_{t\in F} tG_xt^{-1}$. $\square$

This criterion applies to many groups, e.g., linear groups with trivial amenable radicals, acylindrically hyperbolic groups with no nontrivial finite normal subgroups, groups with nonzero ℓ_2-Betti numbers and no nontrivial finite normal subgroups, etc. However, there are C*-simple groups that do not satisfy the above criterion (e.g. the non-solvable Baumslag–Solitar groups). See [B+] for more information.

Theorem 19 ([B+]). *Let N be a normal subgroup of G. Then, G is* C*-*simple if and only if both N and $C_G(N)$ are* C*-*simple. In particular,* C*-*simplicity is preserved under extensions.*

Recall that $C_G(N) = \{s \in G : st = ts \text{ for all } t \in N\}$ is the centralizer of N in G. If N is normal in G, then so is $C_G(N)$. It is rather easy to show that if $\mathrm{C}^*_{\mathrm{r}}(G)$ is simple, then $\mathrm{C}^*_{\mathrm{r}}(N)$ has no nontrivial G-invariant closed ideal, but that $\mathrm{C}^*_{\mathrm{r}}(N)$ has no nontrivial closed ideal at all ultimately come from the following fact.

Lemma 20 ([Gl, Proposition II.4.3]). *Let N be a normal subgroup of G. Then, the N-action on the Furstenberg boundary $\partial_{\mathrm{F}}N$ uniquely extends to a G-action on $\partial_{\mathrm{F}}N$. In particular, $\partial_{\mathrm{F}}N$ is a G-boundary.*

Proof. Let σ be an automorphism of a group N. Then by universality of the Furstenberg boundary, it "extends" to a homeomorphism, still denoted by σ, on $\partial_{\mathrm{F}}N$ such that $\sigma(sx) = \sigma(s)\sigma(x)$ for $s \in N$ and $x \in \partial_{\mathrm{F}}N$. Now, let σ be the conjugation action of G on N, given by $\sigma_s(a) = sas^{-1}$ for $s \in G$ and $a \in N$. This extends to a G-action σ on $\partial_{\mathrm{F}}N$ such that $\sigma_s(ax) = \sigma_s(a)\sigma_s(x)$ for every $s \in G$, $a \in N$, and $x \in \partial_{\mathrm{F}}N$. Let $s \in N$. Then, $x \mapsto s^{-1}\sigma_s(x)$ is a continuous N-map on $\partial_{\mathrm{F}}N$ and hence it has to be the identity map by Lemma 4. Thus σ is the extension of the original N-action to G. Similarly, for any another extension σ', the map $x \mapsto \sigma_s^{-1}(\sigma'_s(x))$ is the identity map for every $s \in G$, i.e., $\sigma' = \sigma$. $\square$

Lemma 21. *Let X be an N-boundary and $U \subset X$ be a nonempty open subset. Then, $\{t \in N : tU \cap U \neq \emptyset\}$ generates N as a subgroup.*

Proof. Let H denote the subgroup generated by $\{t \in N : tU \cap U \neq \emptyset\}$. Then, HU is a nonempty open subset of X such that $tHU \cap HU = \emptyset$ for all $t \in N \setminus H$. Since X is a minimal compact N-space, $\{tHU : t \in N/H\}$ is a finite clopen partition of X. This gives rise to a continuous N-map from X onto N/H. Since X is strongly proximal, N/H is a one-point space, i.e., $H = N$. □

Lemma 22. *Let N be a normal subgroup of G. Assume that N is* C*-*simple. Then, $s \in G$ belongs to $C_G(N)$ if and only if its action on $\partial_{\mathrm{F}} N$ is not topologically free.*

Proof. The 'only if' direction is trivial. To prove the converse, let $s \in G$ be an element which acts as identity on a nonempty open subset $U \subset \partial_{\mathrm{F}} N$. Then, for every $t \in H$ such that $tU \cap U \neq \emptyset$, one has $sts^{-1} = t$ on $U \cap t^{-1}U$. By C*-simplicity of N and Theorem 14, this implies that $sts^{-1} = t$. Since such t's generate N by Lemma 21, we conclude $s \in C_G(N)$. □

Proof of Theorem 19. For brevity, let $K = C_G(N)$ and $L = NK$. Consider the diagonal G-action on $X := \partial_{\mathrm{F}} N \times \partial_{\mathrm{F}} K \times \partial_{\mathrm{F}}(G/L)$. Here G acts on $\partial_{\mathrm{F}} N$ and $\partial_{\mathrm{F}} K$ by Lemma 20 and on $\partial_{\mathrm{F}}(G/L)$ through G/L. We note that N (resp. K) acts non-trivially only on the first (resp. second) coordinate. It is not hard to see X is a G-boundary. We claim that $G_{(x,y,z)}$ is amenable for every $(x, y, z) \in X$. Indeed, both $G_{(x,y,z)} \cap L = N_x K_y$ and $G_{(x,y,z)}/(G_{(x,y,z)} \cap L) \subset (G/L)_z$ are amenable by Lemma 9. First, assume that both N and K are C*-simple. We claim that X is topologically free and hence G is C*-simple by Theorem 14. Let $s \in G$ be an element whose action on X is not topologically free. Then s belongs to K by Lemma 22 and so $s = 1$ by C*-simplicity of K. This proves the claim. Next, assume that G is C*-simple. Then, by Theorem 14 the G-action on X is topologically free. It follows that the N-action on $\partial_{\mathrm{F}} N$ is topologically free. By Theorem 14 again, N is C*-simple, and the same for $C_G(N)$. □

Example 23. Thompson's group T is the group of all piecewise-linear homeomorphisms of $S^1 = \mathbb{R}/\mathbb{Z}$ such that (1) they have finitely many breakpoints, (2) all breakpoints have dyadic rational coordinates, and (3) all slopes are integral powers of 2. The group T is non-amenable (it contains free groups) and simple (in particular $R(G) = \mathbf{1}$). It is not difficult to see that S^1 is a T-boundary which is not topologically free. (Observe that there is a sequence g_n in T such that $g_n x \to 0$ for every $x \in S^1$.) The stabilizer subgroup at 0 is Thompson's group F. Hence the T-space T/F is identified with the T-orbit of 0, which is the set of diadic rational numbers $\mathbb{Z}[\frac{1}{2}] \cap [0, 1)$.

It is a big open problem whether F is amenable or not. Haagerup–Olesen ([HO], see also [BJ]) relates this problem to C*-simplicity of T as follows. Suppose F is amenable. The action $T \curvearrowright T/F$ induces the unitary representation $\pi \colon T \curvearrowright \ell^2(T/F)$. This representation extends to a continuous representation of $C^*_{\mathrm{r}}(T)$, because we have assumed F is amenable. It is easy to find nontrivial elements $a, b \in T$ such that $\mathrm{supp}_{T/F}(a) \cap \mathrm{supp}_{T/F}(b) = \emptyset$, where $\mathrm{supp}_{T/F}(a) = \{x \in T/F : ax \neq x\}$. The operators $\pi(a)$ and $\pi(b)$ commute and $\pi((1-a)(1-b)) = 0$. Hence $(1-\lambda_a)(1-\lambda_b)$ generates a closed proper ideal of $C^*_{\mathrm{r}}(T)$. In conclusion, we have seen that if F is amenable, then T is not C*-simple. This conclusion also follows from Theorem 14. The Haagerup–Olesen scheme says if G is a group which has an amenable subgroup H and nontrivial elements $a, b \in G$ such that $\mathrm{supp}_{G/H}(a) \cap \mathrm{supp}_{G/H}(b) = \emptyset$, then G is not C*-simple. No matter whether T is C*-simple or not, it seems reasonable to believe that there is such a group G whose amenable radical is trivial (and so $R(G) = \mathbf{1}$ will not imply C*-simplicity).

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RIMS, Kyoto University, 606-8502 Japan

Email address: narutaka@kurims.kyoto-u.ac.jp