II_1 factors with at most one Cartan subalgebra

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Introduction

geared for rigidity phenomena

Travel supported by JSPS

What do we classify?

Γ	countable discrete group
(X, μ)	standard probability measure space
$\Gamma \curvearrowright (X, \mu)$	(ergodic) measure preserving action

$\Gamma \curvearrowright X$ is said to be *ergodic* if

$$A \subset X \text{ and } \Gamma A = A \Rightarrow \mu(A) = 0, 1.$$

We only consider either

- $(X, \mu) \cong ([0, 1], \text{Lebesgue})$ and $\Gamma \curvearrowright X$ is *essentially-free* i.e. $\mu(\{x : gx = x\}) = 0 \ \forall g \in \Gamma \setminus \{1\}$;

or

- $X = \{\text{pt}\}$.

How do we classify?

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

GP		vN
Γ	$\longrightarrow$	$\mathrm{vN}(\Gamma)$

To what extent do vN/OE remember OE/GA/GP?

Group measure space constructions

$$\Gamma \curvearrowright (X,\mu) \quad \text{p.m.p.} \qquad \longleftrightarrow \qquad \begin{array}{l} \sigma\colon \Gamma \curvearrowright L^\infty(X,\mu) \\ \sigma_g(f)(x) = f(g^{-1}x) \\ \int \sigma_g(f)\,d\mu = \int f\,d\mu \end{array}$$

The unitary element $u_g = \sigma_g \otimes \lambda_g \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$ satisfies

$$u_g\, f\, u_g^* = \sigma_g(f)$$

for all $f \in L^\infty(X,\mu)$, identified with $f \otimes 1 \in \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma))$.
We encode the information of $\Gamma \curvearrowright X$ into a single vN algebra

$$\mathrm{vN}(X \rtimes \Gamma) := \{\sum_{g\in\Gamma}^{\text{finite}} f_g\, u_g : f_g \in L^\infty(X)\}'' \subset \mathbb{B}(L^2(X) \otimes \ell_2(\Gamma)).$$

$\mathrm{vN}(X \rtimes \Gamma)$ is same as the crossed product vN algebra $L^\infty(X) \rtimes \Gamma$.

Group measure space constructions

$\mathrm{vN}(X \rtimes \Gamma)$ is a vN algebra of type II_1, with the trace τ given by

$$\tau\Big(\sum_g f_g\, u_g\Big) = \Big\langle \sum_g f_g\, u_g\, (\mathbf{1} \otimes \delta_1), (\mathbf{1} \otimes \delta_1) \Big\rangle = \int f_1\, d\mu.$$

(It follows $\tau(xy) = \tau(yx)$.)

The subalgebra $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ has a special property.

Definition

A von Neumann subalgebra $A \subset M$ is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates M as a von Neumann algebra.

Orbit Equivalence Relation

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Singer, Dye, Krieger, Feldman–Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta\colon (X,\mu) \to (Y,\nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*\colon L^\infty(Y,\nu) \ni f \mapsto f \circ \theta \in L^\infty(X,\mu)$$

extends to a $*$-isomorphism

$$\pi\colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda\theta(x) \qquad \text{for } \mu\text{-a.e. } x.$$

Lack of rigidity

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Hakeda–Tomiyama, Sakai 1967)

$\mathrm{vN}(X \rtimes \Gamma)$ *is injective (amenable)* $\Leftrightarrow \Gamma$ *is amenable.*

E.g. Solvable groups and subexponential groups are amenable.
Non-abelian free groups $\mathbb{F}_r$ are not.

Theorem (Connes 1974, Ornstein–Weiss, C–Feldman–W 1981)

Amenable **vN** *and* **OE** *are unique modulo center.*

Lack of rigidity

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Connes–Jones 1982)

OE $\longrightarrow$ **vN** *is not one-to-one,*
i.e. $\exists$ *a* II_1*-factor with non-conjugate Cartan subalgebras.*

Example (Oz–Popa 2008)

$$\mathrm{vN}\Big((\varprojlim(\mathbb{Z}/k_n\mathbb{Z})^2) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))\Big)$$

has at least two Cartan subalg $L^\infty(\varprojlim(\mathbb{Z}/k_n\mathbb{Z})^2)$ and $\mathrm{vN}(\mathbb{Z}^2)$.

Lack of rigidity

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

GP		vN
Γ	$\longrightarrow$	$\mathrm{vN}(\Gamma)$

Theorem (Connes 1975)

$\exists$ *a* II_1*-factor which is not* $*$*-isomorphic to its complex conjugate.*

Theorem (Voiculescu 1994)

$\mathrm{vN}(\mathbb{F}_r)$ *does not have a Cartan subalgebra.*

Rigidity

GA		OE		vN
$\Gamma \curvearrowright (X,\mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Furman 1999, (Monod–Shalom,) Popa, Kida, Ioana)

Some **OE** *fully remembers* **GA**. *E.g.*, $\mathrm{SL}(3,\mathbb{Z}) \curvearrowright \mathbb{T}^3$.

Theorem (Oz–Popa 2007, 2008)

Some **vN** *fully remembers* **OE**, *i.e.*, $\exists$ *a (non-amenable)* II_1*-factor with a unique Cartan subalgebra up to unitary conjugacy.*

Note: Popa (2000) proved $\mathrm{vN}(\mathbb{Z}^2) \subset \mathrm{vN}(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))$ is a unique "Cartan subalgebra with the relative property (T)."

Open problems

GA: $\Gamma \curvearrowright (X, \mu)$ $\longrightarrow$ **OE**: $L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$ $\longrightarrow$ **vN**: $\mathrm{vN}(X \rtimes \Gamma)$

GP: Γ $\longrightarrow$ **vN**: $\mathrm{vN}(\Gamma)$

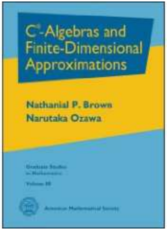
Problem

- Is there **vN** which fully remembers **GA**?
- Is there **vN** which fully remembers **GP**?
- $\mathrm{vN}(\mathbb{F}_r) \ncong \mathrm{vN}(\mathbb{F}_s)$?

Note: Popa (2004) proved $\mathrm{vN}([0,1]^\Gamma \rtimes \Gamma) \cong \mathrm{vN}(Y \rtimes \Lambda)$ implies $(\Gamma \curvearrowright [0,1]^\Gamma) \cong (\Lambda \curvearrowright Y)$ provided that Λ has the property (T). Further results by Popa and Vaes.

From **vN** to **OE**

NEW!



Groups with the Haagerup property

Definition

A 1-cocycle of a loc. cpt group G consists of a conti. unitary rep $(\pi, \mathcal{H})$ and a conti. map $b \colon G \to \mathcal{H}$ such that

$$\forall g, h \in G, \quad b(gh) = b(g) + \pi_g b(h).$$

(i.e., $\theta_g \xi = \pi_g \xi + b(g)$ defines an affine isometric action θ on $\mathcal{H}$.
Schönberg: $\phi_t(g) = e^{-t\|b(g)\|^2}$ is a semigroup of positive type functions.)
The 1-cocycle b is *proper* if $\|b(g)\| \to \infty$ as $g \to \infty$.
A group G has the *Haagerup property* if it admits a proper 1-cocycle $(\pi, \mathcal{H}, b)$. The group G has the *property* (HH) if in addition π can be taken non-amenable (i.e., no $\mathrm{Ad}\pi$-invariant state on $\mathbb{B}(\mathcal{H})$).

Groups with the property (HH)

Observation

A group G with the property (HH) *is not inner-amenable.*
In particular, (infinite amenable) $\times \Gamma$ *does not have* (HH).

Proof.

Let $(\pi, \mathcal{H}, b)$ be a proper 1-cocycle, and suppose that $\exists$ a singular $\mathrm{Ad}G$-invariant state μ on $C_b(G)$. Define a u.c.p. map $\mathbb{B}(\mathcal{H}) \ni x \mapsto f_x \in C_b(G)$ by $f_x(g) = \|b(g)\|^{-2}\langle xb(g), b(g)\rangle$.
Let $h \in G$ be fixed. Since $\lim_g \|b(g)\| = \infty$ and

$$\|b(h^{-1}gh) - \pi_h^{-1}b(g)\| = \|b(h^{-1}) + \pi_{h^{-1}g}b(h)\| \le 2\|b(h)\|,$$

one has $(\mathrm{Ad}h)(f_x) - f_{\pi_h x \pi_h^*} \in C_0(G)$.
It follows that the state $x \mapsto \mu(f_x)$ is $\mathrm{Ad}\pi$-invariant. $\square$

Groups with the property (HH)

Observation

A group G with the property (HH) is not inner-amenable.
In particular, (infinite amenable) $\times \Gamma$ *does not have* (HH).

The converse...

Theorem (Haagerup 1978, De Cannière–H. 1985, Cowling 1983)

The connected simple Lie groups $\mathrm{SO}(n,1)$ with $n \geq 2$ and $\mathrm{SU}(n,1)$ have the property (HH). In particular, lattices of products of $\mathrm{SO}(n,1)$ with $n \geq 2$ and $\mathrm{SU}(n,1)$ have the property (HH).
Moreover, they have the complete metric approximation property.

The Haagerup property and normal amenable subgroups

Proposition

Suppose Γ *has* (H) *and CMAP, and* $\exists \Delta \triangleleft \Gamma$ *infinite normal amenable.*
Then $\exists$ *a mean on* Δ *which is both* Δ*- and* $\mathrm{Ad}\,\Gamma$*-invariant.*
In particular, Γ *is inner amenable and does not have* (HH).

Counterexample by de Cornulier, Stalder and Valette: $(\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{F}_2$.

Proof.

Regard $\phi_t(g) = e^{-t\|b(g)\|^2}$ as multipliers acting on $C^*_\lambda(\Gamma) \subset \mathbb{B}(\ell^2(\Gamma))$.
Since $\phi_t \to 1$, one has $\|\phi_t \circ \mathrm{Ad}_{\lambda(g)} - \mathrm{Ad}_{\lambda(g)} \circ \phi_t\| \to 0$ for every $g \in \Gamma$.
Since Δ is amenable, the trivial character $\tau_0 \colon C^*_\lambda(\Delta) \to \mathbb{C}$ is continuous.
Thus, states $\omega_t = \tau_0 \circ \phi_t$ on $C^*_\lambda(\Delta)$ satisfy $\omega_t(\lambda(h)) \to 1$ for every $h \in \Delta$
and $\|\omega_t \circ \mathrm{Ad}_{\lambda(g)} - \omega_t\| \to 0$ for every $g \in \Gamma$. Thanks to the CMAP,
$\exists$ a finite approximation of ϕ_t and $\omega_t(x) = \langle x\eta_t, \eta_t\rangle$ with $\eta_t \in \ell^2(\Gamma)_{\mathrm{pos}}$.
Then, $|\eta_t|^2 \in \mathrm{Prob}(\Gamma)$ is approximately Δ- and $\mathrm{Ad}\,\Gamma$-invariant. $\square$

At Most One Cartan Subalgebra

The previous proposition says if Γ has the property (HH) and the CMAP, then it does not admit an infinite normal amenable subgroup.

Theorem **A** (Oz–Popa 2008)

Let Γ be a countable group with the property (HH) and the CMAP. Then, $\mathrm{vN}(\Gamma)$ has no Cartan subalgebra. Moreover, if $\Gamma \curvearrowright X$ is profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra in $\mathrm{vN}(X \rtimes \Gamma)$.

Definition

An ergodic action $\Gamma \curvearrowright X$ is *profinite* if $X = \varprojlim \Gamma/\Gamma_n$ for some finite index subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$; or equivalently $\exists A_1 \subset A_2 \subset \cdots \subset L^\infty(X)$ finite-dim Γ-invariant vN subalgebras with dense union. $(A_n = \ell_\infty(\Gamma/\Gamma_n).)$

$$\mathrm{vN}(X \rtimes \Gamma) = \Big(\bigcup \mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma)\Big)'' \cong \Big(\bigcup \mathbb{M}_{[\Gamma:\Gamma_n]}(\mathrm{vN}(\Gamma_n))\Big)''.$$

Weak Compactness

The proof of Theorem **A** is in principle similar to the case of groups, but requires a notion of weak compactness which substitutes inner amenability, and spectral analysis of the quantum Markov semigroup associated with a closable derivation (Sauvageot, et al. and Peterson).

Theorem (Oz–Popa 2007)

Suppose that M has CMAP and A is an amenable vN subalgebra. Then, $A \subset M$ is weakly compact in the following sense: $\exists\ \eta_n \in L^2(A \bar{\otimes} \bar{A})_+$ *such that*

- $\|\eta_n - (u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{U}(A)$;
- $\|\eta_n - \mathrm{Ad}(u \otimes \bar{u})\eta_n\|_2 \to 0$ *for every* $u \in \mathcal{N}(A)$;
- $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n \rangle$ *for every* $x \in M$.

From **OE** to **GA**

Orbit Equivalence Relation

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$

Theorem (Singer, Dye, Krieger, Feldman–Moore 1977)

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be ess-free p.m.p. actions, and

$$\theta \colon (X, \mu) \to (Y, \nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^* \colon L^\infty(Y, \nu) \ni f \mapsto f \circ \theta \in L^\infty(X, \mu)$$

extends to a $*$-isomorphism

$$\pi \colon \mathrm{vN}(Y \rtimes \Lambda) \to \mathrm{vN}(X \rtimes \Gamma)$$

if and only if θ preserves the orbit equivalence relation:

$$\theta(\Gamma x) = \Lambda \theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

From OE to Cocycle (after Zimmer)

Suppose $(\Gamma \curvearrowright X) \cong_{\mathrm{OE}} (\Lambda \curvearrowright Y)$, i.e. $\exists\, \theta \colon X \xrightarrow{\sim} Y$ such that

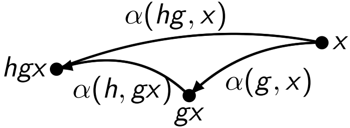
$$\theta(\Gamma x) = \Lambda\theta(x) \qquad \text{for } \mu\text{-a.e. } x.$$

Define $\alpha \colon \Gamma \times X \to \Lambda$ by

$$\theta(gx) = \alpha(g,x)\theta(x).$$

Then, α satisfies the cocycle identity:

$$\alpha(h,gx)\alpha(g,x) = \alpha(hg,x).$$



A cocycle α is a *homomorphism* if ess. independent of the second variable.
Cocycles α and β are *equivalent* if $\exists\, \phi \colon X \to \Lambda$ such that

$$\beta(g,x) = \phi(gx)\alpha(g,x)\phi(x)^{-1}.$$

Theorem (Zimmer)

$(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ *if and only if* α *is equivalent to a homomorphism.*

From Cocycle to Group Action

Theorem (Cocycle Superrigidity)

With some assumption on $\Gamma \curvearrowright X$ (and not on Λ), any cocycle

$$\alpha \colon \Gamma \times X \to \Lambda$$

is equivalent to a homomorphism β.

Applied to the Zimmer cocycle, one obtains (virtual) isomorphism $(\Gamma \curvearrowright X) \cong (\Lambda \curvearrowright Y)$ via the homomorphism $\beta \colon \Gamma \to \Lambda$.

Examples

- Γ higher rank lattice $+$ Λ simple Lie group (Zimmer)
- Γ Kazhdan (T) / product $+$ $\Gamma \curvearrowright X$ Bernoulli (Popa)
- Γ Kazhdan (T) $+$ $\Gamma \curvearrowright X$ profinite (Ioana)

New von Neumann Rigidity

By adapting Ioana's arguments, we obtain a cocycle superrigidity result for some profinite actions of property (τ) groups with residually-finite targets. There are groups with the property (HH) and the property (τ).

Corollary

Let $\Gamma_i = \mathrm{PSL}(2, \mathbb{Z}[\sqrt{2}])$ and $p_1 < p_2 < \cdots$ be prime numbers. Let $\Gamma = \Gamma_1 \times \Gamma_2$ act on $X = \varprojlim \mathrm{PSL}(2, (\mathbb{Z}/p_1 \cdots p_n\mathbb{Z})[\sqrt{2}])$ by the left-and-right translation. Let $\Lambda \curvearrowright Y$ be any (free ergodic prob.m.p.) action of a residually-finite group Λ such that $\mathrm{vN}(X \rtimes \Gamma) \cong \mathrm{vN}(Y \rtimes \Lambda)$. Then, $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are virtually isomorphic.

GA		OE		vN
$\Gamma \curvearrowright (X, \mu)$	$\longrightarrow$	$L^\infty(X) \subset \mathrm{vN}(X \rtimes \Gamma)$	$\longrightarrow$	$\mathrm{vN}(X \rtimes \Gamma)$