

## $II_1$ factors with at most one Cartan subalgebra

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# What do we classify?

$\Gamma$  countable discrete group  
 $(X, \mu)$  standard **probability** measure space  
 $\Gamma \curvearrowright (X, \mu)$  **measure preserving** action

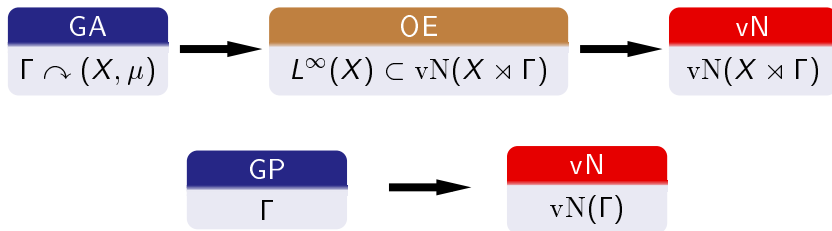
We only consider either

- $(X, \mu) \cong ([0, 1], \text{Lebesgue})$  and  
 $\Gamma \curvearrowright X$  is **essentially-free** i.e.  $\mu(\{x : sx = x\}) = 0 \ \forall s \in \Gamma \setminus \{1\}$ ;  
 or
- $X = \{\text{pt}\}$ .

In passing, recall that  $\Gamma \curvearrowright X$  is **ergodic** if

$$A \subset X \text{ and } \Gamma A = A \Rightarrow \mu(A) = 0, 1.$$

# How do we classify?



To what extent do vN/OE  
 remember OE/GA/GP?

# Group measure space constructions

$$\Gamma \curvearrowright (X, \mu) \quad \text{p.m.p.} \quad \longleftrightarrow \quad \begin{aligned} \sigma &: \Gamma \curvearrowright L^\infty(X, \mu) \\ \sigma_s(f)(x) &= f(s^{-1}x) \\ \int \sigma_s(f) d\mu &= \int f d\mu \end{aligned}$$

We encode the information of  $\Gamma \curvearrowright X$  into a single  $*$ -algebra

$$\mathcal{A}(X \rtimes \Gamma) := \left\{ \sum_{s \in \Gamma}^{\text{finite}} f_s s : f_s \in L^\infty(X, \mu) \right\},$$

which is generated by the group algebra  $\mathbb{C}\Gamma$  and the function algebra  $L^\infty(X)$  with the relationship

$$s f s^{-1} = \sigma_s(f).$$

## Group measure space constructions

Hence,  $(\sum_s f_s s)(\sum_t g_t t) = \sum_{s,t} f_s \sigma_s(g_t) st.$

$\mathcal{A}(X \rtimes \Gamma)$  is a pre-Hilbertian algebra:

$$\|\sum_s f_s s\|_2 = \sqrt{\sum_s \|f_s\|_2^2}.$$

Denote  $\mathcal{H} = \overline{\mathcal{A}(X \rtimes \Gamma)}^{\|\cdot\|_2} \cong L^2(X, \mu) \otimes_2 \ell_2(\Gamma)$  and define

$$\text{vN}(X \rtimes \Gamma) := \text{WOT-cl}(\mathcal{A}(X \rtimes \Gamma)) \subset \mathbb{B}(\mathcal{H}).$$

$\text{vN}(X \rtimes \Gamma)$  is often written as  $L^\infty(X) \rtimes \Gamma$  and has a finite trace  $\tau$ , given by  $\tau(x) = \langle x 1, 1 \rangle$ . (It follows  $\tau(xy) = \tau(yx)$ .)

# Group measure space constructions

$$\begin{aligned} L^\infty(X) &\subset \text{vN}(X \rtimes \Gamma) \\ &= \text{vN}((u \otimes \lambda)(\Gamma), L^\infty(X) \otimes \mathbb{C}1) \\ &\subset \mathbb{B}(L^2(X, \mu) \otimes \ell_2(\Gamma)) \end{aligned}$$

and  $L^\infty(X)$  is a *Cartan subalgebra* of  $\text{vN}(X \rtimes \Gamma)$ .

## Definition

A von Neumann subalgebra  $A \subset M$  is called a *Cartan subalgebra* if it is a maximal abelian subalgebra such that the normalizer

$$\mathcal{N}(A) = \{u \in M : \text{unitary} \quad uAu^* = A\}$$

generates  $M$ .

# Orbit Equivalence Relation

Theorem (Singer, Dye, Krieger, Feldman-Moore 1977)

Let  $\Gamma \curvearrowright X$  and  $\Lambda \curvearrowright Y$  be *ess-free p.m.p. actions*, and

$$\theta: (X, \mu) \rightarrow (Y, \nu)$$

be an isomorphism. Then, the isomorphism

$$\theta^*: L^\infty(Y, \nu) \ni f \mapsto f \circ \theta \in L^\infty(X, \mu)$$

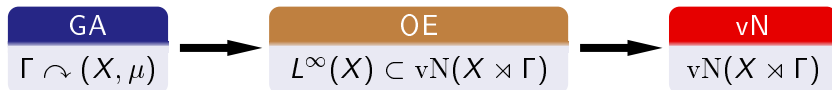
extends to a  $*$ -isomorphism

$$\pi: \text{vN}(Y \rtimes \Lambda) \rightarrow \text{vN}(X \rtimes \Gamma)$$

if and only if  $\theta$  preserves the *orbit equivalence* relation:

$$\theta(\Gamma x) = \Lambda \theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

# Lack of rigidity



Theorem (Connes 1974, Ornstein-Weiss, C-Feldman-W 1981)

*Amenable vN and OE are unique modulo center.*

Theorem (Connes-Jones 1982)

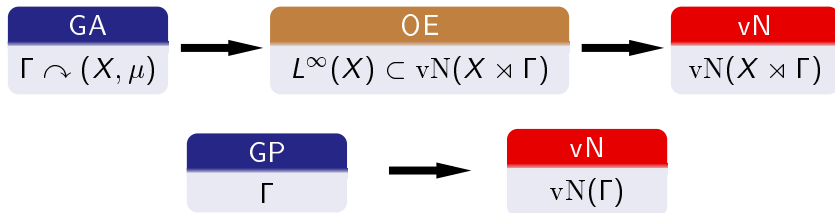
**OE  $\longrightarrow$  vN is not one-to-one,**  
*i.e.  $\exists$  a  $II_1$ -factor with non-conjugate Cartan subalgebras.*

Theorem (Dykema 1993)

$vN(\Gamma_1 * \Gamma_2) \cong vN(\mathbb{F}_2)$  for any infinite amenable groups  $\Gamma_1$  and  $\Gamma_2$ .



# Lack of rigidity



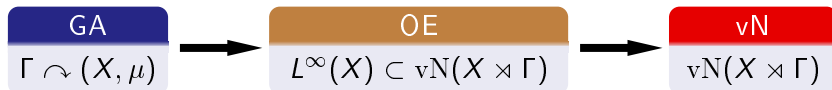
Theorem (Connes 1975)

$\exists$  a  $II_1$ -factor which is not  $*$ -isomorphic to its complex conjugate.

Theorem (Voiculescu 1994)

$vN(\mathbb{F}_r)$  does not have a Cartan subalgebra.

# Rigidity



Theorem (Furman 1999, (Monod-Shalom,) Popa, Kida, Ioana)

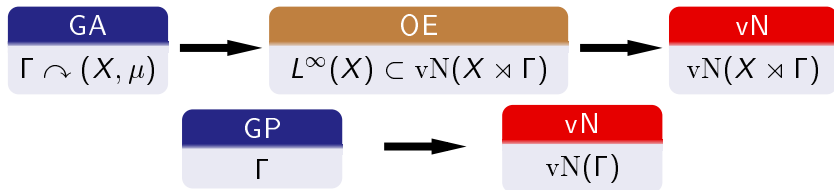
*Some **OE** fully remembers **GA**.*

Theorem (Oz-Popa)

*Some **vN** fully remembers **OE**, i.e.*

*$\exists$  a (non-amenable)  $\text{II}_1$ -factor with a unique Cartan subalgebra, unique **up to unitary conjugacy**.*

# Open problems



## Problem

- Is there **vN** which fully remembers **GA**?
- Is there **vN** which fully remembers **GP**?
- $vN(\mathbb{F}_2) \not\cong vN(\mathbb{F}_3)$  ?

## OE to Cocycle (after Zimmer)

Suppose  $(\Gamma \curvearrowright X) \cong_{\text{OE}} (\Lambda \curvearrowright Y)$ , i.e.  $\exists \theta: X \xrightarrow{\sim} Y$  such that

$$\theta(\Gamma x) = \Lambda \theta(x) \quad \text{for } \mu\text{-a.e. } x.$$

Define  $\alpha: X \times \Gamma \rightarrow \Lambda$  by

$$\theta(x) = \alpha(x, s)\theta(s^{-1}x).$$

Then,  $\alpha$  satisfies the cocycle identity:

$$\alpha(x, s)\alpha(s^{-1}x, t) = \alpha(x, st).$$

Cocycles  $\alpha$  and  $\beta$  are *equivalent* if  $\exists \phi: X \rightarrow \Lambda$  such that

$$\beta(x, s) = \phi(x)\alpha(x, s)\phi(s^{-1}x)^{-1}.$$

# Cocycle Superrigidity

One can recover **GA** from **OE** if one has

## Theorem (Cocycle Superrigidity)

*With some assumption on  $\Gamma \curvearrowright X$  (and not on  $\Lambda$ ), **any** cocycle*

$$\alpha: \Gamma \times X \rightarrow \Lambda$$

*is equivalent to a cocycle  $\beta$  which is independent on  $x \in X$ .*

## Examples

- $\Gamma$  higher rank lattice +  $\Lambda$  simple Lie group (Zimmer)
- $\Gamma$  Kazhdan (T) / product +  $\Gamma \curvearrowright X$  malleable (Popa)
- $\Gamma$  Kazhdan (T) +  $\Gamma \curvearrowright X$  profinite (Ioana)

# Popa's formulation

$$\Gamma \curvearrowright X$$

$$\alpha: X \times \Gamma \rightarrow \Lambda$$

$$\alpha(x, s)\alpha(s^{-1}x, t) = \alpha(x, st)$$



$$\sigma: \Gamma \curvearrowright L^\infty(X)$$

$$\alpha: \Gamma \rightarrow L^\infty(X, \text{vN}(\Lambda))$$

$$\cong L^\infty(X) \bar{\otimes} N$$

$$\alpha_s(x) = \alpha(x, s)$$

$$\alpha_s \sigma_s(\alpha_t) = \alpha_{st}$$

Since  $\sigma_s(f) = s f s^{-1}$  in  $\text{vN}(X \rtimes \Gamma)$ ,

$$\Gamma \ni s \mapsto \alpha_s s \in \text{vN}(X \rtimes \Gamma) \bar{\otimes} N$$

is a group homomorphism which extends to an inclusion

$$\Theta: \text{vN}(\Gamma) \hookrightarrow \text{vN}(X \rtimes \Gamma) \bar{\otimes} N.$$

# Profinite Action

## Definition

An ergodic action  $\Gamma \curvearrowright X$  is *profinite* if  $X = \varprojlim \Gamma/\Gamma_n$  for some finite index subgroups  $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ ;  
 or equivalently  $\exists A_1 \subset A_2 \subset \cdots \subset L^\infty(X)$  finite dimensional  $\Gamma$ -invariant vN-subalgebras with dense union. ( $A_n = \ell_\infty(\Gamma/\Gamma_n)$ .)

$$\mathrm{vN}(X \rtimes \Gamma) = \left( \bigcup \mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \right)'' \cong \left( \bigcup \mathbb{M}_{[\Gamma:\Gamma_n]}(\mathrm{vN}(\Gamma_n)) \right)''.$$

## What's behind Ioana's Cocycle Superrigidity

$$\Theta: \mathrm{vN}(\Gamma) \hookrightarrow \mathrm{vN}(X \rtimes \Gamma) \bar{\otimes} N = \left( \bigcup (\mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \bar{\otimes} N) \right)''$$



Because of the Kazhdan property (T), for a large  $n$ ,  
 $\Theta(\mathrm{vN}(\Gamma))$  is almost contained in  $\mathrm{vN}((\Gamma/\Gamma_n) \rtimes \Gamma) \bar{\otimes} N$ .

# Complete Metric Approximation Property

## Definition

A group  $\Gamma$  has the CMAP if  $\exists f_n$  such that

- $f_n: \Gamma \rightarrow \mathbb{C}$  finitely supported,
- $f_n \rightarrow 1$  pointwise,
- $\|m_{f_n}\|_{cb} \leq 1$ .

The multiplier  $m_f: vN(\Gamma) \rightarrow vN(\Gamma)$  is defined by  $m_f(s) = f(s)s$ .

Besides amenable groups, the following groups have the CMAP.

Theorem (De Cannière-Haagerup 1985, Cowling-Haagerup 1989)

*Free groups  $\mathbb{F}_r$  have the CMAP.*

*Discrete subgroups of  $SO(n, 1)$  and  $SU(n, 1)$  have the CMAP.*



# Groups with CMAP

## Theorem A (Oz-Popa)

*Suppose  $\Gamma$  CMAP and  $\exists \Lambda \triangleleft \Gamma$  infinite normal amenable subgroup. Then,  $\Lambda$  has an invariant mean which is  $\text{Ad}(\Gamma)$ -invariant. In particular,  $\Gamma$  is inner-amenable.*

## Proof (Assuming $\Lambda$ is abelian).

Recall  $vN(\Lambda) \cong L^\infty(\widehat{\Lambda})$  via the Fourier transform  $\ell_2(\Lambda) \cong L^2(\widehat{\Lambda})$ . Let  $\tau_0: C(\widehat{\Lambda}) \rightarrow \mathbb{C}$  be the evaluation at the trivial character 1.  $f: \Lambda \rightarrow \mathbb{C}$  fin. supp.  $\Rightarrow \tau_0 \circ m_f \cong \widehat{f} \in L^1(\widehat{\Lambda})$  and  $\|\widehat{f}\|_1 = \|m_f\|_{cb}$ . Take  $(f_n)$  as in Definition. Then  $\forall s \quad \|m_{f_n} - m_{f_n} \circ \text{Ad}_s\|_{cb} \rightarrow 0$ . Hence, if  $\ell_2(\Lambda) \ni \xi_n \xrightarrow{\text{Fourier}} |\widehat{f_n}|_\Lambda|^{1/2} \in L^2(\widehat{\Lambda})$ , then  $|\xi|^2 \in \ell_1(\Lambda)$  is approximately  $\Lambda$ -invariant and approximately  $\text{Ad}(\Gamma)$ -invariant.  $\square$

# Groups with CMAP

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## Corollary

*The lamplighter group*

$$(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r = \left( \bigoplus_{\mathbb{F}_r} (\mathbb{Z}/2\mathbb{Z}) \right) \rtimes \mathbb{F}_r$$

*does not have the CMAP.*

## Theorem (de Cornulier-Stalder-Valette)

*The lamplighter group  $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_r$  has the Haagerup property.*

# von Neumann algebra with CMAP

## Definition

A finite vN-algebra  $M$  has the CMAP if  $\exists \phi_n$  such that

- $\phi_n: M \rightarrow M$  finite rank,
- $\phi_n \rightarrow \text{id}_M$  pointwise-ultraweak,
- $\|\phi_n\|_{\text{cb}} \leq 1$ .

## Examples

- $\Gamma$  has CMAP  $\Leftrightarrow \text{vN}(\Gamma)$  has CMAP (Haagerup)
- CMAP inherits to a vN-subalgebra (assuming finiteness).
- $\Gamma$  has CMAP and  $\Gamma \curvearrowright X$  profinite  $\Rightarrow \text{vN}(X \rtimes \Gamma)$  has CMAP.  
(Note:  $\text{vN}(X \rtimes \Gamma)$  can be non-( $\Gamma$ ).)

# Upgrading Theorem A

Use  $\mu: P \otimes \bar{P} \ni \sum a_k \otimes \bar{b}_k \mapsto \tau(\sum a_k b_k^*) \in \mathbb{C}$  instead of  $\tau_0$ .

## Theorem A+ (Oz-Popa)

Suppose that  $M$  has CMAF and  $P$  is an amenable vN-subalgebra. Then,  $\exists \eta_n \in L^2(P \bar{\otimes} \bar{P})_+$  such that

- $\|\eta_n - (u \otimes \bar{u})\eta_n\|_2 \rightarrow 0$  for every  $u \in \mathcal{U}(P)$ ;
- $\|\eta_n - \text{Ad}(u \otimes \bar{u})\eta_n\|_2 \rightarrow 0$  for every  $u \in \mathcal{N}(P)$ ;
- $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle \eta_n, (1 \otimes \bar{x})\eta_n \rangle$  for every  $x \in M$ .

We say  $P \subset M$  is **weakly profinite** if the above conclusion holds.

If  $M = P \rtimes \Gamma$  and  $\exists P_1 \subset P_2 \subset \cdots \subset P$  finite dim.  $\Gamma$ -invariant vN-subalgebras with dense union, then  $P \subset M$  is weakly profinite with  $\eta_n = \mu_n^{1/2} \in L^2(P_n \bar{\otimes} \bar{P}_n)_+$ .

# Main Results

## Theorem B (Oz-Popa)

*Suppose that  $M = Q \rtimes \mathbb{F}_r$  and that  $P \subset M$  is weakly profinite. Then, either one of the following occurs*

- *a nonzero corner of  $P$  is unitarily conjugated into  $Q$ ;*
- *$\mathcal{N}(P)''$  is amenable relative to  $Q$ .*

## Corollary

- $P \subset \text{vN}(\mathbb{F}_r)$  diffuse amenable  $\Rightarrow \mathcal{N}(P)''$  amenable.
- $Q$  CMA  $\Rightarrow Q \bar{\otimes} \text{vN}(\mathbb{F}_r)$  has no Cartan subalgebra.
- $\mathbb{F}_r \curvearrowright X$  profinite  $\Rightarrow L^\infty(X) \subset \text{vN}(X \rtimes \mathbb{F}_r)$  is the unique Cartan subalgebra.

# Proof in the case of $P \subset \text{vN}(\mathbb{F}_r)$ diffuse amenable

Let  $a_1, \dots, a_r \in M = \text{vN}(\mathbb{F}_r)$  be the standard unitary generators, and  $M_1 = \langle b_1, \dots, b_r \rangle$  be a copy of  $\text{vN}(\mathbb{F}_r)$ .

For  $t \in \mathbb{R}$ , define a  $*$ -homomorphism  $\alpha_t: M \rightarrow M * M_1$  by

$$\alpha_t(a_k) = a_k \exp(t \log b_k).$$

Observe that  $E_M \circ \alpha_t$  is the Haagerup multiplier on  $M$  associated with  $\mathbb{F}_r \ni s \mapsto \gamma_t^{|s|} \in \mathbb{R}$ , where  $\gamma_t = \tau(\exp(t \log b_k)) = \frac{\sin(\pi t)}{\pi t}$ .

For a given finite subset  $\mathfrak{F} \subset \mathcal{N}(P)$ , choose  $t > 0$  small enough so that  $\alpha = \alpha_t$  satisfies  $\alpha(u) \approx u$  for all  $u \in \mathfrak{F}$ .

Since  $\eta_n \in L^2(P \bar{\otimes} \bar{P})$  are “almost supported on diagonal,”

$((E_M^\perp \circ \alpha) \otimes 1)\eta_n$  is a non-null sequence, almost  $\text{Ad}(\mathfrak{F})$ -invariant.

But  $L^2(M * M_1) \ominus L^2(M) \cong \bigoplus L^2(M) \bar{\otimes} L^2(M)$  as an  $M$ -bimodule, this implies amenability of  $\mathcal{N}(P)''$ . □