Elementary amenable groups are quasidiagonal

Narutaka OZAWA （小澤 登高）
Joint work with M. Rørdam and Y. Sato (arXiv:1404.3462)

Research Institute for Mathematical Sciences, Kyoto University

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Quasidiagonality

Definition (Halmos 1968)

An operator $T \in \mathbb{B}(\mathcal{H})$ is **quasidiagonal** (QD) if there is a sequence of finite rank projections $P_n \nearrow 1$ such that $\|TP_n - P_nT\| \to 0$;
or equivalently, if $T = D + K$ for some block diagonal D and compact K.

A (separable) C^*-algebra $A \subset \mathbb{B}(\mathcal{H})$ is QD if there is a sequence of finite rank projections $P_n \nearrow 1$ such that $\|aP_n - P_na\| \to 0$ for all $a \in A$.

Thanks to Voiculescu's theorem (1976), quasidiagonality is a property of a C^*-algebra A, i.e., it does not depend on faithful rep's $A \subset \mathbb{B}(\mathcal{H})$ as long as $A \cap \mathbb{K}(\mathcal{H}) = \{0\}$ (this will be assumed throughout the talk).

Basic Question

Which C^*-algebra A is QD? What are the permanence properties?

All stably finite nuclear "**classifiable**" C^*-algebras are QD. By the recent work of Matui–Sato on the Toms–Winter conjecture, it has become clearer that QD shall play an important rôle in the classification theory.

Implications of quasidiagonality

Theorem

Every unital QD C^*-algebra A is stably finite, has zero Fredholm indices, and has an **amenable tracial state**.

Definition/Theorem (Connes 1976 and Kirchberg 1993)

A tracial state τ on $A \subset \mathbb{B}(\mathcal{H})$ is said to be **amenable** if it satisfies the following equivalent conditions.

(i) τ extends to a state φ on $\mathbb{B}(\mathcal{H})$ which is A-central, i.e. $\varphi(ax) = \varphi(xa)$ for $a \in A$ and $x \in \mathbb{B}(\mathcal{H})$.

(ii) There is a sequence of finite rank projections $P_n \nearrow 1$ such that
$$\frac{\|aP_n - P_n a\|_{\mathrm{HS}}}{\|P_n\|_{\mathrm{HS}}} \to 0 \text{ and } \tau(a) = \lim_n \frac{\mathrm{Tr}(P_n a P_n)}{\mathrm{Tr}(P_n)} \text{ for all } a \in A.$$

Problem

Is every nuclear C^*-algebra with a faithful trace QD? How about $\mathcal{R}$?

Rosenberg's conjecture

Consider the reduced group C^*-alg $\mathrm{C}^*_\lambda(\Gamma) \subset \mathbb{B}(\ell_2(\Gamma))$. If Γ is amenable and $(F_n)_n$ is a Følner sequence, then the projections P_{F_n} on $\ell_2(\Gamma)$ satisfy

$$\frac{\|aP_n - P_n a\|_{\mathrm{HS}}}{\|P_n\|_{\mathrm{HS}}} \to 0 \text{ and } \frac{\mathrm{Tr}(P_n a P_n)}{\mathrm{Tr}(P_n)} \to \tau(a) \text{ for } a \in \mathrm{C}^*_\lambda(\Gamma),$$

but not $\|aP_n - P_n a\| \to 0$. Hence the trace τ is amenable if Γ is amenable.

Conversely, if there is a $\mathrm{C}^*_\lambda(\Gamma)$-central state φ on $\mathbb{B}(\ell_2(\Gamma))$, then its restriction $\varphi|_{\ell_\infty(\Gamma)}$ to the diagonal is an invariant mean and Γ is amenable.

Corollary (Rosenberg 1987)

$$\mathrm{C}^*_\lambda(\Gamma) \text{ quasidiagonal} \Longrightarrow \Gamma \text{ amenable.}$$

Rosenberg's conjecture: The converse is also true.

Main Theorem (Oz., Rørdam, and Sato: arXiv:1404.3462)

Rosenberg's conjecture holds for all **elementary amenable** groups.

Elementary amenable groups

Main Theorem (Oz., Rørdam, and Sato: arXiv:1404.3462)

If Γ is elementary amenable, then $\mathrm{C}^*_\lambda(\Gamma)$ is QD (and AF-embeddable).

This subsumes the previous results by Carrion–Eckhardt–Dadarlat 2013 (some solvable groups) and by Eckhardt 2014 (all nilpotent groups; in fact he proved that **every** unitary rep of a nilpotent group is QD).

Definition (M. M. Day 1957)

The class **EG** of elementary amenable groups is the smallest class that contains all **finite** and **abelian** groups and is closed under the taking (i) **subgroups**, (ii) **quotients**, (iii) **extensions**, and (iv) **inductive limits**.

⚠ There are amenable groups which are not elementary amenable.

- Grigorchuk's intermediate growth group: residually finite and QD.
- Topological full groups of Cantor minimal systems (Matui 06 + Juschenko–Monod 13): LEF (Grigorchuk–Medynets) and QD (CED, Kerr).

Scheme of the proof

Definition

The class **EG** of elementary amenable groups is the smallest class that contains all **finite** and **abelian** groups and is closed under the taking (i) **subgroups**, (ii) **quotients**, (iii) **extensions**, and (iv) **inductive limits**.

Theorem (Chou 1980 and Osin 2002)

Let **P** be a class of groups. Then **EG** $\subset$ **P** provided that it contains $\{1\}$ and is closed under taking (a) **extensions by finite and cyclic groups** and (b) **directed limits**.

(a) says: $\Gamma \triangleleft \Lambda$ with $\Gamma \in$ **P** and $\Lambda/\Gamma \in \{$finite or cyclic$\} \Longrightarrow \Lambda \in$ **P**.
It is easy to see **QD** is closed under finite extensions and directed limits.

Remaining Problem (Note: An extension by $\mathbb{Z}$ is a semidirect product.)

Suppose that $\Gamma \in$ **QD** and $\alpha \in \mathrm{Aut}(\Gamma)$. Does it follow $\Gamma \rtimes_\alpha \mathbb{Z} \in$ **QD**?

Note: $\mathrm{C}^*_\lambda(\Gamma \rtimes_\alpha \mathbb{Z}) = \mathrm{C}^*_\lambda(\Gamma) \rtimes_\alpha \mathbb{Z}$ and α preserves the trace τ.

Classification theory of nuclear C^*-algebras

When is $\mathrm{C}^*_\lambda(\Gamma) \rtimes \mathbb{Z}$ or more generally $A \rtimes \mathbb{Z}$ quasidiagonal?

Pimsner (1983):	$C(X) \rtimes \mathbb{Z}$ is QD (AF-embeddable) if stably finite.
N. P. Brown (1998):	$\mathrm{AF} \rtimes \mathbb{Z}$ is QD (AF-embeddable) if stably finite.
Matui (2002):	(simple $\mathrm{A}\mathbb{T}$ of RR0) $\rtimes \mathbb{Z}$ is QD (AF-embeddable).

In order to apply Matui's theorem, we invoke the following compilation of the results from the classification theory.

Theorem (Matui–Sato, Winter, Lin–Niu, Rørdam, Elliott)

Let A and B be **monotracial "squab" C^*-algebras** (Separable Simple QD Unital Amenable C^*-algebra in the Bootstrap class).

- Then, $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ if their K-theoretic invariants agree.
- In particular, $A \otimes \mathcal{U}$ is an $\mathrm{A}\mathbb{T}$-algebra of RR0 and $A \rtimes \mathbb{Z}$ is QD.

Note: $\left(\bigotimes_\Gamma \mathbb{M}_{2^\infty}\right) \rtimes \Gamma$ is simple, monotracial, and in the Bootstrap class, because it's an amenable groupoid C^*-algebra (Tu 1999). Moreover,

$$\left(\bigotimes_{\Gamma\rtimes\mathbb{Z}} \mathbb{M}_{2^\infty}\right) \rtimes (\Gamma \rtimes \mathbb{Z}) = \Big(\left(\bigotimes_{\Gamma\times\mathbb{Z}} \mathbb{M}_{2^\infty}\right) \rtimes \Gamma\Big) \rtimes \mathbb{Z} \cong \Big(\left(\bigotimes_{\Gamma} \mathbb{M}_{2^\infty}\right) \rtimes \Gamma\Big) \rtimes \mathbb{Z}.$$

Conclusion

Since it is unclear whether **QD** is closed under semidirect products by $\mathbb{Z}$, we introduce an *ad hoc* class **PQ** $:= \{\Gamma : \left(\bigotimes_\Gamma \mathbb{M}_{2^\infty}\right) \rtimes \Gamma \text{ is QD}\} \subset$ **QD**. Thanks to the classification theory of squab C^*-algebras, it is closed under extensions by finite and cyclic groups and directed limits. So, **EG** $\subset$ **PQ**.

Main Theorem (Oz., Rørdam, and Sato: arXiv:1404.3462)

If Γ is elementary amenable, then $\mathrm{C}^*_\lambda(\Gamma)$ is QD (and AF-embeddable).

Moreover, amenable "locally **PQ**" groups are **PQ** and so **PQ** contains Grigorchuk's groups and top full groups of Cantor minimal systems.

More Problem

For which Γ, is the full group C^*-algebra $\mathrm{C}^*(\Gamma)$ quasidiagonal?

- $\mathrm{C}^*(F_d)$ is QD (residually finite dimensional, Choi 1980).
- $\mathrm{C}^*(F_d \times F_d)$ is QD (N. P. Brown). (⚠ RFD $\Leftrightarrow$ Kirchberg's Conjecture)
- $\mathrm{C}^*(\Gamma)$ is not QD if Γ is a simple (T) group. Probably $\mathrm{SL}(3, \mathbb{Z})$ neither.