Finite Dimensional Representations from Random Walks

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2018 Spring Probability Workshop,
Institute of Mathematics, Academia Sinica, Taipei,
2018 June 05

We connect **geometry** (quasi-isometry, random walks, etc.) of a finitely generated group to **algebra** (∃ a virtually-$\mathbb{Z}$ quotient) of it via **analysis**.

$$G = \langle S \rangle \text{ with finite generating subset } S = S^{-1}$$

$$\leadsto |x| := \min\{n : x \in S^n\} \text{ and } d(x, y) := |x^{-1}y|$$

$$\gamma_G(n) := |\text{Ball}(n)| = |\{x : |x| \leq n\}| \text{ growth}$$

These are (up to a certain equivalence) indep. of $S$, in fact a QI invariant.

A map $f : (X, d_X) \to (Y, d_Y)$ is a **quasi-isometry** (QI) if $\exists K, L > 0$

$$\frac{1}{K}d_X(x, y) - L \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + L \text{ and } Y \subset N_L(f(X)).$$

For example, if $G_0 \leq_{\text{finite index}} G$, then $G_0 \cong_{\text{QI}} G$.

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**Theorem** (Gromov 1981)

If $G$ has polynomial growth ($\exists d \quad \gamma_G(n) \leq n^d$), then it is virtually nilpotent.

**Proof:** By induction on $d$.

It suffices to show $\exists$ a virtually-$\mathbb{Z}$ quotient: $G \geq_{\text{finite index}} G_0 \twoheadrightarrow \mathbb{Z}$.

$\therefore \ker q$ is f.g. and has polynomial growth of degree $\leq d - 1$. 
Further motivation: Grigorchuk’s Conjecture

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**Theorem (Gromov 1981)**

If \( G \) has polynomial growth (\( \exists d \ \gamma_G(n) \leq n^d \)), then it is virtually nilpotent.

**Proof:** By induction on \( d \).

It suffices to show \( \exists \) a virtually-\( \mathbb{Z} \) quotient: \( G \cong_{\text{finite index}} G_0 \xrightarrow{q} \mathbb{Z} \).

\[ \because \text{ker } q \text{ is f.g. and has polynomial growth of degree } \leq d - 1. \]

**Grigorchuk’s Gap Conjecture (1990)**

If \( \gamma_G(n) \ll e^{\sqrt{n}} \) (or \( \exp n^{0.01} \)), then \( G \) has polynomial growth.

There are several empirical evidences, but here’s an optimistic heuristic:

Fix a symmetric probability measure \( \mu \) with \( \text{supp } \mu = S \) and consider the random walk \( X_n = s_1 \cdots s_n, \ s_i \ \mu\text{-i.i.d.} \)

If \( \gamma_G(n) \ll e^{\sqrt{n}} \), then the \( \mu\text{-RW} \) is maybe **diffusive**, e.g., \( \mathbb{E}[|X_n|] \leq \sqrt{n} \).

In turn, as we will see, this probably implies \( G \) has a virtually-\( \mathbb{Z} \) quotient.
How to find a $v\mathbb{Z}$ quotient?

... It suffices to find a finite-dim repn with an infinite image.

**Theorem (Tits Alternative 1972)**

If $G \leq \text{GL}(n, F)$ is a finitely generated infinite amenable subgroup, then $G$ is virtually solvable and has a virtually-$\mathbb{Z}$ quotient.

Shalom’s idea (2004): Use reduced cohomology to get a non-trivial finite-dimensional representation.

Given an orthogonal repn $\pi : G \curvearrowright \mathcal{H}$ (which need not be finite-dim)

$b : G \to \mathcal{H}$ cocycle \(\iff\) $b(gt) = b(g) + \pi_g b(t)$ for $\forall g, t \in G$

- e.g., coboundary \(b_\nu(g) = \nu - \pi_g \nu\), where $\nu \in \mathcal{H}$
- harmonic \(\iff\) $\sum_t b(gt)\mu(t) = b(g)$ for $\forall g \in G$ (or just $g = e$)
- e.g., $\forall$ harm. cob. is zero: $\nu - \sum \mu(g)\pi_g \nu = 0 \leadsto \nu = \pi(g)\nu$ for $\forall g$.

$Z^1(G, \pi) := \{\text{cocycles}\}$ is a Hilbert space w.r.t.

- $\|b\|^2 := \sum_t \|b(t)\|^2 \mu(t)$

\(\overline{H}^1(G, \pi) := Z^1(G, \pi) / B^1(G, \pi) \cong B^1(G, \pi)^\perp = \{\text{harmonic cocycles}\}\)
**Shalom’s property $H_{\text{FD}}$**

**Theorem (Mok ’95, Korevaar–Schoen ’97, Shalom ’99 ▶ )**

If $G$ is a f.g. infinite amenable group, then $\exists \pi$ s.t. $H^1(G, \pi) \neq 0$.

In general, $\pi$ decomposes as

$$\pi = \bigoplus (\text{fd repns}) \oplus (\text{no nonzero fd subrepns})$$

Accordingly

$$b = b_{\text{a.p.}} \oplus b_{\text{w.m.}}$$

**Obs:** $G$ f.g. amenable $\exists b$ harmonic with $b_{\text{a.p.}} \neq 0 \Rightarrow \exists v\cdot\mathbb{Z}$ quotient.

:** If $|\pi(G)| = \infty$, then use Tits Alternative.

If $|\pi(G)| < \infty$, then $\ker\pi \leq_{f.i.} G$ and $b|_{\ker\pi}$ is a $\neq 0$ additive character.

**Shalom (2004)**

$\triangleright$ Gromov $\uparrow$

polynomial growth $\triangleright$ Oz. ’15

$v$-nilpotent $\longrightarrow H_{\text{FD}}$ ($\forall$ harmonic cocycle is a.p.)

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Groups with $H_{\text{FD}}$

**Theorem (Shalom 2004)**

Amenable + $H_{\text{FD}}$ is a quasi-isometry invariant.

**Examples of groups with $H_{\text{FD}}$ (Shalom 2004)**

Polycyclic groups, $\text{BS}(1, n)$, Lamplighter $\mathbb{Z} \wr (\mathbb{Z}/2)$, Kazhdan $(T)$, ...

**Conjecture (Gromov ?): Virtual polycyclicity is a QI invariant.**

(Malcev–Mostow Theorem: $G$ is v-polycyclic iff it is virtually isomorphic to a (uniform) lattice in a simply connected solvable Lie group.)

**Non-examples of groups with $H_{\text{FD}}$**

$\mathbb{Z}^3 \wr (\mathbb{Z}/2)$, $\mathbb{Z} \wr \mathbb{Z}$, f.g. (amenable) torsion/simple groups, $F_r$, ...

**Open Problem**

$\mathbb{Z}^2 \wr (\mathbb{Z}/2)$, $\text{EL}(n, R)$ for nonunital $R$, ...
Criterion for a cocycle to be a.p./w.m. via RW

\[ X_n = s_1 \cdots s_n, \quad s_i \text{-i.i.d.} \]

\( b \) harmonic, i.e.,
\[ \sum_t \mu(t)b(gt) = b(g) + \sum_t \mu(t)\pi_g b(t) = b(g) \text{ for } \forall g \]
\[ \iff b(X_n) \text{ martingale i.e., } \mathbb{E}[b(X_{n+1}) | X_1, \ldots, X_n] = b(X_n) \]
\[ \implies \mathbb{E}[\|b(X_n)\|^2] = n\|b\|^2 \text{ for } \forall n \]

Proposition (Martingale Central Limit Theorem)

\[ \forall v \in \mathcal{H} \quad \langle \frac{1}{\sqrt{n}} b(X_n), v \rangle \xrightarrow{\text{dist}} N(0, q(v)) \]

Compute \( q(v) = \lim_n \mathbb{E}[\langle \frac{1}{\sqrt{n}} b(X_n), v \rangle^2] = \lim_n \frac{1}{n} \mathbb{E}[\langle (b \otimes b)(X_n), v \otimes v \rangle] \).

\[ \mathbb{E}[(b \otimes b)(X_n)] = \mathbb{E}[(b \otimes b)(X_{n-1}Z)] \text{ here } Z \text{ is an indep copy of } X_1 \]
\[ = \mathbb{E}[(b \otimes b)(X_n) + (\pi \otimes \pi)(X_{n-1})(b \otimes b)(Z)] \]
\[ = \mathbb{E}[(b \otimes b)(X_{n-1})] + T^{n-1}w \]
\[ = \cdots = (1 + T + \cdots + T^{n-1})w, \]

where \( T = \sum_g \mu(g)(\pi \otimes \pi)(g) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) a self-adjoint contraction and \( w = \sum_t \mu(t)(b \otimes b)(t) \in \mathcal{H} \otimes \mathcal{H} \).
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\[ q(v) = \lim_n \langle \frac{1}{n}(1 + T + \cdots + T^{n-1})w, v \otimes v \rangle \]
\[ = \langle E_T(\{1\})w, v \otimes v \rangle = \langle Sv, v \rangle, \]

where \( E_T(\{1\}) \) coincides with the orth projection onto \((\mathcal{H} \otimes \mathcal{H})(\pi \otimes \pi)(G)\) and \( S \) is the Hilbert–Schmidt op assoc with \( E_T(\{1\})w \in (\mathcal{H} \otimes \mathcal{H})(\pi \otimes \pi)(G). \)

\( \Rightarrow \) \( S \) is positive, compact, and \( \text{Ad } \pi(G) \)-invariant.
Criterion for a cocycle to be a.p./w.m. via RW, cont’d

Proposition (Martingale Central Limit Theorem)

\[ \forall \nu \in \mathcal{H} \quad \left\langle \frac{1}{\sqrt{n}} b(X_n), \nu \right\rangle \xrightarrow{\text{dist}} N(0, q(\nu)) \]

where \( q(\nu) = \left\langle S\nu, \nu \right\rangle \) for some positive compact Ad \( \pi(G) \)-inv operator \( S \).

Eigenspaces of \( S \) with nonzero eigenvalues are \( \pi(G) \)-invariant finite-dimensional subspaces of \( \mathcal{H} \).

\( \lambda_1, \lambda_2, \ldots \) nonzero eigenvalues; \( \nu_1, \nu_2, \ldots \) orthonormal eigenvectors

\[ \leadsto \left\langle \frac{1}{\sqrt{n}} b(X_n), \nu_i \right\rangle \rightarrow \lambda_i^{1/2} g_i, \quad g_i \text{ i.i.d. } N(0, 1) \]

\[ \| \frac{1}{\sqrt{n}} b(X_n) \|^2 = \sum_i \left| \left\langle \frac{1}{\sqrt{n}} b(X_n), \nu_i \right\rangle \right|^2 + (\text{missing part due to ker } S) \]

Theorem (Erschler–O. 2016)

\[ \forall \text{ harmonic cocycle } b \quad \| \frac{1}{\sqrt{n}} b(X_n) \|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta \]

where \( \theta \geq 0 \) is the constant s.t. \( \sum_i \lambda_i + \theta = \| b \|^2 \).

\[ b = b_{\text{a.p.}} \oplus b_{\text{w.m.}} \text{ with } \| b_{\text{a.p.}} \|^2 = \sum_i \lambda_i \text{ and } \| b_{\text{w.m.}} \|^2 = \theta. \]
Diffusive random walk and $H_{FD}$

**Theorem (Erschler–O. 2016)**

\[ \| \frac{1}{\sqrt{n}} b(X_n) \|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta \quad \text{and so} \quad b = b_{\text{a.p.}} \iff \theta = 0. \]

Since $\| b(x) \| \leq K|x|$ for $K = \max_{g \in S} \| b(g) \|$, one obtains

**Corollary**

$G$ has $H_{FD}$, i.e., $b = b_{\text{a.p.}}$ for $\forall b$ harmonic ($\sim \exists \text{ v.-}\mathbb{Z}$ quotient), provided

\[ \forall c > 0 \quad \limsup_{n \to \infty} \mathbb{P}(|X_n| \leq c \sqrt{n}) > 0 \]

One has $0 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$. How about the opposite implications?

0. **Controlled Følner condition:** $\exists \delta, K > 0$ such that for infinitely many $n$

\[ \exists F \subset \text{Ball}(n) \text{ satisfying } |\mathcal{N}_{\delta n}(F)| \leq K |F| \]

Polycyclic groups as well as poly.gro. groups satisfy this (R. Tessera).

1. $\forall c > 0 \quad \limsup_{n \to \infty} \mathbb{P}(\max_{k=1,\ldots,n} |X_k| \leq c \sqrt{n}) > 0$

3. $\exists C > 0 \quad \limsup_{n \to \infty} \mathbb{P}(|X_n| \leq C \sqrt{n}) > 0$

J. Brieussel & T. Zheng (2017): $H_{FD} \not\Rightarrow 3$. 

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Theorem (Mok ’95, Korevaar–Schoen ’97, Shalom ’99 ◀)

If $G$ is a f.g. infinite amenable group, then $\exists$ non-zero harmonic cocycle.

Proof in the case $G$ is amenable and $\mu^{*1/2}$ exists.

Consider $c_m(g) := \mu^{*m/2} - g\mu^{*m/2} \in \ell_2(G)$ and $b_m(g) := c_m(g)/\|c_m\|$. 

\[
\|c_m\|^2 = \sum_g \mu(g)\|\mu^{*m/2} - g\mu^{*m/2}\|^2 = 2(\mu^m(e) - \mu^{m+1}(e))
\]

Fix a free ultrafilter $\mathcal{U}$ and put $b_\mathcal{U}(g) := [b_m(g)]_m \in \ell_2(G)^\mathcal{U}$. Then, $b_\mathcal{U}$ is a normalized cocycle, which is moreover harmonic, since 

\[
\|\sum_g \mu(g)c_m(g)\|^2 = \|\mu^{*m/2} - \mu^{*m/2+1}\|^2
\]

\[
= \mu^m(e) - 2\mu^{m+1}(e) + \mu^{m+2}(e) \ll \|c_m\|^2.
\]

$b_\mathcal{U}$ may depend on the choice of an ultrafilter $\mathcal{U}$.

Thus, if $G$ is a f.g. amenable without $v\mathbb{Z}$ quotient, then one has

\[
\sup_{\mathcal{U}} \lim_{n \to \infty} \mathbb{E} \left| \frac{\|b_\mathcal{U}(X_n)\|^2}{n} - 1 \right|^2 = \lim_{n \to \infty} \limsup_{m \to \infty} \mathbb{E} \left| \frac{\mu^m(X_n) - \mu^{m+n}(e)}{\mu^m(e) - \mu^{m+n}(e)} \right| = 0.
\]