Finite Dimensional Representations from Random Walks

Narutaka OZAWA （小澤 登高）

based on a joint work with A. Erschler

Research Institute for Mathematical Sciences, Kyoto University

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A. Erschler and N. Ozawa; Finite-dimensional representations constructed from random walks. Comment. Math. Helv., to appear. arXiv:1609.08585

Introduction: Gromov's theorem

We connect **geometry** (quasi-isometry, random walks, etc.) of a finitely generated group to **algebra** ($\exists$ a virtually-$\mathbb{Z}$ quotient) of it via **analysis**.

$G = \langle S \rangle$ with finite generating subset $S = S^{-1}$

$\rightsquigarrow |x| := \min\{n : x \in S^n\}$ and $d(x, y) := |x^{-1}y|$

$\gamma_G(n) := |\,\mathrm{Ball}(n)| = |\{x : |x| \le n\}|$ growth

These are (up to a certain equivalence) indep. of S, in fact a QI invariant.

A map $f\colon (X, d_X) \to (Y, d_Y)$ is a quasi-isometry (QI) if $\exists K, L > 0$

$\frac{1}{K} d_X(x, y) - L \le d_Y(f(x), f(y)) \le K d_X(x, y) + L$ and $Y \subset \mathcal{N}_L(f(X))$.

For example, if $G_0 \le_{\text{finite index}} G$, then $G_0 \cong_{\mathrm{QI}} G$.

Theorem (Gromov 1981)

If G has polynomial growth ($\exists d\ \gamma_G(n) \preceq n^d$), then it is virtually nilpotent.

Proof: By induction on d.

It suffices to show $\exists$ a virtually-$\mathbb{Z}$ quotient: $G \ge_{\text{finite index}} G_0 \overset{q}{\twoheadrightarrow} \mathbb{Z}$.

$\because \ker q$ is f.g. and has polynomial growth of degree $\le d - 1$. $\square$

Further motivation: Grigorchuk's Conjecture

$\gamma_G(n) := |\,\mathrm{Ball}(n)| = |\{x : |x| \leq n\}|$

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Grigorchuk's Gap Conjecture (1990)

If $\gamma_G(n) \ll e^{\sqrt{n}}$ (or $\exp n^{0.01}$), then G has polynomial growth.

There are several empirical evidences, but here's an optimistic heuristic:
Fix a symmetric probability measure μ with $\operatorname{supp}\mu = S$ and consider the random walk $X_n = s_1 \cdots s_n$, $\quad s_i$ μ-i.i.d.
If $\gamma_G(n) \ll e^{\sqrt{n}}$, then the μ-RW is maybe diffusive, e.g., $\mathbb{E}[|X_n|] \preceq \sqrt{n}$.
In turn, as we will see, this probably implies G has a virtually-$\mathbb{Z}$ quotient.

How to find a v-$\mathbb{Z}$ quotient ?

$\cdots$ It suffices to find a finite-dim repn with an infinite image.

Theorem (Tits Alternative 1972)

If $G \leq \mathrm{GL}(n, F)$ is a finitely generated infinite amenable subgroup, then G is virtually solvable and has a virtually-$\mathbb{Z}$ quotient.

Shalom's idea (2004): Use reduced cohomology to get a non-trivial finite-dimensional representation.

Given an orthogonal repn $\pi\colon G \curvearrowright \mathcal{H}$ (which need not be finite-dim)

$b\colon G \to \mathcal{H}$ cocycle $\overset{\text{def}}{\Leftrightarrow}$ $b(gt) = b(g) + \pi_g b(t)$ for $\forall g, t \in G$

e.g., coboundary $b_v(g) = v - \pi_g v$, where $v \in \mathcal{H}$

harmonic $\overset{\text{def}}{\Leftrightarrow}$ $\sum_t b(gt)\mu(t) = b(g)$ for $\forall g \in G$ (or just $g = e$)

e.g., $\forall$ harm. cob. is zero: $v - \sum \mu(g)\pi_g v = 0 \rightsquigarrow v = \pi(g)v$ for $\forall g$.

$Z^1(G, \pi) := \{\text{cocycles}\}$ is a Hilbert space w.r.t.

$$\|b\|^2 := \sum_t \|b(t)\|^2 \mu(t)$$

$$\overline{H^1}(G, \pi) := Z^1(G, \pi) \,/\, \overline{B^1(G, \pi)} \cong B^1(G, \pi)^\perp = \{\text{harmonic cocycles}\}$$

Shalom's property H_{FD}

Theorem (Mok '95, Korevaar–Schoen '97, Shalom '99 ▶)

If G is a f.g. infinite amenable group, then $\exists\pi$ s.t. $\overline{H^1}(G,\pi) \neq 0$.

In general, π decomposes as

$$\pi = \overbrace{\bigoplus(\text{fd repns})}^{\text{almost periodic repn}} \oplus \overbrace{(\text{no nonzero fd subrepns})}^{\text{weakly mixing repn}}$$

Accordingly

$$b = b_{\mathrm{a.p.}} \oplus b_{\mathrm{w.m.}}$$

Obs: G f.g. amenable $\exists b$ harmonic with $b_{\mathrm{a.p.}} \neq 0 \Rightarrow \exists$ v-$\mathbb{Z}$ quotient.

$\because$ If $|\pi(G)| = \infty$, then use Tits Alternative.
If $|\pi(G)| < \infty$, then $\ker\pi \leq_{\mathrm{f.i.}} G$ and $b|_{\ker\pi}$ is a $\neq 0$ additive character.

Shalom (2004) :

$$\begin{tikzcd} \text{v-nilpotent} \arrow[r, Rightarrow] & H_{\mathrm{FD}} \\ \text{polynomial growth} \arrow[u, Rightarrow, "\text{Gromov}"] \arrow[ur, Rightarrow, "\text{Oz. '15}"'] & \end{tikzcd}$$

($\forall$ harmonic cocycle is a.p.)

Groups with H_{FD}

Theorem (Shalom 2004)

Amenable $+$ H_{FD} is a quasi-isometry invariant.

Examples of groups with H_{FD} (Shalom 2004)

Polycyclic groups, $\mathrm{BS}(1,n)$, Lamplighter $\mathbb{Z} \wr (\mathbb{Z}/2)$, Kazhdan (T),...

Conjecture (Gromov ?): Virtual polycyclicity is a QI invariant.

(Malcev–Mostow Theorem: G is v-polycyclic iff it is virtually isomorphic to a (uniform) lattice in a simply connected solvable Lie group.)

Non-examples of groups with H_{FD}

$\mathbb{Z}^3 \wr (\mathbb{Z}/2)$, $\mathbb{Z} \wr \mathbb{Z}$, f.g. (amenable) torsion/simple groups, F_r,...

Open Problem

$\mathbb{Z}^2 \wr (\mathbb{Z}/2)$, $\mathrm{EL}(n, R)$ for nonunital R, ...

Criterion for a cocycle to be a.p./w.m. via RW

$X_n = s_1 \cdots s_n, \quad s_i$ μ-i.i.d

b harmonic, i.e., $\sum_t \mu(t) b(gt) = b(g) + \sum_t \mu(t)\pi_g b(t) = b(g)$ for $\forall g$

$$\Leftrightarrow b(X_n) \text{ martingale i.e., } \mathbb{E}[\, b(X_{n+1}) \mid X_1, \ldots, X_n \,] = b(X_n)$$

$$\Rightarrow \mathbb{E}[\, \|b(X_n)\|^2 \,] = n\|b\|^2 \ \text{ for } \forall n$$

Proposition (Martingale Central Limit Theorem)

$$\forall v \in \mathcal{H} \qquad \langle \tfrac{1}{\sqrt{n}} b(X_n), v\rangle \xrightarrow{\text{dist}} N(0, q(v))$$

Compute $q(v) = \lim_n \mathbb{E}[\, \langle \frac{1}{\sqrt{n}} b(X_n), v\rangle^2 \,] = \lim_n \frac{1}{n} \mathbb{E}[\, \langle (b \otimes b)(X_n), v \otimes v\rangle \,]$.

$$\begin{aligned}
\mathbb{E}[(b \otimes b)(X_n)] &= \mathbb{E}[\, (b \otimes b)(X_{n-1}Z) \,] \qquad \text{here } Z \text{ is an indep copy of } X_1 \\
&= \mathbb{E}\Big[\, (b \otimes b)(X_n) + (\pi \otimes \pi)(X_{n-1})(b \otimes b)(Z) \,\Big] \\
&= \mathbb{E}[(b \otimes b)(X_{n-1})] + T^{n-1} w \\
&= \cdots = (1 + T + \cdots + T^{n-1}) w,
\end{aligned}$$

where $T = \sum_g \mu(g)(\pi \otimes \pi)(g) \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ a self-adjoint contraction and $w = \sum_t \mu(t)(b \otimes b)(t) \in \mathcal{H} \otimes \mathcal{H}$.

Criterion for a cocycle to be a.p./w.m. via RW

Proposition (Martingale Central Limit Theorem)

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$$\begin{aligned}
q(v) &= \lim_n \langle \tfrac{1}{n}(1 + T + \cdots + T^{n-1}) w, v \otimes v \rangle \\
&= \langle E_T(\{1\}) w, v \otimes v \rangle = \langle S v, v \rangle,
\end{aligned}$$

where $E_T(\{1\})$ coincides with the orth projection onto $(\mathcal{H} \otimes \mathcal{H})^{(\pi \otimes \pi)(G)}$ and S is the Hilbert–Schmidt op assoc with $E_T(\{1\}) w \in (\mathcal{H} \otimes \mathcal{H})^{(\pi \otimes \pi)(G)}$.

$\rightsquigarrow$ S is positive, compact, and $\mathrm{Ad}\, \pi(G)$-invariant.

Criterion for a cocycle to be a.p./w.m. via RW, cont'd

Proposition (Martingale Central Limit Theorem)

$\forall v \in \mathcal{H} \quad \langle \frac{1}{\sqrt{n}} b(X_n), v \rangle \xrightarrow{\text{dist}} N(0, q(v))$
where $q(v) = \langle Sv, v \rangle$ for some positive compact $\operatorname{Ad}\pi(G)$-inv operator S.

Eigenspaces of S with nonzero eigenvalues are $\pi(G)$-invariant finite-dimensional subspaces of $\mathcal{H}$.
$\lambda_1, \lambda_2, \ldots$ nonzero eigenvalues; $v_1, v_2, \ldots$ orthonormal eigenvectors

$$\rightsquigarrow \langle \frac{1}{\sqrt{n}} b(X_n), v_i \rangle \to \lambda_i^{1/2} g_i, \quad g_i \text{ i.i.d. } N(0,1)$$

$$\|\frac{1}{\sqrt{n}} b(X_n)\|^2 = \sum_i |\langle \frac{1}{\sqrt{n}} b(X_n), v_i \rangle|^2 + (\text{missing part due to } \ker S)$$

Theorem (Erschler–O. 2016)

$\forall$ harmonic cocycle $b \quad \|\frac{1}{\sqrt{n}} b(X_n)\|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta$
where $\theta \geq 0$ is the constant s.t. $\sum_i \lambda_i + \theta = \|b\|^2$.
$b = b_{\mathrm{a.p.}} \oplus b_{\mathrm{w.m.}}$ with $\|b_{\mathrm{a.p.}}\|^2 = \sum_i \lambda_i$ and $\|b_{\mathrm{w.m.}}\|^2 = \theta$.

Diffusive random walk and H_{FD}

Theorem (Erschler–O. 2016)

$\|\frac{1}{\sqrt{n}}b(X_n)\|^2 \xrightarrow{\text{dist}} \sum_i \lambda_i g_i^2 + \theta$ and so $\quad b = b_{\mathrm{a.p.}} \Leftrightarrow \theta = 0.$

Since $\|b(x)\| \leq K|x|$ for $K = \max_{g\in S}\|b(g)\|$, one obtains

Corollary

G has H_{FD}, i.e., $b = b_{\mathrm{a.p.}}$ for $\forall b$ harmonic ($\rightsquigarrow \exists$ v.-$\mathbb{Z}$ quotient), provided

2 $\quad \forall c > 0 \quad \limsup_{n\to\infty} \mathbb{P}(|X_n| \leq c\sqrt{n}) > 0$

One has 0 $\overset{\text{EZ}}{\Rightarrow}$ 1 $\Rightarrow$ 2 $\Rightarrow$ 3. How about the opposite implications?

0 Controlled Følner condition: $\exists\delta$, $K > 0$ such that for infinitely many n

$$\exists F \subset \mathrm{Ball}(n) \text{ satisfying } |\mathcal{N}_{\delta n}(F)| \leq K|F|$$

Polycyclic groups as well as poly.gro. groups satisfy this (R. Tessera).

1 $\forall c > 0 \quad \limsup_{n\to\infty} \mathbb{P}(\max_{k=1,\ldots,n}|X_k| \leq c\sqrt{n}) > 0$

3 $\exists C > 0 \quad \limsup_{n\to\infty} \mathbb{P}(|X_n| \leq C\sqrt{n}) > 0$

J. Brieussel & T. Zheng (2017): $H_{\mathrm{FD}} \not\Rightarrow$ 3.

Epilogue: Beyond H_{FD}?

Theorem (Mok '95, Korevaar–Schoen '97, Shalom '99 ◀)

If G is a f.g. infinite amenable group, then $\exists$ non-zero harmonic cocycle.

Proof in the case G is amenable and $\mu^{*1/2}$ exists.

Consider $c_m(g) := \mu^{*m/2} - g\mu^{*m/2} \in \ell_2(G)$ and $b_m(g) := c_m(g)/\|c_m\|$.

$\rightsquigarrow \|c_m\|^2 = \sum_g \mu(g)\|\mu^{*m/2} - g\mu^{*m/2}\|_2^2 = 2(\mu^{*m}(e) - \mu^{*m+1}(e))$

Fix a free ultrafilter $\mathcal{U}$ and put $b_{\mathcal{U}}(g) := [\, b_m(g)\,]_m \in \ell_2(G)^{\mathcal{U}}$.

Then, $b_{\mathcal{U}}$ is a normalized cocycle, which is moreover harmonic, since

$$\begin{aligned}\|\textstyle\sum_g \mu(g)c_m(g)\|^2 &= \|\mu^{*m/2} - \mu^{*m/2+1}\|_2^2 \\ &= \mu^{*m}(e) - 2\mu^{*m+1}(e) + \mu^{*m+2}(e) \ll \|c_m\|^2.\end{aligned}$$

□

⚠ $b_{\mathcal{U}}$ may depends on the choice of an ultrafilter $\mathcal{U}$.

Thus, if G is a f.g. amenable without v-$\mathbb{Z}$ quotient, then one has

$$\sup_{\mathcal{U}} \lim_{n\to\infty} \mathbb{E}\left|\frac{\|b_{\mathcal{U}}(X_n)\|^2}{n} - 1\right|^2 = \lim_{n\to\infty} \limsup_{m\to\infty} \mathbb{E}\left|\frac{\mu^{*m}(X_n) - \mu^{*m+n}(e)}{\mu^{*m}(e) - \mu^{*m+n}(e)}\right| = 0.$$