An entropic proof of cutoff on Ramanujan graphs

Narutaka Ozawa (小澤登高)

RIMS, Kyoto University

Wales MPPM Seminar

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Abstract: I will talk about a simple **functional analysis** proof of E. Lebetzky and Y. Peres's theorem (2016) that the simple random walk on a Ramanujan graph exhibits cutoff phenomenon.
Based on my preprint `arXiv:2009.00837`.

G a finite d-regular conn. non-bipartite graph

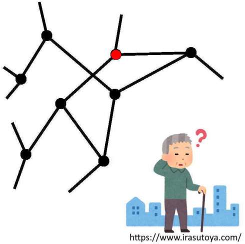
$X^t : (\Omega, \mathbb{P}) \to G$ simple random walk (SRW) on G starting at $X^0 = x_0 \in G$

$\mathbb{P}[X^t = x] = \sum_y P(x,y)\mathbb{P}[X^{t-1} = y]$

$P(x,y) = 1/d$ if $x \sim y$, $= 0$ otherwise

$\mu^t(x) = \mathbb{P}[X^t = x]$ the distribution of X^t

$\mu^t \to \pi$ the uniform distribution on G



Example (of the kind I'm most interested in)

$\Gamma = \mathrm{SL}(3,\mathbb{Z})$ res. finite grp with a sym. gen. subset $S = \{I \pm E_{i,j} : i \neq j\}$

$G_n = \mathrm{SL}(3, \mathbb{Z}/n\mathbb{Z})$; $x \sim xs$ for $x \in G_n$ and $s \in S$.

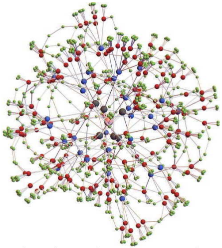
$X^t = s_1 \cdots s_t$ with s_i indep. uniformly distributed on S; $\mu^t = (\mu_S)^{*t}$

$\{G_n\}_n$ is an **expander** family of $|S|$-regular **transitive** graphs, $|G_n| \to \infty$

$\Gamma \twoheadrightarrow G_n$ is injective on the ball of radius $c \log n$ (**injectivity radius**)

When do the random walkers X^t notice they live in a finite world?

Throughout: G a finite d-regular conn. non-bipartite graph, d fixed
1 is a simple eigenvalue of P with the eigenvector $\mathbf{1}$.
$\rho_G := \|P|_{\{\mathbf{1}\}^\perp \cap \ell_2 G}\| < 1$ the reduced spectral radius
We say a family $\{G\}$ of graphs is an expander family if it has **uniform spectral gap**: $\sup_G \rho_G < 1$.



Lubetzky and Peres, GAFA '16

Expander property (Dodziuk '84 and Alon–Milman '85)
For every $A \subset G$, $|\partial A| \geq (1-\rho_G)|A|(1-\frac{|A|}{|G|})$.

$\rightsquigarrow$ If 1% of the population are infected by Disease X and they spread it, 99% will be infected shortly.

- Random d-regular graphs are expanders (Barzdin–Kolmogorov '67).
- Finite quotients of a property (T) group are expanders (Margulis '74).
- Finite quotients of an amenable group are never expanders.

A sample theorem on expanders (by Gillman '98)
X^t SRW on G with the initial distribution $X^0 \sim \pi$ (**NB!** NOT $X^0 = x_0$).
Then for any $f \in \ell_\infty G$ with $m = \frac{1}{|G|}\sum_x f(x)$ and $\|f\|_\infty \leq 1$, one has

$$\mathbb{P}[\,|\tfrac{1}{T}\sum_{t=1}^{T} f(X^t) - m| > \varepsilon] < 2\exp(-\tfrac{1-\rho}{64}\varepsilon^2 T).$$

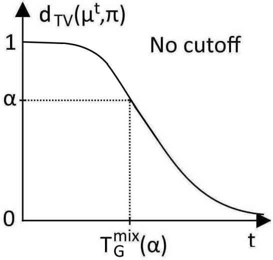
Cutoff phenomena

When do the random walkers X^t notice they live in a finite world?

$$d_{\mathrm{TV}}(\eta,\zeta) = \max\{|\eta(A)-\zeta(A)| : A \subset G\} = \tfrac{1}{2}\|\eta-\zeta\|_1 \in [0,1]$$

$T_G^{\mathrm{mix}}(\alpha) := \min\{t : d_{\mathrm{TV}}(\mu^t,\pi) < \alpha\}$ **total variation mixing time**

Do the random walkers X^t notice it simultaneously?



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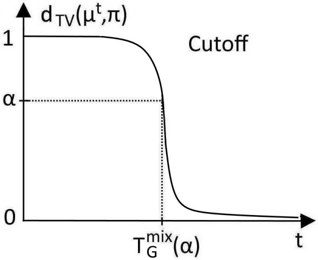

We say a family $\{G\}$ of graphs exhibits cutoff (Aldous–Diaconis '86) if

$$\forall \alpha,\alpha' \in (0,1) \qquad \lim_{|G|\to\infty} \frac{T_G^{\mathrm{mix}}(\alpha)}{T_G^{\mathrm{mix}}(\alpha')} = 1.$$

Conjecture (Peres): **Transitive** expander graphs should exhibit cutoff.

Theorem (Lubetzky–Peres GAFA '16)

Any family of **asymptotically Ramanujan** graphs exhibits cutoff.

- Alternative proofs by Hermon '17 and by Bordenave–Lacoin '18.

Q. How good can expanders be?

A. The **Alon–Boppana bound** for any graphs: $\liminf_{|G|\to\infty} \rho_G \geq \frac{2\sqrt{d-1}}{d} =: \rho_d$.

- ρ_d is the **spectral radius** of the SRW on the d-regular tree (Kesten '59).

We say a family $\{G\}$ is asymptotically Ramanujan if $\lim_{|G|\to\infty} \rho_G = \rho_d$.

- First examples were discovered by Lubotzky, Phillips, and Sarnak '88.
- Random d-regular graphs are asymptotically Ramanujan (Friedman '08).

Q. How fast can random walks mix?

A. Entropic bound: $\forall\alpha \quad \liminf_{|G|\to\infty} \frac{T_G^{\mathrm{mix}}(\alpha)}{\log|G|} \geq \frac{1}{h_d}$ with $h_d = \frac{d-2}{d}\log(d-1)$.

- h_d is the **asymptotic entropy** of the SRW on the d-regular tree.

Cutoff at the entropic time

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- h_d is the asymptotic entropy of the SRW on the d-regular tree.

$\because$ μ^t is concentrated on the ball of radius $\approx \frac{d-2}{d} t$ which has cardinality at most $d(d-1)^{\frac{d-2}{d}t-1}$ which is proportional to $|G|$ at $t = T_G^{\text{mix}}(\alpha)$.

Theorem (Lubetzky–Peres GAFA '16)

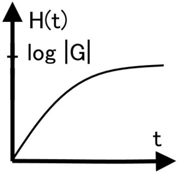
Asymptotically Ramanujan graphs satisfy $\forall \alpha \quad \lim_{|G|\to\infty} \frac{T_G^{\text{mix}}(\alpha)}{\log|G|} = \frac{1}{h_d}$.

Shannon entropy: $H(\nu) := -\sum_x \nu(x)\log\nu(x)$

Entropy growth: $H(t) := H(\mu^t)$ ($\nearrow \log|G|$ if $|G|<\infty$)

Asymptotic entropy: $h := \lim_t \frac{1}{t}H(t)$ ($=0$ if $|G|<\infty$)

SRW on the d-regular tree: $h = h_d = \frac{d-2}{d}\log(d-1)$



Shannon–McMillan–Breiman theorem (Derriennic, Kaimanovich–Vershik):

$$\frac{1}{t}\left|-\log\mu^t(X^t) - H(t)\right| \to 0 \text{ a.s.}$$

Theorem (Entropic characterization of cutoff)

An expander family $\{G\}$ exhibits cutoff iff $\forall\delta>0\ \forall\varepsilon>0$,

one has $H(T_G^{\text{mix}}(1-\varepsilon)) > (1-\delta)\log|G|$ for G large enough.

$\because \forall\nu\ \forall t \le T_G^{\text{mix}}(\varepsilon)\quad H(P\nu)-H(\nu) \ge \frac{1}{4}(1-\rho_G)d_{\text{TV}}(\nu,\pi)^2 > c$ $\square$

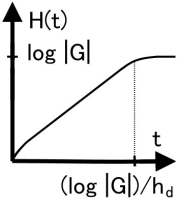
Corollary (Quantitative SMB implies cutoff)

Suppose $\forall\delta>0\ \forall\varepsilon>0\ \exists T_0$ s.t. $\forall G\ \forall t\ge T_0$

$$\mu^t(\mathcal{N}_{\delta\log|G|}(\{x\in G : -\log\mu^t(x) < H(t)+\delta t\})) > 1-\varepsilon.$$

Then $\{G\}$ exhibits cutoff.

By Theorem and the entropic bound for T_G^{mix}, it suffices to show $H(t) - H(t-1) \approx h_d$ for $t \le T_G^{\text{mix}}(1-\varepsilon)$.



$H(t) = -\sum_x \mu^t(x)\log(\mu^t(x)) = -\mathbb{E}[\log \mu^t(X^t)]$

$H(t)-H(t-1) = -\mathbb{E}[\log f_t],\ f_t := \mu^t(X^t)/\mu^{t-1}(X^{t-1})$

$$\langle P(\mu^{t-1})^{1/2}, (\mu^t)^{1/2}\rangle = \sum_{x,y} P(y,x)\mu^{t-1}(x)\left(\frac{\mu^t(y)}{\mu^{t-1}(x)}\right)^{1/2} = \mathbb{E}[f_t^{1/2}].$$

Hence $\mathbb{E}[f_t^{1/2}] \le \|P(\mu^{t-1})^{1/2}\|_2 \le \rho_G + \delta(\varepsilon)$ for $t \le T_G^{\text{mix}}(1-\varepsilon)$.

$\tilde{X}^t$ SRW on the d-regular tree that covers X^t: $f_t \approx \mathbb{E}[\tilde{f}_t \mid X^{t-1}, X^t]$

By concavity of the square root, $\mathbb{E}[\tilde{f}_t^{1/2}] \le \mathbb{E}[f_t^{1/2}] \le \rho_d + \delta(G,\varepsilon)$.

$$\tilde{f}_t \approx \begin{cases} d-1 & \text{with prob } 1/d \text{ (when the step } \tilde{X}_t \text{ is toward the origin)} \\ 1/(d-1) & \text{with prob } (d-1)/d \end{cases}$$

$-\mathbb{E}[\log \tilde{f}_t] \approx -\frac{1}{d}\log(d-1) + \frac{d-1}{d}\log(d-1) = \frac{d-2}{d}\log(d-1) = h_d$

$\mathbb{E}[\tilde{f}_t^{1/2}] \approx \frac{1}{d}\sqrt{d-1} + \frac{d-1}{d}\frac{1}{\sqrt{d-1}} = \frac{2\sqrt{d-1}}{d} = \rho_d$ BINGO!

$\rightsquigarrow\ \mathbb{E}[f_t^{1/2}] \approx \mathbb{E}[\tilde{f}_t^{1/2}]\ \rightsquigarrow\ f_t \approx \tilde{f}_t\ \rightsquigarrow\ -\mathbb{E}[\log f_t] \approx h_d.$ $\square$

Kaimanovich and Vershik's proof of the Shannon–McMillan–Breiman theorem for the SRW on a (infinite) transitive graph:

$$\begin{aligned}\log \mu^t(X^t) &= \log \frac{P^t(X^t, X^0)}{P^{t-1}(X^t, X^1)} + \cdots + \log \frac{P^{t-k}(X^t, X^k)}{P^{t-k-1}(X^t, X^{k+1})} + \cdots \\ &= \log g_t + \cdots + \log S^k(g_{t-k}) + \cdots ,\end{aligned}$$

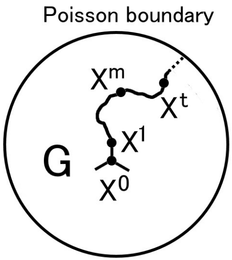
where

$$g_t := \frac{P^t(X^t, X^0)}{P^{t-1}(X^t, X^1)} = \frac{1/d}{\mathbb{P}[\, X^1 \mid X^t \,]}$$

and S is the time (Bernoulli) shift operator on $(\Omega, \mathbb{P}) = \prod_{\mathbb{N}}(\{1, \ldots, d\}, \mu_d)$.
By the MCT, $g_t \to g_\infty$ and

$$\forall \varepsilon > 0 \; \exists m \text{ s.t. } g_\infty \approx_\varepsilon \mathbb{E}[\, g_\infty \mid X_1, \ldots, X_m \,].$$

Hence the CLT implies $-\frac{1}{t} \log \mu^t(X^t) \to h$ a.s.



(Naive) Conjecture: For any transitive expander graph G, one has $g_t \approx \mathbb{E}[\, g_t \mid X_1, \ldots, X_m \,]$ with $m = o(\log |G|)$.