Noncommutative real algebraic geometry of Kazhdan’s property (T)

Narutaka OZAWA (小澤 登高)

Research Institute for Mathematical Sciences, Kyoto University

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Noncommutative Real Algebraic Geometry

NCRAG: equations and inequalities in an algebra over \( \mathbb{R} \) (or \( \mathbb{C}, \ldots \)).

**Hilbert’s 17th Pb:** \( f \in \mathbb{R}[x_1, \ldots, x_d], f \geq 0 \) on \( \mathbb{R}^d \)
(E. Artin 1927) \( \implies f = \sum g_i^2 \) for some \( g_1, \ldots, g_n \in \mathbb{R}(x_1, \ldots, x_d) \).

\( \mathcal{A} \) a unital \(*\)-algebra together with \( \mathcal{A}^+ \). E.g., \( \mathbb{M}_n(\mathbb{R}), \mathcal{B}(\mathcal{H}), \mathbb{R}[\Gamma], \ldots \)

\[
\mathcal{A}^h = \{ x \in \mathcal{A} : x^* = x \} \\
\cup \\
\mathcal{A}^+ = \Sigma^2 \mathcal{A} := \{ \sum_i x_i^* x_i \}
\]

\( ||x|| := \inf\{ R \geq 0 : x^* x \leq R^2 1 \} \in [0, \infty] \).

\( \ni \) This is a \( \mathcal{C}^* \)-norm.

We call \( \mathcal{A} \) a **semi-pre-\( \mathcal{C}^* \)**-algebra if all elements are bounded.

\( \mathcal{C}^*(\mathcal{A}) \): the univ env \( \mathcal{C}^* \)-alg for the positive \(*\)-rep’s on Hilbert spaces.

\( \ni \) \( \mathcal{C}^*(\mathbb{R}[\Gamma]) = \mathcal{C}^*(\Gamma) \) the full group \( \mathcal{C}^* \)-algebra.

**Theorem (Hahn–Banach + Gelfand–Naimark–Segal)**

The “inclusion” \( \iota : \mathcal{A} \hookrightarrow \mathcal{C}^*(\mathcal{A}) \) is isometric and \( \iota(a) \geq 0 \iff a \in \mathcal{A}^+ \).
Topology on real vector spaces

\[ \mathcal{A} : \text{a semi-pre-\(C^{*}\)-algebra with } \mathcal{A}^+ = \Sigma^2 \mathcal{A} = \{ \sum_i x_i^* x_i \}. \]

**Theorem (HB+GNS)**

The "inclusion" \( \iota : \mathcal{A} \hookrightarrow \mathcal{C}^*(\mathcal{A}) \) is isometric and \( \iota(a) \geq 0 \iff a \in \overline{\mathcal{A}^+}. \)

For every \( \mathbb{R} \) vector space \( V \), we consider the finest locally convex topology. For a cone \( V^+ \subset V \),

\[ e \in \text{int } V^+ \text{ if } \forall v \in V \exists R > 0 \text{ s.t. } v + Re \in V^+, \text{ and} \]

\[ \overline{V^+} = \{ v \in V : v + \epsilon e \in V^+ \text{ for } \forall \epsilon > 0 \}. \]

(archimedean closure of \( V^+ \).)

\( \mathcal{A} \) is a semi-pre-\( C^{*} \)-algebra \( \iff 1 \in \text{int } \mathcal{A}^+ \)

\[ \implies \overline{\mathcal{A}^+} = \{ a \in \mathcal{A}^h : a + \epsilon 1 \in \mathcal{A}^+ \text{ for } \forall \epsilon > 0 \}. \]

**Problem:** Is \( \Sigma^2 \mathbb{R}[^\Gamma] \) (or \( \Sigma^2 \mathbb{C}[^\Gamma] \)) closed?

- **YES** if \( \Gamma = \mathbb{Z} \) (Fejér 1916), \( F_d \) (Schmüdgen 80s), \( \mathbb{Z}^2 \) (Scheiderer 06).
- **NO** if \( \Gamma \supset \mathbb{Z}^3 \) (Scheiderer 00).
- How about hyperbolic groups? \( F_d \times F_d \)?
Kazhdan’s property (T)

\( \Gamma \) has (T)\( \iff \forall \text{ orth rep } (\pi, \mathcal{H}) \text{ and } \forall \nu \in \mathcal{H}, \text{ if } \nu \text{ is almost } \Gamma\text{-invariant, then } \nu \text{ is close to a } \Gamma\text{-invariant vector.} \)

\( \iff \exists S \subset \Gamma \text{ finite, } \exists K > 0 \text{ such that } \)

\[
d(\nu, \mathcal{H}^\Gamma) \leq K \max_{g \in S} ||\nu - \pi(g)\nu||.
\]

\( \sim \Gamma = \langle S \rangle \text{ and } K^{-1} \text{ is called the Kazhdan constant for } (\Gamma, S). \)

Non-examples: infinite amenable groups, groups that act on a tree.

Example (Kazhdan 1967)

\( \text{SL}(3, \mathbb{Z}) \text{ and } S = \{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ldots \}, \text{ or any lattice of a s.c. Lie group of rank } \geq 2. \)

\( \sim \)\( X_m := \text{Cayley}(\text{SL}(3, \mathbb{Z}/m\mathbb{Z}), S) \) form a sequence of expanders.

Explicit estimate of the Kazhdan constant: Burger 91 and Shalom 99.

OPEN PROBLEM: \( \text{Aut}(F_d), d \geq 4 \sim \) Product Replacement Algorithm
Laplacian

Let $\Gamma = \langle S \rangle$ with $S$ finite and symmetric.

\[
\Delta := \frac{1}{2|S|} \sum_{g \in S} (1 - g)^*(1 - g) = 1 - \frac{1}{|S|} \sum_{g \in S} g \in \mathbb{R}[\Gamma].
\]

Since $\langle \pi(\Delta) v, v \rangle = \frac{1}{2|S|} \sum \|v - \pi(g)v\|^2$, $v$ is $\Gamma$-invariant iff $\pi(\Delta)v = 0$. $v$ is almost $\Gamma$-inv iff $\pi(\Delta)v \approx 0$.

Hence,

$\Gamma$ has (T) $\iff$ $\exists \kappa > 0$ s.t. $\text{Sp}(\pi(\Delta)) \subset \{0\} \cup [\kappa, \infty)$ for $\forall (\pi, \mathcal{H})$

$\iff \exists \kappa > 0$ s.t. $\Delta^2 - \kappa \Delta \geq 0$ in $C^*(\Gamma)$.

$\iff \exists \kappa > 0$ s.t. $\forall \epsilon > 0 \Delta^2 - \kappa \Delta + \epsilon 1 \in \Sigma^2 \mathbb{R}[\Gamma]$. ❍ No Good! ❍

Theorem (Oz. 2013)

$\Gamma$ has (T) iff $\exists \kappa > 0$ s.t. $\Delta^2 - \kappa \Delta \in \Sigma^2 \mathbb{R}[\Gamma]$ (or $\Sigma^2 \mathbb{Q}[\Gamma]$).

Note that ($\Leftarrow$) is trivial. Explicit formula for $(\text{SL}(3, \mathbb{Z}), S)$ has been found by Netzer and Thom with Matlab in May 2014.

$\Delta^2 - \kappa \Delta = \sum_{i=1}^{n} \xi_i^* \xi_i$ with $\kappa = \frac{1}{120}$, $n = 80$, and $\text{supp } \xi_i \subset (S \cup \{1\})^2$.

$\leadsto$ Kazhdan const $\geq 1/16$. (Previous record by a human $\sim 1/700$.)
Algorithm that detects (T)

Theorem (Oz. 2013)

Γ has (T) iff \( \exists \kappa > 0 \) s.t. \( \Delta^2 - \kappa \Delta = \sum_{i=1}^{n} \xi_i^* \xi_i \) for some \( \xi_1, \ldots, \xi_n \in \mathbb{R}[\Gamma] \).

Corollary (Shalom 00)

Every (T) group is a quotient of finitely presented (T) group.

Corollary

There is a “good” algorithm that detects (T). \( \rightsquigarrow \) Implementation?

Input: \( \Gamma = \langle S \mid R \rangle \). \( \rightsquigarrow Q : F_S \rightarrow \Gamma \).

Step 1: Take \( R > 0 \) and \( \sim \) on \( B_R = B_R(F_S) \) s.t. \( x \sim y \Rightarrow Q(x) = Q(y) \).

Step 2: Use Semi-Definite Program to compute

\[
\sup \{ \kappa \geq 0 : \exists (a_{x,y})_{x,y} \in \mathbb{M}_{B_R}(\mathbb{R})^+ \text{ s.t. } \Delta^2 - \kappa \Delta = \sum a_{x,y} x^{-1} y \mod \sim \}.
\]

If this value is > 0, then \( \Gamma \ has \ (T) \). Else repeat with a larger \( R \)...

\( \heartsuit \) The problem Is(T)? is semidecidable, but not decidable (Silberman), i.e., there is no a priori estimate of \( R \) that works.
Proof of Theorem

\[ \omega : \mathbb{R}[\Gamma] \to \mathbb{R} \] the unit character, \( \omega(\sum a_g g) = \sum a_g. \)

\( I[\Gamma] = \ker \omega, \quad \Delta = \frac{1}{2|S|} \sum_{g \in S} (1 - g)^*(1 - g) \in \Sigma^2 \mathbb{R}[\Gamma] \cap I[\Gamma] = \Sigma^2 I[\Gamma]. \)

**Lemma.** \( \Delta \in \text{int} \Sigma^2 I[\Gamma] \) in \( I[\Gamma]^h. \)

**Proof.**

Lemma claims that for \( \forall f \in I[\Gamma]^h \exists R > 0 \) s.t. \( f \leq R \Delta \) (write \( f \preceq \Delta \)).

Since \( I[\Gamma]^h \) is spanned by \( 2 - x - x^{-1} = (1 - x)^*(1 - x) \),

it suffices to show \( (1 - x)^*(1 - x) \preceq \Delta \) for all \( x \in \Gamma. \)

This is true for \( x \in S \). Moreover, if true for \( x \) and \( y \), then true for \( xy \).

\[
(1 - xy)^*(1 - xy) = (1 - x + x(1 - y))^*(1 - x + x(1 - y)) \preceq \Delta
\]

\[
\leq 2(1 - x)^*(1 - x) + 2(1 - y)^*(1 - y) \preceq \Delta.
\]

Consequently, \( \Sigma^2 I[\Gamma] = \{ f \in I[\Gamma]^h : f + \epsilon \Delta \in \Sigma^2 I[\Gamma] \text{ for } \forall \epsilon > 0 \}. \)

\( \Gamma \) has (T) \( \iff \exists \kappa > 0 \) s.t. \( \Delta^2 - \kappa \Delta \in \Sigma^2 \mathbb{R}[\Gamma] \cap I[\Gamma] \)

\( \implies \Delta^2 - (\kappa - \epsilon) \Delta \in \Sigma^2 I[\Gamma] \text{ for } \epsilon \in (0, \kappa) \)

\[ \Box \]
Proof of Theorem, completed

Γ has (T) \iff \exists \kappa > 0 \text{ s.t. } \Delta^2 - \kappa \Delta \in \Sigma^2 R[\Gamma] \cap I[\Gamma] = \Sigma^2 I[\Gamma]
\implies \Delta^2 - (\kappa - \epsilon) \Delta \in \Sigma^2 I[\Gamma] \text{ for } \epsilon \in (0, \kappa)

Lemma. \quad \Sigma^2 R[\Gamma] \cap I[\Gamma] = \Sigma^2 I[\Gamma].

Proof.

Since \Sigma^2 R[\Gamma] \cap I[\Gamma] is closed and contains \Sigma^2 I[\Gamma], the inclusion \supset is clear.
For the converse inclusion \subset, let \( f \in \Sigma^2 R[\Gamma] \cap I[\Gamma] \) be given.
By HB, it suffices to show \( \phi(f) \geq 0 \) for \forall \phi: I[\Gamma] \to \mathbb{R} \) positive.
The function \( \psi(g) = \phi(1 - g) \) is conditionally negative type on \( \Gamma \),
i.e., for \forall \xi \in I[\Gamma], one has
\[
\psi(\xi^* \xi) = \sum_{x,y} \xi_x \xi_y \psi(x^{-1}y) = \sum_{x,y} \xi_x \xi_y \phi(1 - x^{-1}y) = -\phi(\xi^* \xi) \leq 0.
\]
By Schoenberg’s theorem, \( \phi_t(x) = \exp(-t\psi(x)) \) is positive type on \( \Gamma \).
Hence \( \phi_t \) is positive on \( \mathbb{R}[\Gamma] \) and \( \phi(f) = \lim_{t \downarrow 0} t^{-1} \phi_t(f) \geq 0 \).

Thank You for Your Attention!