Quantum Correlations and Tsirelson's Problem

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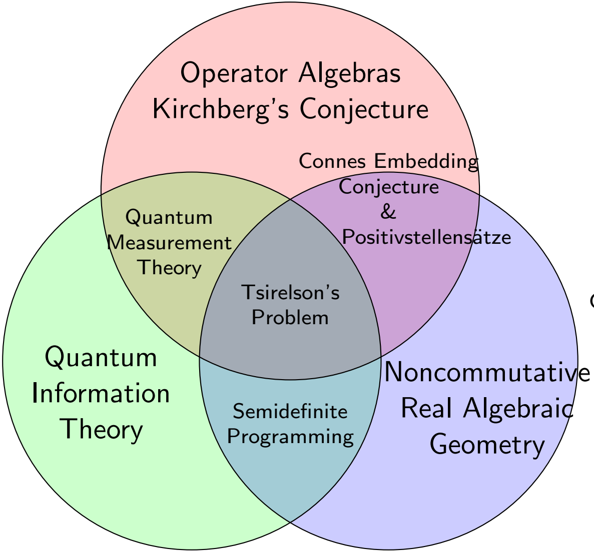
About the Connes Embedding Conjecture—algebraic approaches—. Jpn. J. Math., **8** (2013).
Tsirelson's problem and asymptotically commuting unitary matrices. J. Math. Phys., **54** (2013).

Connes's Embedding Conjecture

[A. Connes; *Classification of injective factors.* Ann. of Math. **104** (1976)]

"We now construct an approximate imbedding of N in R. Apparently such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups."

Overview of today's talk



Connes Embedding Conjecture (1976)

$$\forall M \ M \hookrightarrow R^\omega ?$$

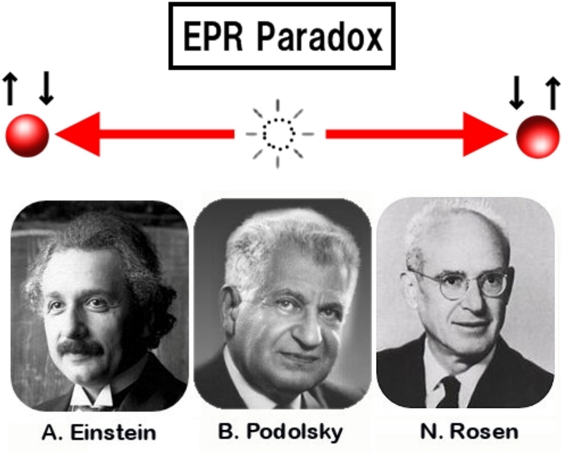
Kirchberg's Conjecture (1993)

$$\mathrm{C}^*\mathbb{F}_d \otimes_{\max} \mathrm{C}^*\mathbb{F}_d = \mathrm{C}^*\mathbb{F}_d \otimes_{\min} \mathrm{C}^*\mathbb{F}_d ?$$

Tsirelson's Problem (1993, 2006)

$$\mathcal{Q}_c = \mathcal{Q}_s ?$$

Quantum information theory



Quantum measurement (von Neumann measurement)

In probability theory, a trial with m outcomes is described by a probability space (X,μ) and a partition $X=\bigsqcup_{i=1}^m X_i$. When one obtains i as an outcome, the ambient probability space changes to $(X_i,\mu(X_i)^{-1}\mu|_{X_i})$.

$\rightsquigarrow$ $P_i=1_{X_i}$ are orthogonal projections on $L^2(X,\mu)$ with $\sum_{i=1}^m P_i=1$.

In quantum theory, a PVM (Projection Valued Measure) with m outcomes is an m-tuple $(P_i)_{i=1}^m$ of orth projections on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^m P_i=1$, and the outcome of a m'ment of a (pure) state $\psi\in\mathcal{H}$, a unit vector, is probabilistic: $(\langle\psi,P_i\psi\rangle)_{i=1}^m\in\mathrm{Prob}([1,\ldots,m])$. When one obtains i as an outcome, the state ψ collapses to $\|P_i\psi\|^{-1}P_i\psi$.

Suppose Alice and Bob have d-PVMs respectively and a shared state:

$$(P_i^k)_{i=1}^m,\ k=1,\ldots,d \text{ and } (Q_j^l)_{j=1}^m,\ l=1,\ldots,d, \text{ and } \psi.$$

Each of them conducts a m'ment of ψ by using one of PVMs they have.

What are the possibilities?

EPR Paradox and Bell Test (CHSH Bell inequality)

Suppose Alice and Bob have d-PVMs respectively and a shared state:

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Each of them conducts a m'ment of ψ by using one of PVMs they have.

$$(\langle \psi, P_i^k \psi \rangle)_{i=1}^m$$

$$(\langle \psi, Q_j^l \psi \rangle)_{j=1}^m$$

ψ

EPR Paradox and Bell Test (CHSH Bell inequality)

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Each of them conducts a m'ment of ψ by using one of PVMs they have.

$\alpha = \{\text{Apple, Grape}\}$
$A = P_{\mathrm{A}} - P_{\mathrm{G}}$
$\alpha' = \{\text{Red, Green}\}$
$A' = P_{\mathrm{R}} - P_{\mathrm{G}}$

In the classical setting,

$$|\mathbb{E}^{\alpha\beta}(AB) + \mathbb{E}^{\alpha\beta'}(AB') + \mathbb{E}^{\alpha'\beta}(A'B) - \mathbb{E}^{\alpha'\beta'}(A'B')| \le 2,$$

because

$$|AB + AB' + A'B - A'B'| \le |B+B'|+|B-B'| \le 2.$$

$\beta = \{\text{Hard, Soft}\}$
$B = Q_{\mathrm{H}} - Q_{\mathrm{S}}$
$\beta' = \{\text{Big, Small}\}$
$B' = Q_{\mathrm{B}} - Q_{\mathrm{S}}$

Does Nature conform this inequality?

EPR Paradox and Bell Test (CHSH Bell inequality)

Suppose Alice and Bob have d-PVMs respectively and a shared state:

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Each of them conducts a m'ment of ψ by using one of PVMs they have.

Suppose we know

Red – Small,

Apple – Soft,

Green – Hard

never occurs. Is

Apple – Small

possible?

Tsirelson's Problem on quantum correlations

We consider the convex sets $\mathcal{C} \subset \mathcal{Q}_s \subset \mathcal{Q}_c \subset \Theta \subset \mathbb{M}_{md}(\mathbb{R}_{\geq 0})$ of the classical and quantum correlation matrices for two **separated** systems:

$$\mathcal{C} = \{ \Big[\int P_i^k Q_j^l \, d\mu \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} (X,\mu) \text{ a (finite) prob space} \\ (P_i^k)_{i=1}^m,\ k=1,\ldots,d, \text{ partitions of } 1_X, \\ (Q_j^l)_{j=1}^m,\ l=1,\ldots,d, \text{ partitions of } 1_X \end{array} \},$$

$$\mathcal{Q}_s = \mathrm{cl}\{ \Big[\langle \psi, (P_i^k \otimes Q_j^l)\psi \rangle \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} \color{red}{\psi \in \mathcal{H} \otimes \mathcal{K} \text{ a state}} \\ (P_i^k)_{i=1}^m,\ k=1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l=1,\ldots,d, \text{ PVMs on } \mathcal{K} \end{array} \},$$

$$\mathcal{Q}_c = \{ \Big[\langle \psi, P_i^k Q_j^l \psi \rangle \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} \color{red}{\mathcal{H} \text{ a Hilbert space, } \psi \in \mathcal{H} \text{ a state}} \\ (P_i^k)_{i=1}^m,\ k=1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l=1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ [P_i^k, Q_j^l] = 0 \text{ for all } i,j \text{ and } k,l \end{array} \},$$

$$\Theta = \{ \Big[\gamma_{i,j}^{k,l} \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} \gamma_{i,j}^{k,l} \geq 0,\ \sum_{i,j} \gamma_{i,j}^{k,l} = 1 \\ \sum_i \gamma_{i,j}^{k,l} \text{ indep of } k,\ \sum_j \gamma_{i,j}^{k,l} \text{ indep of } l \end{array} \}.$$

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$$\mathcal{Q}_s = \mathrm{cl}\{ \Big[\langle \psi, (P_i^k \otimes Q_j^l)\psi \rangle \Big]_{\substack{k,l \\ i,j}} : \begin{array}{l} \psi \in \mathcal{H} \otimes \mathcal{K} \text{ a state} \\ (P_i^k)_{i=1}^m,\ k = 1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l = 1,\ldots,d, \text{ PVMs on } \mathcal{K} \end{array} \},$$

$\mathcal{H}$ a Hilbert space, $\psi \in \mathcal{H}$ a state

$\mathcal{Q}_c = \{ \Big[\langle$ … $\mathcal{H}$, … $\mathcal{H}$, $\}$,

$\Theta = \{ \big[\gamma_{i,j}^{k,l} \big]_{\substack{k,l \\ i,j}} :$ … $\sum_i \gamma_{i,j}^{k,l}$ indep of k, $\sum_j \gamma_{i,j}^{k,l}$ indep of l $\}$.

> $\mathcal{C} \neq \mathcal{Q}_s$ by CHSH Bell inequality (1969)
>
> $$|A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2| \leq 2$$
>
> for commuting variables $-1 \leq A_i, B_j \leq 1$.

Tsirelson's Problem on quantum correlations

We consider the convex sets $\mathcal{C} \subset \mathcal{Q}_s \subset \mathcal{Q}_c \subset \Theta \subset \mathbb{M}_{md}(\mathbb{R}_{\geq 0})$ of the classical and quantum correlation matrices for two **separated** systems:

(X, μ) a (finite) prob space

$\mathcal{C} =$ … },

$$\mathcal{Q}_c \neq \Theta \text{ by Cirel'son's Quantum Bell inequality (1980)}$$

$$|A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2| \leq 2\sqrt{2}$$

for operators $-1 \leq A_i, B_j \leq 1$ with $[A_i, B_j] = 0$.

$\mathcal{Q}_s =$ … $]_{i,j}$ … $\mathcal{H}$, },
$(Q_j^l)_{j=1}^m$, $l = 1, \ldots, d$, PVMs on $\mathcal{K}$

$$\mathcal{Q}_c = \left\{ \left[\langle \psi, P_i^k Q_j^l \psi \rangle \right]_{\substack{k,l \\ i,j}} : \begin{array}{c} \color{red}{\mathcal{H} \text{ a Hilbert space, } \psi \in \mathcal{H} \text{ a state}} \\ (P_i^k)_{i=1}^m,\ k = 1, \ldots, d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l = 1, \ldots, d, \text{ PVMs on } \mathcal{H}, \\ [P_i^k, Q_j^l] = 0 \text{ for all } i,j \text{ and } k,l \end{array} \right\},$$

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Tsirelson's Problem on quantum correlations

We consider the convex sets $\mathcal{C} \subset \mathcal{Q}_s \subset \mathcal{Q}_c \subset \Theta \subset \mathbb{M}_{md}(\mathbb{R}_{\geq 0})$ of the clas

> Note that $\mathcal{Q}_c$ becomes same as $\mathcal{Q}_s$ if we restrict the Hilbert spaces $\mathcal{H}$ appearing in the definition of $\mathcal{Q}_c$ to fin-dim ones.

$\mathcal{C}$

$$\Big]_{i,j} \quad (Q_j^l)_{j=1}^m,\ l = 1, \ldots, d, \text{ partitions of } 1_X$$

$$\mathcal{Q}_s = \mathrm{cl}\Big\{ \Big[\langle \psi, (P_i^k \otimes Q_j^l) \psi \rangle \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} \psi \in \mathcal{H} \otimes \mathcal{K} \text{ a state} \\ (P_i^k)_{i=1}^m,\ k = 1, \ldots, d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l = 1, \ldots, d, \text{ PVMs on } \mathcal{K} \end{array} \Big\},$$

$$\mathcal{Q}_c = \Big\{ \Big[\langle \psi, P_i^k Q_j^l \psi \rangle \Big]_{\substack{k,l \\ i,j}} : \begin{array}{c} \mathcal{H} \text{ a Hilbert space, } \psi \in \mathcal{H} \text{ a state} \\ (P_i^k)_{i=1}^m,\ k = 1, \ldots, d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l = 1, \ldots, d, \text{ PVMs on } \mathcal{H}, \\ [P_i^k, Q_j^l] = 0 \text{ for all } i, j \text{ and } k, l \end{array} \Big\},$$

Tsirelson's Problem: $\mathcal{Q}_c = \mathcal{Q}_s$?

Quantum correlation and C^*-algebras

Quantum correlation matrices are related to the C^*-algebra

$$\ell_\infty^m * \cdots * \ell_\infty^m \quad (d\text{-fold unital full free product}),$$

which is isomorphic to the full group C^*-algebra $\mathrm{C}^*(\Gamma)$ of $\Gamma = \mathbb{Z}_m^{*d}$.
Denote by p_i^k the standard basis of projections in the k-th copy of ℓ_∞^m.
Also $p_i^k := p_i^k \otimes 1$ and $q_j^l := 1 \otimes p_j^l$ in $\mathrm{C}^*(\Gamma) \otimes \mathrm{C}^*(\Gamma)$. Then, one has

$$\mathcal{Q}_c = \{\Big[\phi(p_i^k q_j^l)\Big]_{\substack{k,l\\ i,j}} : \phi \text{ a state on } \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma)\}$$

and

$$\mathcal{Q}_s = \{\Big[\phi(p_i^k q_j^l)\Big]_{\substack{k,l\\ i,j}} : \phi \text{ a state on } \mathrm{C}^*(\Gamma) \otimes_{\min} \mathrm{C}^*(\Gamma)\}.$$

Theorem (Kirchberg 1993, Fritz and Junge et al. 2010, Oz. 2013)

The following conjectures are equivalent.

- Tsirelson's problem has an affirmative answer: $\mathcal{Q}_c = \mathcal{Q}_s$ for all m, d.
- Kirchberg's Conjecture: $\mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma) = \mathrm{C}^*(\Gamma) \otimes_{\min} \mathrm{C}^*(\Gamma)$ holds.
- Connes's Embedding Conjecture: $M \hookrightarrow R^\omega$ for for every II_1 factor M.

Slightly interacting systems

We consider the quantum correlation of **slightly interacting** systems. When Alice and Bob conduct m'ment of a state ψ at the same time, the probability of the outcome (i,j) is given by $\langle\psi,(P_i \bullet Q_j)\psi\rangle$, where $P \bullet Q = (PQP + QPQ)/2$. Thus we consider

$$\mathcal{Q}_\varepsilon = \mathrm{cl}\Big\{\Big[\langle\psi,(P_i^k \bullet Q_j^l)\psi\rangle\Big]_{\substack{k,l\\ i,j}} : \begin{array}{c} \dim\mathcal{H} < +\infty,\ \psi\in\mathcal{H} \text{ a state} \\ (P_i^k)_{i=1}^m,\ k=1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m,\ l=1,\ldots,d, \text{ PVMs on } \mathcal{H}, \\ \|[P_i^k, Q_j^l]\| \le \varepsilon \text{ for all } i,j \text{ and } k,l \end{array}\Big\}.$$

! Surprisingly, it makes no difference if we allow the Hilbert spaces $\mathcal{H}$ to be infinite-dimensional, thanks to the fact $\mathrm{C}^*(\Gamma\times\Gamma)$ is quasi-diagonal.

Theorem (Oz. 2013)

$$\bigcap_{\varepsilon>0}\mathcal{Q}_\varepsilon = \mathcal{Q}_c.$$

C^*-algebras

Group C^*-algebras

Recall $\Gamma = \mathbb{Z}_m^{*d}$, where $m, d \in \{2, 3, \ldots, \infty\}$ s.t. $m + d > 4$, and

$$\mathrm{C}^*(\Gamma \times \Gamma) = \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma) = \mathrm{C}^*((p_i^k)_{i=1}^m, (q_j^l)_{j=1}^m),$$

$$\mathcal{Q}_c = \{ \left[\phi(p_i^k q_j^l)\right]_{\substack{k,l \\ i,j}} : \phi \text{ a state on } \mathrm{C}^*(\Gamma \times \Gamma)\}.$$

The group Γ, or $\mathrm{C}^*(\Gamma)$, is RFD (Residually Finite Dimensional), i.e. every unitary rep is weakly contained in the closure of the finite-dim rep's. (NB! Residually finite doesn't imply RFD in general, e.g., $\mathrm{SL}(3, \mathbb{Z})$.)

Kirchberg's conjecture $\iff$ $\Gamma \times \Gamma$ is RFD.

In fact, for $\Gamma \supset \Gamma_0 \overset{\pi}{\twoheadrightarrow} \Lambda$, the unitary rep of $\Gamma \times \Gamma$ on $\ell_2(\Gamma \underset{\pi}{\times} \Gamma)$ is weakly contained in the closure of the finite-dim rep's **iff** the group vN algebra $\mathrm{vN}(\Lambda)$ satisfies the Connes Embedding Conjecture: $\mathrm{vN}(\Lambda) \hookrightarrow R^\omega$.

Note: If Λ is sofic, then $\mathrm{vN}(\Lambda) \hookrightarrow R^\omega$.
Is the converse possibly true...???

Quasi-diagonality

Recall $\Gamma = \mathbb{Z}_m^{*d}$, where $m, d \in \{2, 3, \ldots, \infty\}$ s.t. $md > 4$, and

Kirchberg's conjecture $\iff$ $\mathrm{C}^*(\Gamma \times \Gamma)$ is RFD.

Theorem (Brown–Oz. 2008)

The group C^*-algebra $\mathrm{C}^*(\Gamma \times \Gamma)$ is QD (quasi-diagonal).

A C^*-algebra A is QD if there are unital completely positive maps $\theta_n \colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ such that $\|\theta_n(a)\theta_n(b) - \theta_n(ab)\| \to 0$ and $\|\theta_n(a)\| \to \|a\|$ for all $a, b \in A$.

Proof for an easier case $\Gamma = \mathbb{F}_d$.

Every unitary rep π of $\mathbb{F}_d$ is homotopic inside $\pi(\mathbb{F}_d)''$ to the trivial rep.

$\rightsquigarrow$ Every unitary rep of $\mathbb{F}_d \times \mathbb{F}_d$ is homotopic to the trivial rep.

The theorem now follows from homotopy invariance of quasi-diagonality (Voiculescu 1991). The proof does not provide explicit finite-dimensional approximants. $\square$

Noncommutative real algebraic geometry

Positivstellensätze

A linear functional $\phi\colon \mathbb{C}[\Gamma] \to \mathbb{C}$ is called a state if $\phi(f * f^*) \geq 0$ and $\phi(\mathbf{1}) = 1$. It is tracial if it moreover satisfies $\tau(f * g) = \tau(g * f)$.

Theorem (Hahn–Banach + GNS)

Let Γ be a discrete group and $f \in \mathbb{C}[\Gamma]$. Then, ❶ $\Leftrightarrow$ ❷ $\Rightarrow$ ❸ $\Leftrightarrow$ ❹.

1. $f \geq 0$ in $\mathrm{C}^*(\Gamma)$, i.e. $\pi(f) \geq 0$ for every unitary rep π.
2. $f + \varepsilon\mathbf{1} \in \{\sum_i g_i * g_i^* : g_i \in \mathbb{C}[\Gamma]\}$ for every $\varepsilon > 0$.
3. $\phi(f) \geq 0$ for every tracial state ϕ on $\mathbb{C}[\Gamma]$.
4. $f + \varepsilon\mathbf{1} \in \{\sum_i g_i * g_i^* : g_i \in \mathbb{C}[\Gamma]\} +$ commutators, for every $\varepsilon > 0$.

When $\Gamma = \mathbb{F}_d$, $\mathrm{C}^*(\Gamma)$ is RFD and it's enough to consider fin-dim π's in ❶.

Theorem (Klep–Schweighofer/Juschenko–Popovych 2008)

Tsirelson's Problem has an affirmative answer iff ❸ for $\Gamma = \mathbb{F}_d$ is equiv to

5. $\mathrm{Tr}(\pi(f)) \geq 0$ for every finite-dimensional unitary rep π of $\mathbb{F}_d$.

Strict Positivstellensätze

Theorem (Hahn–Banach + GNS)

Let Γ be a discrete group and $f \in \mathbb{C}[\Gamma]$. Then, ❶ $\Leftrightarrow$ ❷.

1. $f \geq 0$ in $\mathrm{C}^*(\Gamma)$, i.e. $\pi(f) \geq 0$ for every unitary rep π.
2. $f + \varepsilon \mathbf{1} \in \{\sum_i g_i * g_i^* : g_i \in \mathbb{C}[\Gamma]\}$ for every $\varepsilon > 0$.

Theorem (Riesz–Fejér, Schmüdgen, Bakonyi–Timotin 2007)

Let $f \in \mathbb{C}[\mathbb{F}_d]$ be s.t. $\operatorname{supp} f \subset EE^{-1}$ for a conn. subset $1 \in E \subset \mathbb{F}_d$. Then, TFAE.

- $f \geq 0$ in $\mathrm{C}^*(\mathbb{F}_d)$, i.e. $\pi(f) \geq 0$ for every finite (dim) unitary rep π.
- $f \in \{\sum_i g_i * g_i^* : g_i \in \mathbb{C}[\Gamma],\ \operatorname{supp} g_i \subset E\}$.

Deeper results from real algebraic geometry:
Scheiderer (2006): "$+\varepsilon\mathbf{1}$" isn't necessary for $\Gamma = \mathbb{Z}^2$, but it's instable.
Scheiderer (2009): "$+\varepsilon\mathbf{1}$" is necessary for $\Gamma \supset \mathbb{Z}^3$.

How about $\mathbb{F}_d \times \mathbb{F}_d$?

Operator System Tensor Product

Consider the operator system

$$\mathcal{S} = \mathrm{span}\{p_i^k : i, k\} = \ell_\infty^m + \cdots + \ell_\infty^m \subset \ell_\infty^m * \cdots * \ell_\infty^m.$$

A map ϕ from $\mathcal{S}$ into $\mathbb{B}(\mathcal{H})$ is completely positive iff its restriction ϕ_k to each copy of ℓ_∞^m is c.p. and $\phi_1(1) = \cdots = \phi_d(1)$.
It follows that the operator system dual $\mathcal{S}^{\mathrm{d}}$ of $\mathcal{S}$ is given by

$$\mathcal{S}^{\mathrm{d}} = \{(f_k)_{k=1}^d \in \bigoplus_{k=1}^d \ell_\infty^m : \sum_i f_k(i) \text{ indep of } k\} \subset \bigoplus_{k=1}^d \ell_\infty^m,$$

and hence

$$\mathcal{Q}_s = \{\Big[\phi(p_i^k \otimes p_j^l)\Big]_{\substack{k,l\\ i,j}} : \phi \text{ a state on } \mathcal{S} \otimes_{\mathrm{min}} \mathcal{S}\}$$
$$= \{\phi \in (\mathcal{S}^{\mathrm{d}} \otimes_{\mathrm{max}} \mathcal{S}^{\mathrm{d}})_{+,1} : \text{ evaluated at } \{p_i^k \otimes p_j^l\}\}.$$

Here $\otimes_{\mathrm{max}}$ denotes the maximal op sys tensor product (Farenick–Paulsen).
A simple calculation shows that ϕ can be realized by finite-dimensional system if it is strictly positive (i.e. faithful on $\mathcal{S} \otimes_{\mathrm{min}} \mathcal{S}$).

Semidefinite programming

The convex sets $\mathcal{Q}_s \subset \mathcal{Q}_c$ are determined by infinitely many **explicit** inequalities. Also, theory of operator systems can describe what is

$$\mathcal{Q}_s = \mathrm{cl}\Big(\bigcup_n \{\Big[\langle\psi, (P_i^k \otimes Q_j^l)\psi\rangle\Big]_{\substack{k,l\\i,j}} : \psi \in \ell_2^n \otimes \ell_2^n \text{ a state}\}\Big).$$

It is unknown whether the closure is necessary, but generic (open dense) elements of $\mathcal{Q}_s$ are realizable by finite-dimensional systems. While,

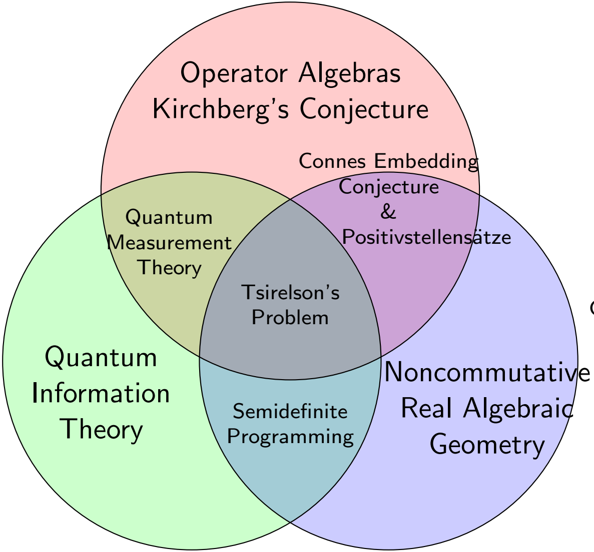
$$\mathcal{Q}_c = \{\Big[\phi(p_i^k q_j^l)\Big]_{\substack{k,l\\i,j}} : \phi \text{ a state on } \mathbb{C}[\mathbb{Z}_m^{*d} \times \mathbb{Z}_m^{*d}]\},$$

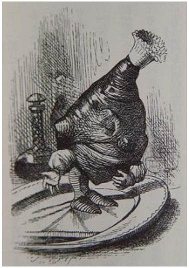
where $\phi\colon \mathbb{C}[\Lambda] \to \mathbb{C}$ is a state (positive type) iff $[\phi(xy^{-1})]_{x,y\in E}$ is positive semidefinite for every finite subset $E \subset \Lambda$.

Instability of $\Gamma \times \Gamma$ probably means infinitely many inequality are necessary, i.e. $\mathcal{Q}_s$ and $\mathcal{Q}_c$ are very likely not semi-algebraic, except for $(m,d) \neq (2,2)$.

Also, $\exists$ infinitely many Bell type inequalities.

Thank you for your attention!





Connes Embedding Conjecture (1976)

$$\forall M \; M \hookrightarrow R^{\omega}?$$

Kirchberg's Conjecture (1993)

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$$\mathcal{Q}_c = \mathcal{Q}_s?$$