Dixmier's Similarity Problem

Narutaka OZAWA
Joint work with Nicolas Monod

Geometry and Analysis,
Kyoto University, 16 March 2011

Dixmier's Similarity Problem

Narutaka OZAWA
Joint work with Nicolas Monod

Geometry and Analysis,
Kyoto University, 16 March 2011

Amenability and von Neumann's Problem

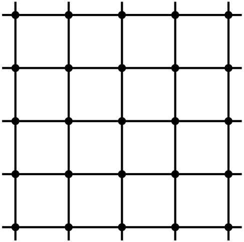
Cayley Graph

graph ≒ countable set Γ & metric $d\colon \Gamma\times\Gamma\to\mathbb{N}\cup\{\infty\}$.

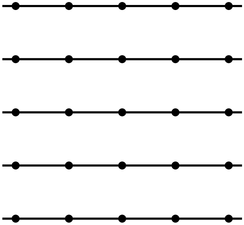
Cayley graph $\mathrm{Cayley}(\Gamma,\mathcal{S})$

Γ countable discrete group, $\mathcal{S}\subset\Gamma$ subset.

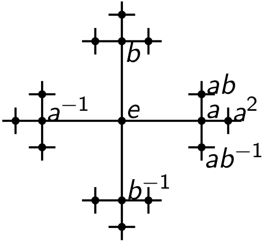
$$\begin{aligned} |x|_{\mathcal{S}} &= \min\{n : \exists s_i\in\mathcal{S}\cup\mathcal{S}^{-1} \text{ s.t. } x = s_1 s_2\cdots s_n\}. \\ d_{\mathcal{S}}(x,y) &= |x^{-1}y|_{\mathcal{S}}. \end{aligned}$$



$\mathbb{Z}^2=\langle(1,0),(0,1)\rangle$



$\mathbb{Z}^2$, $\mathcal{S}=\{(1,0)\}$



$\mathbb{F}_2=\langle a,b\rangle$

Expansion Constant, Amenability

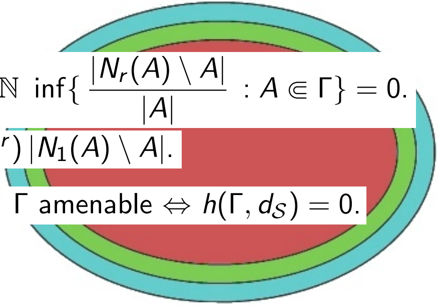
$N_r(A) := \{x \in \Gamma : d(x, A) \le r\}$, r-neighborhood of $A \subset \Gamma$

Expansion constant $$h(\Gamma) = \inf\{ \frac{|N_1(A) \setminus A|}{|A|} : \emptyset \ne A \subset \Gamma \text{ finite } \}.$$

Definition

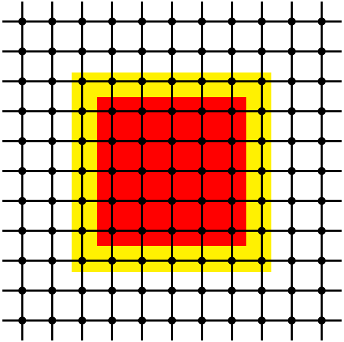
A graph Γ is amenable $\iff$ $h(\Gamma) = 0$.
A discrete group Γ is amenable $\iff$ $h(\Gamma, d_{\mathcal{S}}) = 0$ for all finite $\mathcal{S} \subset \Gamma$.

- $|N_r(A)| \ge (1 + h(\Gamma))^r\, |A|$.
- Γ amenable, $\deg(\Gamma) \le d \Rightarrow \forall r \in \mathbb{N}\ \inf\{ \frac{|N_r(A) \setminus A|}{|A|} : A \Subset \Gamma\} = 0.$
  $\because |N_r(A) \setminus A| \le (1 + d + \cdots + d^r)\, |N_1(A) \setminus A|$.
- If $\Gamma = \langle \mathcal{S} \rangle$ finitely generated, then Γ amenable $\Leftrightarrow h(\Gamma, d_{\mathcal{S}}) = 0$.



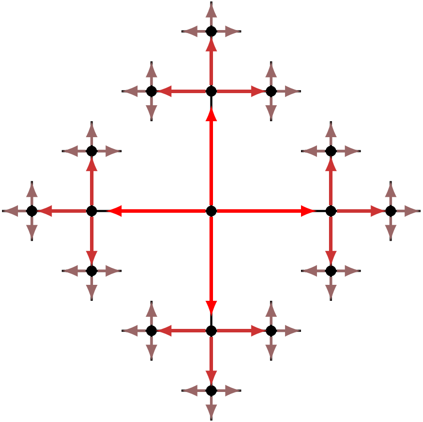
Examples of (Non-)Amenable Groups

$\mathbb{Z}^d$ amenable



More generally, finite groups and solvable groups are amenable. Moreover, subgroups, extensions, directed unions, . . .

$h(d\text{-regular tree}) = d - 2$



$\Gamma \supset \mathbb{F}_2 \implies \Gamma$ non-amenable.

Pyramid Scheme

Γ non-amenable graph $\iff \exists$ successful pyramid scheme.

Theorem (Benjamini–Schramm 1997)

$$h(\Gamma) \geq d \iff \exists \text{ spanning forest } F \subset \Gamma \text{ s.t. } h(F) = d.$$

Here, a *spanning forest* of a set Γ is a graph structure on Γ whose connected components are trees.

von Neumann's Problem

$$\Gamma \text{ non-amenable group} \implies \mathbb{F}_2 \hookrightarrow \Gamma\ ?$$

Tits Alternative (1972)

Γ f.g. linear group $\implies$

- $\exists$ solvable subgroup $\Lambda \leq \Gamma$ of finite index $\rightsquigarrow \Gamma$ amenable

or

- $\mathbb{F}_2 \hookrightarrow \Gamma$

.

Burnside Groups

von Neumann's Problem

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma \text{ ?}$$

Free Burnside group: $B(m,n) := \mathbb{F}_m / \langle\langle x^n : x \in \mathbb{F}_m \rangle\rangle$.

Burnside's Problem (1902): $|B(m,n)| < \infty$ for every $m, n < \infty$?

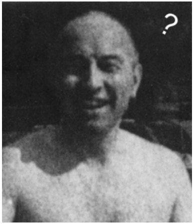
- Yes! for $n = 2, 3, 4, 6$ (Burnside 1902, Sanov 1940, Hall 1958).
- Yes! for linear Burnside groups (Burnside 1905).
- No! for $n \gg 1$ odd (Adian–Novikov 1968).

Obviously, $\mathbb{F}_2 \not\hookrightarrow B(m,n)$. $\rightsquigarrow$ A counterexample to vN's problem?

Theorem (Ol'shanskii 1980, Adian 1982)

Every free Burnside group $B(m,n)$ is non-amenable, for $n \gg 1$ odd.
In particular, **No!** to von Neumann's problem.

Uniformly Bounded Representations
and
Dixmier's Problem



Sz.-Nagy's Theorem (1947)

$\mathbb{B}(\mathcal{H})$ = the space of bounded linear operators on a Hilbert space $\mathcal{H}$.
For $T \in \mathbb{B}(\mathcal{H})$, $\quad \|T\| = \sup\{\, \|Th\| : h \in \mathcal{H},\ \|h\| \leq 1 \,\} < +\infty$.
An isometric isomorphism on $\mathcal{H}$ is called *unitary*: $\|T\| = \|T^{-1}\| = 1$.

Theorem (Sz.-Nagy 1947)

$$\exists S \in \mathbb{B}(\mathcal{H})^{-1} \text{ s.t. } S^{-1}TS \text{ unitary} \iff \sup_{n\in\mathbb{Z}} \|T^n\| < \infty.$$

Proof of ($\Leftarrow$).

Let $F_n := [-n, n] \cap \mathbb{Z}$ and define a new inner product $\langle\,\cdot,\cdot\,\rangle_T$ on $\mathcal{H}$ by

$$\langle h, k\rangle_T := \operatorname*{Lim}_n \frac{1}{|F_n|} \sum_{m\in F_n} \langle T^m h, T^m k\rangle.$$

Then, T becomes unitary on $(\mathcal{H}, \langle\,\cdot,\cdot\,\rangle_T)$.
Let $C = \sup_{n\in\mathbb{Z}} \|T^n\|$. Since $C^{-1}\|h\| \leq \|h\|_T \leq C\|h\|$,
$\exists S \in \mathbb{B}(\mathcal{H})$ s.t. $\|S\| \leq C$, $\|S^{-1}\| \leq C$ & $S^{-1}TS$ unitary. □

Uniformly Bounded Representations

Definition

A representation $\pi\colon \Gamma \to \mathbb{B}(\mathcal{H})$ is called *uniformly bounded* if

$$|\pi| \ := \ \sup_{g\in\Gamma} \|\pi(g)\| \ < \ \infty.$$

Recall: A group Γ is amenable $\Leftrightarrow \exists F_n$ s.t. $\dfrac{|F_n g \triangle F_n|}{|F_n|} \to 0$ for all $g \in \Gamma$.

Theorem (Day, Dixmier, Nakamura–Takeda 1950)

Every unif bdd representation π of an amenable group is *unitarizable* (aka similar to a unitary repn), i.e., $\exists S$ s.t. $S\pi(\,\cdot\,)S^{-1}$ is unitary.

Proof.

Define a new inner product $\langle\,\cdot,\,\cdot\,\rangle_\pi$ on $\mathcal{H}$ by

$$\langle h, k\rangle_\pi \ := \ \operatorname*{Lim}_n \frac{1}{|F_n|}\sum_{g\in F_n}\langle \pi(g)h, \pi(g)k\rangle.$$

$\vdots$

□

Unitarizable Groups

Definition

A group Γ is called *unitarizable* if every unif bdd repn of Γ is unitarizable.

Dixmier's Problem (1950)

1. Are all groups unitarizable?
2. In case (1) is not true, does unitarizability characterize amenability?

Answer: No! for $\Gamma = \mathrm{SL}(2,\mathbb{R})$ (Ehrenpreis–Mautner 1955).

Corollary (by Induction)

$$\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma \text{ not unitarizable.}$$

$$\because \mathbb{F}_2 \supset \mathbb{F}_\infty \to \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\pi} \mathbb{B}(\mathcal{H}).$$

Leinert 1979, Mantero–Zappa, Pytlik–Szwarc, Bożejko–Fendler 1991, . . .

Review

$$\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma \text{ non-amenable.}$$

von Neumann's Problem ✓

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma \text{ ?}$$

$$\Gamma \text{ amenable} \implies \Gamma \text{ unitarizable.}$$

Dixmier's Problem

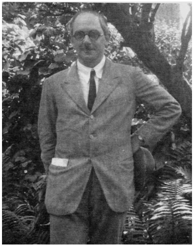
$$\Gamma \text{ unitarizable} \implies \Gamma \text{ amenable?}$$

We know:

- $\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma$ not unitarizable.
- $\exists$ non-amenable Γ s.t. $\mathbb{F}_2 \not\hookrightarrow \Gamma$, e.g. $B(m,n)$.

Pisier 2006, 2007: strongly unitarizable $\implies$ amenable.

Littlewood and Random Forests



Derivations and Uniformly Bounded Representations

$\lambda\colon \Gamma \to \mathbb{B}(\ell_2\Gamma)$ the left regular representation: $\lambda_x\delta_y = \delta_{xy}$.

A map $D\colon \Gamma \to \mathbb{B}(\ell_2\Gamma)$ is called a *derivation* if it satisfies the Leibniz rule:

$$D(xy) \;=\; \lambda(x)D(y) + D(x)\lambda(y).$$

$$D \text{ derivation} \iff \pi_D\colon \Gamma \ni x \mapsto \begin{bmatrix} \lambda(x) & D(x) \\ 0 & \lambda(x) \end{bmatrix} \in \mathbb{B}(\ell_2\Gamma \oplus \ell_2\Gamma) \text{ is a repn.}$$

If $D_T(x) = [T, \lambda(x)]$ is an inner derivation assoc. with $T \in \mathbb{B}(\mathcal{H})$, then

$$\pi_{D_T}(x) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{bmatrix} \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}.$$

Theorem

- π_D uniformly bounded $\iff$ D uniformly bounded.
- π_D unitarizable $\iff$ D inner.

Littlewood and Pyramid Scheme

Definition (p-Littlewood function; $p = 1$)

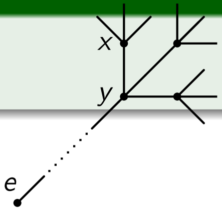
$f\colon \Gamma \to \mathbb{C}$ belongs to $T_1(\Gamma)$ if $\exists\, A, B\colon \Gamma \times \Gamma \to \mathbb{C}$ s.t.

- $f(x^{-1}y) = A(x,y) + B(x,y)$ for all $x, y \in \Gamma$,
- $\sup_x \sum_y |A(x,y)| + \sup_y \sum_x |B(x,y)| < \infty$.

$$\|f\|_{T_1(\Gamma)} := \inf\{\, \|A\|_{1,1} + \|B\|_{\infty,\infty} \,\}.$$

Example

$\Gamma = \mathbb{F}_d$, $2 \le d \le \infty$, $f(x) = \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{if } |x| \neq 1 \end{cases}$.



$$A = \{(x,y) : |x^{-1}y| = 1,\ |x| > |y|\},$$
$$B = \{(x,y) : |x^{-1}y| = 1,\ |x| < |y|\}.$$

Bożejko–Fendler's Construction (1991)

For $f \in T_1(\Gamma)$ with $f(x^{-1}y) = A(x,y) + B(x,y)$, define $D\colon \Gamma \to \mathbb{B}(\ell_2\Gamma)$ by
$$D(g) := [A, \lambda(g)] = -[B, \lambda(g)].$$
One has $\|D(g)\|_{1,1} \leq 2\|A\|_{1,1}$ & $\|D(g)\|_{\infty,\infty} \leq 2\|B\|_{\infty,\infty}$.
Consequently, $\|D(g)\|_{2,2} \leq \|A\|_{1,1} + \|B\|_{\infty,\infty}$. $\rightsquigarrow$ D is unif bdd.

Theorem

D inner $\implies$ $f \in \ell_2\Gamma$.

In particular, if Γ is unitarizable, then $T_1(\Gamma) \subset \ell_2\Gamma$.

Proof.

$$\begin{aligned} D \text{ inner} &\implies \exists T \text{ s.t. } D(g) = T\lambda(g) - \lambda(g)T \\ &\implies \big(g \mapsto \langle D(g)\delta_e, \delta_e\rangle\big) \in \ell_2\Gamma \\ &\implies \big(g \mapsto A(e,g) - A(g,e)\big) \in \ell_2\Gamma \\ &\implies \big(g \mapsto A(g,e)\big) \in \ell_2\Gamma \\ &\implies f \in \ell_2\Gamma. \end{aligned}$$
□

Bożejko–Fendler's Construction (1991)

For $f \in T_1(\Gamma)$ with $f(x^{-1}y) = A(x,y) + B(x,y)$, define $D\colon \Gamma \to \mathbb{B}(\ell_2\Gamma)$ by
$$D(g) \ := \ [A, \lambda(g)] \ = \ -[B, \lambda(g)].$$
One has $\|D(g)\|_{1,1} \le 2\|A\|_{1,1}$ & $\|D(g)\|_{\infty,\infty} \le 2\|B\|_{\infty,\infty}$.
Consequently, $\|D(g)\|_{2,2} \le \|A\|_{1,1} + \|B\|_{\infty,\infty}$. $\rightsquigarrow$ D is unif bdd.

Theorem

D inner $\implies$ $f \in \ell_2\Gamma$.

In particular, if Γ is unitarizable, then $T_1(\Gamma) \subset \ell_2\Gamma$.

Corollary (Bożejko–Fendler 1991)

$$\mathbb{F}_n \hookrightarrow \Gamma \implies \exists D\colon \Gamma \to \mathbb{B}(\ell_2\Gamma) \text{ non-inner unif bdd derivation.}$$

Random Forests

$$\mathrm{Forest}(\Gamma) := \{ \text{ spanning forest of } \Gamma \}.$$

Example (Free subgroup $\mathbb{F}_n = \langle g_1, \dots, g_n \rangle \hookrightarrow \Gamma$)

$\mathrm{Cayley}(\Gamma, \{g_1, \dots, g_n\}) \in \mathrm{Forest}(\Gamma)$. Moreover, it is left Γ-invariant.

Theorem (Benjamini–Schramm 1997)

$$h(\Gamma) \geq d \iff \exists\, F \in \mathrm{Forest}(\Gamma) \text{ s.t. } h(F) = d.$$

Definition

A *random forest* on Γ means a Γ-invariant prob measure on $\mathrm{Forest}(\Gamma)$.

Random Forests

$$\mathrm{Forest}(\Gamma) := \{ \text{spanning forest of } \Gamma \}.$$

Example (Free subgroup $\mathbb{F}_n = \langle g_1, \ldots, g_n \rangle \hookrightarrow \Gamma$)

$\mathrm{Cayley}(\Gamma, \{g_1, \ldots, g_n\}) \in \mathrm{Forest}(\Gamma)$. Moreover, it is left Γ-invariant.

Theorem (Benjamini–Schramm 1997)

$$h(\Gamma) \geq d \iff \exists\, F \in \mathrm{Forest}(\Gamma) \text{ s.t. } h(F) = d.$$

Definition

A *random forest* on Γ means a Γ-invariant prob measure on $\mathrm{Forest}(\Gamma)$.

Random Forests

$$\mathrm{Forest}(\Gamma) := \{ \text{ spanning forest of } \Gamma \}.$$

Example (Free subgroup $\mathbb{F}_n = \langle g_1, \ldots, g_n \rangle \hookrightarrow \Gamma$)

$\mathrm{Cayley}(\Gamma, \{g_1, \ldots, g_n\}) \in \mathrm{Forest}(\Gamma)$. Moreover, it is left Γ-invariant.

Definition

A *random forest* on Γ means a Γ-invariant prob measure on $\mathrm{Forest}(\Gamma)$.

Example (Free Minimal SF (Lyons–Peres–Schramm 2006))

$E :=$ Edge set of $\mathrm{Cayley}(\Gamma, \mathcal{S})$ $(= \Gamma \times \mathcal{S})$. $\Gamma \curvearrowright E$.

$\Omega = [0,1]^E$, $\mu = m^{\otimes E}$, $\Gamma \curvearrowright (\Omega, \mu)$ Bernoulli shift.

The push-out measure of $\Theta \colon \Omega \to 2^E \cap \mathrm{Forest}(\Gamma)$;

$\Theta(\omega) = \{ e \in E : e \text{ is not the maximum of } \omega \text{ in any cycle containing } e \}$.

Gaboriau–Lyons's Theorem
Measure Group Theoretic Solution to vN's Problem

$$[\Lambda,\Gamma] := \{\alpha\colon \Lambda\to\Gamma \mid f(e)=e\} \subset \Gamma^\Lambda.$$

Then, $\alpha\in[\Lambda,\Gamma]$ is a group homomorphism $\iff \alpha(x) = \alpha(xs)\alpha(s)^{-1}$.

So, we define $\Lambda \curvearrowright [\Lambda,\Gamma]$ by $(s\cdot\alpha)(x) := \alpha(xs)\alpha(s)^{-1}$.

Definition

A *random homomorphism* means a Λ-invariant prob measure on $[\Lambda,\Gamma]$.

Theorem (Gaboriau–Lyons 2009)

Γ non-amenable $\iff \exists$ random embedding $\mathbb{F}_2 \hookrightarrow \Gamma$.

A positive "answer" to von Neumann's problem!!

Digression of Measure Group Theory

$$[\Lambda,\Gamma] \ := \ \{\alpha\colon \Lambda \to \Gamma \mid f(e) = e\} \subset \Gamma^\Lambda.$$

Then, $\alpha \in [\Lambda,\Gamma]$ is a group homomorphism $\iff \alpha(x) = \alpha(xs)\alpha(s)^{-1}$.

So, we define $\Lambda \curvearrowright [\Lambda,\Gamma]$ by $(s\cdot\alpha)(x) \ := \ \alpha(xs)\alpha(s)^{-1}$.

Definition

A *random homomorphism* means a Λ-invariant prob measure on $[\Lambda,\Gamma]$.

Ornstein–Weiss 1980:	Γ ∞ amenable	$\iff$	$\Gamma \cong_{\mathrm{ran}} \mathbb{Z}$.
Gaboriau–Lyons 2009:	Γ non-amenable	$\iff$	$\mathbb{F}_2 \hookrightarrow_{\mathrm{ran}} \Gamma$.
Jones–Schmidt 1987:	Γ $\neg$ property (T)	$\iff$	$\Gamma \twoheadrightarrow_{\mathrm{ran}} \mathbb{Z}$.
Furman 1999:	$\Gamma \cong_{\mathrm{ran}} \mathrm{SL}(3,\mathbb{Z})$	$\iff$	$\Gamma \leq_{\mathrm{vir.lattice}} \mathrm{SL}(3,\mathbb{R})$
Kida 2006:	$\Gamma \cong_{\mathrm{ran}} \mathrm{MCG}(\Sigma)$	$\iff$	$\Gamma \cong_{\mathrm{vir}} \mathrm{MCG}(\Sigma)$.

Review 2

von Neumann's Problem ✓

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma \text{ ?}$$

Dixmier's Problem

$$\Gamma \text{ unitarizable} \implies \Gamma \text{ amenable ?}$$

Theorem (Gaboriau–Lyons)

$$\Gamma \text{ non-amenable} \implies \exists \text{ random embedding } \mathbb{F}_2 \hookrightarrow \Gamma.$$

Theorem (Bożejko–Fendler)

$$\mathbb{F}_2 \hookrightarrow \Gamma \implies \exists D \colon \Gamma \to \mathbb{B}(\ell_2\Gamma) \text{ non-inner unif bdd derivation.}$$

i.e., bounded cohomology group $H^1_b(\Gamma, \mathbb{B}(\ell_2\Gamma)) \neq 0$.

Now are our browes bound with Victorious Wreathes[♠]

The *wreath product* of a group A by Γ is defined as

$$A \wr \Gamma \ := \ (\bigoplus_\Gamma A) \rtimes \Gamma.$$

Here $((a_x)_x, g) \cdot ((b_x)_x, h) := ((a_x b_{g^{-1}x})_x, gh)$.

Gaboriau–Lyons's theorem + Induction of $D \in H^1_b(\mathbb{F}_2, \mathbb{B}(\ell_2\mathbb{F}_2)) \rightsquigarrow$

Theorem (Monod–Ozawa)

Γ amenable $\iff$ $A \wr \Gamma$ unitarizable for some/any infinite abelian A.

Corollary

Free Burnside groups $B(m, n)$ are not unitarizable, for $n \gg 1$ composite.

Proof.

$B(m, pq) \supset B(\infty, pq) \twoheadrightarrow (\bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}) \wr B(2, q)$. □

[♠] Shakespeare, *Richard III*, 1:1.

Now are our browes bound with Victorious Wreathes♠

The *wreath product* of a group A by Γ is defined as

$$A \wr \Gamma \ := \ (\bigoplus_\Gamma A) \rtimes \Gamma.$$

Here $((a_x)_x, g) \cdot ((b_x)_x, h) := ((a_x b_{g^{-1}x})_x, gh)$.

Gaboriau–Lyons's theorem + Induction of $D \in H^1_b(\mathbb{F}_2, \mathbb{B}(\ell_2\mathbb{F}_2)) \rightsquigarrow$

Theorem (Monod–Ozawa)

Γ amenable $\iff$ $A \wr \Gamma$ unitarizable for some/any infinite abelian A.

Corollary

Free Burnside groups $B(m,n)$ are not unitarizable, for $n \gg 1$ composite.

Proof.

$B(m,pq) \supset B(\infty,pq) \twoheadrightarrow (\bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}) \wr B(2,q)$. □

♠Shakespeare, *Richard III*, 1:1.