Survey on Weak Amenability

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Fejér's theorem

$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Let $f \in C(\mathbb{T})$ and $f \sim \sum_{k=-\infty}^{\infty} c_k z^k$ be the Fourier expansion.

Does the partial sum $s_m := \sum_{k=-m}^{m} c_k z^k$ converges to f?

Not necessarily! But,

Theorem (Fejér)

The Cesáro mean $\frac{1}{n} \sum_{m=1}^{n} s_m$ converges to f uniformly.

Note: For $\varphi_n(k) = (1 - \frac{|k|}{n}) \vee 0$, one has $\frac{1}{n} \sum_{m=1}^{n} s_m = \sum_{k=-\infty}^{\infty} \varphi_n(k) c_k z^k$.

(Reduced) Group C^*-algebra

Γ a countable discrete group
$\lambda\colon \Gamma \curvearrowright \ell_2\Gamma$ the left regular representation

$$(\lambda_s\xi)(x) = \xi(s^{-1}x),$$
$$\lambda(f)\xi = f * \xi \quad \text{for } f = \sum f(s)\delta_s \in \mathbb{C}\Gamma$$

$C^*_\lambda\Gamma$ the C^*-algebra generated by $\lambda(\mathbb{C}\Gamma) \subset \mathbb{B}(\ell_2\Gamma)$

For $\Gamma = \mathbb{Z}$, Fourier transform $\ell_2\mathbb{Z} \cong L^2\mathbb{T}$ implements

$$C^*_\lambda\mathbb{Z} \cong C(\mathbb{T}), \qquad \lambda(f) \leftrightarrow \textstyle\sum_{k\in\mathbb{Z}} f(k)z^k.$$

Fejér's theorem means that multipliers

$$m_{\varphi_n}\colon \lambda(f) \mapsto \lambda(\varphi_n f)$$

converge to the identity on the group C^*-algebra $C^*_\lambda\mathbb{Z}$,
where $\varphi_n(k) = (1 - \frac{|k|}{n}) \vee 0$ has finite support and is positive definite.

Amenability

Definition

An **approximate identity** on Γ is a sequence (φ_n) of finitely supported functions such that $\varphi_n \to \mathbf{1}$. A group Γ is **amenable** if there is an approximate identity consisting of positive definite functions.

$$\begin{aligned} \varphi \text{ positive definite } &\Leftrightarrow\ m_\varphi \text{ is completely positive on } C^*_\lambda\Gamma \\ &\Rightarrow\ m_\varphi \text{ is completely bounded and } \|m_\varphi\|_{\mathrm{cb}} = \varphi(1). \end{aligned}$$

A group Γ is **weakly amenable** (or has the Cowling–Haagerup property) if there is an approximate identity (φ_n) such that $\sup \|m_{\varphi_n}\|_{\mathrm{cb}} < \infty$.
Definition for locally compact groups is similar.

Fejér's theorem implies that $\mathbb{Z}$ is amenable. In fact, all abelian groups and finite groups are amenable. The class of amenable groups is closed under subgroups, quotients, extensions, and limits.

Cowling–Haagerup constant

Recall that a group Γ is **weakly amenable** if there is an approximate identity (φ_n) such that $C := \sup \|m_{\varphi_n}\|_{\mathrm{cb}} < \infty$. The optimal constant $C \geq 1$ is called the Cowling–Haagerup constant and denoted by $\Lambda_{\mathrm{cb}}(\Gamma)$.

- $\Lambda_{\mathrm{cb}}(\Gamma)$ is equal to the CBAP constant for $C^*_\lambda\Gamma$ and the W*CBAP constant for the group von Neumann algebra $\mathcal{L}\Gamma$:
$$\Lambda_{\mathrm{cb}}(\Gamma) = \Lambda_{\mathrm{cb}}(C^*_\lambda\Gamma) = \Lambda_{\mathrm{cb}}(\mathcal{L}\Gamma).$$
- If $\Lambda \leq \Gamma$, then $\Lambda_{\mathrm{cb}}(\Lambda) \leq \Lambda_{\mathrm{cb}}(\Gamma)$.
- If Γ is amenable, then $\Lambda_{\mathrm{cb}}(\Gamma) = 1$.
- $\mathbb{F}_2$ is not amenable, but is weakly amenable and $\Lambda_{\mathrm{cb}}(\mathbb{F}_2) = 1$.
- $\Lambda_{\mathrm{cb}}(\Gamma_1 \times \Gamma_2) = \Lambda_{\mathrm{cb}}(\Gamma_1)\,\Lambda_{\mathrm{cb}}(\Gamma_2)$ (Cowling–Haagerup 1989).
- If $\Lambda_{\mathrm{cb}}(\Gamma_i) = 1$, then $\Lambda_{\mathrm{cb}}(\Gamma_1 * \Gamma_2) = 1$ (Ricard–Xu 2006).
- Λ_{cb} is invariant under measure equivalence.
- OPEN: Is weak amenability preserved under free products?

Herz–Schur multipliers

Theorem (Grothendieck, Haagerup, Bożejko–Fendler)

For a function φ on Γ and $C \geq 0$, the following are equivalent.

- The multiplier
$$m_\varphi \colon \lambda(f) \mapsto \lambda(\varphi f)$$
is completely bounded on $C^*_\lambda\Gamma$ and $\|m_\varphi\|_{\mathrm{cb}} \leq C$.
- The Schur multiplier
$$M_\varphi \colon [A_{x,y}]_{x,y\in\Gamma} \mapsto [\varphi(y^{-1}x)A_{x,y}]_{x,y\in\Gamma}$$
is bounded on $\mathbb{B}(\ell_2\Gamma)$ and $\|M_\varphi\| \leq C$.
- There are a Hilbert space $\mathcal{H}$ and $\xi, \eta \in \ell_\infty(\Gamma, \mathcal{H})$ such that
$$\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle$$
and $\|\xi\|_\infty \|\eta\|_\infty \leq C$.

A function φ satisfying the above conditions is called a Herz–Schur multiplier, and the optimal constant $C \geq 0$ is denoted by $\|\varphi\|_{\mathrm{cb}}$.

An application to convolution operators on $\ell_p\Gamma$

If φ is a Herz–Schur multiplier, then the Schur multiplier

$$M_\varphi \colon [A_{x,y}]_{x,y\in\Gamma} \mapsto [\varphi(y^{-1}x)A_{x,y}]_{x,y\in\Gamma}$$

is bounded on $\mathbb{B}(\ell_p\Gamma)$ for all $p \in [1,\infty]$. Indeed, suppose

$$\varphi(y^{-1}x) = \langle \xi(x), \eta(y)\rangle \quad \text{for} \quad \xi, \eta \in \ell_\infty(\Gamma, \mathcal{H}).$$

Then, using $\mathcal{H} \hookrightarrow L_p$ and $\mathcal{H} \hookrightarrow L_q$, we define $V_\xi \colon \ell_p\Gamma \to \ell_p\Gamma \otimes L_p$ by $V_\xi\delta_x = \delta_x \otimes \xi(x^{-1})$, and likewise V_η. One has $V_\eta^*(A\otimes 1)V_\xi = M_\varphi(A)$.

Now we wonder

$$\begin{aligned}\mathbb{B}(\ell_p\Gamma)\cap\rho(\Gamma)' &= \{\lambda(f) : \lambda(f) \text{ is bounded on } \ell_p\Gamma\}\\ &\overset{?}{=} \text{SOT-cl}\{\lambda(f) : f \text{ is finitely supported}\}.\end{aligned}$$

No counterexample is known.

Theorem (von Neumann, Cowling)

It is true if $p = 2$ or Γ is weakly amenable (or has the weaker property **AP**).

Indeed, if Γ is weakly amenable, then $m_{\varphi_n}(\lambda(f)) \to \lambda(f)$ in SOT.

Examples of Herz–Schur multipliers

Every **coefficient** of a uniformly bounded representation $(\pi, \mathcal{H})$ of Γ is a Herz–Schur multiplier: For every $\xi, \eta \in \mathcal{H}$, the function

$$\varphi(s) = \langle \pi(s)\xi, \eta \rangle$$

on Γ has $\|\varphi\|_{\mathrm{cb}} \leq \sup \|\pi(x)\|^2 \|\xi\| \|\eta\|$. Indeed, one has

$$\varphi(y^{-1}x) = \langle \pi(x)\xi, \pi(y^{-1})^* \eta \rangle.$$

Conversely, every Herz–Schur multiplier on an **amenable** group Γ is a coefficient of some unitary representation. Indeed, if Γ is amenable, then the unit character τ_0 is continuous on $C^*_\lambda \Gamma$ and

$$\|\omega_\varphi \colon C^*_\lambda \Gamma \ni \lambda(f) \mapsto \sum \varphi(s) f(s) \in \mathbb{C}\| = \|\tau_0 \circ m_\varphi\| \leq \|m_\varphi\|.$$

For the GNS rep'n $\pi \colon C^*_\lambda \Gamma \to \mathbb{B}(\mathcal{H})$, there are $\xi, \eta \in \mathcal{H}$ such that

$$\langle \pi(\lambda(s))\xi, \eta \rangle = \omega_\varphi(\lambda(s)) = \varphi(s).$$

Relation to Dixmier's and Kadison's problems

Thus, for any group Γ one has

$$B(\Gamma) \subset UB(\Gamma) \subset B_2(\Gamma),$$

where

$B(\Gamma)$	the space of coefficients of unitary representations
$UB(\Gamma)$	the space of coefficients of uniformly bounded representations
$B_2(\Gamma)$	the space of Herz–Schur multipliers

Theorem

- (Bożejko 1985) A group Γ is amenable iff $B(\Gamma) = B_2(\Gamma)$.
- $B(\Gamma) \subsetneq UB(\Gamma)$ if $\mathbb{F}_2 \hookrightarrow \Gamma$. ¿Is this true for any non-amenable Γ?
- (Haagerup 1985) A Herz–Schur multiplier need not be a coefficient of a uniformly bounded representation, i.e., $UB(\mathbb{F}_2) \subsetneq B_2(\mathbb{F}_2)$.
- A uniformly bounded representation π of Γ extends on the full group C^*-algebra $C^*\Gamma$ if all of its coefficients belong to $B(\Gamma)$.

Examples of weakly amenable subgroups 1

Theorem (De Cannière–Haagerup, Cowling, Co.–Ha., Ha. 80s)

For a simple connected Lie group G, one has

$$\Lambda_{\mathrm{cb}}(G) = \begin{cases} 1 & \text{if } G = \mathrm{SO}(1,n) \text{ or } \mathrm{SU}(1,n), \\ 2n-1 & \text{if } G = \mathrm{Sp}(1,n), \\ +\infty & \text{if } \mathrm{rk}_{\mathbb{R}}(G) \geq 2, \text{ e.g., } \mathrm{SL}(3,\mathbb{R}) \end{cases} \quad \begin{matrix} \text{Haagerup} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{property (T)} \end{matrix} .$$

For a lattice $\Gamma \leq G$, one has $\Lambda_{\mathrm{cb}}(\Gamma) = \Lambda_{\mathrm{cb}}(G)$.

The idea of the proof: If $G = PK$ with P amenable and K compact, then for any bi-K-invariant function φ on G, one has

$$\|\varphi\|_{\mathrm{cb}} = \|\varphi|_P\|_{\mathrm{cb}} = \|C^*(P) \ni \lambda(f) \mapsto \int f\varphi\,d\mu \in \mathbb{C}\|_{C^*(P)^*}.$$

Further results: If $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$, then G and its lattices even fail the AP. (Lafforgue–de la Salle 2010 and Haagerup–de Laat 2012).

Examples of weakly amenable subgroups 2

Theorem (Oz. 2007, Oz.–Popa 2007, Oz. 2010)

- Hyperbolic groups are weakly amenable.
- If G is weakly amenable and $N \triangleleft G$ is an amenable closed normal subgroup, then there is an $\mathrm{Ad}(G)$-invariant N-invariant states on $L^\infty(N)$, or equivalently G is co-amenable in $G \ltimes N$.

  $\rightsquigarrow$ $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ are not w.a. (Haagerup 1988), nor any wreath product $\Delta \wr \Gamma$ with $\Delta \neq \mathbf{1}$ and Γ non-amenable.

Proof of Corollary (non weak amenability of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$).

Consider $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{Z}^2$. Then, the stabilizer of every non-neutral element is amenable. (The stabilizer of $\left[\begin{smallmatrix} m \\ 0 \end{smallmatrix}\right] \in \mathbb{Z}^2$, $m \neq 0$, is $\{\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]\}$.)
If P is amenable, then $\Gamma \curvearrowright \ell_2(\Gamma/P)$ is weakly contained in $\Gamma \curvearrowright \ell_2(\Gamma)$.
Hence, any Γ-invariant mean on $\mathbb{Z}^2$ has to be concentrated on $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$.
$\rightsquigarrow$ No mean on $\mathbb{Z}^2$ is at the same time Γ-invariant and $\mathbb{Z}^2$-invariant. $\square$

Applications to von Neumann algebras

For a vN subalgebra $M \leq N$ which is a range of a conditional expectation,

$$\Lambda_{\mathrm{cb}}(M) \leq \Lambda_{\mathrm{cb}}(N).$$

In particular, any non-weakly amenable von Neumann algebra, e.g. $\mathcal{L}(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2)$, does not embed into an weakly amenable II_1-factor.

Theorem (Oz.–Popa 2007, Oz. 2010)

Let M be an weakly amenable finite von Neumann algebra and $P \leq M$ be an amenable von Neumann subalgebra. Then P is weakly compact in M, or equivalently, there is a state ω on $\mathbb{B}(L^2(P))$ such that $\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \mathcal{U}(P) \cup \sigma(\mathcal{N}_M(P))$, where

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : uPu^* = P\}$$

is the normalizer of P, acting on $L^2(P)$ by conjugation.

Applications to strong solidity and uniqueness of Cartan subalgebras by Oz.–Popa (2007), Chifan–Sinclair (2011) and Popa–Vaes (2011-12).

Some more examples

Theorem

The following groups have the Cowling–Haagerup constant 1.

- Amenable groups, $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ (Haagerup and his friends).
- Free products of groups with $\Lambda_{\mathrm{cb}} = 1$ (Ricard–Xu).
- Finite-dim CAT(0) cube complex groups (Guentner–Higson, Mizuta).
- $\mathrm{SL}(2, K)$ as a discrete group (Guentner–Higson–Weinberger); in particular, limit groups.
- Baumslag–Solitar groups (Gal).

Counterexample:

$$\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \cong (\mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}^2) *_{\mathbb{Z}^2} (\mathbb{Z}/6\mathbb{Z} \ltimes \mathbb{Z}^2)$$

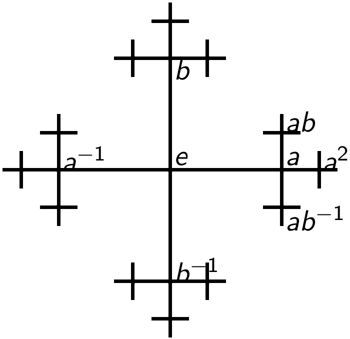
is not weakly amenable.

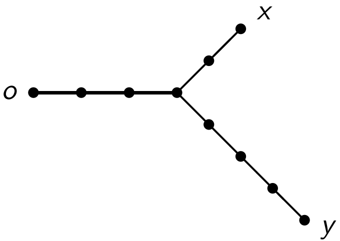
OPEN: relatively hyperbolic groups, mapping class groups, $\tilde{A}_2$-groups, ...

Proofs of (non-)weak amenability

Cayley graph of $\mathbb{F}_2$

The Cayley graph of $\mathbb{F}_2 = \langle a, a^{-1}, b, b^{-1} \rangle$ is a tree with the metric d.





$$d(x, y) = \|\chi_{[o,x]} - \chi_{[o,y]}\|_2^2 = 6$$

Theorem (Haagerup 1978)

$\sqrt{d}$ is a Hilbert space metric, and r^d is positive definite for $r \in [0, 1]$.

Weak amenability of hyperbolic groups

Theorem (Pytlik–Szwarc 1986 and Oz. 2007)

For any *hyperbolic* graph $\mathbf{K}$ of bounded degree, there is $C \geq 1$ satisfying: For every $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the function

$$\theta_z \colon \mathbf{K} \times \mathbf{K} \ni (x, y) \longmapsto z^{d(x,y)} \in \mathbb{C}$$

is a bounded Schur multiplier on $\mathbb{B}(\ell_2\mathbf{K})$ with

$$\|\theta_z\|_{\mathrm{cb}} \leq C \, \frac{|1-z|}{1-|z|}.$$

Moreover $z \mapsto \theta_z$ is holomorphic.

If $\Gamma \curvearrowright \mathbf{K}$ properly, then $\varphi_r(g) = r^{d(go,o)}$ is an approximate identity on Γ, and Γ is weakly amenable.

OPEN: Is φ_z a coefficient of a uniformly bounded representation?

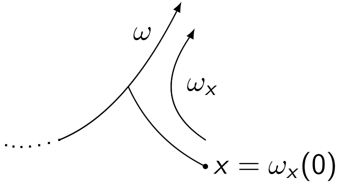
To prove Theorem, one needs to factorize z^d as

$$z^{d(x,y)} = \langle \xi_z(x), \eta_z(y) \rangle, \quad \xi_z, \eta_z \in \ell_\infty(\mathbf{K}, \mathcal{H}).$$

Proof for trees

Let $\mathbf{K}$ be a tree and fix an infinite geodesic ω in $\mathbf{K}$.

For every $x \in \mathbf{K}$, let ω_x be the unique geodesic that starts at x and eventually flows into ω.



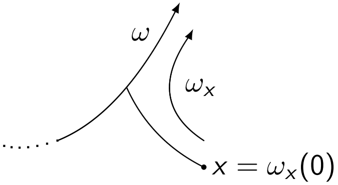
For $z \in \mathbb{D}$ and $x, y \in \mathbf{K}$, define

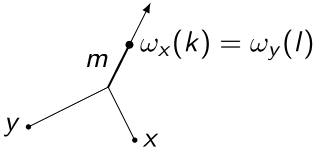
$$\xi_z(x) = \sqrt{1 - z^2} \sum_{k=0}^{\infty} z^k \delta_{\omega_x(k)} \in \ell_2\mathbf{K}, \quad \text{and} \quad \eta_z(y) = \overline{\xi_z(y)}.$$

Then, one has

$$\|\xi_z(x)\|_2^2 = \|\eta_z(y)\|_2^2 = |1 - z^2| \sum_{k \geq 0} |z|^{2k} = \frac{|1 - z^2|}{1 - |z|^2} \leq \frac{|1 - z|}{1 - |z|}.$$

Proof for trees, cont'd





$$\xi_z(x) = \sqrt{1-z^2}\sum_{k=0}^{\infty} z^k \delta_{\omega_x(k)} \in \ell_2\mathbf{K}, \quad \text{and} \quad \eta_z(y) = \overline{\xi_z(y)}.$$

For $x, y \in \mathbf{K}$, one has

$$\begin{aligned} \langle \xi_z(x), \eta_z(y)\rangle &= (1-z^2)\sum_{k,l\geq 0} z^{k+l}\langle \delta_{\omega_x(k)}, \delta_{\omega_y(l)}\rangle \\ &= (1-z^2)\sum_{m\geq 0} z^{d(x,y)+2m} \qquad = z^{d(x,y)}. \end{aligned}$$

$\square$

Non weakly amenable groups

Theorem (Oz.–Popa 2007, Oz. 2010)

If G is weakly amenable and $N \triangleleft G$ is an amenable closed normal subgroup, then there is an $\mathrm{Ad}(G)$-invariant N-invariant states on $L^\infty(N)$.

Proof (Assuming $\Gamma = G$ is discrete, N is abelian and $\Lambda_{\mathrm{cb}}(\Gamma) = 1$.)

Let φ_n be fin. supp. functions on Γ s.t. $\varphi_n \to 1$ and $\sup \|m_{\varphi_n}\| = 1$.
Then, for every $s \in \Gamma$, one has $\lim \|m_{\varphi_n} - m_{\varphi_n} \circ \mathrm{Ad}(s)\| = 0$.
Indeed, if $\varphi_n(y^{-1}x) = \langle \xi_n(x), \eta_n(y) \rangle$ for $\xi, \eta \colon \Gamma \to \mathcal{H}$ of norm 1, then $\xi_n(x) \approx \eta_n(x) \approx \xi_n(xs)$ uniformly for x, and likewise for η.
Let $\tau_0 \colon C^*_\lambda N \to \mathbb{C}$ be the unit character and $\omega_n = \tau_0 \circ m_{\varphi_n} \colon C^*_\lambda N \to \mathbb{C}$.
Recall $C^*_\lambda N \cong C(\widehat{N})$ via the Fourier transform $\ell_2 N \cong L^2(\widehat{N})$.
Then, $\omega_n \in (C^*_\lambda N)^*$ is nothing but $\widehat{\varphi_n|_N} \in L^1(\widehat{N})$ and $\|\widehat{\varphi_n|_N}\|_1 = \|\omega_n\|$.
Thus, $(\widehat{\varphi_n|_N})$ is an approximately Γ-invariant approximate unit for $L^1(\widehat{N})$.
Consequently, for $\zeta_n \in \ell_2 N$ that corresponds to $\left|\widehat{\varphi_n|_N}\right|^{1/2} \in L^2(\widehat{N})$, $|\zeta_n|^2 \in \ell_1 N$ is approximately $\mathrm{Ad}(\Gamma)$- and N-invariant. $\square$