Talk on Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d \geq 3$

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July 9, 2010

1 Introduction

In this talk we are going to discuss the mass critical nonlinear Schrodinger initial value problem

$$\begin{aligned} iu_t + \Delta u &= \mu |u|^{4/d} u, \\ u(0,x) &= u_0. \end{aligned} \tag{1.1}$$

The case $\mu = 1$ is called the defocusing case, $\mu = -1$ is the focusing case. A solution to (1.1) in fact gives an entire family of solutions to (1.1) since if $u(t,x)$ solves (1.1) on the interval $[0, T_0]$ with initial data u_0, then

$$u_{\lambda}(t,x) = \frac{1}{\lambda^{d/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$$

is a solution to (1.1) on $[0, \lambda^2 T_0]$ with initial data $\frac{1}{\lambda^{d/2}} u_0(\frac{x}{\lambda})$.

$$\|u_0\|_{L_x^2(\mathbf{R}^d)} = \|\frac{1}{\lambda^{d/2}} u_0(\frac{x}{\lambda})\|_{L_x^2(\mathbf{R}^d)}. \tag{1.2}$$

We can also apply the Galilean transform. If $u(t,x)$ solves (1.1), then

$$e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u(t, x - 2t\xi_0) \tag{1.3}$$

solves (1.1). This transformation has the effect of shifting a solution in frequency by a fixed amount, and also shifting the solution in space by $x - 2t\xi_0$.

A solution to (1.1) conserves the quantities mass,

$$M(u(t)) = \int |u(t,x)|^2 dx, \tag{1.4}$$

and energy,

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t,x)|^{\frac{2(d+2)}{d}} dx. \tag{1.5}$$

The solution to

$$\begin{aligned} iv_t + \Delta v &= 0, \\ v(0,x) &= v_0, \end{aligned} \tag{1.6}$$

is given by

$$v(t,x) = e^{it\Delta} v_0. \tag{1.7}$$

Moreover, the solution to

$$\begin{aligned} iv_t + \Delta v &= F(t), \\ v(0,x) &= v_0, \end{aligned} \tag{1.8}$$

is given by Duhamel's formula,

$$v(t,x) = e^{it\Delta} v_0 - i \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau. \tag{1.9}$$

This talk is going to focus on $d \geq 3$. Taking the Fourier transform,

$$\mathcal{F}(e^{it\Delta} u_0)(\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi). \tag{1.10}$$

The solution to the free Schrodinger equation,

$$e^{it\Delta} u_0 = \frac{C(d)}{t^{d/2}} \int e^{-i\frac{|x-y|^2}{4t}} u_0(y) dy, \tag{1.11}$$

also obeys the dispersive estimate

$$\|e^{it\Delta} u_0\|_{L_x^\infty(\mathbf{R}^d)} \lesssim \|u_0\|_{L_x^1(\mathbf{R}^d)}. \tag{1.12}$$

Therefore, by [17], (1.9), (1.10), and (1.12), when $d \geq 3$, a pair (p,q) is called an admissible pair if $\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q})$ and $p \geq 2$. If (p,q), $(\tilde{p}, \tilde{q})$ are also admissible pairs then a solution v to

$$\begin{aligned} iv_t + \Delta v &= F, \\ v(0,x) &= v_0, \end{aligned} \tag{1.13}$$

obeys the Strichartz estimates

$$\|v\|_{L_t^p L_x^q(I \times \mathbf{R}^d)} \lesssim \|v_0\|_{L_x^2(\mathbf{R}^d)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I \times \mathbf{R}^d)}. \tag{1.14}$$

Therefore, if u is a solution to (1.1),

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R} \times \mathbf{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbf{R}^d)} + \|u\|^{1+4/d}_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R} \times \mathbf{R}^d)}. \tag{1.15}$$

For $\|u_0\|_{L_x^2(\mathbf{R}^d)} \leq \epsilon_0$, ϵ_0 sufficiently small, this proves global well-posedness by Picard iteration. We also define scattering.

Definition 1.1 *A solution to* (1.1) *is said to scatter to a free solution if there exist* $u_\pm \in L^2(\mathbf{R}^d)$ *such that*

$$\lim_{t \to \infty} \|u(t,x) - e^{it\Delta} u_+\|_{L_x^2(\mathbf{R}^d)} = 0, \tag{1.16}$$

and

$$\lim_{t \to -\infty} \|u(t,x) - e^{it\Delta} u_-\|_{L_x^2(\mathbf{R}^d)} = 0. \tag{1.17}$$

The solution to (1.1) is also scattering for small initial data. Since

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R} \times \mathbf{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbf{R}^d)}$$

when $\|u_0\|_{L_x^2(\mathbf{R}^d)} \leq \epsilon_0$, for any $k > 0$, there exists $T(k)$ such that

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T(k),\infty))} \leq 2^{-k}. \tag{1.18}$$

$$\|e^{-iT_k\Delta} u(T_k) - e^{-iT_{k+1}\Delta} u(T_{k+1})\|_{L_x^2(\mathbf{R}^d)}$$

$$= \| \int_{T_k}^{T_{k+1}} e^{-i\tau\Delta} |u(\tau)|^{4/d} u(\tau) \|_{L_x^2(\mathbf{R}^d)} \lesssim \|u\|^{1+4/d}_{L_{t,x}^{\frac{2(d+2)}{d}}([T(k),\infty))} \leq 2^{-k}.$$

Then let

$$u_+ = \lim_{k \to \infty} u(T_k). \tag{1.19}$$

We can similarly define u_-.

Now define the quantity

$$A(m) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R} \times \mathbf{R}^d)} : \|u(t)\|_{L_x^2(\mathbf{R}^d)} = m\}. \tag{1.20}$$

If $A(m) = C(m) < \infty$, then (1.1) is globally well-posed and scattering for $\|u_0\|_{L_x^2(\mathbf{R}^d)} = m$. This is because we can partition $\mathbf{R}$ into $\sim C(m)^{\frac{2(d+2)}{d}}$ subintervals with $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \leq \epsilon_0$ on each separate subinterval.

Now take one such subinterval $[a, b]$. By Duhamel's principle, the solution on $[a, b]$ has the form

$$e^{i(t-a)\Delta} u(a) - i \int_a^t e^{i(t-\tau)\Delta} |u(\tau)|^{4/d} u(\tau) d\tau. \tag{1.21}$$

Moreover,

$$\| \int_a^t e^{i(t-\tau)\Delta} |u(\tau)|^{4/d} u(\tau) d\tau \|_{L_{t,x}^{\frac{2(d+2)}{d}}([a,b] \times \mathbf{R}^d)} \lesssim \epsilon_0^{1+4/d},$$

so the linear solution $e^{i(t-a)\Delta} u(a)$ will dominate the solution to (1.1) over the time interval $[a, b]$. This idea will be a very important notion at several points throughout the argument.

Making a perturbative argument, we can prove A is a continuous function. Therefore, $\{m : A(m) < \infty\}$ is a nonempty open set and therefore the set $\{m : A(m) = \infty\}$ has a least element. We will define m_0 to be this least element. Then a solution u to (1.1) with

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R} \times \mathbf{R}^d)} = \infty$$

and

$$\|u(t)\|_{L_x^2(\mathbf{R}^d)} = m_0$$

is called a minimal mass blowup solution. Such a solution must possess a number of additional properties, in particular it must be concentrated in both frequency and space.

Lemma 1.1 *If a minimal mass blowup solution u exists on a time interval I, then there exist functions $x(t), \xi(t) : I \to \mathbf{R}^d$, $N(t) : I \to (0, \infty)$, such that for every $\eta > 0$ there exists $C(\eta)$ such that*

$$\int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta \tag{1.22}$$

$$\int_{|x - x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx < \eta \tag{1.23}$$

Proof: See [24].

Furthermore, to prove $A(m) < \infty$ for all m, it suffices to exclude the minimal mass blowup scenarios

1. $N(t) \sim t^{-1/2}$, on $(0, \infty)$,
2. $N(t) \equiv 1$,
3. $N(t) \leq 1$, $\liminf_{t \to \pm\infty} N(t) = 0$.

See [18] for details.

To prove

$$A(m) < \infty \tag{1.24}$$

for all $m < \infty$, it therefore suffices to exclude the three minimal mass blowup scenarios (1) - (3). Because we are dealing with the nonradial case, we need to understand how $\xi(t)$ moves around on the maximum interval I.

Lemma 1.2 *If J is an interval with $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)} \leq \epsilon_0$, then for $t_1, t_2 \in J$, $|\xi(t_1) - \xi(t_2)| \lesssim N(t_1) + N(t_2)$.*

Proof: Recall that for the interval $J = [a, b]$, the linear evolution $e^{i(t-a)\Delta} u(a)$ dominates. Therefore, the balls

$$\{|\xi - \xi(t_1)| \leq C(\frac{m_0^2}{1000}) N(t_1)\} \tag{1.25}$$

and

$$\{|\xi - \xi(t_2)| \leq C(\frac{m_0^2}{1000})N(t_2)\} \tag{1.26}$$

must intersect. Therefore, $|\xi(t_1) - \xi(t_2)| \lesssim N(t_1) + N(t_2)$. $\square$

Since the linear solution dominates over the interval J the scale cannot change too rapidly, and thus we also have $N(t_1) \sim N(t_2)$.

2 Scenario 1:

To deal with this scenario, we will adopt the arguments from [19] in the radial case. There are two additional complications that arise from the nonradial case. The first complication is that in the radial case $\xi(t) \equiv 0$, while in the nonradial case this might not be so. We quote the theorem

Theorem 2.1 *If $u(t,x)$ is a minimal mass blowup solution to* (1.1), *then*

$$\int_{T_1}^{T_2} N(t)^2 dt \lesssim \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T_1,T_2]\times \mathbf{R}^d)}^{\frac{2(d+2)}{d}} \lesssim 1 + \int_{T_1}^{T_2} N(t)^2 dt. \tag{2.1}$$

Proof: See [19]. $\square$

This implies that for any k,

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([2^k,2^{k+1}]\times \mathbf{R}^d)} \lesssim 1. \tag{2.2}$$

This in turn implies $|\xi(2^k) - \xi(2^{k+1})| \lesssim 2^{-k/2}$. Thus the limit

$$\lim_{k\to\infty} \xi(2^k) = \xi_\infty \tag{2.3}$$

exists, and moreover $|\xi(2^k) - \xi_\infty| \lesssim 2^{-k/2}$. Now make a Galilean transformation that maps ξ_∞ to the origin. This implies that after making a Galilean transformation and modifying $C(\eta)$ by a fixed constant,

$$\int_{|\xi|\geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta. \tag{2.4}$$

The arguments in [19] then prove a minimal mass self-similar solution $u(t) \in H_x^{1+4/d-}(\mathbf{R}^d)$, which in the defocusing case $N(t) \to \infty$ contradicts conservation of energy (1.5). This is accomplished via proving additional regularity by induction on H_x^s, starting with H_x^ϵ for some $\epsilon > 0$. In order to put $u(t) \in H_x^\epsilon$, [19] used a restriction estimate specialized to the radial case. This estimate is obviously not available in the nonradial case.

In point of fact, in order to start the induction in [19], it is enough to show

$$a_k = \sup_{t \in (0,\infty)} \|P_{>t^{-1/2}2^k} u(t)\|_{L_x^2(\mathbf{R}^d)} \tag{2.5}$$

is rapidly decreasing in k. The solution to (1.1) can be split, $u = v + w$, where v and w solve the coupled equations

$$\begin{aligned} iv_t + \Delta v &= 0, \\ v(1,x) &= P_{>N} u(1), \end{aligned} \tag{2.6}$$

$$\begin{aligned} iw_t + \Delta w &= |u|^{4/d} u, \\ w(1,x) &= P_{\leq N} u(1). \end{aligned} \tag{2.7}$$

We must have

$$\int_0^1 |\frac{d}{dt}\langle w, w\rangle| dt \geq \|P_{>N} u(1)\|_{L_x^2(\mathbf{R}^d)}^2, \tag{2.8}$$

or some of the mass will stick to low frequencies as $N(t) \nearrow \infty$, which gives a contradiction.

$$\frac{d}{dt}\langle w, w\rangle = -2\langle i|u|^{4/d} v, w\rangle.$$

Now let

$$\mathcal{M}(A) = \sup_{T \in (0,\infty)} \|P_{>AT^{-1/2}} u(T)\|_{L_x^2(\mathbf{R}^d)}. \tag{2.9}$$

We prove that for some $\sigma(d) > 0$,

$$\mathcal{M}(2^k) \lesssim \mathcal{M}(2^{\frac{k}{2d}})^{2+2/d} + 2^{-k\sigma}. \tag{2.10}$$

Thus we prove $\mathcal{M}(2^k)$ is rapidly decreasing. By interpolation, for

$$\mathcal{S}(A) = \sup_{T>0} \|P_{>AT^{-1/2}} u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbf{R}^d)}, \tag{2.11}$$

and

$$\mathcal{N}(A) = \sup_{T>0} \|P_{>AT^{-1/2}}(|u|^{4/d}u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbf{R}^d)}, \tag{2.12}$$

$\mathcal{S}(2^k)$ and $\mathcal{N}(2^k)$ are rapidly decreasing in k. Then following the arguments in [19] we can prove $u(t) \in H_x^{1+4/d-}(\mathbf{R}^d)$. This excludes the $N(t) \sim t^{-1/2}$ case.

3 $N(t) \equiv 1$:

In this talk we are going to exclude the $N(t) \equiv 1$ case. To simplify the talk, we will deal with the case $\xi(t) \equiv 0$ only. In dealing with the case $N(t) \equiv 1$, $d \geq 3$, we make use of the interaction Morawetz estimate proved in [8], [23],

$$\begin{aligned}\int_{-T}^{T}\int_{\mathbf{R}^d\times\mathbf{R}^d}&(-\Delta\Delta a(x,y))|u(t,x)|^2|u(t,y)|^2dxdydt \\ &\lesssim \|u\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbf{R}^d)}\|u\|^3_{L_t^\infty L_x^2([-T,T]\times\mathbf{R}^d)}.\end{aligned} \tag{3.1}$$

With $a(x,y) = |x-y|$. When $d = 3$, $(-\Delta\Delta a(x,y)) = C\delta(|x-y|)$, and when $d \geq 4$,

$$(-\Delta\Delta a(x,y)) = \frac{C(d)}{|x-y|^3}.$$

For all $d \geq 3$,

$$\int_{-T}^{T} N(t)^3 dt \lesssim \int_{-T}^{T}\int_{\mathbf{R}^d\times\mathbf{R}^d}(-\Delta\Delta a(x,y))|u(t,x)|^2|u(t,y)|^2dxdydt.$$

This can be seen more clearly for $d \geq 4$ since most of the mass is concentrated around $|x - x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}$ and $\frac{1}{|x-y|^3} \gtrsim N(t)^3$ when $|x - x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}$ and $|y - x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}$.

If we had $u_0 \in H_x^1(\mathbf{R}^d)$, then by conservation of energy and (3.1) this would imply

$$\int_{-T}^{T} N(t)^3 dt \lesssim 1, \tag{3.2}$$

giving a contradiction for T sufficiently large when $N(t) \equiv 1$. Instead of proving $u(t) \in H_x^1(\mathbf{R}^d)$ for any t, we will localize the solution u to low frequencies. Let I be the Fourier multiplier

$$\widehat{If}(\xi) = \phi(\frac{\xi}{CN})\hat{f}(\xi), \tag{3.3}$$

with $\phi \in C_0^\infty(\mathbf{R}^d)$, ϕ radial, and

$$\phi = \begin{cases} 1, & |\xi| \leq 1; \\ 0, & |\xi| > 2. \end{cases} \tag{3.4}$$

Make a Galilean transformation so that $\xi(0) = 0$ and choose C sufficiently large so that $|\xi(t)| << CN$ when $t \in [-N, N]$. By (1.22), this implies

$$\|Iu\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbf{R}^d)} \lesssim o(N). \tag{3.5}$$

So if

$$\partial_t(Iu) = i\Delta(Iu) - i|Iu|^{4/d}(Iu),$$

then we could apply the exact same arguments as found in [10], [23], and prove

$$\begin{aligned} &\int_{-N}^{N}\int_{\mathbf{R}^d\times\mathbf{R}^d}(-\Delta\Delta a(x,y))|Iu(t,x)|^2|Iu(t,y)|^2 dxdydt \\ &\lesssim \|Iu\|_{L_t^\infty \dot{H}_x^1([-N,N]\times\mathbf{R}^d)}\|Iu\|^3_{L_t^\infty L_x^2([-N,N]\times\mathbf{R}^d)} \lesssim o(N), \end{aligned} \tag{3.6}$$

giving a contradiction for N sufficiently large. But because $I(|u|^{4/d}u) \neq |Iu|^{4/d}Iu$,

$$\partial_t(Iu) = i\Delta(Iu) - i|Iu|^{4/d}(Iu) + i|Iu|^{4/d}(Iu) - iI(|u|^{4/d}u), \tag{3.7}$$

and

$$\begin{aligned} &\int_{-N}^{N}\int_{\mathbf{R}^d\times\mathbf{R}^d}(-\Delta\Delta a(x,y))|Iu(t,x)|^2|Iu(t,y)|^2 dxdydt \\ &\lesssim \|Iu\|_{L_t^\infty \dot{H}_x^1([-N,N]\times\mathbf{R}^d)}\|Iu\|^3_{L_t^\infty L_x^2([-N,N]\times\mathbf{R}^d)} + \mathcal{E}, \end{aligned} \tag{3.8}$$

$\mathcal{E}$ is an error term. It suffices to prove $\mathcal{E} \lesssim o(N)$. To prove this, it suffices to prove that for any $N_j \leq N$,

$$\|P_{>N_j}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([-N,N]\times\mathbf{R}^d)} \lesssim \frac{N^{1/2}}{N_j^{1/2}}. \tag{3.9}$$

We prove (3.9) by induction. When $N(t) \equiv 1$, $\|u\|^{\frac{2(d+2)}{d}([-N,N]\times\mathbf{R}^d)}_{L_{t,x}^{\frac{2(d+2)}{d}}} \sim N$. Therefore we can partition $[-N, N]$ into $\sim N$ subintervals J_l with

$\|u\|_{L^{2(d+2)/d}_{t,x}} = \epsilon_0$. By Strichartz estimates and conservation of mass, this proves

$$\|u\|_{L^2_t L_x^{\frac{2d}{d-2}}([-N,N]\times\mathbf{R}^d)} \lesssim N^{1/2},$$

which takes care of $N_j \leq 1$.

Next, divide $[-N,N]$ into $\sim \frac{N}{N_j}$ subintervals, with $|\xi(t_1) - \xi(t_2)| \leq \frac{N_j \eta}{1000}$, $\eta > 0$ is a small constant to be chosen later. For simplicity, for the rest of the talk we will concentrate on $d = 3$. Take one such interval, $[a,b]$. By Duhamel's formula,

$$u(t) = e^{i(t-a)\Delta}u(a) - i\int_a^t e^{i(t-\tau)\Delta}|u(\tau)|^{4/3}u(\tau)d\tau. \tag{3.10}$$

$$\|P_{|\xi-\xi(t)|>N_j}u\|_{L^2_t L^6_x([a,b]\times\mathbf{R}^3)} \leq \|P_{|\xi-\xi(a)|>\frac{N_j}{2}}u\|_{L^2_t L^6_x([a,b]\times\mathbf{R}^3)}$$

$$\lesssim 1 + \|P_{|\xi-\xi(a)|>\frac{N_j}{2}}(|u|^{4/3}u)\|_{L^2_t L_x^{6/5}([a,b]\times\mathbf{R}^3)}.$$

Without loss of generality suppose $\xi(a) = 0$.

$$(|u|^{4/d}u) = (|u_{\leq \eta N_j}|^{4/d}(u_{\leq \eta N_j}))$$

$$+O(|u_{>\eta N_j}||u_{|\xi-\xi(t)|>C_0}|^{4/d}) + O(|u_{>\eta N_j}||u_{|\xi-\xi(t)|\leq C_0}|^{4/d}).$$

Using [28] and induction we can prove

$$\|P_{>N_j}(|u_{\leq\eta N_j}|^{4/d}u_{\leq\eta N_j})\|_{L^2_t L_x^{\frac{2d}{d+2}}} \leq C\eta^{1/2}\frac{N^{1/2}}{N_j^{1/2}}. \tag{3.11}$$

Next, choose $C_0(\epsilon)$ sufficiently large so that

$$\|u_{>C_0}\|_{L^\infty_t L^2_x} \leq \epsilon(\eta).$$

$$\||u_{>\eta N_j}||u_{>C_0}|^{4/d}\|_{L^2_t L_x^{\frac{2d}{d+2}}} \leq C\eta^{-1/2}N_j^{-1/2}N^{1/2}\epsilon(\eta)^{4/d}. \tag{3.12}$$

Similarly, choose a cutoff function $\chi(x - x(t))$, $\chi \equiv 1$ for $|x - x(t)| \leq C_0$.

$$\||u_{>\eta N_j}||u_{\leq C_0}|^{4/d}(1-\chi(t))\|_{L^2_t L_x^{\frac{2d}{d+2}}} \leq C\eta^{-1/2}N_j^{-1/2}N^{1/2}\epsilon(\eta)^{4/d}. \tag{3.13}$$

Finally, we use a bilinear estimate to attack

$$\| |u_{>\eta N_j}| |u_{\leq C_0}|^{4/d} \chi(t)) \|_{L_t^2 L_x^{\frac{2d}{d+2}}}. \tag{3.14}$$

This term is the "main term", since the mass is concentrated in both space and frequency. If $\hat{u}_0$ is supported on $|\xi| \leq M$ and $\hat{v}_0$ is supported on $|\xi| \geq N$, $M << N$,

$$\|(e^{it\Delta}u_0)(e^{it\Delta}v_0)\|_{L^2_{t,x}(\mathbf{R}\times\mathbf{R}^d)} \lesssim \frac{M^{(d-1)/2}}{N^{1/2}} \|u_0\|_{L^2_x(\mathbf{R}^d)} \|v_0\|_{L^2_x(\mathbf{R}^d)}. \tag{3.15}$$

We partition $[a,b]$ into $\sim N_j$ small intervals with $\|u\|_{L^{10/3}_{t,x}(J_l \times \mathbf{R}^3)} \leq \epsilon_0$. Then the linear solution dominates over each small interval.

$$\begin{aligned} &\| |u_{>\eta N_j}| |u_{\leq C_0}|^{4/3} \chi(t) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_l \times \mathbf{R}^3)} \\ \lesssim &\|(u_{\leq C_0})(u_{>\eta N_j})\|_{L^2_{t,x}(J_l\times\mathbf{R}^3)} \|\chi(t)\|_{L_t^\infty L_x^6(J_l\times\mathbf{R}^d)} \|u\|_{L_t^\infty L_x^2(J_l\times\mathbf{R}^3)} \\ \lesssim &C_0^{3/2} \frac{N^{1/2}}{\eta^{1/2} N_j^{1/2}}. \end{aligned}$$

Therefore, by induction, when $d = 3$,

$$\begin{aligned} &\|u_{>N_j}\|_{L_t^2 L_x^6([-N,N]\times\mathbf{R}^3)} \\ \leq &C(d)C\eta^{1/2}(\frac{N}{N_j})^{1/2} + C(d)C\epsilon(\eta)^{4/3}\eta^{-1/2}(\frac{N}{N_j})^{1/2} + C(d)C_0(\epsilon)^{3/2}(\frac{N}{N_j})^{1/2}. \end{aligned} \tag{3.16}$$

We choose η sufficiently small so that $C(d)\eta^{1/2} << 1$. Then we choose $\epsilon(\eta)$ sufficiently small so that $C(d)\eta^{-1/2}\epsilon(\eta)^{4/3} << 1$. Finally, choose C such that $C(d)C_0(\epsilon)^{3/2} << C$ to close the induction. We make a similar argument for $d \geq 4$.

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