Normal forms and the *upside-down* I-method: growth of higher Sobolev norms for periodic NLS

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with
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Nonlinear Schrödinger equation (NLS)

We consider one-dimensional periodic defocusing NLS:

$$\text{(NLS)} \qquad \begin{cases} iu_t - u_{xx} + |u|^{2p}u = 0 \\ u\big|_{t=0} = u_0 \in H^s(\mathbb{T}), \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R},$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $p \in \mathbb{N}$

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- NLS is a Hamiltonian PDE:
$$u_t = i\frac{\partial H}{\partial \bar{u}},$$
with Hamiltonian
$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 + \frac{1}{2p+2}\int_{\mathbb{T}} |u|^{2p+2}$$

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- NLS also conserves the L^2-norm and the momentum

$\Longrightarrow$ a priori control on the H^1-norm of solutions

Bourgain '93 proved local well-posedness of NLS (in subcritical sense):

- in $L^2(\mathbb{T})$ for the cubic NLS ($p = 1$),
- in $H^s(\mathbb{T})$, $s > 0$, for the quintic NLS ($p = 2$),
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Goal: Fix $s > 1$. Establish upperbounds on growth of H^s-norms of solutions (*without* using complete integrability if $p = 1$)

Motivation: Growth of Sobolev norms may be interpreted as a manifestation of *low-to-high frequency cascade*, and thus establishing upper- and lower-bounds is a physically relevant study.

- Exponential bound

  -By iterating local theory: $\|u(t+\tau)\|_{H^s} \leq C\|u(t)\|_{H^s}$, we obtain

$$\|u(t)\|_{H^s} \lesssim C_1 e^{C_2|t|},$$

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- Polynomial bound: Bourgain '96

  -Suppose that there exists $\delta = \delta(s,p) > 0$ such that an improved iteration bound holds:

$$\|u(t+\tau)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-\delta}$$

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  $\Longrightarrow$ This implies

$$\|u(t)\|_{H^s} \lesssim C(1+|t|)^{\frac{1}{\delta}}$$

  where $C = C(s, p, u_0)$

  Staffilani '97: (nonhomogeneous) cubic NLS with $\delta^{-1} = (s-1)+$

- *Upside-down I-method*: Sohinger '10
  - For (nonhomogeneous) cubic NLS ($p = 1$),

$$\|u(t)\|_{H^s} \lesssim (1 + |t|)^{\frac{1}{2}s+}$$

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  - This idea can be applied to other powers. Colliander-Kwon-O '10

  For $p \geq 3$,

$$\|u(t)\|_{H^s} \lesssim (1 + |t|)^{2(s-1)+}$$

  For $p = 1$, the same as $p = 2$.

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Theorem Colliander-Kwon-O '10

Fix $s > 1$. Let u_0 in $H^s(\mathbb{T})$, and u be the global solution to NLS with initial condition u_0.

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(c) $p = 1$: Then, by explicitly computing the terms appearing in the first few steps of NF reduction, we obtain

$$\|u(t)\|_{H^s} \lesssim (1 + |t|)^{\frac{4}{9}(s-1)+}.$$

improvement: $\frac{1}{2}(s-1)+ \Longrightarrow \frac{4}{9}(s-1)+$

- Our argument is closely related to Bourgain '94 on "NF reduction and *I*-method" for establishing GWP of (quintic) NLS in low regularity setting

Upside-down I-method: Fix $s > 1$.

- $\mathcal{D}$ = Fourier multiplier operator with multiplier $m : \mathbb{Z} \to \mathbb{R}$, where

$$m(n) = \begin{cases} 1, & |n| \leq N \\ \left(\frac{|n|}{N}\right)^{s-1}, & |n| > N. \end{cases}$$

  i.e. $\mathcal{D}$ is basically a differentiation operator of order $s-1$.
  Moreover, it satisfies

$$\|\mathcal{D}q\|_{H^1} \leq \|q\|_{H^s} \leq N^{s-1}\|\mathcal{D}q\|_{H^1}.$$

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- *Suppose* that we have $\left|\frac{d}{dt}H(\mathcal{D}q)(t)\right| \lesssim N^{-\beta+}$ for $|t| \leq T$, *assuming* $\|\mathcal{D}q(t)\|_{H^1} \lesssim 1$. Then, we have

$$\|\mathcal{D}q(t)\|_{H^1}^2 \sim H(\mathcal{D}q(t)) \leq H(\mathcal{D}q(0)) + CTN^{-\beta+}, \qquad |t| \leq T.$$

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  By choosing $N \sim T^{\frac{1}{\beta}+}$, we can gurantee $\|\mathcal{D}q(t)\|_{H^1} \lesssim 1$ for $|t| \leq T$.

$$\Longrightarrow \|q(t)\|_{H^s} \lesssim N^{s-1}\|\mathcal{D}q(t)\|_{H^1} \lesssim T^{\frac{1}{\beta}(s-1)+}, \qquad |t| \leq T.$$

  Therefore, we conclude that

$$\|q(t)\|_{H^s} \lesssim (1+|t|)^{\frac{1}{\beta}(s-1)+}.$$

Write the Hamiltonian H as

$$H(q) = H(q, \bar{q}) = \sum_n n^2 |q_n|^2 + \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}}$$

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Differentiating in time, we obtain

$$\begin{aligned}
\frac{d}{dt} H(\mathcal{D}q) &= \frac{\partial H}{\partial q}(\mathcal{D}q) \cdot \mathcal{D}q_t + \frac{\partial H}{\partial \bar{q}}(\mathcal{D}q) \cdot \overline{\mathcal{D}q}_t \\
&= i \sum_n m(n)^2 n^2 \bigg(\bar{q}_n \frac{\partial N}{\partial \bar{q}_n}(q) - q_n \frac{\partial N}{\partial q_n}(q) \bigg) \\
&+ i \sum_n m(n) n^2 \bigg(q_n \frac{\partial N}{\partial q_n}(\mathcal{D}q) - \bar{q}_n \frac{\partial N}{\partial \bar{q}_n}(\mathcal{D}q) \bigg) \\
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&\quad + i \sum_n m(n) n^2 \bigg(q_n \frac{\partial N}{\partial q_n}(\mathcal{D}q) - \bar{q}_n \frac{\partial N}{\partial \bar{q}_n}(\mathcal{D}q) \bigg) \\
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\end{aligned}$$

In the following, we apply the upside-down I-method *after* "simplifying" H into a new Hamiltonian of the form

$$\mathcal{H}(q) = \underbrace{H_0(q)}_{\text{same quadratic part}} + \mathcal{N}(q).$$

Normal form reduction

Normal form reduction is a sequence of symplectic transformations, transforming the nonlinear part $H_1(q) = \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}}$ of the Hamiltonian into expressions involving only *nearly-resonant* monomials for the form

$$q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}$$

where $n_1 - n_2 + \cdots + n_{2r-1} - n_{2r} = 0$ and $|D(\bar{n})| < K$ with

$$D(\bar{n}) := n_1^2 - n_2^2 + \cdots + n_{2r-1}^2 - n_{2r}^2$$

and some large $K > 0$, plus a (non-resonant) error.

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and some large $K > 0$, plus a (non-resonant) error.

Goal: Iterate this procedure so that the transformed Hamiltonian $\mathcal{H}$ consists of the quadratic part $H_0 = \sum_n n^2 |q_n|^2$, the (nearly) resonant part $\mathcal{N}_0$, and the error $\mathcal{N}_r$:

$$\mathcal{H}(q) = H_0(q) + \mathcal{N}_0(q) + \mathcal{N}_r(q).$$

(Then, we apply the upside-down I-method to $\mathcal{H}$.)

Consider (a part of) a Hamiltonian obtained at some stage of this process:

$$\widetilde{H}(q, \bar{q}) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}$$

Divide $\widetilde{H}$ into the resonant part $\widetilde{H}_0$ and non-resonant part $\widetilde{H}_1$, according to

$$\begin{cases} \text{resonant:} & |D(\bar{n})| \leq K \Longrightarrow \widetilde{H}_0 \\ \text{non-resonant:} & |D(\bar{n})| > K \Longrightarrow \widetilde{H}_1. \end{cases}$$

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Now, introduce a Lie transform $\Gamma = \Gamma_F$ to eliminate $\widetilde{H}_1$. Define $F \sim$ "$D^{-1}\widetilde{H}_1$" by

$$F(q,\bar{q}) = \sum_{\substack{n_1-n_2+\cdots-n_{2r}=0\\ |D(\bar{n})|>K}} \frac{c(\bar{n})}{D(\bar{n})} q_{n_1}\bar{q}_{n_2}\cdots q_{n_{2r-1}}\bar{q}_{n_{2r}}.$$

Then, F satisfies the following homological equation:

$$\{H_0, F\} = -\widetilde{H}_1,$$

where $H_0(q) = \sum_n n^2|q_n|^2$ and the Poisson bracket $\{\cdot,\cdot\}$ is defined by

$$\{H_1, H_2\} = i\sum_n \left[\frac{\partial H_1}{\partial q_n}\frac{\partial H_2}{\partial \bar{q}_n} - \frac{\partial H_1}{\partial \bar{q}_n}\frac{\partial H_2}{\partial q_n}\right].$$

Note analogy with I-method & resonant decomposition by CKSTT '08

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Lemma: Chain rule

Let Γ_t be as above. Then, for a smooth function G, we have

$$\frac{d}{dt}(G \circ \Gamma_t) = \{G, F\} \circ \Gamma_t.$$

Proof.

By Chain Rule, we have

$$\begin{aligned} \frac{d}{dt}(G \circ \Gamma_t) &= \frac{\partial G}{\partial q}(q(t)) \cdot q_t + \frac{\partial G}{\partial \bar{q}}(q(t)) \cdot \bar{q}_t \\ &= i\frac{\partial G}{\partial q} \cdot \frac{\partial F}{\partial \bar{q}} - i\frac{\partial G}{\partial \bar{q}} \cdot \frac{\partial F}{\partial q} = \{G(q(t)), F(q(t))\} \end{aligned}$$

since $\overline{\frac{\partial F}{\partial \bar{q}}} = \frac{\partial F}{\partial q}$. $\square$

Let $\Gamma := \Gamma_1$. Then, by the Taylor series expansion of $G \circ \Gamma_t$ centered at $t = 0$,

$$G \circ \Gamma = \sum_{k=0}^{\infty} \frac{1}{k!} \{G, F\}^{(k)},$$

where $\{G, F\}^{(k)}$ denotes the k-fold Poisson bracket of G with F, i.e.

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Suppose $H = H_0 + \widetilde{H}$. Then, the transformed Hamiltonian $H' = H \circ \Gamma$ is given by

$$\begin{aligned} H' &= H \circ \Gamma = H_0 \circ \Gamma + \widetilde{H}_0 \circ \Gamma + \widetilde{H}_1 \circ \Gamma \\ &= H_0 + \widetilde{H}_0 + \underbrace{\widetilde{H}_1 + \{H_0, F\}}_{=0} + \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \end{aligned}$$

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Hence, we have eliminated the non-resonant part $\widetilde{H}_1$ by the Lie transform Γ.

$\Longrightarrow$ Define the resonant part $\widetilde{H}_0'$ and the non-resonant part $\widetilde{H}_1'$ of H' by

$$\begin{aligned} \widetilde{H}_0' &:= \widetilde{H}_0 + \text{resonant part of } \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \\ \widetilde{H}_1' &:= \text{non-resonant part of } \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \end{aligned}$$

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- At each step, the lowest degree among the monomials in the non-resonant part increases at least by two, since $\deg F \geq 4$.

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- Γ acts boundedly on bounded subsets of $H^s(\mathbb{T})$, $s > \frac{1}{2}$. From Sobolev embedding,

$$\begin{aligned}\left\|\frac{\partial F}{\partial \bar{q}}\right\|_{H^s} &\lesssim \sup_{\|p\|_{L^2}=1} K^{-1} \sum_{n} |c(\bar{n})||p_{n_1}||q_{n_2}| \cdots |q_{n_{2r-1}}||q_{n_{2r}}| \\ &\lesssim \sup_{\|p\|_{L^2}=1} \|p\|_{L^2} \|q\|_{L^2} \|q\|^{2r-2}_{H^{\frac{1}{2}+}} \leq \|q\|^{2r-1}_{H^{\frac{1}{2}+}}.\end{aligned}$$

Remarks:

- At each step, the lowest degree among the monomials in the non-resonant part increases at least by two, since $\deg F \geq 4$.
- Γ acts boundedly on bounded subsets of $H^s(\mathbb{T})$, $s > \frac{1}{2}$. From Sobolev embedding,

$$\Big\| \frac{\partial F}{\partial \bar{q}} \Big\|_{H^s} \lesssim \sup_{\|p\|_{L^2}=1} K^{-1} \sum_{n} |c(\bar{n})||p_{n_1}||q_{n_2}| \cdots |q_{n_{2r-1}}||q_{n_{2r}}|$$
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$$\|\Gamma q\|_{L^2} = \|q\|_{L^2}, \qquad \|\Gamma q\|_{H^1} \sim \|q\|_{H^1}.$$

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(L^2-norm preservation) By definition, we have

$$\Gamma_t q = q(t) = q(0) + i \int_0^t \frac{\partial F}{\partial \bar{q}}(q(t'))dt',$$

Let $M(q) = \|q\|^2_{L^2} = \sum_n |q_n|^2$. Then, by Chain rule, we have

$$\frac{d}{dt} M(q(t)) = \{M(q(t)), F(q(t))\} = 0.$$

Goal: By a *finite* sequence of Lie transforms, we transform H into $\mathcal{H}$ of the form

$$\mathcal{H}(q) = \underbrace{H_0(q)}_{\text{quadratic}} + \underbrace{\mathcal{N}_0(q)}_{\text{resonant}} + \underbrace{\mathcal{N}_r(q)}_{\text{error}},$$

assuming that $q = \{q_n\}_{n\in\mathbb{Z}}$ satisfies the following L^2- and H^1-bounds:

(L2) $$\|q\|_{L^2} \leq C_1,$$

(H1) $$\|q\|_{H^1} \leq C_2.$$

Remark: Regard the phase space element q above as really $\mathcal{D}q \in H^1$ for $q \in H^s$.

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We need to define the "norm" $\|\cdot\|$ to measure a size of a Hamiltonian.
Given $\mathcal{N}(q,\bar{q}) = \sum_{n_1-n_2+\cdots-n_{2r}=0} c(\bar{n}) q_{n_1}\bar{q}_{n_2}\cdots q_{n_{2r-1}}\bar{q}_{n_{2r}}$, define the "size" of $\mathcal{N}$ by

$$\|\mathcal{N}\| = \sup_{*} \sum_{n} |c(\bar{n})||q^{(1)}_{n_1}||q^{(2)}_{n_2}|\cdots|q^{(2r)}_{n_{2r}}|$$

where the supremum is taken over factors $q^{(j)}$, $1 \leq j \leq 2r$ such that

- all factors satisfy (L2)
- all except at most two factors also satisfy (H1).

i.e. the supremum is taken over all the factors, allowing two *exceptional* ones.

Proposition: "Algebra" property

Let H_1 and H_2 be homogeneous Hamiltonians. Then, we have

$$\|\{H_1, H_2\}\| \lesssim \|H_1\|\|H_2\|.$$

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Idea of proof: It suffices to prove

$$\Big\| \sum_n \frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} \Big\| \lesssim \|H_1\|\|H_2\|.$$

Consider three cases, depending on the location of the two *exceptional* factors.

(i) both exceptional factors appear in $\partial H_1/\partial q_n$

(ii) exactly one exceptional factor appears in each of $\partial H_1/\partial q_n$ and $\partial H_2/\partial \bar{q}_n$

(iii) both exceptional factors appear in $\partial H_2/\partial \bar{q}_n$

For fixed N (to be chosen in terms of T in the next section), we set $K = N^\delta$. Assume that at some stage of the process, the Hamiltonian is of the form

$$(*) \qquad \mathcal{H}(q) = \sum_n n^2|q_n|^2 + \mathcal{N}_0(q) + \mathcal{N}_1(q) + \mathcal{N}_r(q),$$

where there exists small $\delta > 0$ such that

- resonant part $\mathcal{N}_0$: $|D(\bar{n})| \leq N^\delta$
- non-resonant part $\mathcal{N}_1$: $|D(\bar{n})| > N^\delta$
- remainder part $\mathcal{N}_r$: $\|\mathcal{N}_r\| < N^{-C}$ for some large $C > 0$
- Moreover, we have $\|\mathcal{N}_0\|$, $\|\mathcal{N}_1\| \lesssim 1$.

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By Sobolev embedding along with (L2) and (H1), we have

$$\Big| \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}} \Big| \leq \|q\|_{L^2}^2 \|q\|_{L^\infty}^{2p} \leq \|q\|_{L^2}^2 \|q\|_{H^{\frac{1}{2}+}}^{2p} \lesssim 1.$$

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Hence, the initial Hamiltonian satisfies the above conditions.

$\Longrightarrow$ We proceed by induction with inductive hypothesis (*).

Non-resonant part $\mathcal{N}_1$ is given by

$$\sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > N^\delta}} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}.$$

As before, define $F(q, \bar{q}) = \displaystyle\sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > N^\delta}} \frac{c(\bar{n})}{D(\bar{n})} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}$

such that $\{F, H_0\} = \mathcal{N}_1$. Then, we have

$$\begin{aligned} \mathcal{H}' = \mathcal{H} \circ \Gamma &= H_0 + \mathcal{N}_0 + \mathcal{N}_1 \\ &\quad + \{H_0, F\} + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} + \mathcal{N}_r \circ \Gamma \\ &= H_0 + \mathcal{N}_0 + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} + \mathcal{N}_r \circ \Gamma. \end{aligned}$$

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$$\|F\| \le N^{-\delta}\|\mathcal{N}_1\| \lesssim N^{-\delta}$$

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- By Proposition $\mathcal{N}_r \circ \Gamma$ is "small": $\|\mathcal{N}_r \circ \Gamma\| \lesssim \|\mathcal{N}_r\| \lesssim N^{-C}$
- Higher order terms with sufficiently high degrees are also small

$\Longleftarrow$ new error part $\mathcal{N}_r'$

Remaing terms $\mathfrak{N}$:

$$\mathfrak{N} := \sum_{k=1}^{M} \frac{1}{k!}\{\mathcal{N}_0, F\}^{(k)} + \sum_{k=1}^{M} \frac{1}{k!}\{\mathcal{N}_1, F\}^{(k)} + \sum_{k=2}^{M} \frac{1}{k!}\{H_0, F\}^{(k)}$$

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Proposition and $\|F\| \lesssim N^{-\delta}$

$$\Longrightarrow \quad \|\{\mathcal{N}_0, F\}^{(k)}\| \lesssim N^{-k\delta}\|\mathcal{N}_1\|$$

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Divide $\mathfrak{N}$ into its resonant part $\mathfrak{N}_0$ and its non-resonant part $\mathfrak{N}_1$, and write the new Hamiltonian $\mathcal{H}'$ as

$$\mathcal{H}' = H_0 + \mathcal{N}_0' + \mathcal{N}_1' + \mathcal{N}_r'$$

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$\implies$ By iterating the process sufficiently many times, we can hide non-resonant part in error part

$\implies$ By a *finite* sequence of Lie transforms, we obtain a new Hamiltonian $\mathcal{H}$:

$$\mathcal{H}(q) = \sum_n n^2|q_n|^2 + \mathcal{N}_0(q) + \mathcal{N}_r(q),$$

where $\|\mathcal{N}_0\| \lesssim 1$ and $\|\mathcal{N}_r\| \lesssim N^{-C}$

Upside-down I-method

Apply the upside-down I-method to the transformed Hamiltonian $\mathcal{H} = H_0 + \mathcal{N}$. Differentiating in time, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}(\mathcal{D}q) &= \frac{\partial \mathcal{H}}{\partial q}(\mathcal{D}q)\cdot \mathcal{D}q_t + \frac{\partial \mathcal{H}}{\partial \bar{q}}(\mathcal{D}q)\cdot \overline{\mathcal{D}q}_t \\
&= -i \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) R(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots \bar{q}_{n_{2r}} \\
&\quad + i \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) D(\bar{n}) \mathcal{D}q_{n_1} \overline{\mathcal{D}q}_{n_2} \cdots \overline{\mathcal{D}q}_{n_{2r}} \\
&\quad + i \sum_n m(n) \bigg(\frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q) \frac{\partial \mathcal{N}}{\partial \bar{q}_n}(q) - \frac{\partial \mathcal{N}}{\partial q_n}(q) \frac{\partial \mathcal{N}}{\partial \bar{q}_n}(\mathcal{D}q) \bigg) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}
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where $R(\bar{n}) := m(n_1)^2 n_1^2 - m(n_2)^2 n_2^2 + \cdots - m(n_{2r})^2 n_{2r}^2$

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Note: $\mathrm{I} + \mathrm{II} = 0$ and $\mathrm{III} = 0$ if $\operatorname{supp} q \subset [-N, N]$

$\Longrightarrow$ we can assume that

$$\max(|n_1|, \dots, |n_{2r}|) > N$$

In the following, we assume that $\mathcal{D}q$ satisfies both (L2) and (H1). Then, we have

(decay) $$\|\mathbb{P}_{\geq N}\mathcal{D}q\|_{H^\sigma} \lesssim N^{-1+\sigma}\|\mathcal{D}q\|_{H^1} \lesssim N^{-1+\sigma}$$

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- **Estimate on I:**
  - $\mathcal{N}_0$-contribution: $\Longleftarrow$ worst term

    Suppose $n_3^* \gg N^{\frac{\delta}{2}}$. In this case, it turns out

$$\begin{aligned} |R(\bar{n})| &= |m(n_1)^2 n_1^2 - m(n_2)^2 n_2^2 + \cdots - m(n_{2r})^2 n_{2r}^2| \\ &\lesssim m(n_1)m(n_2)n_3^* n_4^*. \end{aligned}$$

    By Hölder, Sobolev, and (decay), we have

$$\begin{aligned} |\,\mathrm{I}\,| &\lesssim \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} |c(\bar{n})| \cdot m(n_1)q_{n_1} \cdot m(n_2)\bar{q}_{n_2} \cdot n_3^*\, n_4^*\, q_{n_3} \cdots \bar{q}_{n_{2r}} \\ &\lesssim \|\mathbb{P}_{\gtrsim N}\mathcal{D}q\|^2_{H^{\frac{1}{2}+}} \big\||\partial_x|\, q\big\|^2_{L^2} \|q\|^{2r-4}_{H^{\frac{1}{2}+}} \\ &\lesssim N^{-1+}\|\mathcal{D}q\|^{2r}_{H^1} \lesssim N^{-1+}. \end{aligned}$$

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  - $\mathcal{N}_r$-contribution: Use $|R(\bar{n})| \lesssim m(n_1^*)^2(n_1^*)^2$ and $\|\mathcal{D}q\|_{H^1} \lesssim 1$

$$\Longrightarrow |\,\mathrm{I}\,| \lesssim \|\mathcal{N}_r\| < N^{-C}$$

Proof of Theorem (a)

Given $u_0 \in H^s$, let $\mathcal{D}q_0 = \Gamma^{-1}\mathcal{D}u_0$. Then , we have

$$\Big|\frac{d}{dt}\mathcal{H}(\mathcal{D}q)(t)\Big| \lesssim N^{-1+} \quad \text{for } |t| \leq T,$$

assuming $\|\mathcal{D}q(t)\|_{H^1} \lesssim 1$. Then, we have

$$\|\mathcal{D}q(t)\|_{H^1}^2 \sim \mathcal{H}(\mathcal{D}q(t)) \leq \mathcal{H}(\mathcal{D}q(0)) + CTN^{-1+}, \qquad |t| \leq T.$$

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By choosing $N \sim T^{1+}$, we can gurantee $\|\mathcal{D}q(t)\|_{H^1} \lesssim 1$ for $|t| \leq T$.

$$\Longrightarrow \|u(t)\|_{H^s} \lesssim N^{s-1}\|\mathcal{D}u(t)\|_{H^1} \sim N^{s-1}\|\mathcal{D}q(t)\|_{H^1} \lesssim T^{(s-1)+}, \qquad |t| \leq T.$$

Therefore, we conclude that

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{(s-1)+}.$$

Improvement for $p \leq 2$: Theorem (b)

So far, we simply used Sobolev embedding.

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Recall the L^6-Strichartz estimate due to Bourgain '93:

$$\text{(L6)} \qquad \|e^{-it\Delta}\phi\|_{L^6(\mathbb{T}^2)} \lesssim C_N \|\phi\|_{L^2}, \quad \operatorname{supp} \widehat{\phi} \subset [-N, N],$$

where $C_N = \exp\left(C \frac{\log N}{\log\log N}\right) \ll N^{0+}$. From (L6), one obtains

$$\max_{a \in \mathbb{Z}} \Bigg| \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ D(\bar{n}) = a}} |c(\bar{n})| |q^{(1)}_{n_1}| |q^{(2)}_{n_2}| \cdots |q^{(6)}_{n_6}| \Bigg| \lesssim (n_1^*)^{0+} \prod_{j=1}^{6} \|q^{(j)}_{n_j}\|_{L^2}$$

by setting $Q^{(j)}(n_j) = e^{itn_j^2} q^{(j)}_{n_j}$, $j \ge 2$, and $Q^{(1)}(n_1) = e^{-ita} e^{itn_1^2} q^{(1)}_{n_1}$.

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Then, one inductively proves estimates for Hamiltonians with higher order nonlinearity, which appear in the process of the normal form reduction.

In the end, one obtains

$$(*) \qquad \max_{a\in\mathbb{Z}} \Bigg| \sum_{\substack{n_1-n_2+\cdots-n_{2r}=0\\ D(\bar{n})=a}} |c(\bar{n})||q^{(1)}_{n_1}||q^{(2)}_{n_2}|\cdots|q^{(2r)}_{n_{2r}}| \Bigg| \lesssim (n_1^*)^{0+} \prod_{j=1}^{2r} \|q^{(j)}_{n_j}\|_{L^2}.$$

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By (*), we have

$$\begin{aligned} |\, \mathrm{I}\, | &\lesssim N^{\delta} \max_{|a|\leq N^\delta} \sum_{\substack{n_1-n_2+\cdots-n_{2r}=0\\ D(\bar{n})=a}} |c(\bar{n})|\cdot m(n_1)q_{n_1}\cdot m(n_2)\bar{q}_{n_2}\cdot n_3^*\, n_4^*\, q_{n_3}\cdots \bar{q}_{n_{2r}} \\ &\lesssim N^{\delta} \|\mathbb{P}_{\gtrsim N}\mathcal{D}q\|^2_{H^{0+}} \big\| |\partial_x|\, q\big\|^2_{L^2} \|q\|^{2r-4}_{L^2} \\ &\lesssim N^{-2+\delta+}\|\mathcal{D}q\|^{2r}_{H^1}. \end{aligned}$$

This provides an improvement.

Cubic NLS: $p = 1$

For cubic NLS, we can improve the result further by explicitly computing the first few terms appearing in the normal form reduction

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Write the Hamiltonian H as

$$H(q) = \sum_n n^2 |q_n|^2 + \sum_{n_1 - n_2 + n_3 - n_4 = 0} q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} =: H_0(q) + H_1(q).$$

- First, divide H_1 into the resonant part $\mathcal{R}$ and non-resonant part $\mathcal{N}$, depending on $D_1(\bar{n}) = 0$ or $\neq 0$, where

  $D_1(\bar{n}) := n_1^2 - n_2^2 + n_3^2 - n_4^2 = -2(n_1 - n_2)(n_3 - n_2)$.

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  $D_1(\bar{n}) := n_1^2 - n_2^2 + n_3^2 - n_4^2 = -2(n_1 - n_2)(n_3 - n_2)$.
- We further split $\mathcal{R}$ into two parts:

$$\mathcal{R} = \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ D_1(\bar{n}) = 0}} q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} = 2 \sum_{n_1} \sum_{n_3} |q_{n_1}|^2 |q_{n_3}|^2 - \sum_n |q_n|^4 =: \mathcal{R}_1 + \mathcal{R}_2.$$

- By the conservation of the L^2-norm, $\mathcal{R}_1 = 2\mu \sum_n |q_n|^2$, with $\mu = (2\pi)^{-1} \int |u|^2 dx$. By a direct computation, one easily sees that $\{\mathcal{R}_1, F\} = 0$ for homogeneous polynomial F

As the first step of the normal form reduction, define F_1 such that $\{H_0, F_1\} = -\mathcal{N}$. i.e.

$$F_1 = \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4}}{-2(n_1 - n_2)(n_3 - n_2)}.$$

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Let Γ_1 be the Lie transform associated with F_1. Then, we have

$$H' := H \circ \Gamma_1 = \underbrace{H_0 + \mathcal{R}_1}_{=:\widetilde{H}_0} + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\} + \tfrac{1}{2}\{\mathcal{N}, F_1\} + \text{h.o.t.}$$

where $\{\mathcal{R}_2, F_1\} = 2i(\mathcal{I}_0 - \overline{\mathcal{I}_0})$, with $\mathcal{I}_0$ given by

$$\begin{aligned} \mathcal{I}_0 &= \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} |q_{n_4}|^2 \bar{q}_{n_4}}{(n_1 - n_2)(n_3 - n_2)} \\ &= \sum_{\substack{n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0 \\ n_2 \neq n_1, n_3 \\ n_4 = n_5 = n_6}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} q_{n_5} \bar{q}_{n_6}}{(n_1 - n_2)(n_3 - n_2)}. \end{aligned}$$

Next, we introduce two more transformations to eliminate the "non-resonant" part of $\{\mathcal{R}_2, F_1\}$ and $\frac{1}{2}\{\mathcal{N}, F_1\}$. Write

$$\{\mathcal{R}_2, F_1\} = \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \{\mathcal{R}_2, F_1\}^{(\mathrm{nr})}$$
$$\{\mathcal{N}, F_1\} = \{\mathcal{N}, F_1\}^{(\mathrm{r})} + \{\mathcal{N}, F_1\}^{(\mathrm{nr})},$$

depending on $|D_2(\bar{n})| \leq N^\beta$ or $|D_2(\bar{n})| > N^\beta$ for some $\beta > 0$ (to be chosen later), where $D_2(\bar{n})$ is defined by

$$D_2(\bar{n}) := n_1^2 - n_2^2 + n_3^2 - n_4^2 + n_5^2 - n_6^2.$$

Now, define F_2 and F_3 such that

$$\{H_0, F_2\} = -\tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{nr})}$$
$$\{H_0, F_3\} = \{\mathcal{R}_2, F_1\}^{(\mathrm{nr})}$$

Let Γ_2 and Γ_3 be the Lie transforms associated with F_2 and F_3. Then,

$$\begin{aligned} H'' := {} & H \circ \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 \\ & = H_0 + \mathcal{R}_1 + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} + \text{h.o.t.} \end{aligned}$$

with

$$\begin{aligned} \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} &= 2i(\mathcal{I}_1 - \overline{\mathcal{I}_1}), \\ \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} &= 2i(\mathcal{I}_2 - \overline{\mathcal{I}_2}), \end{aligned}$$

where $\mathcal{I}_1$ is the resonant part of $\mathcal{I}_0$ and $\mathcal{I}_2$ is given by

$$\mathcal{I}_2 = \sum_{\substack{n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0 \\ n_2 \neq n_1, n_3 \\ n_5 \neq n_4, n_6 \\ |D_2(\bar{n})| \leq N^\beta}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} q_{n_5} \bar{q}_{n_6}}{(n_1 - n_2)(n_3 - n_2)}.$$

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After this point, we perform the (usual) normal form reductions on the higher order terms . In particular, we use $|D(\bar{n})| \geq N^\delta$ or $< N^\delta$ to distinguish the resonant and non-resonant terms.

After a finite number of iterations, we obtain

$$\mathcal{H} = \widetilde{H}_0 + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} + \underbrace{\mathcal{N}_0 + \mathcal{N}_r,}_{=\text{ h.o.t.}}$$

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- Higher order terms have an extra factor of $N^{-\beta}$: $|c(\bar{n})| < N^{-\beta}$

  $\Longrightarrow$ *larger* β is better (more decay)

- Sums $\{\mathcal{R}_2, F_1\}^{(\mathrm{r})}$ and $\{\mathcal{N}, F_1\}^{(\mathrm{r})}$ are restricted to $|D_2(\bar{n})| \leq N^{\beta}$

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- $\{\mathcal{R}_2, F_1\}^{(\mathrm{r})}$ and $\{\mathcal{N}, F_1\}^{(\mathrm{r})}$ have $(n_1 - n_2)(n_3 - n_2)$ in the denominators

  $\Longrightarrow$ either $(n_1 - n_2)(n_3 - n_2)$ is large or sum is restricted!!

This provides an improvement with $\beta = \frac{1}{4}$.

Remarks and comments

- For cubic NLS, further improvement can be probably achieved by computing more terms in an explicit manner. However, the computation becomes very cumbersome. Moreover, iterating such computation would require exploiting fine (integrable) structure of cubic NLS. In such a case, one may as well use uniform H^k-bounds directly obtained from integrability.
- We plan to use normal form reduction and (upside-down) I-method on gKdV for low regularity GWP as well as upperbounds on growth of high Sobolev norms.
- Can we combine normal form reduction with the idea in Takaoka-Tsutsumi '04 and Nakanishi-Takaoka-Tsutsumi '10 of reducing the nonlinear (nearly) resonant effect to prove local well-posedness of a PDE?
- In applying the I-method, one can add a correction term to non-resonant part for improvement. Colliander-Keel Staffilani-Takaoka-Tao '08. This can be viewed as the first step of normal form reduction.
- The idea used in Germain-Masmoudi-Shatah, Babin-Ilyin-Titi, Kwon-Oh, etc. can be regarded as a version of normal form reduction.