A Blow-up problem related to the Euler equations of incompressible inviscid fluid motion

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Abstract

We study the blow-up of a certain system of ODEs which are coupled in such a way that the "total mass" is preserved. The system of ODEs is a model proposed by the second author and J. Zhu in order to demonstrate the importance of the convection term in the Proudman-Johnson equation, which describes the motion of incompressible fluid. In the present paper, we derive a necessary and sufficient condition for blow-up of solutions, and we provide long time or near blow-up time asymptotic behavior.

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1 Introduction

We study the blow-up phenomena of the following problem, for $u = u(x,t)$,

$$\begin{cases} \dot{u} = f(u) - \int_0^1 f(u(y,t))\,dy, & x \in [0,1],\ t > 0, \\ u(x,0) = u_0(x), & x \in [0,1] \end{cases} \tag{1}$$

where $\dot{} = \frac{d}{dt}$, and $f : \mathbf{R} \to [0,\infty)$ and $u_0 : [0,1] \to \mathbf{R}$ satisfy the following:

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(F) $f(\cdot)$ is continuously differentiable, even, and strictly convex; Further, $0 = f(0) < f(s)$ for $s \neq 0$, $\sup_{s>0} \frac{sf'(s)}{f(s)} < \infty$, and $\int_1^\infty \frac{ds}{f(s)} < \infty$.

(U) $u_0(x)$ is defined everywhere in $[0, 1]$, bounded and measurable, and $\int_0^1 u_0(x)dx = 0$.

Problem (1) arises from what is called the Proudman–Johnson equation, which is written as

$$f_{txx} + ff_{xxx} - f_x f_{xx} = \nu f_{xxxx} \qquad (0 < x < 1), \tag{2}$$

where ν is the viscosity of the fluid. This equation is derived from the Navier-Stokes equations for incompressible viscous fluid, and its solution represents an exact (but unbounded) solution of the Navier-Stokes equations. The well-posedness of the equation is known only partly, see [1, 2, 3, 4] and the references therein. By some reason stated in [4], the convection term ff_{xxx} does not seem to play an important role in the well-posedness. So, an equation where the convection term is neglected were considered in [4] and compared with (2). The equation then becomes: $f_{txx} - f_x f_{xx} = \nu f_{xxxx}$, which can be integrated once and we obtain

$$u_t = \nu u_{xx} + u^2 - \gamma.$$

Here $u = \frac{1}{2} f_x$ and γ depends only on t. Boundary conditions can be associated in many ways but we assume here the periodic boundary condition, by which we have $\int_0^1 u(t, x)dx = 0$. This constraint on u determines the integral constant γ. We then have

$$u_t = \nu u_{xx} + u^2 - \int_0^1 u(t, x)^2 dx. \tag{3}$$

Some properties of the equation were studied in [4] but the case where $\nu = 0$, which is nothing but (1) with $f(u) = u^2$, were examined only in special cases.

The following proposition was proved in [4]:

Proposition 1 *Assume that $f(u) = u^2$ and that u_0 is a piecewise constant function with zero mean. Let $\Omega := \{x \in [0, 1] \mid u_0(x) = \max\ u_0\}$ and $\Omega^c = [0, 1] \setminus \Omega$. Then the following* **(i)** *and* **(ii)** *hold true:*

(i) *If $|\Omega| < \frac{1}{2}$, the solution to (1) blows up in finite time;*

(ii) *If $|\Omega| \geq \frac{1}{2}$, the solution to (1) exists globally in time and the solution satisfies*

$$\lim_{t \to \infty} u(x, t) = \begin{cases} 0 & \text{if } |\Omega| > 1/2, \\ Q\{I_\Omega(x) - I_{\Omega^c}(x)\} & \text{if } |\Omega| = 1/2, \end{cases} \tag{4}$$

where $|\ |$ denotes the Lebesgue measure, Q is a positive constant, and I_A stands for the characteristic function of the set A.

They were unable to determine asymptotic behavior when general initial data, not necessarily piecewise constant, were assumed. Also, there was uncertainty in the case **(i)**. In fact, based on numerical experiments, the second author and J. Zhu made the following speculation [4]: *a solution to (1) with piecewise constant initial data blows up in such a way that the maximum of $u(\cdot,t)$ tends to infinity while all non–maximum values of $u(\cdot,t)$ becomes negative eventually.*

The purpose of this paper is to prove the speculation under some assumption, and to extend Proposition 1 to general initial data u_0 and nonlinear f.

Remark 1.1 *As for the set of the initial data, we take $L^\infty(0,1)$. It should, however, be noted that we henceforth do not employ the usual convention that two functions differing only on the set of zero measure are regarded as identical. Even two initial data differing only at a point give different solutions for* (1). *We therefore consider all the bounded measurable function defined everywhere in* $[0,1]$ *and the norm is defined* $\|f\|_\infty = \sup_{0\le x\le 1}|f(x)|$, *where* sup, *not* ess sup, *is used. Even though we follow this unusual rule, the following remark is important: it is enough to consider one initial datum among those initial functions which differ from one another on a set of measure zero. In fact, for two functions u_0 and v_0 differing only on a set of measure zero, it holds that $\{x\ ;\ u(x,t)\neq v(x,t)\} = \{x\ ;\ u_0(x)\neq v_0(x)\}$ for all t, where u and v denote solutions corresponding to the initial data u_0 and v_0, respectively. Accordingly, once the solution*

$$\dot{u} = f(u) - \int_0^1 f(u(y,t))dt \quad \& \quad u(y,0) = u_0(y)$$

is known, we can compute $v(x,t)$ since $\int_0^1 f(u(x,t))dx \equiv \int_0^1 f(v(x,t))dx$.

Because of the fact stated in Remark 1.1, we assume without losing generality that $\sup u_0 = \operatorname{esssup} u_0$ and $\inf u_0 = \operatorname{ess\,inf} u_0$. Since the right hand side of (1) is Lipschitz continuous mapping in L^∞, (1) admits a local (in time) solution for any bounded initial data u_0. The solution can be extended as long as it is bounded.

We now prepare the following symbols. We denote by $[0,T^*)$ the maximal existence interval of (1). We define

$$m_0 = \inf\{u_0(\cdot)\}, \quad M_0 = \sup\{u_0(\cdot)\}, \tag{5}$$

$$\Omega = \{x\in[0,1] \mid u_0(x) = M_0\}, \quad \Omega^c = [0,1]\setminus\Omega, \quad \mu^* = \tfrac{|\Omega|}{|\Omega^c|}\ , \tag{6}$$

$$m(t) = \inf\{u(\cdot,t)\}, \quad M(t) = \sup\{u(\cdot,t)\} \quad \forall\, t\in[0,T^*), \tag{7}$$

$$q(t) = \textstyle\int_0^1 f(u(y,t))\,dy. \tag{8}$$

Our result is summarized as follows:

Theorem A *Assume* **(F)** *and* **(U)**. *Then*

$$T^* = \infty \iff |\Omega| \ge \tfrac{1}{2}.$$

In addition, the following hold:

$$\lim_{t\nearrow T^*} \frac{u(x,t)}{M(t)} = I_\Omega(x) - \mu^* I_{\Omega^c}(x) \quad \forall x\in[0,1], \tag{9}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{M(t)}^{\infty}\frac{ds}{f(\mu^* s)-f(s)}=|\Omega^c| \qquad \textit{if } |\Omega|>\tfrac{1}{2}, \tag{10}$$

$$\lim_{t\to\infty} M(t)=Q \qquad \textit{if } |\Omega|=\tfrac{1}{2}\ , \tag{11}$$

$$\lim_{t\nearrow T^*}\frac{1}{t-T^*}\int_{M(t)}^{\infty}\frac{ds}{f(\mu^* s)-f(s)}=|\Omega^c| \qquad \textit{if } |\Omega|<\tfrac{1}{2}, \tag{12}$$

where Q is a non–negative constant, and $Q=0 \iff \int_{\Omega^c}\log(M_0-u_0(x))\,dx=-\infty.$

Theorem B *If $|\Omega|=0$, (9) can be strengthened as follows:*

(i) *If $\int_0^{T^*}\frac{q(t)}{|m(t)|}\,dt<\infty$, there exist a finite m^* and a function $u^*:\Omega^c\to[m^*,\infty)$ such that*

$$\lim_{t\nearrow T^*} m(t)=m^*, \quad \lim_{t\nearrow T^*} u(x,t)=u^*(x)\ \forall x\in\Omega^c, \quad \int_0^1 u^*(x)\,dx=0.$$

(ii) *If $\int_0^{T^*}\frac{q(t)}{|m(t)|}\,dt=\infty$, then $\lim_{t\nearrow T^*} m(t)=-\infty$ and the following holds:*

(iia) *if $\int_0^{T^*}\frac{f(m)}{|m|}\,dt=\infty$, then $\lim_{t\nearrow T^*}(u(x,t)-m(t))=0 \quad \forall x\in\Omega^c$.*

(iib) *if $\int_0^{T^*}\frac{f(m)}{|m|}dt<\infty$, then there exists $v^*:\Omega^c\to[0,\infty)$, such that*

$$\lim_{t\nearrow T^*}(u(x,t)-m(t))=v^*(x)>0 \quad \forall x\in\{y\in[0,1]\mid m_0<u_0(y)<M_0\}.$$

When $f=u^2$, (10)–(12) can be written as

$$\begin{aligned}
&\text{if } |\Omega|>\tfrac{1}{2}, && \lim_{t\to\infty} tM(t)=\tfrac{|\Omega^c|}{|\Omega|-|\Omega^c|}\ ;\\
&\text{if } |\Omega|=\tfrac{1}{2}, && \lim_{t\to\infty} M(t)=\tfrac{1}{2}\exp\left\{2\int_{\Omega^c}\log(M_0-u_0(y))dy\right\}\ ;\\
&\text{if } |\Omega|<\tfrac{1}{2}, && \lim_{t\nearrow T^*}(t-T^*)M(t)=\tfrac{|\Omega^c|}{|\Omega|-|\Omega^c|}.
\end{aligned}$$

Also, when $f=u^2$ and $|\Omega|=0$, we have

$$\begin{aligned}
\int_0^{T^*}\tfrac{q}{|m|}dt=\infty &\iff \int_0^1\tfrac{dx}{M_0-u_0(x)}=\infty,\\
\int_0^{T^*}\tfrac{f}{|m|}dt=\infty &\iff \int_0^1\log(M_0-u_0(x))=-\infty,\\
u^*(x)&=\left\{\tfrac{1}{M_0-u_0(x)}-\int_0^1\tfrac{dy}{M_0-u_0(y)}\right\}\ \exp\left(2\int_0^1\log(M_0-u_0(y))dy\right)\ ,\\
v^*(x)&=\left\{\tfrac{1}{M_0-u_0(x)}-\tfrac{1}{M_0-m_0}\right\}\ \exp\left(2\int_0^1\log(M_0-u_0(y))dy\right).
\end{aligned}$$

We thus obtained a rather complete description of asymptotic behavior of the solutions when **(F)** is assumed. However, for general f, we have only partial results (not presented here) which are not enough to provide necessary and sufficient conditions (in terms of initial data u_0) relating conclusions (i), (iia), and (iib).

2 Preliminary

For each $\alpha \in (-\infty, M_0]$, we define $w(\alpha, t)$ as the solution to the initial value problem

$$\dot{w} = f(w) - q(t) \quad \text{for} \quad t \in [0, T^*), \qquad w(\alpha, 0) = \alpha. \tag{13}$$

The properties of the solution are summarized as follows:

Lemma 2.1 *Let M_0, m_0, M, m, μ^*, and q be defined as in (5)–(8). We then have*

(a) *For all $t \in [0, T^*)$, $m(t) = w(m_0, t)$, $M(t) = w(M_0, t)$, and $\int_0^1 u(y, t)\, dy = 0$.*

(b) *If $\alpha_1 < \alpha_2 \le M_0$, then $w(\alpha_1, t) < w(\alpha_2, t)$ for all $t \in [0, T^*)$. In addition, if $u \not\equiv 0$, then that $w(\alpha, \hat{t}) \le 0$ for some $\hat{t} \in [0, T^*)$ implies $w(\alpha, t) < 0$ for all $t \in (\hat{t}, T^*)$.*

(c) *If $u_0 = M_0[I_\Omega - \mu^* I_{\Omega^c}]$ for some $M_0 > 0$, then $u(x, t) = M(t)[I_\Omega - \mu^* I_{\Omega^c}]$, where*

1. *if $\mu^* = 1$, i.e., $|\Omega| = \frac{1}{2}$, then $M(t) \equiv M_0$,*
2. *if $\mu^* > 1$, i.e., $|\Omega| > \frac{1}{2}$, then $M(t) \le M_0$ and*

$$\int_{M(t)}^{M_0} \frac{ds}{f(\mu^* s) - f(s)} = |\Omega^c|\, t, \quad \forall t \in [0, \infty).$$

3. *if $\mu^* < 1$, i.e., $|\Omega| < \frac{1}{2}$, then $M(t)$ blows up in finite time and*

$$\int_{M_0}^{M(t)} \frac{ds}{f(s) - f(\mu^* s)} = |\Omega^c|\, t, \quad \forall t \in [0, \infty).$$

The proof is straightforward and is omitted. Note that the case (c) is known in [4] and that $\mu^* < 1$ implies $f(s) - f(\mu^* s) \ge (1 - \mu^*) f(s)$.

Suppose now that the initial data is different from the one in the case (c) of the lemma above. In view of Remark 1.1, we may then assume that the set $\{x \in [0,1] \mid m_0 < u_0(x) < M_0\}$ has positive measure.

Lemma 2.2 *Let*

$$\theta(x, t) = \tfrac{M(t) - u(x,t)}{M(t) - m(t)}, \qquad \mu(t) = \tfrac{|m(t)|}{M(t)}. \tag{14}$$

Then $\mu = \frac{\int_0^1 (1-\theta) dx}{\int_0^1 \theta dx}$ and for all $x \in [0, 1]$ and $t \in [0, T^*)$,

$$\dot{\theta} \;=\; \tfrac{1}{M - m}\Big\{\theta f(m) + (1 - \theta) f(M) - f(\theta m + (1 - \theta) M)\Big\}\,. \tag{15}$$

Consequently, the following holds:

(i) *$\theta(x, \cdot) \equiv 0$ if $x \in \Omega$, $\theta(x, \cdot) \equiv 1$ if $u_0(x) = m_0$, and $\theta(x, \cdot) \in (0, 1)$ and $\dot{\theta}(x, \cdot) > 0$ if $u_0(x) \in (m_0, M_0)$;*

(ii) *there exists* $\theta^* : [0,1] \to [0,1]$ *such that as* $t \nearrow T^*$,

$$\begin{array}{rcl} \theta(x,t) & \nearrow & \theta^*(x) \qquad \forall\, x \in \{y \mid m_0 < u_0(y) < M_0\}, \\ \mu(t) & \searrow & \frac{1-\int_0^1 \theta^*(y)\,dy}{\int_0^1 \theta^*(y)dy} \geq \frac{|\Omega|}{|\Omega^c|} = \mu^*. \end{array}$$

Proof. The identity $\mu = \frac{\int_0^1 (1-\theta)dx}{\int_0^1 \theta dx}$ follows from $u = \theta\, m + (1-\theta)\, M$ and $\int_0^1 u(y,t)dy = 0$. The differential equation (15) follows by a direct differentiation.

Since f is strictly convex and $f(0) = 0$ is the global minimum, one sees that $\theta f(m) + (1-\theta)f(M) - f(\theta m + (1-\theta)M) > 0$ if $\theta \in (0,1)$. The assertion (i) and (ii) thus follows from (15).

∎

Lemma 2.3 *Let* θ *and* μ *be defined as in* (14). *Then*

$$\dot{\theta} = \theta(1-\theta)\kappa, \tag{16}$$

$$\frac{\min\{1,\mu\}}{1+\mu}\left\{\frac{f(M)}{M} + \frac{f(m)}{|m|}\right\} \leq \kappa \leq f'(M) + |f'(m)|. \tag{17}$$

Proof. Since $u = \theta\, m + (1-\theta)M$, we have $\kappa = \frac{f(m)-f(u)}{u-m} + \frac{f(M)-f(u)}{M-u} \leq |f'(m)| + f'(M)$ since f is convex and $f(0) = 0$ is the global minimum of f.

To find the lower bound for κ, we notice that $f(u)/u \leq f(M)/M$ for $u \in (0,M]$ and $f(u)/|u| \leq f(m)/|m|$ when $u \in [m,0)$. Replacing $f(u)$ by $uf(M)/M$ when $u \geq 0$ and by $uf(m)/m$ when $u \leq 0$ we then find that $\kappa \geq \frac{\min\{1,\mu\}}{1+\mu}(f(M)/M + f(m)/|m|)$. This completes the proof.

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Lemma 2.4 $\displaystyle\int_0^{T^*} \frac{f(M)+f(m)}{M+|m|}\,dt = \infty.$

Proof. Integrating $\frac{d}{dt}\log(M-m) = \frac{f(M)-f(m)}{M-m}$ gives

$$M(t) + |m(t)| = (M_0 + |m_0|)\exp\left\{\int_0^t \frac{f(M)-f(m)}{M+|m|}d\tau\right\}. \tag{18}$$

If the assertion of the lemma is not true, then both $\int_0^{T^*} \frac{f(M)}{M+|m|}$ and $\int_0^{T^*} \frac{f(m)}{M+|m|}$ are finite, whence, by (18), there exist positive constants c_1 and c_2 such that $c_1 \leq M + |m| \leq c_2$ for all $t \in [0,T^*)$. In particular, $T^* = \infty$. Since $\frac{f(M)+f(m)}{M+|m|} \geq \frac{2}{M+|m|}f((M+|m|)/2)$ is bounded from below by a positive constant, the integral for $\frac{f(M)+f(m)}{M+|m|}$ cannot be convergent. This is a contradiction and we are done.

∎

Lemma 2.5 *Let θ^* be as in* Lemma 2.2(ii). *Then $\theta^* = I_{\Omega^c}$ and hence $\mu(t) \searrow \mu^*$ as $t \nearrow T^*$. In addition, when $|\Omega| \neq \frac{1}{2}$,*

$$\lim_{t\nearrow T^*} \frac{\dot{M}}{f(M) - f(-\mu^* M)} = |\Omega^c|. \tag{19}$$

Proof. Obviously $\theta^*(x) = 0$ if $x \in \Omega$. Suppose now that $x \in \Omega^c$. If $\theta^*(x) < 1$, then $\theta^*(z) \leq \theta^*(x) < 1$ for all $z \in \{y \in [0,1] \mid u_0(y) \geq u_0(x)\}$. It then follows that $\mu(0) \geq \mu(t) \geq (1 - \theta^*(x)) \times \text{measure}\{y \in [0,1] | u_0(y) \geq u_0(x)\} > 0$. (Here $\operatorname{ess\,sup} u_0 = \sup u_0$ is used.) Hence from Lemmas 2.3 and 2.4 we see that $\int_0^{T^*} \kappa\, dt = \infty$. Since

$$\int_{\theta(x,0)}^{\theta(x,t)} \frac{d\theta}{\theta(1-\theta)} = \int_0^t \kappa\, dt,$$

we have $\theta^*(x) = 1$. Thus, $\theta^*(x) = 1$ for all $x \in \Omega^c$.

Once we know θ^*, we obtain $\mu(t) \searrow \int_0^1 (1-\theta^*) dx / \int_0^1 \theta^* dx = |\Omega|/|\Omega^c| = \mu^*$ as $t \nearrow T^*$.

As $u = \theta m + (1-\theta) M = M[1 - \theta - \mu\theta]$ and $q = \int_0^1 f(u)\, dx$, we can write, when $\mu^* \neq 1$,

$$\frac{\dot{M}}{f(M) - f(-\mu^* M)} = |\Omega^c| - \int_{\Omega^c} \frac{f(M[1-\theta-\theta\mu]) - f(-\mu^* M)}{f(M) - f(-\mu^* M)}\, dx\ .$$

The assertion of the Lemma then follows from Lebesgue's dominated convergence theorem and the fact that $1 - \theta - \mu\theta \to -\mu^*$ as $t \nearrow T^*$ for all $x \in \Omega^c$. Here we need the assumption that $\sup_{s>0} \frac{sf'(s)}{f(s)} < \infty$, which implies $\lim_{\eta\to 1} \sup_{s>0} \frac{f(\eta s)}{f(s)} = 1$.

■

3 Proof of Theorems A and B

We consider four different cases

$$|\Omega| > \tfrac{1}{2}, \quad |\Omega| = \tfrac{1}{2}, \quad \tfrac{1}{2} > |\Omega| > 0, \quad |\Omega| = 0.$$

3.1 The case $|\Omega| \geq \frac{1}{2}$

Lemma 3.1 *Assume that $|\Omega| \geq \frac{1}{2}$. Then, $M(t)$ is non-increasing, $m(t)$ is non-decreasing, and $T^* = +\infty$. If $|\Omega| > \frac{1}{2}$, then $M(t)$ is strictly decreasing and $m(t)$ is strictly increasing.*

Proof. From $0 = \int_0^1 u(y,t)\, dy \geq M|\Omega| + m|\Omega^c|$, we have $|m(t)| \geq \mu^* M(t)$. As $\mu^* \geq 1$, $q \leq |\Omega| f(M) + |\Omega^c| f(m)$ so that $\dot{m} = f(m) - q \geq \{f(m) - f(M)\}|\Omega| \geq 0$ for all $t \in [0, T^*)$. Hence, $M(t) < |m(t)| < m_0$ for all $t \in (0, T^*)$ and $T^* = \infty$.

Since f is convex, $\frac{1}{|\Omega^c|} \int_{\Omega^c} f(u)\, dy \geq f(\frac{1}{|\Omega^c|} \int_{\Omega^c} u\, dy) = f(-\mu^* M) = f(M)$, so that $\dot{M} = |\Omega^c| f(M) - \int_{\Omega^c} f(u) \leq 0$ for all $t > 0$.

We now have $m_0 \leq m(t) < 0 < M(t) \leq M_0$ for all t and $T^* = +\infty$ follows. Statements in the case of $|\Omega| = \frac{1}{2}$ is obvious now. Also, when $|\Omega| > 1/2$, we have $\mu^* > 1$, so the previous derivation gives us $\dot{m} > 0 > \dot{M}$.

■

Theorem 3.1 *Assume that* $|\Omega| > \frac{1}{2}$. *Then* $T^* = \infty$, $M(t) < |m(t)|$ *for all* $t \geq 0$, $m(t) \nearrow 0$ *as* $t \nearrow \infty$, *and*

$$\lim_{t\to\infty} \frac{u(x,t)}{M(t)} = I_\Omega(x) - \mu^* I_{\Omega^c}(x) \qquad \forall x \in [0,1], \tag{20}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_{M(t)}^{\infty} \frac{ds}{f(\mu^* s) - f(s)} = |\Omega^c| \,. \tag{21}$$

Proof. By the proceeding lemma, it is sufficient to prove (20) and (21). Using Lemma 2.5, we obtain $u/M = 1 - \theta - \theta\mu \to 1 - \theta^* - \theta^*\mu^* = I_\Omega - \mu^* I_{\Omega^c}$ as $t \to T^*$. This proves (20). The formula (21) follows from L'Hospital's rule and (19).

■

Remark 3.1 *To prove* $T^* = \infty$, *we only need* f *to be even and non–decreasing in* $[0, \infty)$.

3.2 The case $|\Omega| = \frac{1}{2}$

Theorem 3.2 *Assume that* $|\Omega| = \frac{1}{2}$. *Then* $T^* = \infty$ *and*

$$\lim_{t\to\infty} \frac{u(x,t)}{M(t)} = I_\Omega(x) - I_{\Omega^c}(x) \quad \forall x \in [0,1], \tag{22}$$

Also, there exists $Q \in [0, M_0]$ *such that*

$$m(t) \nearrow -Q \quad \textit{and} \quad M(t) \searrow Q \qquad \textit{as} \;\; t \nearrow \infty. \tag{23}$$

In addition,

$$Q = 0 \qquad \Longleftrightarrow \qquad \int_{\Omega^c} \log(M_0 - u_0(y))\, dy = -\infty. \tag{24}$$

In the special case that $f(u) = u^2$, Q *is explicitly given by*

$$Q = \tfrac{1}{2} \exp\left(2 \textstyle\int_{\Omega^c} \log[M_0 - u_0(y)]\, dy\right). \tag{25}$$

Proof. By Lemma 3.1, we know that $T^* = \infty$. The limit (22) is proved as in (20). Lemma 3.1 shows that there exist a Q such that (23) holds.

We next prove (24). First of all, since $|m| = \mu M > \mu^* M = M$,

$$-\frac{d}{dt} \log(M - m) = \frac{f(m) - f(M)}{M - m} = \frac{f(|m|) - f(M)}{|m| - M} \, \frac{|m| - M}{M - m}.$$

As f is convex, $\frac{f(|m|)-f(M)}{|m|-M} \leq f'(|m|)$. Also, as $\sup_{s>0} \frac{sf'(s)}{f(s)} \leq p$ for some $p > 0$, $\frac{f(|m|)-f(M)}{|m|-M} \geq \frac{f(|m|)}{|m|} \geq \frac{1}{p} f'(|m|)$.

As $|\Omega| = |\Omega^c| = \frac{1}{2}$, we derive from $|\Omega| M + \int_{\Omega^c} u dx = 0$ and $u = \theta m + (1-\theta) M$ that $\frac{|m|-M}{M-m} = 2\int_{\Omega^c}(1-\theta)\, dx$. Hence,

$$\frac{2}{p} f'(|m|) \int_{\Omega^c} (1-\theta) dx \le -\frac{d}{dt} \log(M-m) \le 2 f'(|m|) \int_{\Omega^c} (1-\theta)\, dx \ .$$

We now use Lemma 2.3 to estimate the integral. From (17) and the fact that $1 = \mu^* < \mu(t) \le \mu(0) = |m_0|/M_0$, we see that there exists a positive constant C depending only on f and $|m_0|/M_0$ such that

$$|f'(m)| \le C\kappa \le C^2 |f'(m)|.$$

Hence, integrating $\frac{d}{dt}\log\theta = \kappa(1-\theta)$ over Ω^c we obtain,

$$\tfrac{2}{pC} \int_{\Omega^c} \frac{d}{dt} \log\theta \, dx \le -\frac{d}{dt}\log(M-m) \le 2C \int_{\Omega^c} \frac{d}{dt}\log\theta\, dx.$$

Integrating over $[0,t]$ we then obtain

$$\tfrac{2}{pC} \int_{\Omega^c} \log\frac{\theta(x,t)}{\theta(x,0)}\, dx \le \log\frac{M_0 - m_0}{M(t)-m(t)} \le 2C \int_{\Omega^c} \log\frac{\theta(x,t)}{\theta(x,0)}\, dx \ .$$

Since as $t \nearrow \infty$, $\theta(x,t) \nearrow 1$ for all $x \in \Omega^c$, (24) then follows by Fatou's lemma and Lebesgue's dominated convergence theorem.

Finally, we consider the special case $f(u) = u^2$. Then for $x \in \Omega^c$, $\frac{d}{dt}\log(M-u) = M+u$. Hence, integrating over t and $x \in \Omega^c$, we then obtain

$$\int_{\Omega^c} \log\frac{M(t)-u(y,t)}{M_0 - u_0(y)}\, dy = 0 \quad \forall t \in [0,\infty) \ .$$

Note that the integral is convergent since $|\log\frac{M(t)-u(y,t)}{M_0-u_0(y)}| = |\int_0^t [M(\tau)+u(y,\tau)]\, d\tau| \le 2\int_0^t |m(\tau)|\, d\tau$. Sending $t \to \infty$, we then obtain by Fatou's Lemma that

$$\lim_{t\to\infty} \log M(t) = \lim_{t\to\infty} \int_{\Omega^c} 2\log\frac{M_0 - u_0(y)}{1 - u(y,t)/M(t)}\, dy = 2\int_{\Omega^c} \log\frac{M_0 - u_0(y)}{2}\, dy \ .$$

This proves (25).

■

3.3 The case $0 < |\Omega| < \frac{1}{2}$

Theorem 3.3 *Assume that* $0 \le |\Omega| < \frac{1}{2}$. *Then* $T^* < \infty$, *and*

$$\lim_{t\nearrow T^*} \frac{u(x,t)}{M(t)} = I_\Omega(x) - \mu^* I_{\Omega^c}(x) \qquad \forall\, x \in [0,1], \tag{26}$$

$$\lim_{t\nearrow T^*} \frac{1}{T^*-t} \int_{M(t)}^\infty \frac{ds}{f(s) - f(-\mu^* s)} = |\Omega^c| \ . \tag{27}$$

Proof. From (19), there exists $\hat{t} \in (0, T^*)$ such that for all $t \in (\hat{t}, T^*)$, $\dot{M} > \frac{1}{2}|\Omega^c|(f(M) - f(\mu^* M)) \geq \frac{1}{2}|\Omega^c|(1-\mu^*)f(M) = (\frac{1}{2} - |\Omega|)f(M) > 0$ since $f(\eta z) \leq \eta f(z)$ for all $\eta \in [0,1]$ and $|z| > 0$. It then follows that

$$T^* - \hat{t} \leq \int_{M(\hat{t})}^{\infty} \frac{ds}{(\frac{1}{2} - |\Omega|)f(s)} < \infty.$$

The limit (26) follows from Lemmas 2.2 and 2.5, and the relation $u/M = 1 - \theta - \theta\mu$, whereas the limit (27) follows from L' Hospital's rule and (19).

■

When $|\Omega| = 0$, (26) only give us $\lim_{t \nearrow T^*} \frac{u}{M} = 0$ in Ω^c. More details will be given later.

3.4 The case $|\Omega| = 0$

Theorem 3.4 *Assume that $|\Omega| = 0$. Then $T^* < \infty$ and*

$$\lim_{t \nearrow T^*} \frac{1}{T^* - t} \int_{M(t)}^{\infty} \frac{1}{f(s)} ds = 1, \tag{28}$$

$$\lim_{t \nearrow T^*} \frac{u(x,t)}{M(t)} = \mu^* = 0 \quad \forall x \in \Omega^c. \tag{29}$$

In addition, the following holds:

(i) *If $\int_0^{T^*} q(t)dt = \infty$, then*

$$\lim_{t \nearrow T^*} m(t) = -\infty, \qquad \lim_{t \nearrow T^*} \frac{u(x,t)}{m(x,t)} = 1 \quad \forall x \in \Omega^c.$$

(ii) *If $\int_0^{T^*} q(t)dt < \infty$, then there exists $m^* \in (-\infty, 0)$ and $u^*(x) : \Omega^c \to [m^*, \infty)$ such that*

$$\lim_{t \nearrow T^*} m(t) = m^*, \qquad \lim_{t \nearrow T^*} u(x,t) = u^*(x) \quad \forall x \in \Omega^c, \qquad \int_0^1 u^*(x)\, dx = 0.$$

Proof. Because of Theorem 3.3, we need only prove the in addition part.

First we show that

$$\limsup_{t \nearrow T^*} u(x,t) < \infty \quad \forall x \in \Omega^c. \tag{30}$$

Note that if $u(x, \cdot)$ is non–positive at some $\hat{t} \in [0, T^*)$, then u is negative for all $t \in (\hat{t}, T^*)$. Hence, if $u_0(x) > 0$, there exists $T(x) \in (0, T^*]$ such that $u(x, \cdot) > 0$ in $[0, T(x))$ and $u(\cdot, t) < 0$ in $(T(x), T^*)$. For all $t \in [0, T(x))$, we can integrate $\dot{u}/f(u) - \dot{M}/f(M) = q/f(M) - q/f(u)$ to obtain

$$\int_{u_0(x)}^{u(x,t)} \frac{ds}{f(s)} - \int_{M_0}^{M(t)} \frac{ds}{f(s)} = \int_0^t \frac{q(\tau)[f(u(x,\tau)) - f(M(\tau)]}{f(M(\tau))f(u(x,\tau))} d\tau < 0.$$

It then follows that $u(x,t) \le v(x)$ where $v(x)$ is defined by

$$\int_{u_0(x)}^{v(x)} \frac{ds}{f(s)} = \int_{M_0}^{\infty} \frac{ds}{f(s)} \quad \forall\, x \in \{y \in [0,1] \mid M_0 > u_0(y) > 0\}.$$

Therefore, (30) holds.

Integrating $\dot m / f(m) = 1 - q/f(m)$, we obtain

$$\int_{m_0}^{m(t)} \frac{ds}{f(s)} = t - \int_0^t \frac{q(\tau)}{f(m(\tau))}\, d\tau \to T^* - \int_0^{T^*} \frac{q(\tau)}{f(m(\tau))}\, ds \text{ as } \ t \nearrow T^*.$$

Hence, $m^* := \lim_{t\nearrow T^*} m(t)$ exists, and $m^* \in [-\infty, 0)$.

Now we consider the case where $\int_0^{T^*} q(t) = \infty$. In this case we must have $m^* = -\infty$, since otherwise, integrating $\dot m \le f(m^*) - q$ over $[0, T^*)$ we would have $m^* \le m_0 + f(m^*)T^* - \int_0^{T^*} q\, dt = -\infty$. Integrating $\dot u = f(u) - q$ gives $u(x,t) = u_0(x) + \int_0^t f(u(s,\tau))\, d\tau - \int_0^t q(\tau)\, d\tau$. Since $u(x,t)$ is uniformly bounded from above, we see that, for each $x \in \Omega^c$, there exists $T(x) \in [0, T^*)$ such that $u(x, T(x)) \le 0$. Now in $[T(x), T^*)$, we have $\frac{d}{dt}(m - u) = f(m) - f(u) \ge 0$, so that $0 \ge m - u \ge m(T(x)) - u(x, T(x))$, i.e., $0 \le 1 - \frac{u(x,t)}{m(t)} \le \frac{m(T(x)) - u(x,T(x))}{m(t)}$ for all $t \ge T(x)$. Sending $t \nearrow T^*$ we then conclude that $\lim_{t\nearrow T^*} \frac{u(x,t)}{m(t)} = 1$. This proves (i).

Next we consider the case where $\int_0^{T^*} q(t)\, dt < \infty$. In this case, we have $\dot m = f(m) - q > -q$, so that $m^* \ge m_0 - \int_0^{T^*} q(\tau)\, d\tau > -\infty$. Thus, for every $x \in \Omega^c$, $u(x,\cdot)$ is bounded in $[0, T^*)$, so that $u(x,t) = u_0(x) + \int_0^t (f(u) - q)\, d\tau \to u_0(x) + \int_0^{T^*} (f(u(x,t)) - q(t))\, dt =: u^*(x)$, as $t \nearrow T^*$.

Finally, we prove that $\int_0^1 u^*(x)\, dx = 0$. Since u is bounded from below, by Fatou's lemma, $\int_0^1 u^*(x)\, dx \le \lim_{t\nearrow T^*} \int_0^1 u(x,t) dx = 0$. Secondly, for each $x \in [0,1]$, $u^*(x) \ge u(x,t) - \int_t^{T^*} q(\tau)\, d\tau$. Hence, $\int_0^1 u^*(x)\, dx \ge -\int_t^{T^*} q(\tau)\, d\tau$. Sending $t \nearrow T^*$ we then conclude that $\int_0^1 u^*(x)\, dx \ge 0$. This completes the proof.

■

Theorem 3.5 *Assume that $|\Omega| = 0$ and denote $m^* = \lim_{t\nearrow T^*} m(t)$. Then*

$$\int_0^{T^*} q(\tau)\, d\tau = \infty \quad \Longleftrightarrow \quad m^* = -\infty \quad \Longleftrightarrow \quad \int_0^{T^*} \frac{q(t)}{|m(t)|} dt = \infty. \tag{31}$$

In addition, when $m^* = -\infty$, the following holds:

(a) *If $\int_0^{T^*} \frac{f(m)}{|m|}\, dt = \infty$, then $\lim_{t\nearrow T^*}(u(x,t) - m(t)) = 0 \quad \forall x \in \Omega^c$.*

(b) *If $\int_0^{T^*} \frac{f(m)}{|m|}\, dt < \infty$, then there exists $v^*(x) : \Omega^c \to [0, \infty)$ such that*

$$\lim_{t\nearrow T^*} (u(x,t) - m(t)) = v^*(x) > 0 \quad \forall x \in \{y \in [0,1] \mid m_0 < u_0(y) < M_0\}.$$

Proof. From $\int_{m_0}^{m} \frac{ds}{f(s)} \leq t < T^*$ we see that there is a positive constant c_0 such that $m(t) \leq -c_0$. Also, integrating $\frac{d}{dt}\log|m| = \frac{f}{m} - \frac{q}{m} \leq \frac{q}{|m|}$ yields $\log\frac{m}{m_0} \leq \int_0^t \frac{q}{|m|}\,d\tau$. Hence,

$$\int_0^{T^*} \frac{q}{|m|}\,dt = \infty \quad \Longrightarrow \quad \int_0^{T^*} q\,dt = \infty \quad \Longrightarrow \quad m^* = -\infty \quad \Longrightarrow \quad \int_0^{T^*} \frac{q}{|m|}dt = \infty,$$

which proves (31).

For any $x \in \Omega^c$ satisfying $u_0(x) > m_0$, let $-\frac{d}{dt}\log(u-m) = \frac{f(m)-f(u)}{u-m} =: K(x,t)$, so that

$$u(x,t) - m(t) = \{u_0(x) - m_0\}\exp\Big\{ -\int_0^t K(x,\tau)\,d\tau\Big\}. \tag{32}$$

Theorem 3.4 defines $T(x) \in [0, T^*)$ as the smallest of those t such that $u(x, T(x)) \leq 0$. Then, in $(T(x), T^*)$, $u(x,t) < 0$ so that, as f is even and convex and $\sup_{s>0}\frac{sf'(s)}{f(s)} \leq p$, $\frac{f(m)}{|m|} \leq K \leq |f'(m)| \leq p\frac{f(m)}{|m|}$. The assertions (a) and (b) then follow from (32). This completes the proof.

∎

Finally, we provide an example that all (i), (ii)(a), and (ii)(b) can happen.

Theorem 3.6 *Assume that $f(u) = u^2$ and that $|\Omega| = 0$. The following holds.*
(i) $\int_0^{T^*} \frac{q}{|m|}\,dt = \infty \iff \int_0^1 \frac{dx}{M_0 - u_0(x)} = \infty$;

In addition, if $\int_0^1 \frac{dx}{M_0-u_0(x)} < \infty$, *then for all* $x \in \Omega^c$,

$$\lim_{t \nearrow T^*} u(x,t) = \left\{\frac{1}{M_0-u_0(x)} - \int_0^1 \frac{dy}{M_0-u_0(y)}\right\} \exp\left(2\int_0^1 \log(M_0 - u_0(y))dy\right). \tag{33}$$

If $\int_0^1 \frac{dx}{M_0-u_0(x)} = \infty$, *then* $\lim_{t\nearrow T^*} m(t) = -\infty$ *and* $\lim_{t\nearrow T^*}\frac{u(x,t)}{m(t)} = 1$ *for all* $x \in \Omega^c$.
(ii) $\int_0^{T^*} \frac{f(m)}{|m|}dt = \infty \iff \int_0^1 \log(M_0 - u_0(x))dx = -\infty$. *More precisely,*

(iia) if $\int_0^1 \log(M_0 - u_0(x))dx = -\infty$, *then* $\lim_{t\nearrow T^*}(u(x,t) - m(t)) = 0$ *for all* $x \in \Omega^c$;

(iib) if $\int_0^1 \log(M_0 - u_0(y))dy > -\infty$, *then*

$$\lim_{t\nearrow T^*}(u(x,t) - m(t)) = \left\{\frac{1}{M_0-u_0(x)} - \frac{1}{M_0-m_0}\right\} \exp\left(2\int_0^1 \log(M_0 - u_0(y))dy\right). \tag{34}$$

Proof. Set $\theta = \frac{M-u}{M-m}$. We have $\frac{d}{dt}\log\theta = u - m$ so that, after integration first in t and then in x, we have

$$\int_{\Omega^c} \log\frac{\theta(x,t)}{\theta(x,0)}dx = -\int_0^t m(\tau)\,d\tau.$$

For any x_1 and x_2 with $u_0(x_1) < u_0(x_2)$, we have, denoting $u_i = u(x_i, t)$ and $\theta_i = \theta(x_i,t)$, $\frac{d}{dt}\log(u_2 - u_1) = u_1 + u_2 = (u_1 - m) + (u_2 - m) + 2m = \frac{d}{dt}\log(\theta_1\theta_2) + 2m$. After integrating in t, we then obtain, denoting $u_{0i} = u_0(x_i)$ and $\theta_{0i} = \theta(x_i, 0)$,

$$\log\frac{u_1-u_2}{u_{01}-u_{02}} = \log\frac{\theta_1\theta_2}{\theta_{01}\theta_{02}} - 2\int_{\Omega^c}\log\frac{\theta}{\theta_0}\,dx.$$

That is, for any $x, y \in \Omega^c$,

$$u(x,t) - u(y,t) = \left\{ \frac{\theta(x,t)\theta(y,t)}{M_0 - u_0(x)} - \frac{\theta(x,t)\theta(y,t)}{M_0 - u_0(y)} \right\} \exp\left(2 \int_{\Omega^c} \log \frac{(M_0 - u_0(z))}{\theta(z,t)} dz \right). \tag{35}$$

(i) Assume that $\int_0^1 \frac{dy}{M_0 - u_0(y)} < \infty$. Then integrating (35) over $y \in \Omega^c$ gives

$$u(x,t) = \int_{\Omega^c} \theta(x,t)\theta(y,t) \left\{ \frac{1}{M_0 - u_0(x)} - \frac{1}{M_0 - u_0(y)} \right\} dy \quad \exp\left(2 \int_{\Omega^c} \log \frac{(M_0 - u_0(z))}{\theta(z,t)} dy \right).$$

Sending $t \nearrow T^*$ and recalling that $\theta \nearrow 1$ pointwise in Ω^c, we then obtain (33).

If $\int_0^{T^*} \frac{q}{|m|} dt < \infty$, then $m^* > -\infty$. Hence, we derive from (35) that $u^*(x) = m^* + \{\frac{1}{M_0 - u_0(x)} - \frac{1}{M_0 - m_0}\} \exp\{2 \int_0^1 \log(M_0 - u_0(z)) dz\}$. As $m^* < 0$ and $\int_0^1 u^*(x) dx = 0$, we must have $\int_0^1 \frac{dx}{M - u_0(x)} < \infty$. This proves the assertion (i).

(iia) Assume that $\int_0^1 \log(M_0 - u_0(x))\, dx = -\infty$. We see from (35) that $\lim_{t \nearrow T^*}(u(x,t) - m(t)) = 0$.

(iib) Assume that $\int_0^1 \log(M_0 - u_0(x)) dx > -\infty$. Sending $t \nearrow T^*$ we obtain (34) from (35). This completes the proof.

■

4 Conclusion

The dynamical system (1) (considered in $L^\infty(0,1)$) has the following property. It has infinite number of steady-states, which are characterized by $|u_0(x)| =$ constant. If $u_0 \in L^\infty$ is a steady-state, then, for all $\lambda \in \mathbf{R}$, λu_0 is a steady-state, too. All the steady-states are unstable. In fact, Theorems A and B shows that any neighborhood of a steady-state contains blow-up solutions.

(1) can be written as

$$\dot{u} = P\left(f(u)\right),$$

where $P = L^2(0,1) \to L^2(0,1)/\mathbf{R}$ is the orthogonal projection. Without the projection, all the solution except for the trivial one $u \equiv 0$ blows up in finite time. We may say that an infinite number of global solutions including steady-states are created by the projection, although almost all solutions blow up in finite time even in the presence of the projection — the set of all the global solutions are of first category.

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