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defined as follows.

$$H^s(S^1) = \left\{ \sum_{n\in\mathbf{Z}} a_n e^{in\sigma} \;\middle|\; a_n = \overline{a_{-n}} \;\; (\forall n), \;\; \sum_{n\in\mathbf{Z}} (1+n^2)^s |a_n|^2 < \infty \right\},$$

where $\mathbf{Z}$ denotes the set of all the integers. This function space can be defined for any real number s, positive or negative. However, we shall use it only for $0 \le s$ in what follows. The symbol $/\mathbf{R}$ implies that only functions with zero means are admitted. Therefore,

$$H^s(S^1)/\mathbf{R} = \left\{ f \in H^s(S^1) \;\middle|\; \int_0^{2\pi} f(\sigma)d\sigma = 0 \right\}.$$

Note that $\omega(0) = 0$ implies that $\int_0^{2\pi} \theta^*(\sigma)d\sigma = 0$, where $\theta^*(\sigma) = \theta(1,\sigma)$. Now we have $\tau(1,\sigma) = H\theta^*$. The equation (1.13) is now written as

$$e^{2H\theta^*}\frac{dH\theta^*}{d\sigma} - pe^{-H\theta^*}\sin\theta^* + q\frac{d}{d\sigma}\left(e^{H\theta^*}\frac{d\theta^*}{d\sigma}\right) = 0 \quad (0 \le \sigma < 2\pi). \tag{1.16}$$

Accordingly the problem is:

Third Formulation. *To find a 2π-periodic function θ^* satisfying* (1.16) *and* $\int_0^{2\pi} \theta^*(\sigma)d\sigma = 0$.

The purpose of the first four chapters is to show the structure of the set of solutions to (1.16). Our first observation is that $\theta^* \equiv 0$ satisfies (1.16) for all $(p,q) \in [0,\infty) \times [0,\infty)$. $\theta^* \equiv 0$ implies $\omega \equiv 0$, hence $df/dz \equiv c$ by (1.12). This means that the flow is uniform: $(u,v) \equiv (c,0)$. Also, (1.15) implies that the wave profile of the solution $\theta^* \equiv 0$ is flat. For this reason, we call $\theta^* \equiv 0$ the trivial solution. What Stokes [166, 167] found is that there are bifurcation points along the branch of this trivial solution. This issue will be discussed in section 1.3.

Remark. The Hilbert transform appears in many branches of applied mathematics. Its mathematical properties are well-known. Some of them play an important role in the existence theorems of gravity waves discussed in Chapter 3. The reader who wishes to learn the properties of the Hilbert transform should consult standard textbooks, for instance [57, 84, 91, 202, 207]

1.2 Waves on fluid of finite depth

Thus far, we have considered only flows of infinite depth for the sake of simplicity. However, it is more natural to consider waves on fluid of finite depth. The work of Levi-Civita [100] was generalized by Struik [168] to the case of finite depth. We now outline his theory.

For the sake of simplicity, we consider only cases where the bottom is a straight horizontal line. In the case where the depth is finite, the conditions (1.5) and (1.7) are retained while (1.8) is replaced by the condition that the bottom bed $\{y = y_0\}$ is a streamline. We write this in the following way:

$$V(x, y_0) \equiv -a \qquad (\ -L/2 < x < L/2\ ) \tag{1.17}$$

where a is a constant. The constant a and the propagation speed c have the same sign, the proof of which goes as follows. We first remark that U is harmonic and that U satisfies Neumann's boundary condition on the bottom and the free boundary (namely the normal derivative of U is zero, which is equivalent to saying that the normal component of the velocity vanishes on the boundary). We then remark that we may assume $c \neq 0w$. To show this, we recall Hopf's maximum principle for for harmonic functions (see, e.g., John, [75]), which says that the outward normal derivative of a harmonic function at its maxima is positive. Now, if $c = 0$, then $U \equiv 0$ by Hopf's maximum principle, hence $df/dz \equiv 0$. Consequently, we have

$$gh(x) - \frac{T}{m}\left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x = s_0 \tag{1.18}$$

where s_0 is a constant. Putting $h_1(x) = h(x) - s_0/g$, multiplying the equality by h_1, and integrating in $x \in (-L/2, L/2)$, we obtain

$$\int_{-L/2}^{L/2}\left(gh_1^2 + \frac{T}{m}\frac{h_{1,x}^2}{\sqrt{1+h_{1,x}^2}}\right)dx = 0,$$

from which follows $h_1 \equiv 0$. Here use has been made of the periodicity of h, hence h_1. Also used is the positivity of g and T/m. Now, $h_1 \equiv 0$ implies that the free boundary is flat and the fluid is at rest. Thus we may assume that $c \neq 0$ without losing generality. If that is the case, then U/c takes its maximum on the side boundary $x = L/2$ and its minimum on $-L/2$. On

the other hand, we have

$$a = \int_{y_0}^{h(L/2)} \frac{\partial V}{\partial y}(L/2, y)dy = \int_{y_0}^{h(L/2)} \frac{\partial U}{\partial x}(L/2, y)dy.$$

The conclusion now follows from Hopf's maximum principle.

Remark. In deriving "$h \equiv$ constant" from (1.18), we have assumed that g and T/m are of the same sign. This assumption is legitimate when we consider the motion of water in the ocean. However, if we consider a water layer which is attached to a ceiling, then g is negative while T/m is positive. If that is the case, the triviality of the solution of the equation (1.18) does not hold. See the problem at the end of this chapter. This is also true if the surface tension is negative while g is positive. Those cases where either g or T/m is negative will be dealt with in Chapter 6.

Since both cases are treated similarly, we henceforth assume that both a and c are positive constants. If we define ζ as in (1.11), then ζ runs in $\eta < |\zeta| < 1$, where $\eta = \exp(-2\pi a/cL)$, and ω is an analytic function in the annulus $\{\eta < |\zeta| < 1\}$. The condition (1.17) implies that the velocity is horizontal on $y = y_0$ and that it is expressed as $\theta = 0$ on $\rho = \eta$. Now the problem is:

Find a function $\omega = \omega(\zeta)$ which is continuous on $\{\eta \le |\zeta| \le 1\}$, is analytic in $\{\eta < |\zeta| < 1\}$, and satisfies the following (1.19) *and* (1.20).

$$e^{2\tau}\frac{\partial \tau}{\partial \sigma} - pe^{-\tau}\sin\theta + q\frac{\partial}{\partial \sigma}\left(e^{\tau}\frac{\partial \theta}{\partial \sigma}\right) = 0 \qquad on \quad \rho = 1, \tag{1.19}$$

$$\theta = 0 \qquad on \quad \rho = \eta. \tag{1.20}$$

In this way the free boundary problem has been transformed to a nonlinear boundary value problem of an analytic function in an annulus. We remark that both Levi-Civita [100] and Struik [168] considered the case where $q = 0$, while Okamoto [129] considered the general case.

As in the case of infinite depth, a further reduction of the equation (1.19) is possible. To this end, the following definition is formulated;