On the interior layer appearing in the similarity solutions of the Navier-Stokes equations

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Dedicated to Professors Masayasu Mimura and Takaaki Nishida
on their sixtieth birthdays

Abstract

We consider a special class of similarity solutions of the stationary Navier-Stokes equations and prove the existence of the solution for all the Reynolds numbers. We further prove that the solution exhibits interior and boundary layers as the Reynolds number tends to $+\infty$ and $-\infty$, respectively.

Key words: Navier-Stokes eqns., interior layer, boundary layer

1 Introduction

We consider the following boundary value problem

$$f'''' - 2ff''' = 0 \qquad (-1 < x < 1) \tag{1}$$

with boundary condition either

$$f(\pm 1) = 0, \qquad f'(\pm 1) = R, \tag{2}$$

or

$$f(-1) = -R, \qquad f(1) = R, \qquad f'(\pm 1) = 0, \tag{3}$$

where prime implies the differentiation and R is a real parameter. This boundary value problem is derived from the Navier-Stokes equations. The purpose of the present paper is to prove (i) the existence of the solution for all real number R, (ii) the existence of boundary layers as $R \to -\infty$, and (iii) the existence of interior layers as $R \to +\infty$.

We first explain how (1) is derived. We consider the Navier-Stokes equations for incompressible viscous fluid motion in the following infinite domain:

$$\Omega = \{(x, y, z)\ ;\ -a < x < a, -\infty < y, z < \infty\},$$

where a is a constant. Following Zhu [32], we assume the following form of the velocity vector:

$$u = f - g, \quad v = -yf_x, \quad w = zg_x, \tag{4}$$

where u, v, and w are x, y, and z components, respectively, of the velocity. f and g are assumed to be functions of (t, x) only; $f = f(t, x)$, $g = g(t, x)$. The subscript implies the differentiation. The velocity $\mathbf{u} = (u, v, w)$ and the pressure p satisfy the Navier-Stokes equations:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \nu \triangle \mathbf{u} - \frac{1}{\rho}\nabla p, &(5)\\ \operatorname{div} \mathbf{u} &= 0. &(6) \end{aligned}$$

The incompressibility condition (6) is automatically satisfied by the ansatz (4). The equation (5) leads to the following coupled equations:

$$\begin{aligned} f_{txx} &= \nu f_{xxxx} + (f_x + g_x) f_{xx} - (f - g) f_{xxx}, &(7)\\ g_{txx} &= \nu g_{xxxx} - (f_x + g_x) g_{xx} - (f - g) g_{xxx}. &(8)\end{aligned}$$

These equations are to be satisfied for $-a < x < a$ and $0 < t$. Note that no approximation is assumed in deriving (7) and (8). Accordingly, any solution of (7) and (8) is what is called the exact solution of the Navier-Stokes equations. The coupled equations (7) and (8) were first derived by Taylor et al. [26] in a different but equivalent form. Later they were used as a blow-up model by [20] and [7].

If the wall located at $x = \pm a$ is assumed to be an elastic membrane and if it is stretched along itself, then the following boundary conditions are imposed:

$$f(t, \pm a) = g(t, \pm a) = 0, \quad f_x(t, \pm a) = \beta_1, \quad g_x(t, \pm a) = \beta_2, \tag{9}$$

where β_1 and β_2 are constants. On the other hand, if injection or suction is permitted with the same rate on two walls, then we have

$$f(t, \pm a) = \pm\beta_1, \quad g(t, \pm a) = \pm\beta_2, \quad f_x(t, \pm a) = g_x(t, \pm a) = 0 \quad (\text{ same sign }). \tag{10}$$

We now assume that $g(t, x) = -f(t, x)$, which forces $\beta_1 = -\beta_2$. This is the case precisely when the flow is axisymmetric. Then it is easily seen that the coupled equations are reduced to the single equation

$$f_{txx} = \nu f_{xxxx} - 2 f f_{xxx} \qquad (-a < x < a). \tag{11}$$

The boundary condition (9) becomes

$$f(t, \pm a) = 0, \qquad f_x(t, \pm a) = \beta_1. \tag{12}$$

By re-defining the variables as

$$x \mapsto a\tilde{x}, \qquad t \mapsto \frac{a^2}{\nu}\tilde{t}, \qquad f \mapsto \frac{\nu}{a}\tilde{f},$$

the equations (11) and (12) becomes, after dropping the tilde,

$$f_{txx} = f_{xxxx} - 2 f f_{xxx} \qquad (-1 < x < 1). \tag{13}$$

and

$$f(t, \pm 1) = 0, \qquad f_x(t, \pm 1) = R, \tag{14}$$

respectively. Here R is defined as

$$R = \frac{\beta_1 a^2}{\nu}$$

and is called the Reynolds number. Since the walls are stretched or contracted, R can take any real numbers, negative or positive. If we consider the steady-states, then we obtain (1) and (2). Similarly we obtain

$$f(t, -1) = -R, \quad f(t, 1) = R, \quad f_x(t, \pm 1) = 0 \tag{15}$$

for the injection/suction case. Here, the Reynolds number R is defined by $R = -a\beta_1/\nu$ with β_1 in (10).

If we assume that $\beta_2 = 0$ and $g \equiv 0$ in (7), (8), and (9) and if only the steady-states are considered, then we have

$$f'''' + f'f'' - ff''' = 0. \tag{16}$$

Depending on the physical situations, many boundary conditions can be associated to this equation. Accordingly quite a lot of studies have appeared for this equation. See, for instance, [4, 6, 11, 18, 19, 23, 28, 29, 31] and references therein. The equation (1) or (16) can also be considered in the semi-infinite interval $0 < x < \infty$; Weyl [30] proved the existence of a solution satisfying certain boundary conditions at $x = 0$ and $x = +\infty$. Taylor et al. [26] considered the stationary solutions of (13) and (15), and showed by numerical computation and asymptotic analysis that some solutions had an interior layer at $x = 0$ as the Reynolds number tends to infinity.

Many exact solutions of the Navier-Stokes equations are known but, as far as we know, none of them has an interior layer except for the boundary value problem considered by [26], though many solutions are known to possess boundary layers. Examples of interior layers for incompressible flows are few in number even in numerical solutions and we think that more searches should be made for interior layers (cf. [12, 21]).

In what follows, we prove mathematically that

- the boundary value problem (1) and (2) has a family of solutions in which an interior layer appears at $x = 0$ as $R \to +\infty$ and boundary layers appears at $x = \pm 1$ as $R \to -\infty$.
- the boundary value problem (1) and (3) has a family of solutions in which an interior layer appears at $x = 0$ as $R \to -\infty$ and boundary layers appears at $x = \pm 1$ as $R \to +\infty$.

Since a mathematically rigorous analysis did not seem to be intended in [26], the analysis in the present paper would be helpful for a better understanding of the Navier-Stokes equations.

From now on, until section 6, we consider (1) and (2). The boundary value problem (1) and (3) is considered in section 7.

2 Numerical experiment

Before going into rigorous analysis, it would be helpful to understand the qualitative properties of the solutions of (1) and (2) which we obtained numerically. We remark that the stability of the solutions which we are going to study is meant in the framework of (13), whence only *the stability among axisymmetric similarity solutions* are considered. If axisymmetry is dropped, a different stability diagram will appear, as is indicated in [20]. Stability among general, non-similar 3D flows is very difficult to consider, due mainly to the unboundedness of the velocity field.

We use the centered finite difference scheme on uniform grids and the resulting system of nonlinear equations are solved by the path-continuation method, see any of [1, 2, 5, 13, 15]. For $-300 \leq R \leq 300$, the interval $[-1, 1]$ is divided evenly into 400 subintervals. For $300 < |R| \leq 600$, it is divided into 800 subintervals.

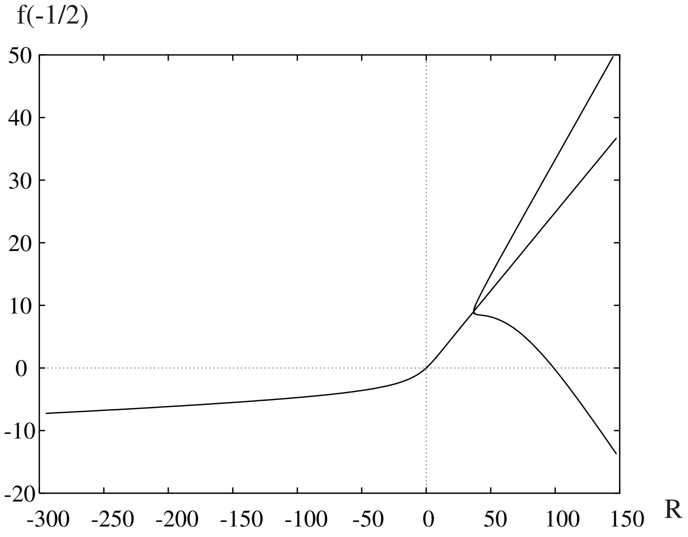

Figure 1: Bifurcation diagram of (1) and (2).

Figure 1 shows the bifurcation diagram of solutions to (1) and (2). Here and hereafter, bifurcation diagrams are drawn in $(R, f(-1/2))$ with the Reynolds number R horizontally. We chose $f(-1/2)$ as the typical magnitude of the solutions. Other values $f(c)$ $(-1 < c < 1)$ may be chosen but we found $f(-1/2)$ more convenient than others from a graphical viewpoint. The branch which pass through the origin ($f \equiv 0, R = 0$) is called the main branch. A secondary branch, which is a supercritical pitchfork, bifurcates at $R \approx 36.8$. No solution other than those on the main branch was found when $R < 0$. Figure 1 shows all the stationary solutions which we have computed. Solutions on the main branch is stable for $-600 < R < 36.8\cdots$. We do not know whether the solutions are stable for all $-\infty < R < 0$ or lose stability at the Reynolds number smaller than -600. The solutions are unstable after they undergo the pitchfork bifurcation at around $R = 36.8$. For $36.8\cdots < R$ the bifurcating solutions are stable. Our experiments show that there exists a Hopf bifurcation point on the main branch at around $R = 386.4\cdots$. (Here time-periodic solutions are considered as solutions to (13) and (14)).

All the solutions f of the main branch are odd functions, while those of the secondary branch are not: see Figure 2.

When $R \to \pm\infty$, the solutions show an interesting asymptotic behavior. We first consider the case where $R \to -\infty$. When R decreases, boundary layers develops, as is shown in Figure 3. At the same time, f'' tends to zero in the interval $[-1+\epsilon, 1-\epsilon]$ for any $\epsilon \in (0, 1)$, which will be proved in section 6 under an additional hypothesis. Solutions on the main branch behave differently when R is increased. Figure 4 shows that $R^{-1}f''$ tends to -2 in $-1 \leq x < 0$ and that $R^{-1}f''$ tends to $+2$ in $0 < x \leq 1$. This will be proved in section 5. Figure 5 is the graph of $R^{-1}f'$, which, together with Figure 4, suggests that $\lim_{R\to+\infty} R^{-1}f' = 2|x| - 1$ uniformly in $-1 \leq x \leq 1$. It would be appropriate to say that the solutions develop an interior layer. Figure 6 shows the graphs of f'' along the secondary branch. It shows a development of a boundary layer at one and only one end of the interval. Note that $-f(-x)$ solves (1) and (2) if $f(x)$ does.

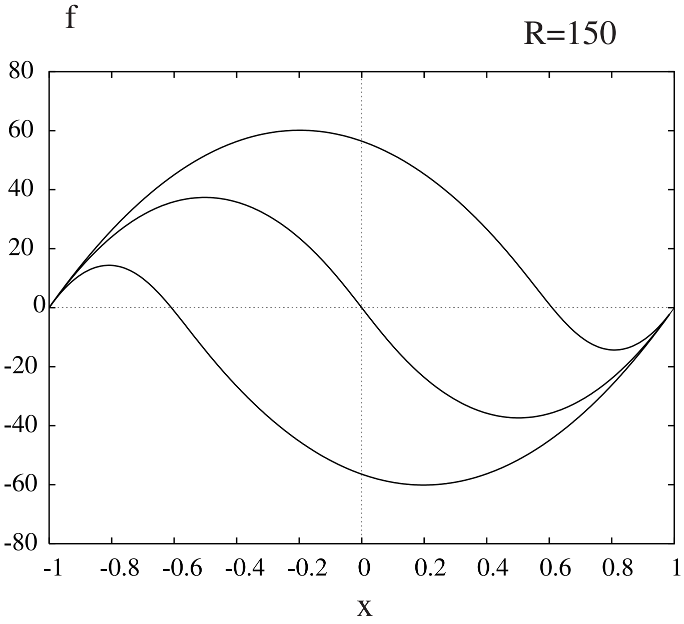

Figure 2: Graphs of the solutions of (1) and (2) at $R = 150$.

Accordingly, it is sufficient to study one of the two bifurcating solutions.

3 Rigorous results

We now define ϕ by $\phi(x) = R^{-1}f(x)$. Then it satisfies

$$\phi'''' - 2R\phi\phi''' = 0 \qquad (-1 < x < 1), \tag{17}$$

$$\phi(\pm 1) = 0, \quad \phi'(\pm 1) = 1. \tag{18}$$

We first prove the following Theorem 1, which is valid for any solution of (17) and (18). We subsequently prove propositions which are valid only for odd solutions of (17) and (18). Methods used in this section are standard and, in particular, modified from the one in Shih [24].

Theorem 1 *Any solution of* (17) *and* (18) *satisfies* $\phi'''(-1) = \phi'''(1)$. ϕ''' *is positive everywhere in* $[-1, 1]$. $\phi'(x) \leq 1$ *for all* x *and* R. $\phi(x) = \frac{1}{2}(x^3 - x)$ *is the only solution if* $R = 0$.

Proof. Note first that the equation (1) can be integrated once and we have

$$\phi''' - 2R\phi\phi'' + R\left(\phi'\right)^2 = \beta, \tag{19}$$

where β is a constant. This equation and (18) gives us $\phi'''(-1) = \phi'''(1)$. Next, we rewrite (17) as

$$\phi'''(x) = A \exp\left(2R\int_{-1}^{x} \phi(y)dy\right), \tag{20}$$

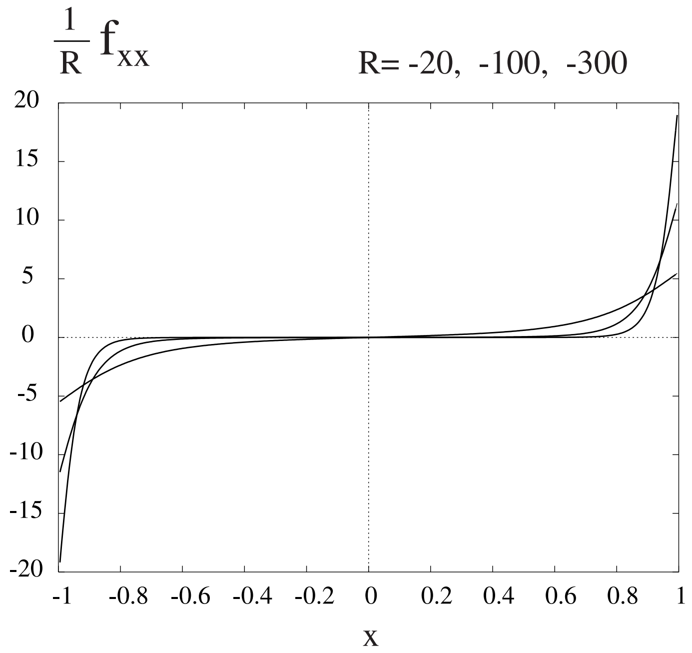

Figure 3: Graphs of the solutions of (1) and (2) on the main branch for $R = -20, -100, -300$.

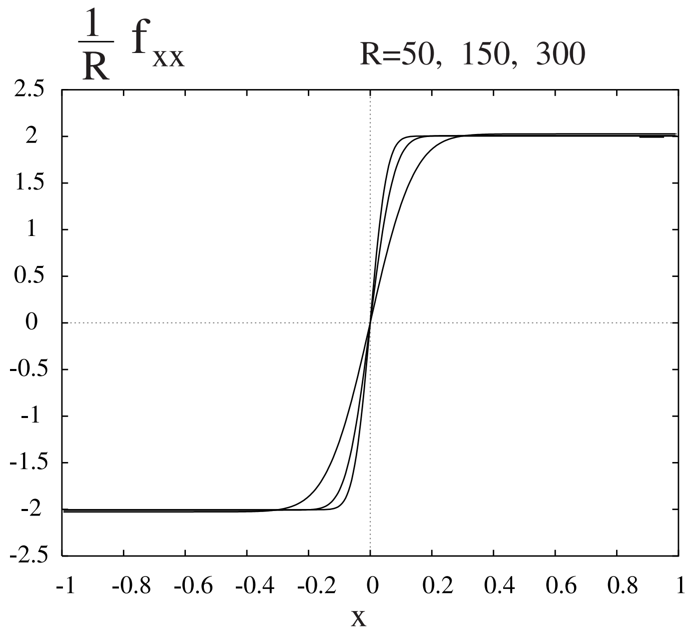

Figure 4: Graphs of the solutions of (1) and (2) on the main branch for $R = 50, 150, 300$.

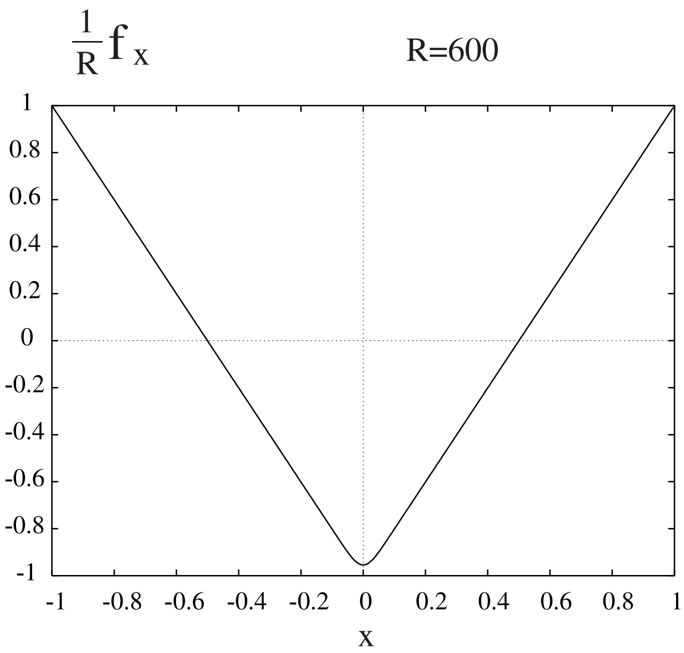

Figure 5: Graphs of $\frac{1}{R}f'$ of (1) and (2) on the main branch when $R = 600$.

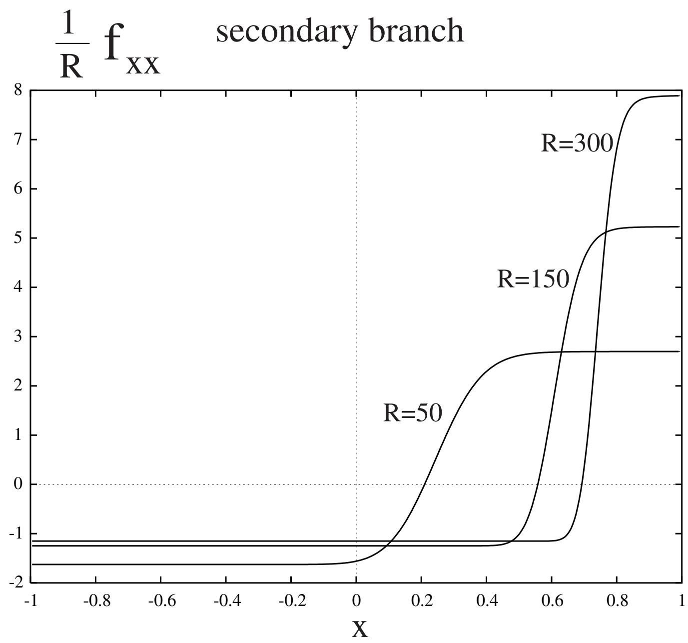

Figure 6: Graphs of the solutions of (1) and (2) on the secondary branch.

where A is a constant. Multiply both sides by $1-x^2$ and integrate it in $-1<x<1$. Then we obtain

$$A\int_{-1}^{1}(1-x^2)\exp\left(2R\int_{-1}^{x}\phi(y)dy\right)dx=\int_{-1}^{1}(1-x^2)\phi'''(x)dx=2\int_{-1}^{1}x\phi''(x)dx=4.$$

This shows that A is positive, whence $\phi'''(x)$ is positive everywhere. This means convexity of ϕ', which, together with (18), shows that $\phi'(x)\le 1$ for all x. If $R=0$, then (17) is linear and solvable; it is easy to verify that $(x^3-x)/2$ is the only solution.

□

From now on we consider rigorous results valid for odd solutions.

Proposition 1 *Suppose that ϕ is odd in x and that ϕ satisfies* (17) *and* (18). *Then ϕ' satisfies $|\phi'(x)|\le 1$ for all $x\in[-1,1]$. ϕ'' is positive for all $x\in(0,1]$. ϕ is negative everywhere in* $(0,1)$.

Proof. Since ϕ is assumed to be odd, it holds that $\phi(0)=\phi''(0)=0$. Note that

$$\phi'''(x)=\phi'''(0)\exp\left(2R\int_{0}^{x}\phi(y)dy\right). \tag{21}$$

As we have already shown, ϕ''' is positive. Then, because of $\phi''(0)=0$, $\phi''(x)>0$ for all $0<x\le 1$. Accordingly we have either that ϕ' is nonnegative everywhere or that $\phi'(0)<0$ and ϕ' has one and only one zero. Since $\phi(0)=\phi(1)=0$, the former cannot be the case. Consequently there exists an x_0 such that ϕ is decreasing in $0<x<x_0$ and that ϕ is increasing in $x_0<x<1$. Therefore ϕ is negative in $0<x<1$.

Now (21) shows that

$$0<\phi'''(1)<\phi'''(0)\qquad(\text{for } R>0)$$

and

$$0<\phi'''(0)<\phi'''(1)\qquad(\text{for } R<0).$$

By (19) we have $\phi'''(0)+R\phi'(0)^2=\phi'''(1)+R$, hence $\phi'(0)^2<1$. Therefore

$$-1<\phi'(0)\le\phi'(x)\le\phi'(1)=1$$

for all $x\in(0,1)$.

□

Theorem 2 *For any $R\in(-\infty,\infty)$, there exists at least one odd solution of* (1) *and* (2).

Proof. It is sufficient to prove that

$$\phi''''-2R\phi\phi'''=0,\qquad(\ 0<x<1\ ) \tag{22}$$

$$\phi(0)=\phi''(0)=\phi(1)=0,\quad \phi'(1)=1. \tag{23}$$

has a solution for any R.

This boundary value problem is converted to an integral equation by the following Green's function:

Lemma 1 *The solution of*

$$\begin{cases} \Phi'''' = f \qquad (0 < x < 1), \\ \Phi(0) = \Phi''(0) = \Phi(1) = \Phi'(1) = 0 \end{cases}$$

is represented as

$$\begin{aligned} \Phi(x) &= -\frac{1}{12}\int_0^x (x-1)^2 y \left\{ x(y^2-3) + 2y^2 \right\} f(y)dy \\ &\quad -\frac{1}{12}\int_x^1 (y-1)^2 x \left\{ y(x^2-3) + 2x^2 \right\} f(y)dy. \end{aligned}$$

This lemma can be verified in an elementary way.

Keeping this lemma in mind, we define Φ by

$$\phi(x) = \Phi(x) + \frac{x^3 - x}{2}.$$

Then the boundary-value problem (22) and (23) is re-written as follows:

$$\Phi'''' = 2R\left(\Phi + \frac{x^3-x}{2}\right)(\Phi''' + 3) \qquad (\ 0 < x < 1\ ), \tag{24}$$

$$\Phi(0) = \Phi''(0) = \Phi(1) = \Phi'(1) = 0. \tag{25}$$

We now use Lemma 1 for $f = 2R\phi\phi'''$. Using $2\phi\phi''' = \{(\phi^2)'' - 3(\phi')^2\}'$, after integrating by parts, we obtain

$$\begin{aligned} \Phi(x) &= \frac{R}{4}\int_0^x (x-1)^2 \left[x(y^2-1) + 2y^2 \right] \left[\left(\phi(y)^2\right)'' - 3\left(\phi'(y)\right)^2 \right] dy \\ &\quad + \frac{R}{12}\int_x^1 x(y-1) \left[(x^2-3)(3y-1) + 4x^2 \right] \left[\left(\phi(y)^2\right)'' - 3\left(\phi'(y)\right)^2 \right] dy, \end{aligned}$$

where ϕ denotes $\Phi(x) + (x^3 - x)/2$. Differentiating in x, we have

$$\begin{aligned} \Phi'(x) &= \frac{R}{4}\int_0^x (x-1) \left[3(x+1)y^2 - 3x + 1 \right] \left[\left(\phi(y)^2\right)'' - 3\left(\phi'(y)\right)^2 \right] dy \\ &\quad + \frac{R}{4}\int_x^1 (y-1) \left[3(y+1)x^2 - 3y + 1 \right] \left[\left(\phi(y)^2\right)'' - 3\left(\phi'(y)\right)^2 \right] dy \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \Phi' &= \frac{3R}{2}(x^2-1)\int_0^1 \phi(y)^2 dy + R\phi(x)^2 \\ &\quad - \frac{3R}{4}\int_0^x (x-1)\left[3(x+1)y^2 - 3x + 1 \right] \left(\phi'(y)\right)^2 dy \\ &\quad - \frac{3R}{4}\int_x^1 (y-1)\left[3(y+1)x^2 - 3y + 1 \right] \left(\phi'(y)\right)^2 dy \\ &\qquad\qquad (\phi = \Phi(x) + (x^3 - x)/2). \end{aligned} \tag{26}$$

In view of this equation, we define a function space and a mapping as follows: let X be defined as

$$X = \left\{ f \in L^2(0,1) \; ; \; \int_0^1 f(x)dx = 0 \right\},$$

where $L^2(0,1)$ denote the usual space of square integrable functions defined over the interval $(0,1)$. For a given $g \in X$, we define

$$G(x) = \int_0^x g(y)dy + \frac{x^3 - x}{2}, \qquad h(x) = g(x) + \frac{3x^2 - 1}{2}.$$

With these functions we define

$$\begin{aligned} F(g)(x) \;\; = \;\; & \frac{3}{2}(x^2-1)\int_0^1 G(y)^2 dy + G(x)^2 \\ & -\frac{3}{4}\int_0^x (x-1)\left[3(x+1)y^2 - 3x + 1\right](h(y))^2\, dy \\ & -\frac{3}{4}\int_x^1 (y-1)\left[3(y+1)x^2 - 3y + 1\right](h(y))^2\, dy. \end{aligned}$$

It can be readily verified that the mapping $g \mapsto F(g)$ defines a continuous mapping from X into itself, and that it is a compact (completely continuous) mapping. Some computation is necessary for verifying $\int_0^1 F(g)(x)dx = 0$. However, this can be done in an elementary way and we leave the proof to the readers. It is also easy to check that a fixed point of RF, i.e., a function g satisfying $g = RF(g)$, solves (24) and (25) by setting $\Phi(x) = \int_0^x g(y)dy$.

Proposition 1 implies that any solution satisfies $\|g\| \leq |R|$, where $\| \;\|$ denotes the maximum norm. This shows that we are in a position to use the Leray-Schauder fixed-point theorem. For instance the following theorem in page 32 of Ladyzhenskaya [16] can be used. We do not go into the details since it is a routine.

Theorem 3 *Let X be a separable Hilbert space and let A be a compact continuous mapping defined on X. Suppose that there exists $\rho > 0$ such that any possible solution of $x = \lambda A(x)$ for any $\lambda \in [0,1]$ satisfies $\|x\| \leq \rho$. Then $x = A(x)$ possesses at least one solution in the ball $\|x\| \leq \rho$.*

□

4 Uniqueness of the odd solution when $R > 0$

We now prove the uniqueness of the odd solution. The idea in this section is borrowed from [19]. We assume that $R > 0$ throughout this section.

We consider (22) and (23). We already know that $\phi(x) < 0$ in $(0,1)$, $\phi'(0) < 0$, $\phi'' > 0$ in $(0,1]$, and $\phi''' > 0$ in $[0,1]$. We now define μ and Φ by $\mu = (\phi'''(0))^{1/4}$ and $\Phi(x) = \mu^{-1}\phi(\mu^{-1}x)$. Then it satisfies

$$\begin{aligned} &\Phi'''' - 2R\Phi\Phi''' = 0 \qquad (\; 0 < x < \mu \;), \\ &\Phi(0) = \Phi''(0) = \Phi(\mu) = 0, \quad \Phi'''(0) = 1, \quad \Phi'(\mu) = \mu^{-2}. \end{aligned}$$

(Note that this Φ has nothing to do with the one appearing in the preceding section.)

We now consider the differential equation

$$\Psi'''' - 2R\Psi\Psi''' = 0 \tag{27}$$

with the initial condition

$$\Psi(0) = 0, \quad \Psi'(0) = -a, \quad \Psi''(0) = 0, \quad \Psi'''(0) = 1. \tag{28}$$

We let a run in $0 < a < \infty$. $\Psi(x)$ is negative for sufficiently small x. Let $\mu = \mu(a) > 0$ denote the first zero of Ψ. Then we have to show that there exists only one a such that $\mu^2\Psi'(\mu) = 1$.

Let us first prove that μ is a well-defined increasing function of $a \in (0, \infty)$. Since

$$\Psi'''(x) = \exp\left(2R\int_0^x \Psi(y)dy\right),$$

we know that $\Psi''(x) > 0$ as long as Ψ exists. Take a small $\delta > 0$ and note that $\Psi''(x) \geq \Psi''(\delta) > 0$ for $\delta \leq x$. Integrating twice, we have

$$\Psi(x) - \Psi(\delta) - \Psi'(\delta)(x-\delta) \geq \frac{1}{2}\Psi''(\delta)(x-\delta)^2.$$

So, Ψ eventually becomes positive and must necessarily have a zero and μ is well-defined. It is not difficult to see that $\Psi'(\mu) > 0$. This shows that μ is a continuous function of a. Let Ψ_1 and Ψ_2 be solutions of (27) and (28) with $a = a_1$ and $a = a_2$, respectively. Suppose that $0 < a_1 < a_2$. Then $w = \Psi_2 - \Psi_1$ satisfies

$$w'''(x) = \exp\left(2R\int_0^x \Psi_2\right) - \exp\left(2R\int_0^x \Psi_1\right)$$

and

$$w(0) = 0, w'(0) < 0, w''(0) = 0, w'''(0) = 0.$$

It is readily verified that $w < 0$ as far as w exists. This implies that $\mu(a_2) > \mu(a_1)$, namely μ is an increasing function of a.

We now integrate (27) to obtain $\Psi''' - 2R\Psi\Psi'' + R(\Psi')^2 =$ constant. The constant is actually equal to $1 + Ra^2$ because of the boundary condition (28). We therefore obtain

$$R\left[\Psi'(\mu)\right]^2 = 1 + Ra^2 - \exp\left(2R\int_0^\mu \Psi(y)dy\right),$$

the right hand side of which is increasing in a. Consequently $\mu^2\Psi'(\mu)$ is an increasing function of a and $\mu^2\Psi'(\mu) - 1 = 0$ can be satisfied at most one a. We now have

Theorem 4 *For all $R \in [0, \infty)$, the solution of* (22) *and* (23) *is unique.*

5 Interior layer

This section is devoted to the proof of the existence of an interior layer. We only consider the odd solutions.

Theorem 5 *For all $x \in (0,1]$, the solution of (22) and (23) satisfies*

$$\phi'''(x) = O\left(\sqrt{R} e^{-\frac{x^4}{6}R}\right) \qquad \text{as } R \to +\infty. \tag{29}$$

Also,

$$\lim_{R\to+\infty} \phi''(x) = 2 \qquad (0 < x \le 1).$$

We first prove the following lemma:

Lemma 2 *For all $R > 0$ and all $0 < \alpha \le 1$, it holds that*

$$2\alpha \le \phi''(\alpha). \tag{30}$$

Proof. We begin with

$$\phi'''(x) = A \exp\left(2R\int_0^x \phi(y)dy\right),$$

where A is a constant. Integrating this in $0 < x < \alpha$, we have

$$\phi''(\alpha) = A\int_0^\alpha \exp\left(2R\int_0^x \phi(y)dy\right)dx.$$

Accordingly it holds that

$$\phi'''(x) = \frac{\phi''(\alpha)}{\int_0^\alpha \exp\left(2R\int_0^x \phi(y)dy\right)dx} \exp\left(2R\int_0^x \phi(y)dy\right). \tag{31}$$

We multiply both sides by $1 - x^2$ and integrate it in $0 < x < 1$. Since

$$\int_0^1 (1-x^2)\phi'''(x)dx = \int_0^1 2x\phi''(x)dx = 2,$$

we obtain

$$\begin{aligned} 2 &= \frac{\phi''(\alpha)}{\int_0^\alpha \exp\left(2R\int_0^x \phi(y)dy\right)dx}\int_0^1 (1-x^2)\exp\left(2R\int_0^x \phi(y)dy\right)dx \\ &\le \frac{\phi''(\alpha)}{\int_0^\alpha \exp\left(2R\int_0^x \phi(y)dy\right)dx}\int_0^1 \exp\left(2R\int_0^x \phi(y)dy\right)dx. \end{aligned}$$

This inequality holds for all $0 < \alpha \le 1$. Since ϕ is negative in $0 < x < 1$, it holds that

$$\begin{aligned} \int_0^1 \exp\left(2R\int_0^x \phi(y)dy\right)dx &= \frac{1}{\alpha}\int_0^\alpha \exp\left(2R\int_0^{z/\alpha} \phi(y)dy\right)dz \\ &\le \frac{1}{\alpha}\int_0^\alpha \exp\left(2R\int_0^z \phi(y)dy\right)dz \end{aligned}$$

We therefore obtain (30).

$\square$

Lemma 3 *For all $R > 0$ and $0 \le x \le 1$, it holds that*

$$\int_0^x \phi(y)dy \le -\frac{x^4}{12}. \tag{32}$$

Proof. We have

$$\begin{aligned}\int_0^x y(x-y)\phi''(y)dy &= -\int_0^x (x-2y)\phi'(y)dy \\ &= x\phi(x) - 2\int_0^x \phi(y)dy \le -2\int_0^x \phi(y)dy, \end{aligned} \tag{33}$$

since ϕ is non-positive. On the other hand, by the previous lemma,

$$\int_0^x y(x-y)\phi''(y)dy \ge 2\int_0^x y^2(x-y)dy = \frac{x^4}{6}.$$

$\square$

We next note that the solution of (22) and (23) satisfies

$$\phi'''(x) = \frac{2}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right)dz} \exp\left(2R\int_0^x \phi(y)dy\right). \tag{34}$$

We now prove that the right hand side of (34) tends to zero for all $0 < x \le 1$. We first note that Proposition 1 implies that $\phi(y) \ge -y$ for all y. Then, assuming that $R > 1$, we have

$$\begin{aligned}\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right)dz &\ge \int_0^{1/\sqrt{R}} (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right)dz \\ &\ge \frac{R-1}{R}\int_0^{1/\sqrt{R}} \exp\left(-2R\int_0^z ydy\right)dz \\ &\ge \frac{R-1}{eR^{3/2}}\end{aligned}$$

which implies that

$$\frac{2}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right)dz}$$

is $O(\sqrt{R})$ as $R \to +\infty$. Accordingly, (29) holds true by Lemma 3 and (34).

We next prove that for all $0 < x \le 1$, $\phi''(x)$ tends to 2 as $R \to +\infty$. By (34), we have

$$\phi''(x) = \frac{2}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right)dz}\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right)dz. \tag{35}$$

for all $0 < x \le 1$. We now fix any $\alpha \in (0, x)$ and divide the integral as follows.

$$\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right)dz = \int_0^\alpha \exp\left(2R\int_0^z \phi(y)dy\right)dz + \int_\alpha^x \exp\left(2R\int_0^z \phi(y)dy\right)dz$$

Then, by Lemma 3, the second term of the right hand side is exponentially small, while the first term is bounded below by a constant times $1/\sqrt{R}$. Therefore,

$$\lim_{R\to\infty} \frac{\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right) dz}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz} = \lim_{R\to\infty} \frac{\int_0^\alpha \exp\left(2R\int_0^z \phi(y)dy\right) dz}{\int_0^\alpha (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz}$$

for any $\alpha \in (0,x)$. (Strictly speaking, we must use lim sup and lim inf since the existence of the limit is not guaranteed. But a modification needed here and below would be obvious.) On the other hand, it is obvious that

$$\begin{aligned}
\int_0^\alpha (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz &\leq \int_0^\alpha \exp\left(2R\int_0^z \phi(y)dy\right) dz \\
&\leq \frac{1}{1-\alpha^2}\int_0^\alpha (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz.
\end{aligned}$$

Therefore

$$\lim_{R\to\infty} \frac{\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right) dz}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz} = 1.$$

if $0 < x \leq 1$.

□

We now prove that

Theorem 6 *If ϕ solves* (22) *and* (23), *then*

$$\lim_{R\to+\infty} \phi'(x) = 2x - 1 \tag{36}$$

uniformly in $x \in [0,1]$.

Proof. To prove this, we integrate (35) to obtain

$$\begin{aligned}
1 - \phi'(x) &= A\int_x^1 \int_0^u \exp\left(2R\int_0^z \phi(y)dy\right) dzdu \\
&= A\left\{(1-x)\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right) dz + \int_x^1 (1-z)\exp\left(2R\int_0^z \phi(y)dy\right) dz\right\},
\end{aligned}$$

where A is a constant defined by

$$A = \frac{2}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz}.$$

This yields

$$2x - 1 - \phi'(x) = 2(1-x)B + A\int_x^1 (x-z)\exp\left(2R\int_0^z \phi(y)dy\right) dz, \tag{37}$$

where

$$B = \frac{\int_0^1 z^2 \exp\left(2R\int_0^z \phi(y)dy\right) dz}{\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz}.$$

It is easy to see that the constant B tends to zero as $R \to +\infty$. Also,

$$
\begin{aligned}
0 \quad &\geq \quad \int_x^1 (x-z) \exp\left(2R \int_0^z \phi(y)dy\right) dz = -\int_0^{1-x} u \exp\left(2R \int_0^{u+x} \phi(y)dy\right) du \\
&\geq \quad -\int_0^1 u \exp\left(2R \int_0^u \phi(y)dy\right) du.
\end{aligned}
$$

Since both B and

$$\frac{\int_0^1 z \exp\left(2R \int_0^z \phi(y)dy\right) dz}{\int_0^1 (1-z^2) \exp\left(2R \int_0^z \phi(y)dy\right) dz}$$

tends to zero as $R \to +\infty$, the right hand side of (37) tends to zero uniformly in $x \in [0,1]$.

□

6 Boundary layer

We consider now the solutions of (22) and (23) with $R < 0$. Although we do not know a uniqueness theorem in this case, we can prove that boundary layers grow as $R \to -\infty$ for solutions satisfying a certain bound which will be explained below.

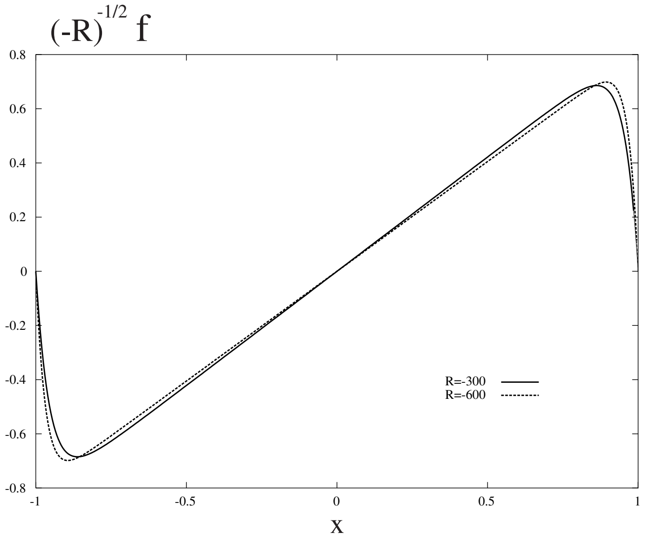

Figure 7: Two solutions of (1) and (2) with $R = -300$ and $R = -600$.

The equation (35) still holds if $R < 0$. Since R is negative now, and since $\phi < 0$ by Proposition 1, the integral on the denominator of (35) is exponentially larger than the integral over $0 < z < x$. Therefore

$$\lim_{R \to -\infty} \phi''(x) = 0 \qquad (0 \leq x < 1)$$

is intuitively obvious. However, care is needed since a scaling different from that in the interior layer holds true in the case of negative R. In fact, Figure 7 shows that f is of the order $O\left(\sqrt{-R}\right)$

(hence $\phi = O(1/\sqrt{-R})$) as $R \to -\infty$. We can nevertheless prove the following under a certain hypothesis;

$$\lim_{R \to -\infty} \phi''(1) = +\infty,$$

$$\lim_{R \to -\infty} \phi''(x) = 0 \qquad (0 \leq x < 1).$$

In order to proceed further, we define a new parameter ϵ by $\epsilon = 1/\sqrt{-R}$. We then assume that there exists a family of solutions of (22) and (23) with $R < 0$ such that

$$c_1 \frac{\epsilon}{1+\epsilon} \leq -\min_{0 \leq x \leq 1} \phi(x) \leq c_2 \frac{\epsilon}{1+\epsilon}. \tag{38}$$

where c_1 and c_2 are positive constants independent of ϵ. If the uniqueness of the odd solutions holds true, then Figure 7 suggests that this hypothesis is true. We can not prove this but we may derive all the information about the boundary layer only by this hypothesis. From now on, we assume (38) until the end of this section.

We now prove that

Lemma 4 *Put* $g(x) = \int_0^x \phi(y)dy$. *Then*

$$\frac{c_1 \epsilon x^2}{2(1+\epsilon)} \leq -g(x) \qquad (0 \leq x \leq 1). \tag{39}$$

Proof. We first note that the positivity of ϕ'' and (38) implies that

$$g(1) = \int_0^1 \phi(y)dy \leq -\frac{c_1 \epsilon}{2(1+\epsilon)}.$$

By (33) and the positivity of ϕ'', we have

$$0 \leq x g'(x) - 2g(x).$$

Namely,

$$\frac{2}{x} \geq \left[\log\left(-g(x)\right)\right]'.$$

Integrating this in $[x, 1]$, we obtain

$$-g(x) \geq -g(1)x^2 \geq \frac{c_1 \epsilon}{2(1+\epsilon)} x^2.$$

□

We now prove the following theorem

Theorem 7 *If* ϕ *is the solution of* (22) *and* (23) *satisfying* (38), *then*

$$\lim_{\epsilon \to 0} \phi''(x) = 0 \qquad (0 \leq x < 1).$$

$$\lim_{\epsilon \to 0} \phi''(1) = +\infty.$$

Proof. We choose any x_0 and x_1 such that $0 \le x_0 < x_1 \le 1$. Let x be arbitrary number in $[0, x_0]$. Then,

$$\int_0^x \exp\left(2R\int_0^z \phi(y)dy\right) dz \le x_0 \exp\left(-2\epsilon^{-2} g(x_0)\right).$$

On the other hand,

$$\begin{aligned}\int_0^1 (1-z^2)\exp\left(2R\int_0^z \phi(y)dy\right) dz &\ge \int_{x_1}^1 (1-z^2)dz \exp\left(-2\epsilon^{-2} g(x_1)\right)\\ &= \hat{c}\exp\left(-2\epsilon^{-2} g(x_1)\right),\end{aligned}$$

where $\hat{c} > 0$ depends only on x_1. Consequently, by (35),

$$\phi''(x) \le \tilde{c}\exp\left(2\epsilon^{-2}\int_{x_0}^{x_1} \phi(y)dy\right) \qquad (\forall x \le x_0),$$

where $\tilde{c} > 0$ depends only on x_0 and x_1. On the other hand, by the convexity of ϕ and the hypothesis (38), there is a constant c_3 depending only on x_0 and x_1 such that

$$\int_{x_0}^{x_1} \phi(y)dy \le -c_3\epsilon.$$

Accordingly, $\phi''(x)$ is exponentially small uniformly in $[0, x_0]$.

In order to prove the divergence of $\phi''(1)$, we note that

$$\phi''(1) = 2\frac{\int_0^1 \exp\left(-2\epsilon^{-2} g(z)\right) dz}{\int_0^1 (1-z^2)\exp\left(-2\epsilon^{-2} g(z)\right) dz}.$$

We now proceed as in the proof of Theorem 5 to obtain the second half of the present theorem.

□

7 The case of porous wall

We here consider (1) and (3). Figure 8 is the bifurcation diagram which we obtained by the same method as in section 2. Computation for $R < 0$ was easy but computation for $R > 0$ encountered a serious difficulty when R exceeds 23.0. The reason is that at $R = 23.0$ the solution has a strong boundary layer as is shown in Figure 9. Figure 10 shows solutions with $R = -50$ and $R = -300$, which suggests us the existence of an interior layer at $x = 0$ as $R \to -\infty$. We could not find a secondary bifurcation, in particular, the solutions with an interior layer are stable (presumably for all $-\infty < R \le 0$). All the solutions we obtained were odd functions.

Let us consider odd solutions of (1) and (3). Accordingly solutions of the following boundary value problem are considered:

$$\phi'''' - 2R\phi\phi''' = 0 \qquad (0 < x < 1) \tag{40}$$

$$\phi(0) = \phi''(0) = 0, \quad \phi(1) = 1, \quad \phi'(1) = 0. \tag{41}$$

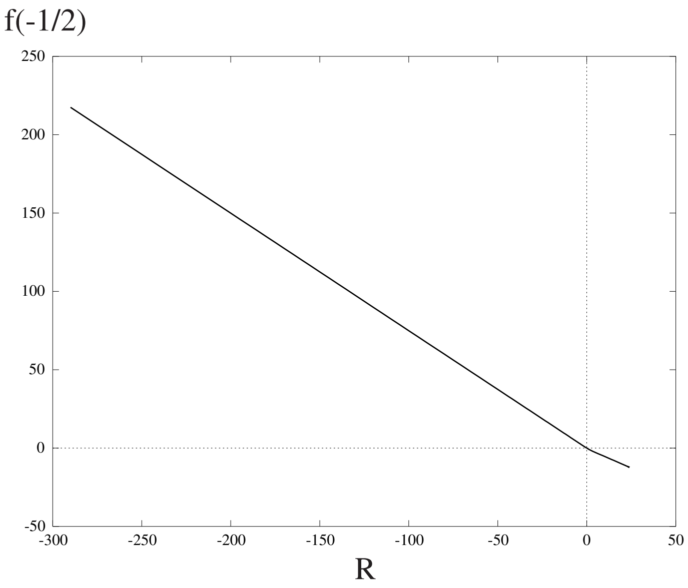

Figure 8: Bifurcation diagram of the solutions of (1) and (3).

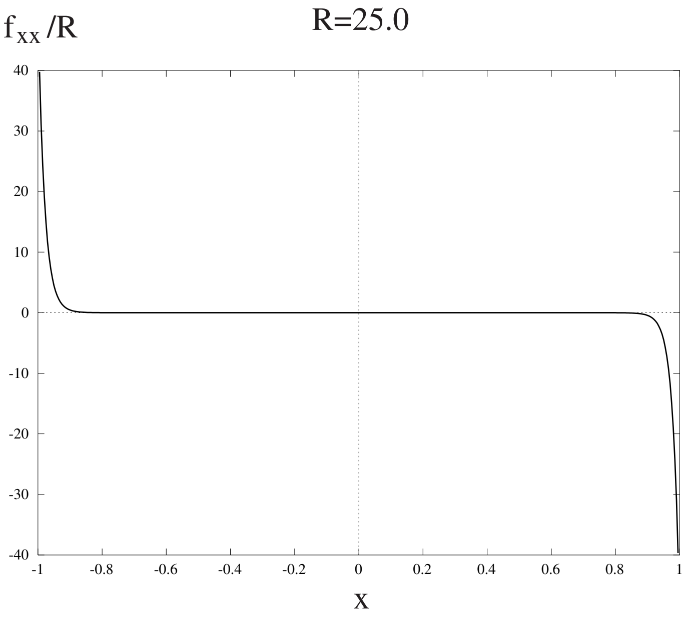

Figure 9: Solution of (1) and (3) with $R = 25$.

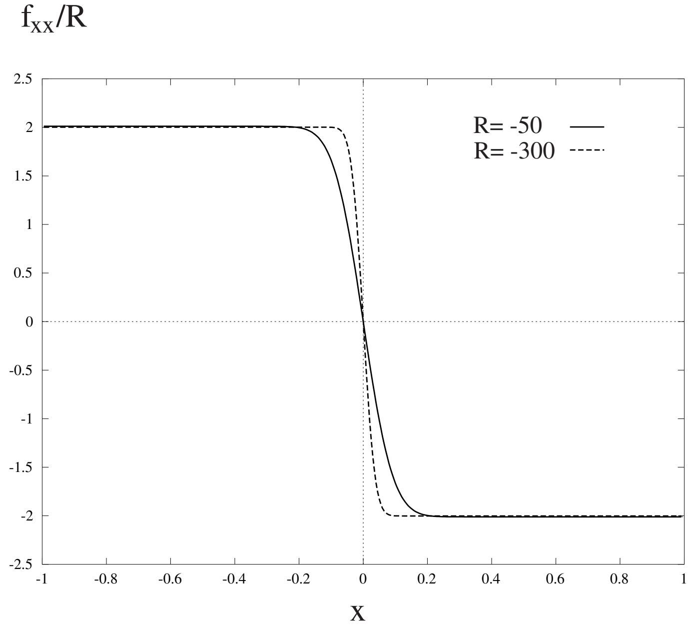

Figure 10: Solutions of (1) and (3) with $R = -50$ and $R = -300$.

The solution satisfies

$$\phi'''(x) = \phi'''(0) \exp\left(2R \int_0^x \phi(y)dy\right) \tag{42}$$

In the same way as in section 5, we can prove the following lemma.

Lemma 5 *For all $x \in (0, 1]$, the solution ϕ satisfies*

$$\phi'''(x) < 0, \quad \phi''(x) < 0, \quad \phi'(x) > 0, \quad 0 < \phi(x) < 1.$$

Proof. Multiplying (42) by $1 - x^2$ and integrating, we obtain, after integration by parts,

$$-2 = \phi'''(0) \int_0^1 (1 - x^2) \exp\left(2R \int_0^x \phi(y)dy\right) dx,$$

which, together with (42), shows that $\phi'''(x) < 0$ for all $x \in [0, 1]$. Other inequalities follow from this and the boundary conditions.

$\square$

This lemma is sufficient to prove the uniqueness of the solution of (40) and (41):

Theorem 8 *For $R \leq 0$, there exists at most one solution to* (40) *and* (41)*.*

Proof is almost the same as the one for Theorem 4 and we omit it.

An existence theorem can be established similarly:

Theorem 9 *For all $-\infty < R < +\infty$, the boundary value problem* (40) *and* (41) *has at least one solution.*

Now we can apply the same method as in section 5 to obtain the following lemma, whose proof is omitted.

Lemma 6 *Suppose that $R < 0$. Then, for all $x \in (0, 1]$, the solution of* (40) *and* (41) *satisfies*

$$\phi''(x) \leq -2x, \qquad \int_0^x \phi(y)dy \geq \frac{x^4}{12}.$$

One can easily notice that this lemma gives us the same asymptotic estimates as in section 5. We only state the result.

Theorem 10 *Let $0 < x \leq 1$ be fixed. Then the solution ϕ of* (40) *and* (41) *satisfies*

$$\lim_{R \to -\infty} \phi''(x) = -2.$$

8 Why the interior layer is so hard to find?

We showed that the boundary value problem (1) with (2) or (3) possesses solutions which display an interior layer. On the other hand, as far as the authors know, the equation (16) has no solution with interior layer under any boundary condition. We give an example, which suggests that an interior layer is rare in the similarity solutions of the Navier-Stokes equations.

We consider the equation(1) with the following boundary condition

$$f(\pm 1) = 0, \quad f'(-1) = sR, \quad f'(1) = R, \tag{43}$$

where s is a new parameter. The case of $s = 1$ is the case we have discussed. When $s = 0$ or $s = 2$, there is a branch emanating from the origin and extending toward $R = +\infty$. If R is fixed to 150, then we have the diagram shown in Figure 11. This shows that there exist three solutions in $1 - \epsilon < s < 1 + \delta$, where ϵ and δ are very small (both are smaller than 0.00002). The interior layer appears only if there are three solutions and only in the solution in the middle. Consequently, (1) and (43) have no interior layer for a vast majority of the values of s. This is puzzling and we have no theoretical explanation for this fact.

9 Conclusion

The existence of a solution was proved for solutions of (1) & (2) and (1) & (3). Their asymptotic behavior as $R \to \pm\infty$ was determined and, in particular, the existence of an interior layer was rigorously proved. We were unable to determine whether the odd solution was unique for (1) & (2) with $R < 0$ and for (1) & (3) with $R > 0$. Whether the boundary layer property can be proved without the hypothesis (38) is left as an open problem. The boundary value problem (1) & (2) was embedded into (1) & (43), and we showed that interior layers rarely appeared.

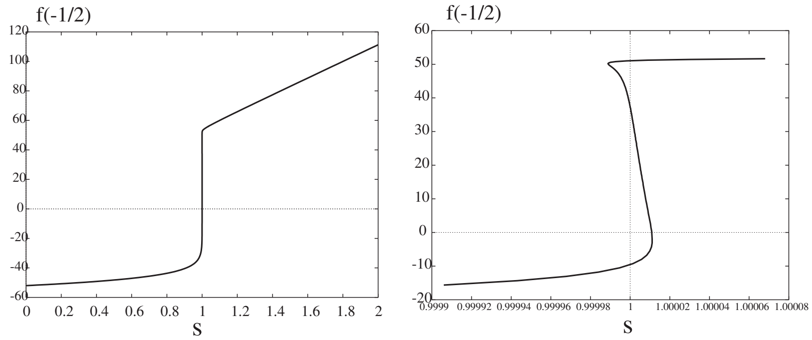

Figure 11: Bifurcation diagram for (1) and (43) with $R = 150$. The figure in the right hand side is a blow-up of the central part of the figure on the left hand side.

References

[1] E.L. Allgower and K. Georg, Numerical Continuation Methods, Springer, (1990).

[2] W.-J. Beyn, Numerical methods for dynamical systems, in Advances in Numerical Analysis, ed. W. Light, Clarendon Press, (1991), pp. 175–236.

[3] J. F. Brady and A. Acrivos, Steady flow in a channel or tube with an accelerating surface velocity. An exact solution to the Navier-Stokes equations with reverse flow, *J. Fluid Mech.*, **112** (1981), 127–150.

[4] S. M. Cox, Analysis of steady flow in a channel with one porous wall, or with accelerating walls, *SIAM J. Appl. Math*, **51** (1991), pp. 429–438.

[5] W. J. F. Govaerts, Numerical Methods for bifurcations of Dynamical Equilibria, SIAM (2000).

[6] R. E. Grundy and H. R. Allen, The asymptotic solution of a family of boundary value problems involving exponentially small terms, *IMA J. Appl. Math.*, **53** (1994), pp. 151–168.

[7] R. E. Grundy and R. McLaughlin, Three-dimensional blow-up solutions of the Navier-Stokes equations, *IMA J. Appl. Math.*, vol 63 (1999), pp. 287–306.

[8] S. P. Hastings, An existence theorem for a problem from boundary layer theory, *Arch. Rat. Mech. Anal.*, **33** (1969), pp 103–109.

[9] L. Howarth, Laminar boundary Layers, in *"Handbuch der Physik"*, vol. VIII/1 (1959), pp 264–350.

[10] K. Hiemenz, Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszlynder, *Dinglers J.*, **326** (1911), 321–4, 344–8, 357–62, 372–6, 391–3, 407–10.

[11] T.-W. Hwang and C.-A. Wang, On multiple solutions for Berman's problem, *Proc. R. Soc. Edinburgh*, **121A** (1992), pp. 219–230.

[12] H. Ikeda, M. Mimura, and H. Okamoto, A singular perturbation problem arising in Oseen's spiral flows, *Japan J. Indust. Appl. Math.*, vol. 18 (2001), pp. 393–403.

[13] H. B. Keller, Lectures on Numerical Methods in Bifurcation Theory (Tata Institute of Fundamental Research No. 79), Springer Verlag (1987).

[14] J. R. King and S. M. Cox, Asymptotic analysis of the steady-state and time-dependent Berman problem, preprint.

[15] M. Kubicek and M. Marek, Computational Methods in Bifurcation Theory and Dissipative Structure, Springer Verlag, (1983).

[16] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach (1963). page 32

[17] C.-C. Lan, On functional-differential equations and some laminar boundary layer problems, *Arch. Rat. Mech. Anal.*, **42** (1971), pp. 24–39.

[18] C. Lu, A. D. MacGillivray, and S. P. Hastings, Asymptotic behavior of solutions of a similarity equation for laminar flows in channels with porous walls, *IMA J. Appl. Math.*, **49** (1992), pp. 139–162.

[19] J. B. McLeod, Laminar flow in a porous channel, in *Asymptotics beyond All Orders*, eds. H. Segur, S. Tanveer, and H. Levine, Plenum Press, pp. 255–266.

[20] M. Nagayama, H. Okamoto, and J. Zhu, On the blow-up of some similarity solutions of the Navier-Stokes equations, to appear in Quaderni di Matematica.

[21] H. Okamoto, Localization of singularities in inviscid limit — numerical examples, in *Proceedings of Navier-Stokes equations: theory and numerical methods*, ed. R. Salvi, Longman, Pitman Research Notes in Mathematics Series vol. 388 (1998), pp. 220–236.

[22] H. Okamoto and J. Zhu, Some similarity solutions of Some similarity solutions of the Navier-Stokes equations and related topics, *Taiwanese J. Math.*, **4** (2000), pp. 65–103.

[23] W. A. Robinson, The existence of multiple solutions for the laminar flow in a uniformly porous channel with suction at both walls, *J. Eng. Math.*, **10** (1976), pp. 23–40.

[24] K.-G. Shih, On the existence of solutions of an equation arising in the theory of laminar flow in a uniformly porous channel with injection, *SIAM J. Appl. Math.*, **47** (1987), pp. 526–533.

[25] F. M. Skalak and C.-Y. Wang, On the nonunique solutions of laminar flow through a porous tube or channel, *SIAM J. Appl. Math.*, **34** (1978), pp. 535–544.

[26] C.L. Taylor, W.H.H. Banks, M.B. Zaturska, and P.G. Drazin, Three dimensional flow in a porous channel, *Quar. J. Mech. Appl. Math.*, **44** (1991), pp. 105–133.

[27] C.-Y. Wang, The three-dimensional flow due to a stretching flat surface, *Phys. Fluid*, **27** (1984), pp. 1915–1917.

[28] E. B. B. Watson, W. H. H. Banks, M. B. Zaturska and P. G. Drazin, On transition to chaos in two-dimensional channel flow symmetrically driven by accelerating walls, *J. Fluid Mech.*, **212** (1990), pp. 451–485

[29] P. Watson, W. H. H. Banks, M. B. Zaturska and P. G. Drazin, Laminar channel flow driven by accelerating walls, *Euro. J. Appl. Math.*, **2** (1991), pp. 359–385.

[30] H. Weyl, On the differential equations of the simplest boundary-layer problems, *Ann. Math.*, **43** (1942), pp. 381–407.

[31] M. B. Zaturska, P. G. Drazin, and W. H. H. Banks, On the flow of a viscous fluid driven along a channel by suction at porous walls, *Fluid Dynam. Res.*, **4** (1988), 151–178.

[32] J. Zhu, Numerical study of stagnation-point flows of incompressible fluid, *Japan J. Indust. Appl. Math.*, **17** (2000), pp. 209–228.