On the blow-up of some similarity solutions of the Navier-Stokes equations

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Dedicated to Professor John G. Heywood
on the occasion of his sixtieth birthday

Abstract

The Navier-Stokes equations for incompressible viscous fluid are simplified, by means of a similarity assumption, to coupled equations defined in a finite interval. We show numerically that some of the solutions can blow up after finite time. We discuss the difference between 2D and 3D cases and present an evidence of richness of 3D equations from a dynamical systems viewpoint.

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1 Introduction

We consider the Navier-Stokes equations for incompressible viscous fluid motion in the following infinite domain:

$$\Omega = \{(x, y, z) \ ; \ -a < x < a, -\infty < y, z < \infty \},$$

where a is a constant. Following Zhu [52], we assume the following form (sometimes called a similarity form) of the velocity vector:

$$u = f - g, \quad v = -yf_x, \quad w = zg_x, \tag{1.1}$$

where u, v, and w are x, y, and z components, respectively, of the velocity. f and g are assumed to be functions of (t, x) only; $f = f(t, x)$, $g = g(t, x)$. The subscript implies the differentiation. The velocity $\mathbf{u} = (u, v, w)$ and the pressure p satisfy the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \triangle \mathbf{u} - \frac{1}{\rho}\nabla p \tag{1.2}$$

$$\operatorname{div} \mathbf{u} = 0. \tag{1.3}$$

The incompressibility condition (1.3) is automatically satisfied by the ansatz (1.1). The equation (1.2) leads to the following coupled equations:

$$f_{txx} = \nu f_{xxxx} + (f_x + g_x)f_{xx} - (f - g)f_{xxx}, \tag{1.4}$$

$$g_{txx} = \nu g_{xxxx} - (f_x + g_x)g_{xx} - (f - g)g_{xxx}. \tag{1.5}$$

These equations are to be satisfied in $-a < x < a$ and $0 < t$. Note that no approximation is assumed in deriving (1.4) and (1.5). Accordingly, any solution of (1.4) and (1.5) is what is called an exact solution of the Navier-Stokes equations. The wall located at $x = \pm a$ is assumed to be fixed. Accordingly, the following boundary conditions are imposed:

$$f(t, \pm a) = g(t, \pm a) = f_x(t, \pm a) = g_x(t, \pm a) = 0. \tag{1.6}$$

The purpose of the present paper is to show numerically that some initial data do not admit a solution global in time. Namely, blow-up occurs for (1.4), (1.5), and (1.6).

The present paper is composed of five sections. In section 2, we explain the background of the problem and our motivation. Numerical results will be presented in section 3. Two special cases of the problem are studied in section 4, where a simpler equation having blow-up solutions will be derived. Section 5 is devoted to the computation of the steady-states.

2 Background

If $g \equiv 0$, then we obtain the following single equation:

$$f_{txx} = \nu f_{xxxx} + f_x f_{xx} - f f_{xxx}. \tag{2.1}$$

In view of the important contribution by Proudman and Johnson [36], this equation was called the Proudman-Johnson equation by Cox [8]. Its first appearance in the literature, however, seems to be in Riabouchinsky [39]. The stationary version of (2.1), i.e., $\nu f_{xxxx} + f_x f_{xx} - f f_{xxx} = 0$, was considered much earlier in Hiemenz [20]. The purpose of [35] was to study the property of the solutions to (2.1) and related equations. One of the unsolved problem for (2.1) was to determine whether any solution of (2.1) with boundary condition

$$f(t, \pm a) = f_x(t, \pm a) = 0 \tag{2.2}$$

can blow up in finite time. In 1989, Childress and others [5] reported, among others, that a blow-up occurred for (2.1) and (2.2) with $\nu > 0$. On the other hand, there are some papers which report that no solution of (2.1) and (2.2) blow up in finite time. We do not know who was the first to find this but Cox [8] already says it clearly in 1991 (see [35] for other information). Based on the consideration in [8] and the numerical experiments in [34, 35, 52], we had believed that none of the solution blow up in finite time but a mathematical proof for global existence of the solutions was found only recently, see [6].

Grundy and McLaughlin [16] showed that some solutions of (2.1) could blow up if the following inhomogeneous boundary condition

$$f(t,-a)=\rho_1, \quad f_{xx}(t,-a)=-\sigma_1, \quad f(t,a)=\rho_2, \quad f_{xx}(t,a)=-\sigma_2, \tag{2.3}$$

where $\rho_i, \sigma_i\,(i=1,2)$ are appropriate constants, were imposed. So, the boundary condition plays an important role for blow-up. Presumably [16] is the first paper to find that an exact solution of the Navier-Stokes equations blows up in finite time, with a kind of uniqueness of the solution. In this sense, the contribution of [16] is very remarkable. Although their evidence for blow-up is convincing, their argument is partly numerical and partly asymptotic analytic. Therefore a rigorous proof of blow-up seems yet to be sought. If the boundary data in (2.3) is increased gradually from zero, the phase portrait of the dynamics changes interestingly. Bifurcation of steady-states happens and boundary layers appear for large Reynolds numbers. This is studied in detail by [15]. With boundary conditions different from (2.3), steady-states, their stability, and appearance of chaotic solutions were computed by [3, 8, 9, 23, 40, 49, 50, 51].

The coupled equations (1.4) and (1.5) were derived (with a slightly different but equivalent notation) by Taylor et al. [46] (f should be replaced by $-f$ in their notation.). With an appropriate boundary condition, they computed steady-states and studied numerically the dynamical behavior of nonstationary solutions. However, the existence or nonexistence of blow-up solutions seems to be out of their interest. Later, Grundy and McLaughlin introduced in [17] a system of coupled equations, which can be transformed, by change of variables, to our equations (1.4) and (1.5). With appropriate inhomogeneous boundary condition analogous to (2.3), some solutions were shown, by numerical and asymptotic methods, to blow up (see [17]). At nearly the same time, Zhu [52] independently found (1.4) and (1.5). Though [46] is the earliest among these papers, the coupled equations are actually a special case of what Lin [26] derived in 1957. (We learned this by [17].)

In this paper we will show, by numerical experiments, that some solutions of (1.4) and (1.5) with the *homogeneous* boundary condition (1.6) blow up in finite time. This shows an interesting difference between two and three dimensional cases. Zhu [52] showed numerically that blow-up occurred for (1.4) and (1.5) with the *homogeneous* boundary condition (1.6). However, he used only one initial data and there seems to be much room for further experiments. In the present paper we test several initial data and, furthermore, explain the blow-up by a hypothesis concerning steady-states.

We finally remark that the blow-up of some similarity solutions of the Navier-Stokes equations is not new. In fact, there are pioneering works by Ohkitani and Craik: Ohkitani [30], Craik [12]. They gave examples of blow-up. However, the solutions in [30] are not unique in the sense that one initial datum may yield many different solutions. This was pointed out by [31]. Craik [10] implicitly mentioned non-uniqueness of his solutions. This lack of uniqueness is caused by the linear (or higher) growth of the velocity vector and/or the pressure. With moderate growth condition on the pressure and/or the velocity gradient, we have uniqueness of the solution, see [13, 22, 31]. The uniqueness conditions assumed in these papers are, of course, violated by the above examples of blow-up. The most serious obstruction to the uniqueness is the unboundedness of the pressure (the pressure associated with our solution is a quadratic function in y and z). Also, the solutions of (2.1) or (1.4) & (1.5) do not meet the criteria of the uniqueness in [13, 22, 31]. Chae and Dubovskii [4] gave a more systematic way to construct blow-up solutions, which, too, allow the non-uniqueness of the solutions.

For mechanical applications, it is important to find a class of functions in which a solution is unique. This is difficult for those solutions mentioned in the preceding paragraph. A solution of the initial-boundary value problem (2.1) or (1.4) & (1.5) can easily be proved to be unique, say, in $C([0,T]; L^2(-a,a))$. In other words, the solution may not be unique among general velocity vector fields but it is unique among the velocity fields of similarity form (1.1).

Recently Ohkitani and Gibbon [29] showed numerically that some similarity solutions blow up in finite time. Their solutions have uniqueness in the same sense as the solution of (1.4),(1.5), and (1.6) is unique. The difference of the equation in [29] and ours is that their equations are defined in a two-dimensional domain while ours are defined in one-dimension. Their velocity field is unbounded in one direction and ours are unbounded in two directions. Accordingly, their problem may be more suited to physics problems. On the other hand, our problem may be more suitable to rigorous analysis.

We finally remark that Okamoto [32] gave some examples of blow-up via Leray's backward similarity equations. Here, again, the uniqueness of the solution to the nonstationary Navier-Stokes equations is not clear.

We have no mathematical proof for our blow-up results concerning the Navier-Stokes equations. The situation changes if we consider the Euler equations. Some similarity solutions which blows up in finite time are known if $\nu = 0$. See [4, 5, 7, 27, 43, 44] and [35].

It has long been conjectured that the Navier-Stokes equations in three-dimensions have a unique smooth solution for all the time and the Euler equations may have many spontaneous singularities. We believe this conjecture is true in the case where the initial data have finite energy. The results here and in [29] show that, if the finiteness of the energy is discarded, there is a case where the solution is unique in some class and at the same time blows up in finite time. This shows the importance of the finiteness of the energy.

We now rewrite the equation in a nondimensinal form. By the following re-definition of the variables

$$x \mapsto a\tilde{x}, \qquad t \mapsto \frac{a^2}{\nu}\tilde{t}, \qquad f \mapsto \frac{\nu}{a}\tilde{f}, \qquad g \mapsto \frac{\nu}{a}\tilde{g},$$

the equations (1.4) and (1.5) becomes, after dropping the tildes,

$$f_{txx} = f_{xxxx} + (f_x + g_x)f_{xx} - (f - g)f_{xxx}, \tag{2.4}$$

$$g_{txx} = g_{xxxx} - (f_x + g_x)g_{xx} - (f - g)g_{xxx}. \tag{2.5}$$

This equation should be satisfied in $0 < t$ and $-1 < x < 1$ together with the boundary condition.

$$f(t, \pm 1) = g(t, \pm 1) = f_x(t, \pm 1) = g_x(t, \pm 1) = 0. \tag{2.6}$$

In what follows, we use the following norm:

$$\|\phi\|_\infty = \max_{-1 \le x \le 1} |\phi(x)|, \qquad \|\phi\| = \left(\int_{-1}^{1} |\phi(x)|^2 dx \right)^{1/2}.$$

L^∞ and L^2 denote the set of all the functions ϕ with $\|\phi\|_\infty < \infty$ and $\|\phi\| < \infty$, respectively.

3 Numerical experiments

The approximation theory of blow-up solutions has a long history and many sophisticated schemes have been proposed (see, for instance Ushijima [47] or Wang [38]). We, however, use a rather simple method to minimize our programming labor and the same technique as in [35] is used in the present paper. Namely we employ the finite difference scheme proposed by Nakagawa [28].

He proposed a fully-explicit scheme but we modify it to a semi-implicit scheme in the sense that the viscous term is explicit but nonlinear term is partly implicit. In doing so we use an idea of Tabata [45]. Since the idea is the same for all the equations considered in the present paper, we write down the finite difference scheme in the simplest case. Namely we consider (2.1), which can be rewritten, after normalization, as follows:

$$\omega_t = \omega_{xx} + f_x \omega - f \omega_x. \tag{3.1}$$

The function f is determined by ω via the Poisson equation:

$$-f_{xx}(t,x) = \omega(t,x) \qquad (-1 < x < 1), \qquad f(t, \pm 1) = 0.$$

In view of this, the equation (3.1) is closed in ω. The boundary condition $f_x(t, \pm 1) = 0$ in (2.2) is equivalent to

$$\int_{-1}^{1} \omega(t,x) dx = \int_{-1}^{1} x\omega(t,x) dx = 0. \tag{3.2}$$

See [52]. Our scheme is the following one.

$$\frac{\omega_k^{n+1} - \omega_k^n}{\Delta t_n} = \frac{\omega_{k-1}^n - 2\omega_k^n + \omega_{k+1}^n}{h^2} + \frac{f_{k+1}^n - f_{k-1}^n}{2h} \omega_k^{n+1} - f_k^n \frac{\omega_{k+1}^n - \omega_{k-1}^n}{2h}. \tag{3.3}$$

Here $h = 2/N$ with N being the number of meshes and ω_k^n is an approximation for $\omega(t_n, -1 + kh)$ with $t_n = \sum_{m=0}^{n-1} \Delta t_m$. We give $\{\omega_k^0\}_{k=0}^N$. Then f_k^0 's are determined by the finite differences with $f_0^0 = f_N^0 = 0$. Then the finite difference scheme (3.3) are to be satisfied for $n = 1$ and $k = 1, \cdots, N-1$. Finally the condition (3.2) is discretized by the trapezoidal rule and the resulting equation determines ω_0^1 and ω_N^1. This process is repeated. The scheme is complete if we give a rule for Δt_m. Following Nakagawa [28], we define Δt_m as follows:

$$\Delta t_m = \min \left\{ \tau, \frac{q}{\Delta f_m} \right\},$$

where Δf_m is defined as

$$\Delta f_m = \max_{1 \le k \le N-1} \left| \frac{f_{k+1}^m - f_{k-1}^m}{2h} \right|,$$

and τ and q are parameters. In order that ω_k^n are well-defined, $\tau > 0$ and $q > 0$ must be sufficiently small.

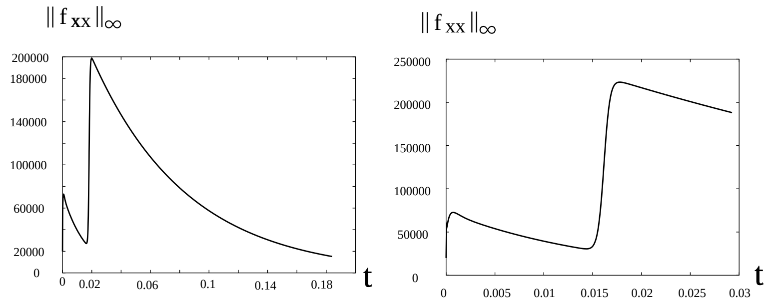

Figure 1: Time evolution of $\|f_{xx}(t,\cdot)\|_\infty$ for (3.1) and (3.2). Initial data; $f_{xx}(0,x) = -10000(5x^3 - 3x)$. The part in $0 < t < 0.03$ is magnified in the right. 300 meshes are used ($N = 300$). $\tau = 0.0001, q = 0.02$

Figure 1 shows how a solution of (3.1) and (3.2) behaves as t increases. There is a sharp increase of $\|f_{xx}(t,\cdot)\|_\infty$ near $t = 0$ and $t = 0.018$ but the solution eventually decays to zero slowly.

We now consider (1.4), (1.5), and (1.6). It was pointed out by Zhu [52] that the solution does not blow up if one of $f_{xx}(0,\cdot)$ and $g_{xx}(0,\cdot)$ is small enough. He therefore set $f(0,x) = g(0,x)$ and obtained Figure 2, which shows how $(\|f_{xx}(t,\cdot)\|_\infty, \|g_{xx}(t,\cdot)\|_\infty)$ changes with t. Although $\max(\|f_{xx}(0,\cdot)\|_\infty, \|g_{xx}(0,\cdot)\|_\infty)$ is smaller than $\|f_{xx}(0,\cdot)\|_\infty$ in Figure 1, the solution seems to blow up.

We tested other initial data and confirm that solutions blow up if both of the initial functions are large enough. Rather than showing these numerical data, we would like to show experiments for a restricted class of solutions which blows up. This is done in the next section.

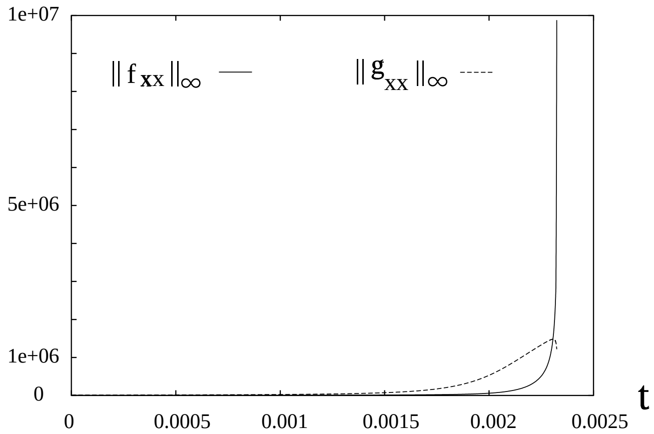

Figure 2: Time evolution of $\|f_{xx}(t,\cdot)\|_\infty$ and $\|g_{xx}(t,\cdot)\|_\infty$. Initial data; $f_{xx}(0,x) = g_{xx}(0,x) = -3000(5x^3 - 3x)$. $\tau = 0.0002$ and $q = 0.03 \times \tau$

4 Special cases

Suppose that the velocity field is axisymmetric with respect to the x-axis. This amount to assuming that $g(t,x) = -f(t,x)$. If we put $g = -f$ in (1.4) and (1.5), then the coupled equations are reduced to the following single equation:

$$f_{txx} = \nu f_{xxxx} - 2ff_{xxx} \qquad (-1 < x < 1) \tag{4.1}$$

with the boundary condition

$$f(t,\pm 1) = f_x(t,\pm 1) = 0. \tag{4.2}$$

The evolution equation (4.1) with the periodic boundary condition has a global solution for any initial data, which was proved in [35]. We remark that the global existence is a consequence of the fact that f_{xx} satisfies the maximum principle ($\omega = -f_{xx}$ satisfies $\omega_t = \nu\omega_{xx} - 2f\omega_x$).

There is another reduction to a scalar equation. Put $g(t,x) = f(t,-x)$. Then (1.4) and (1.5) are reduced to the following equation:

$$f_{txx} = \nu f_{xxxx} + 2H[f_x]f_{xx} - 2H[f]f_{xxx}, \tag{4.3}$$

where the operator H is defined as follows:

$$H[\phi] = \frac{\phi(t,x) - \phi(t,-x)}{2}.$$

If $f(0,x)$ is odd in x, then the solution to (4.3) and (4.2) remains to be odd in x for all t. In this case (4.3) is the same as (4.1). On the other hand, evenness of f is not preserved by (4.3) and some solutions lead to blow-up as is shown in Figure 3. The graphs of f_x are plotted in Figure 4, which seems to indicate that f_x blows up everywhere in $-1 < x < 1$.

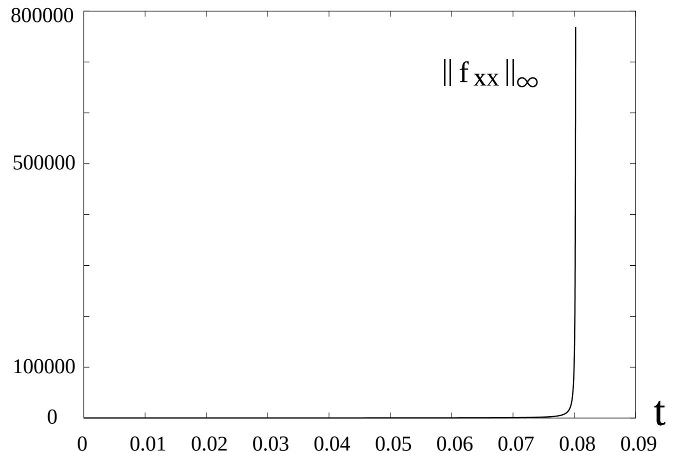

Figure 3: The time evolution of $\|f_{xx}(t,\cdot)\|_\infty$ for (4.3) and (4.2). $f(0,x) = 100(3x^2-1)$. $N = 1000$, $\tau = 0.0001$, $q = 0.4 \times \tau$

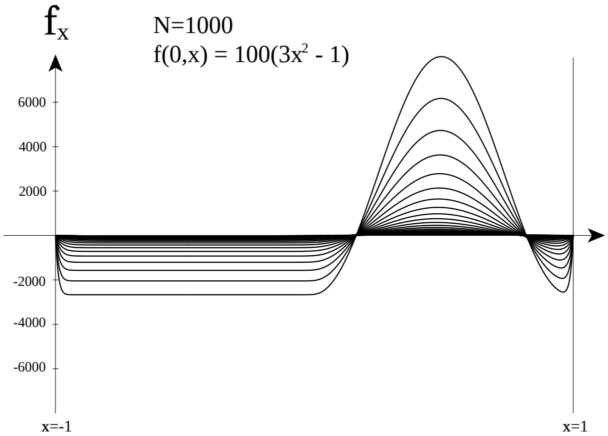

Figure 4: Plots of the graphs of $f_x(t, \cdot)$. The same data as in Figure 3.

5 Steady-state

One of the major differences between the coupled equations (1.4) & (1.5) and the single equation (2.1) is the existence or nonexistence of nontrivial steady-states. The equation (2.1) with the boundary condition (2.2) has no stationary solution other than $f \equiv 0$ (see [35]). On the other hand we will show numerically that there exist nontrivial stationary solutions of (1.4), (1.5) and (1.6). The existence of nontrivial steady-states is an indirect evidence that the dynamics of (1.4) & (1.5) may have blow-up solutions. In fact, we can understand why blow-up is favored by (1.4)–(1.6) but not by (2.1) & (2.2) in the following way.

It is easy to prove that (1.4),(1.5), and (1.6) admit a unique strong solution for short time for any $(f(0,\cdot), g(0,\cdot)) \in L^2(-1,1) \times L^2(-1,1)$. It is also easy to prove that the solution exists globally in time and $\lim_{t\to\infty} \|f(t,\cdot)\| = \lim_{t\to\infty} \|g(t,\cdot)\| = 0$, if the initial data are small. In the case of (2.1) with the boundary condition (2.2), the zero state $f \equiv 0$ attracts every solution ([6]). Namely the whole $L^2(-1,1)$ is the basin of $f \equiv 0$. On the other hand, the basin of $(f,g) = (0,0)$ is a proper subset of $L^2(-1,1) \times L^2(-1,1)$. The steady-states are divides in the sense that the boundary of the basin contains them. We actually found a steady-state which had a one-dimensional unstable manifold such that, on one side of the unstable manifold, the solution blew up in finite time and, on the other side of the manifold, the solution was attracted by $(f,g) = (0,0)$. The situation is illustrated by Figure 5.

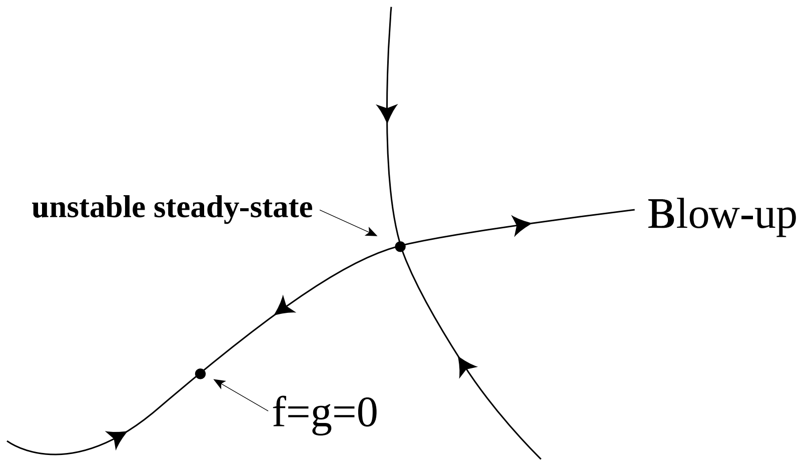

Figure 5: The steady-state as a divide.

In order to obtain nontrivial solutions, it is convenient to consider the following boundary value problem:

$$\nu f_{xxxx} + (f_x + g_x) f_{xx} - (f-g) f_{xxx} = 0, \quad (-a < x < a) \tag{5.1}$$
$$\nu g_{xxxx} - (f_x + g_x) g_{xx} - (f-g) g_{xxx} = 0, \quad (-a < x < a) \tag{5.2}$$
$$f(\pm a) = g(\pm a) = 0, \quad f_x(\pm a) = \beta_1, \quad g_x(\pm a) = \beta_2, \tag{5.3}$$

where β_1 and β_2 are prescribed constants. The physical meaning is that the walls are impermeable and are stretched or contracted along itself as elastic membranes.

We do not know who was the first to discover this system of equations but it dates back at least to 1959, when Howarth found the equations (5.1) and (5.2), see (13.9) and (13.10) in page 315 of [19] (Set the constant c in the page be 1, substitute $(-f, g)$ in place of (f, g), and differentiate the resulting equation.). However, he considered (5.1) and (5.2) in $0 < x < \infty$. Later [18, 25] gave mathematical proofs which guaranteed the existence of the solution with suitable boundary conditions at $x = 0$ and $x = \infty$. A similar problem was considered by Wang [48]. The equations (5.1) and (5.2) were studied by [46] in a finite interval with a boundary condition $f(\pm a) = \rho_\pm$, $g(\pm a) = \sigma_\pm$, $f_x(\pm a) = g_x(\pm a) = 0$. We are unaware of a paper which proves the existence of the solution of (5.1)–(5.3).

By the following nondimensionalization,

$$x \mapsto ax, \qquad f \mapsto \nu a^{-1} f, \qquad g \mapsto \nu a^{-1} g,$$

the boundary value problem above is written as

$$f_{xxxx} + (f_x + g_x) f_{xx} - (f - g) f_{xxx} = 0, \quad (-1 < x < 1) \quad (5.4)$$

$$g_{xxxx} - (f_x + g_x) g_{xx} - (f - g) g_{xxx} = 0, \quad (-1 < x < 1) \quad (5.5)$$

$$f(\pm 1) = g(\pm 1) = 0, \quad f_x(\pm 1) = R_1, \quad g_x(\pm 1) = R_2, \qquad (5.6)$$

where

$$R_1 = \frac{\beta_1 a^2}{\nu}, \qquad R_2 = \frac{\beta_2 a^2}{\nu}$$

are nondimensional parameters, which we call Reynolds numbers. Note that R_1 and R_2 can take any real numbers; positive, zero, or negative.

We now define another nondimensional parameter s by $R_2 = sR_1$ and consider the problem (5.4)–(5.6) as a bifurcation parameter with two parameters $R = R_1$ and s.

Our strategy to obtain nontrivial solutions of (5.4)–(5.6) with $R_1 = R_2 = 0$ is as follows. We start with $f \equiv 0, g \equiv 0, R_1 = R_2 = 0$ and use the path-continuation method (see for instance, [1], [2], [14], [21], [24], or [37]). With s fixed to 1, we compute solutions for $-400 \le R_1 \le 100$. From the solution $f, g, R_1 = R_2 = -400$ or $f, g, R_1 = R_2 = 100$, we fix $R = R_1$ and let s change from 1 to -1. We then have solutions, for instance $f, g, R_1 = -400, -1 \le s \le 1$. From there, we let s be fixed and let $R = R_1$ vary from -400 to 0. Then the continued solution may get back to the trivial solution but there is a chance that it is connected to a nontrivial solution. We actually found many solutions at $R_1 = -400$, and some of them became nontrivial solutions as $|R_1| + |R_2| \to 0$. Once we obtain a nontrivial solution, then we can repeat the process above and we may have other solutions.

In the case where $s = 0$, we solve

$$f_{xxxx} + (f_x + g_x)f_{xx} - (f - g)f_{xxx} = 0, \quad (-1 < x < 1) \quad (5.7)$$

$$g_{xxxx} - (f_x + g_x)g_{xx} - (f - g)g_{xxx} = 0, \quad (-1 < x < 1) \quad (5.8)$$

$$f(\pm 1) = g(\pm 1) = 0, \quad f_x(\pm 1) = R, \quad g_x(\pm 1) = 0, \quad (5.9)$$

We computed solutions in the range $-400 \le R \le 100$ through the finite difference discretization with 800 equal meshes. The result is shown in Figure 6. It is very important to note that some of the branches pass through those points where $R = 0$. This shows that there are many nontrivial solutions for $R_1 = R_2 = 0$.

Remark. If $g \equiv 0$, then the boundary value problem (5.7) – (5.9) is reduced to

$$f_{xxxx} + f_x f_{xx} - f f_{xxx} = 0, \quad (\ -1 < x < 1 \ ) \quad (5.10)$$

$$f(\pm 1) = 0, \quad f_x(\pm 1) = R, \quad (5.11)$$

Solutions of this boundary value problem was computed by [3, 49]. One can consider a slightly more general problem by imposing the

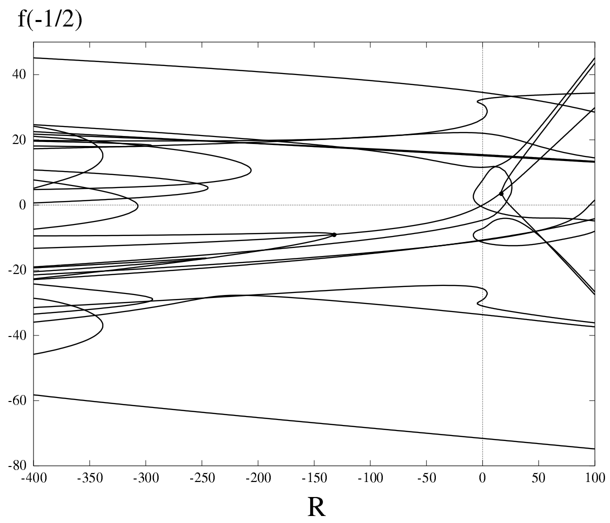

Figure 6: Bifurcation diagram of (5.7)–(5.9). There are many intersections in the figure but most of them are artificial ones and not bifurcation points. There are only two bifurcation points, which are marked by dots.

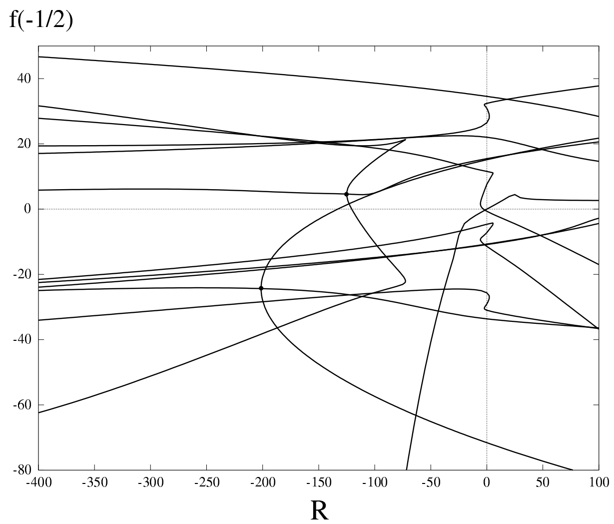

Figure 7: Bifurcation diagram of (5.4)–(5.6). $s = 1$. The dot represents the bifurcation point.

boundary condition $f_x(1) = R, f_x(-1) = R'$, which was studied in [50]. The boundary-value problem of (5.10) with the boundary condition $f(\pm 1) = R_{\pm}, f_x(\pm 1) = 0$ has been the subject of many researches. See [8, 9, 40, 41, 42, 49, 51] and references therein.

In the case where $s = 1$, the bifurcation diagram is shown in Figure 7. When $s = -1$, we obtained Figure 8. Figure 9 is the bifurcation diagram showing solutions with $R = -400, -1 \leq s \leq 1$.

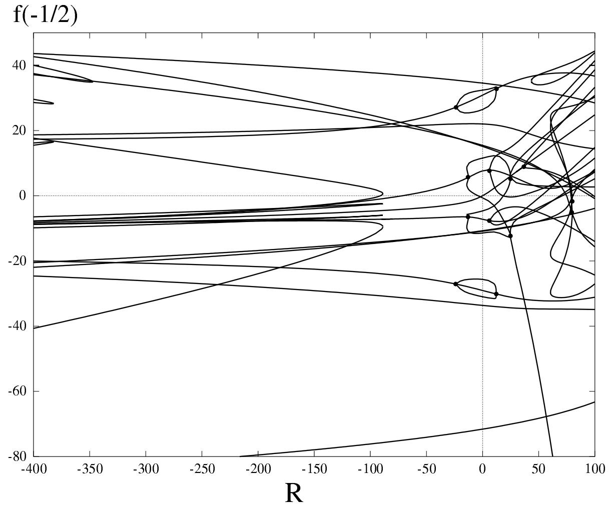

Figure 8: Bifurcation diagram of (5.4)–(5.6). $s = -1$. The dots represent the bifurcation points.

In this way, we found 20 nontrivial solutions in total for (5.4)–(5.6) with $R_1 = R_2 = 0$. We here note the following symmetry:

$$(f(x), g(x)) \mapsto (g(-x), f(-x)), \qquad (f(x), g(x)) \mapsto (-g(x), -f(x)),$$
$$(f(x), g(x)) \mapsto (-f(-x), -g(-x)).$$

These mappings and the identity map constitute the group $\mathbf{Z}_2 \times \mathbf{Z}_2$, with which the boundary value problem (5.4)–(5.6) with $R_1 =$

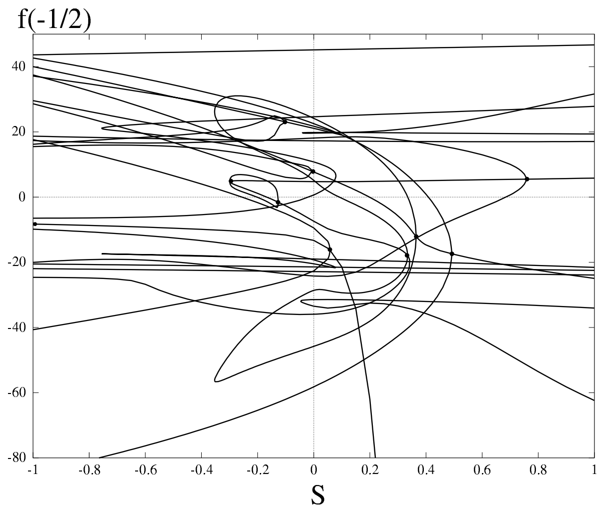

Figure 9: Bifurcation diagram of (5.4)–(5.6). $-1 \leq s \leq 1$, $R = -400$.

$R_2 = 0$ is equivariant. Accordingly, $(-g(x), -f(x))$ is a solution if $(f(x), g(x))$ is, etc. With this symmetry we can see that only six solutions are mutually independent and other 14 solutions are obtained from the six solutions through the action of $\mathbf{Z}_2 \times \mathbf{Z}_2$. The six solutions are depicted in Figure 10. The solution given in the left bottom of Figure 10 has the property which we are looking for: it has one-dimensional unstable manifold. $g(x) = f(-x)$ is satisfied by this solution and it is a steady-state of (4.3). We therefore consider it in the dynamics of (4.3).

In order to verify what we predicted in section 4, we performed the following experiments. We set $N = 1600$ and computed the steady-states $\hat{f} = (f_0, f_1, \cdots, f_N)$. We linearized the equation at $\hat{f}$ and computed the eigenvector $\hat{e} = (e_0, e_1, \cdots, e_N)$. Its was normalized in such a way that $\max_k |e_k| = 1$. We chose $\hat{f} + \epsilon\hat{e}$ as initial data with appropriate ϵ. Then we found that the solution blows up in finite time if it starts from $\hat{f} + \epsilon\hat{e}$ with $\epsilon = 0.2, 0.1, 0.05$ and that it decays to zero if $\epsilon = -0.2, -0.1, -0.05$. The results in the case of $\epsilon = 0.2$ are shown in Figures 11 and 12. Note that the graph of $f(t, \cdot)$ when t is near the blow-up time is almost flat in $0 < x < 1$. This and Figure 4 suggest that a kind of asymptotic analysis of the blow-up is possible. We however leave it to future works.

6 Conclusion

We have studied coupled equations, which are a three-dimensional extension of the Proudman-Johnson equation. By a finite difference method, we have shown that some solutions blow up in finite time. The phase portrait of the dynamical system generated by the equations is better understood by steady-states. We found many nontrivial steady-states, which suggests that the dynamical system is a rather complicated one. A scalar, nonlinear, nonlocal equation of one variable — (4.3) — is derived and blow-up solutions were obtained numerically.

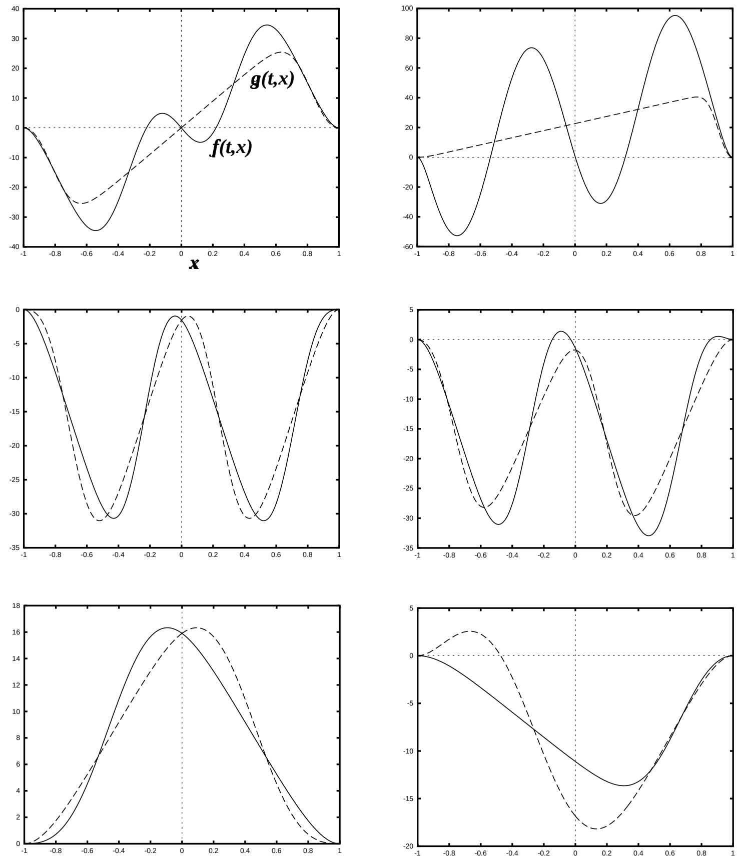

Figure 10: Graphs of f and g of (5.4)–(5.6) with $R_1 = R_2 = 0$. The solid lines are the graphs of f and the broken lines are those of g.

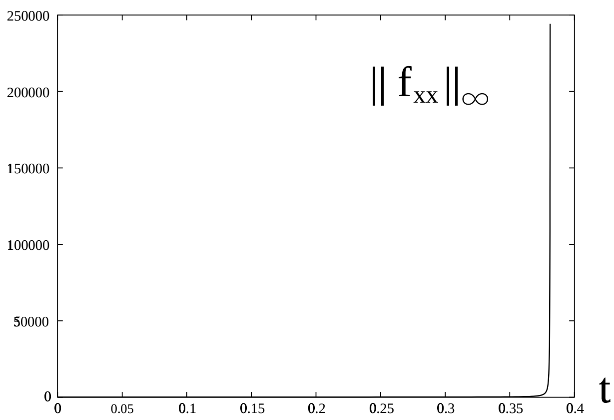

Figure 11: Time evolution of $\|f\|_\infty$ for (4.3) and (4.2) with $\hat{f} + 0.2\hat{e}$ as its initial data.

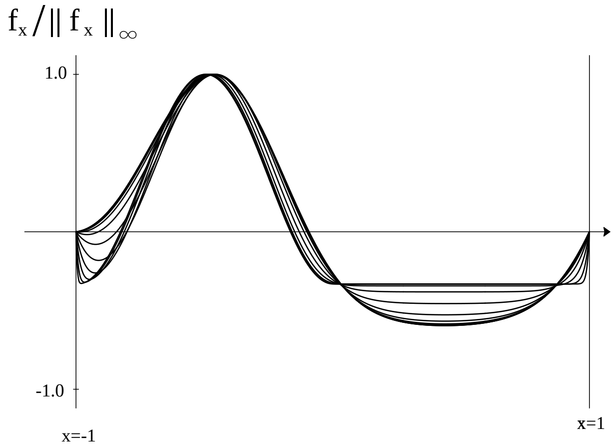

Figure 12: Plots of $f_x(t, x)/\|f_x(t, \cdot)\|_\infty$ for the experiments in Figure 11.

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