Blow-up problems modeled from the strain-vorticity dynamics

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Dedicated to the memory of the late Professor Tosio Kato

Abstract

Model equations are derived from what we call the strain-vorticity dynamics of the incompressible viscous fluid motion. The global existence and blow-up are examined for them and we see that the L^∞ norm of the vorticity plays an important role. Blow-up solutions are obtained as self-similar solutions.

1 Introduction

One of the open questions about the Navier-Stokes equations is the problem on the existence global-in-time or blow-up in-finite-time of the solutions in three dimensions. Since this is a notoriously difficult problem, many attempts have been made to extract the essence of the 3D mechanism and simplify the problem. The present paper is one of those which consider the problem by means of models.

Special solutions of the Navier-Stokes equations for incompressible viscous fluid are obtained by the following ansatz:

$$\mathbf{u} = \Big(-\gamma_1(t)x + u(t,x,y),\ -\gamma_2(t)y + v(t,x,y),\ (\gamma_1(t)+\gamma_2(t))z \Big),$$

where $\mathbf{u}$ is the velocity field, t denotes time, and (x,y,z) denotes a point in three dimensional space $\mathbf{R}^3$. The x and y components of the vorticity $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{u}$ turns out to vanish and the z component is $v_x - u_y$, which is denoted by ω. Then ω, after being substituted into the Navier-Stokes equations, satisfies

$$\omega_t + (-\gamma_1 x + u)\omega_x + (-\gamma_2 y + v)\omega_y - (\gamma_1+\gamma_2)\omega = \nu\triangle\omega, \tag{1}$$

where the subscripts t, x, y imply the differentiation. Since u and v satisfy $u_x + v_y = 0$, they are given as

$$u(t,x,y) = \frac{-1}{2\pi}\int_{\mathbf{R}^2} \frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}\omega(t,\xi,\eta)d\xi d\eta,$$
$$v(t,x,y) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}\omega(t,\xi,\eta)d\xi d\eta.$$

Therefore, the equation (1) can be viewed as an equation of ω only. This is a nonlinear, nonlocal equation of ω, and can be solved once γ_1 and γ_2 are prescribed. It is customary to call the scalar function ω vorticity. The parameters γ_1 and γ_2 are called the strain-rates.

If $\gamma_1 \equiv \gamma_2$, then there exist axisymmetric solutions, where $\omega = \omega(t, r)$ with $r = \sqrt{x^2 + y^2}$. The velocity and the vorticity are related indirectly by

$$u = -f(t,r)\sin\theta, \quad v = f(t,r)\cos\theta, \quad \omega = \frac{1}{r}(rf)_r\,.$$

With $\gamma(t) = \gamma_1(t) = \gamma_2(t)$, the vorticity satisfies the following equation:

$$\omega_t - \gamma(t)\,(r\omega_r + 2\omega) = \nu\frac{1}{r}\,(r\omega_r)_r \qquad (0 \le r < \infty). \tag{2}$$

Equations (1) and (2) are known for many decades, originally due to Burgers, see [4, 6, 7, 8].

The equations above are derived above by modeling what is called Burgers' vortex tube. We can consider the vortex sheet as well; in that case, we start with the following ansatz:

$$\mathbf{u} = (-\gamma(t)x, v(t,x), \gamma(t)z)\,.$$

The vorticity $\omega = v_x$ satisfies

$$\omega_t - \gamma(t)(x\omega_x + \omega) = \nu\omega_{xx} \qquad (-\infty < x < \infty). \tag{3}$$

Equations (2) and (3) can be solved with respect to ω once we know the strain-rate γ. There is no way of specifying $\gamma(t)$ without resorting to a kind of hypothesis. There are many papers in which γ is regarded as a specified constant ([4, 6, 7, 8]). Moffatt [8] considered the case where $\gamma(t)$ is given as a singular function $c/(T-t)$, where c and T are positive constants, and he concluded the same blow-up asymptotics for ω as the one for $\gamma(t)$. This choice of $\gamma(t)$ makes the equation (2) non-autonomous. In general we may assume that γ is determined by ω through a functional relation $\gamma = F(\omega)$ and make (2) autonomous.

We introduce two specific examples of $F(\omega)$ in the next section. For those models, steady-states are found in section 3, and some blow-up solutions of similarity form is obtained in section 4. Global existence of the solutions in some cases are proved in section 5. Vortex sheet models are considered in section 6. Finally, concluding remarks are given in section 7.

2 Models

The assumption $\gamma = F(\omega)$ can be interpreted as follows. In general 3D flows, the vorticity is highly localized if ν is small. With this in mind, we assume that many vortex tubes and other vortical structures are distributed in the 3D space. We then focus on a vortex tube located on the z-axis. This vortex tube influences other vortical structures which are distant from the vortex tube. They, in turn, apply a force on the vortex tube by inducing a strained velocity field, which is given as $(-\gamma_1(t)x, -\gamma_2(t)y, (\gamma_1(t)+\gamma_2(t))z)$. The magnitude of the strain-rate is determined by the magnitude of the vortex tube. A similar interpretation can be given to the vortex sheet.

To choose the strain-rates more specifically, we recall that the strain-rate tensor $S(x) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)_{1\leq i,j\leq 3}$ has the following integral representation (see e.g. [1])

$$S(x) = \frac{3}{8\pi}\mathrm{P.V.}\int_{\mathbf{R}^3}[y\otimes(y\times\omega(x+y)) + (y\times\omega(x+y))\otimes y]\frac{dy}{|y|^5},$$

where P.V. denotes the principal value. We observe that (i) the strain-rate tensor has the same dimension as $\omega(x)$ and (ii) it is a linear functional of $\omega(x)$. Those properties should be reflected in any models.

We here propose two hypotheses on the relation between the strain-rate and the vorticity. The first one is obtained by assuming

$$\gamma(t) = \mu\|\omega(t)\|_p, \tag{4}$$

where $\|\ \|_p$ denotes the L^p norm and μ is a constant. L^p norm is defined as

$$\|f\|_p = \left(2\pi\int_0^\infty |f(r)|^p r dr\right)^{1/p}$$

for $1 \leq p < \infty$ and

$$\|f\|_\infty = \sup_{0\leq r<\infty}|f(r)|$$

for $p = \infty$.

The presence of the constant μ is to adjust the dimension of both sides: μ has dimension $L^{-2/p}$. Accordingly, we are assuming that a length-scale is prescribed. If $p = \infty$, then γ and $\|\omega\|_\infty$ are of the same dimension, hence there is no need to introduce the constant. But, for $1 \leq p < \infty$, the constant is necessary, although this is an unidentified parameter. Now the problem is to study the properties of the solution ω of

$$\omega_t = \mu\|\omega(t)\|_p\,(r\omega_r + 2\omega) + \nu\frac{1}{r}\,(r\omega_r)_r \qquad (0\leq r<\infty, 0<t), \tag{5}$$

$$\omega(0,r) = \omega_0(r). \tag{6}$$

The second hypothesis is obtained by postulating absence of a typical length scale, which can be accomplished by a suitable combination of L^p norm and L^1 norm:

$$\gamma(t) = \|\omega(t)\|_p^{p/(p-1)}\|\omega(t)\|_1^{-1/(p-1)} \tag{7}$$

for $1 < p < \infty$. Here, the dimensions of both sides are the same, and we do not need to introduce a new dimensional parameter. The evolution equation is now

$$\omega_t = \|\omega(t)\|_p^{p/(p-1)}\|\omega(t)\|_1^{-1/(p-1)}\,(r\omega_r + 2\omega) + \nu\frac{1}{r}\,(r\omega_r)_r\,. \tag{8}$$

By similar hypotheses on vortex sheet, we get to

$$\omega_t = \mu \|\omega(t)\|_p \, (x\omega_x + \omega) + \nu \omega_{xx} \qquad (-\infty < x < \infty, 0 < t) \tag{9}$$

and

$$\omega_t = \|\omega(t)\|_p^{p/(p-1)} \|\omega(t)\|_1^{-1/(p-1)} \, (x\omega_x + \omega) + \nu \omega_{xx}. \qquad (-\infty < x < \infty, 0 < t) \tag{10}$$

Note that the positivity is preserved in the sense that $\omega_0(r) \geq 0$ everywhere implies $\omega(t, r) \geq 0$ for all t and r. (This can be verified most easily by looking at (16) and (17) in section 5.) *We thereby consider only those initial data which are nonnegative and smooth everywhere and decay sufficiently rapidly to zero as r or $|x| \to \infty$.* Note also that the circulation is preserved;

$$\int_0^\infty \omega(t, r) r dr \equiv \int_0^\infty \omega_0(r) r dr,$$

which can be easily verified. We have therefore obtained a very important proposition that $\|\omega\|_1$ is conserved. If $p = 1$, the equation (5) becomes linear and no blow-up occurs. Also, the equation (8) is written as

$$\omega_t = \lambda \|\omega(t)\|_p^{p/(p-1)} \, (r\omega_r + 2\omega) + \nu \frac{1}{r} \, (r\omega_r)_r \, , \tag{11}$$

where λ is a constant depending on the initial data.

Circulation is also preserved for vortex sheet and we have

$$\int_{-\infty}^{+\infty} \omega(t, x) dx \equiv \int_{-\infty}^{+\infty} \omega_0(x) dx.$$

Remark. When $p = \infty$, (7) is the same hypothesis as (4) with $p = \infty$. The relation (7), however, tends to a nontrivial relation as $p \to 1$:

$$\lim_{p \to 1} \frac{\|\omega\|_p^{p/(p-1)}}{\|\omega\|_1^{1/(p-1)}} = \|\omega\|_1 \exp\left(\frac{1}{\|\omega\|_1} \int_0^\infty 2\pi r |\omega| \log |\omega| dr \right).$$

We do not have any result for this hypothesis.

Remark. Nonlinear evolution equations which contains nonlinear terms represented by L^p norms are not new, see for instance [2, 10]. We, however, could not find (5), (8), (9), or (10) in references.

3 Steady-state

Equations (5) and (8) possess a steady-state known as Burgers' vortex tube, which is given by $\omega(t, r) = A \exp(-ar^2)$, where a and A satisfy

$$A = \frac{2a\nu}{\mu} \left(\frac{ap}{\pi} \right)^{1/p} \quad (\, 1 \leq p < \infty \,), \qquad A = \frac{2a\nu}{\mu} \quad (\, p = \infty \,)$$

for (5) and

$$A = 2a\nu p^{1/(p-1)}$$

for (8). The constant a and A are determined if $1 \le p < \infty$ and if we specify the value of the circulation:

$$\Gamma = 2\pi \int_0^\infty \omega(t,r) r dr.$$

Similarly, (9) and (10) possess a steady-state given by $\omega(t,x) = A\exp(-ax^2)$, where

$$A = \frac{2a\nu}{\mu}\left(\frac{ap}{\pi}\right)^{1/(2p)} \quad \text{and } A = 2a\nu p^{1/(2(p-1))},$$

respectively for (9) and (10).

4 Similarity solution

We first consider the equation (5). We assume the solution of the following form:

$$\omega(t,r) = (T-t)^{-\alpha}\phi(r/\sqrt{T-t}).$$

Then it turns out that $\alpha = 1 + p^{-1}$ and ϕ satisfies

$$\alpha\phi(\xi) + \frac{1}{2}\xi\phi'(\xi) = \mu\|\phi\|_p\left(2\phi + \xi\phi'(\xi)\right) + \nu\frac{1}{\xi}\left(\xi\phi'(\xi)\right)', \tag{12}$$

where $\xi = r/\sqrt{T-t}$.

If $p = \infty$, then

$$\nu\left(\phi''(\xi) + \frac{1}{\xi}\phi'(\xi)\right) + (2A-1)\left(\phi(\xi) + \frac{1}{2}\xi\phi'(\xi)\right) = 0, \tag{13}$$

where

$$A = \mu \sup_{0\le\xi<\infty}\phi(\xi).$$

The equation (13) can be integrated and we have

$$\nu\xi\phi'(\xi) + \frac{2A-1}{2}\xi^2\phi(\xi) = k,$$

where k is a constant. It turns out that a solution which is bounded near $\xi = 0$ can be obtained only if $k = 0$ and the solution is given by

$$\phi(\xi) = \frac{A}{\mu}\exp\left(-\frac{2A-1}{4\nu}\xi^2\right),$$

where $1/2 < A$ is assumed. For this blow-up solution, we have $\gamma = \mu\|\omega(t)\|_\infty = A(T-t)^{-1}$. Thus, we may say that what was assumed in [8] can be derived from our hypothesis $\gamma = \mu\|\omega(t)\|_\infty$.

Now, we have found an explicit blow-up solution for $p = \infty$. Since there is no blow-up for $p = 1$, it would be an interesting question to determine which p permits blow-up solutions and which p does not.

The equation (12) does not seem to admit a solution if $1 \leq p < \infty$. Let us search a solution ϕ, which decays sufficiently rapidly at $r = \infty$ and is positive everywhere. The equation (12) can be written as

$$\nu \left(\xi \phi'\right)' + A(\xi^2 \phi)' = \frac{1}{2} \xi^{2-2\alpha} \left(\xi^{2\alpha} \phi\right)', \tag{14}$$

where

$$A = \mu \left(2\pi \int_0^\infty \phi(\xi)^p \xi d\xi\right)^{1/p}.$$

By integrating this equation, we obtain

$$\nu \xi \phi'(\xi) + \left(A - \frac{1}{2}\right) \xi^2 \phi(\xi) = \frac{1}{p} \int_0^\xi \phi(\eta) \eta d\eta.$$

By letting $\xi \to \infty$, we have

$$\int_0^\infty \phi(\eta) \eta d\eta = 0,$$

which is impossible for positive ϕ.

It is therefore natural to suspect that the dynamical system (5) admits blow-up solutions if $p = \infty$ but not if $1 \leq p < \infty$. This is actually true and will be proved in the next section.

We now look for similarity solutions of (8): we have $\alpha = 1$ and

$$\phi(\xi) + \frac{1}{2} \xi \phi'(\xi) = \|\phi\|_p^{p/(p-1)} \|\phi\|_1^{-1/(p-1)} \left(2\phi + \xi \phi'(\xi)\right) + \nu \frac{1}{\xi} \left(\xi \phi'(\xi)\right)'. \tag{15}$$

Its solution is

$$\phi(\xi) = K p^{1/(p-1)} \exp\left(-\frac{2K - 1}{4\nu} \xi^2\right),$$

where K is a constant satisfying $K > 1/2$. Therefore (8) has blow-up solutions for all $p \in (1, \infty)$. The strain-rate satisfies $\gamma(t) = C/(T - t)$ with a positive constant C.

It is not easy for us to determine whether blow-up occurs or not for general initial data. If a comparison theorem such as the one below holds, any initial date which is larger everywhere than the self-similar blow-up solution blows up in finite time. But we do not know whether this is true or not.

If ω and ζ are two solutions of (5) *such that $\omega(0, r) \leq \zeta(0, r)$ for all $r \in [0, \infty)$. Then $\omega(t, r) \leq \zeta(t, r)$ for all t and r?*

Souplet [10] proved a comparison theorem for nonlocal parabolic equations, but we could not apply his theorem; we could not verify, in the case of our equations, one of the assumption appearing in his theorem.

5 Energy estimates

We prove in this section the global existence of the solutions of (5) for $1 \le p < \infty$.

Before deriving a priori estimates necessary for the global existence, some facts about the local existence should be noted. Solutions local-in-time is constructed by the successive approximations; for $n = 1, 2, \cdots$

$$\omega_t^{(n+1)} = \mu \|\omega^{(n)}(t)\|_p \left(r\omega_r^{(n+1)} + 2\omega^{(n+1)} \right) + \nu \frac{1}{r} \left(r\omega_r^{(n+1)} \right)_r$$

In doing so, we need to estimate the solutions of the following linear equation:

$$\Omega_t = A(t)\left(r\Omega_r + 2\Omega\right) + \nu \frac{1}{r}\left(r\Omega_r\right)_r, \tag{16}$$

where $A(t)$ is a given function of t. This linear equation can be solved easily by a trick originally due to Lundgren (see [4, 7]). The trick is to use the following change of variables:

$$\Omega(t,r) = a(\tau)^2 u(\tau, a(\tau) r), \tag{17}$$

where

$$a(\tau) = \exp\left(\int_0^t A(s)ds\right), \qquad \frac{d\tau}{dt} = \exp\left(2\int_0^t A(s)ds\right).$$

By this transformation, we have $u_\tau = \nu(u_{\rho\rho} + \frac{1}{\rho}u_\rho)$, where $\rho = a(\tau)r$. Accordingly, necessary estimates for Ω are derived from well-known ones for the linear diffusion equation.

In this way, we can prove that, for all $\omega_0 \in L^1 \cap L^p$, there exists $T > 0$ such that the solution of (5) exists and unique in $C^0([0,T]; L^1 \cap L^p)$.

We now consider a priori estimates, which are necessary for global existence. Multiplying (5) by $2\pi p \omega(t,x)^{p-1} r$ and integrating by parts, we have

$$\frac{d}{dt}\|\omega(t)\|_p^p = 2(p-1)\mu\|\omega(t)\|_p^{p+1} - 2\pi\nu(p-1)p\int_0^\infty \omega^{p-2}\omega_r^2 r dr. \tag{18}$$

We then use the following theorem due to Gagliardo and Nirenberg:

Theorem 1 *Let n be a positive integer. For $1 \le \alpha, \beta, \gamma \le \infty$, we define s by*

$$\frac{1}{\alpha} = s\left(\frac{1}{\gamma} - \frac{1}{n}\right) + \frac{1-s}{\beta}.$$

We assume that $0 \le s \le 1$. If $n \ge 2$, we also assume either $\alpha \ne \infty$ or $\gamma \ne n$. Then there exists a constant c such that the following inequality holds true for any f defined in $\mathbf{R}^n$:

$$\|f\|_\alpha \le c\|f\|_\beta^{1-s}\|\nabla f\|_\gamma^s, \tag{19}$$

where

$$\|f\|_\alpha = \left(\int_{\mathbf{R}^n} |f(x)|^\alpha dx\right)^{1/\alpha}.$$

Proof of this theorem can be found in many textbooks on functional analysis or partial differential equations. See, e.g., [3] or [5].

From now on, the symbol c is used to denote various positive constant independent of t. It represents different values in different places. We now use the Gagliardo-Nirenberg theorem for $n = 2$ and $f(x) = g(r), r = |x|$ to obtain

$$\left(\int_0^\infty |g(r)|^\alpha r dr\right)^{1/\alpha} \le c \left(\int_0^\infty |g(r)|^\beta r dr\right)^{(1-s)/\beta} \left(\int_0^\infty |g'(r)|^\gamma r dr\right)^{s/\gamma}. \qquad (20)$$

This equation is then applied to the solution of (5). We put $g(r) = \omega(t,r)^{p/2}$ and $\alpha = 2, \beta = 2/p$, $\gamma = 2$. Here $2 \ge p$ is assumed. Then $s = (p-1)/p$ and

$$\|\omega(t)\|_p \le c\|\omega(t)\|_1^{1/p} \left(\int_0^\infty \omega(t,r)^{p-2}\omega_r(t,r)^2 r dr\right)^{(p-1)/p^2}.$$

Since L^1-norm of $\omega(t)$ is non-increasing, we have

$$\|\omega(t)\|_p^{p^2/(p-1)} \le c\int_0^\infty \omega(t,r)^{p-2}\omega_r(t,r)^2 r dr,$$

where c is independent of t. We therefore obtain

$$\frac{d}{dt}\|\omega(t)\|_p^p \le 2(p-1)\mu\|\omega(t)\|_p^{p+1} - c\|\omega(t)\|_p^{p^2/(p-1)},$$

where c is independent of t. Note that $p+1 < p^2/(p-1)$. From this inequality, it is easy to derive the boundedness of $\omega(t)$ in L^p. Therefore we have proved

Theorem 2 *Consider* (5) *and assume that* $1 \le p \le 2$. *If* $\omega(0,\cdot) \in L^1 \cap L^p$, *then the solution exists globally in time.*

The restriction $p \le 2$ is actually unnecessary. This is in fact the consequence of the following lemma.

Lemma 1 *Consider* (5) *and assume that* $\omega(0) \in L^1 \cap L^p$. *Let* $1 \le q \le p$ *and assume that* $M \equiv \sup_{0<t}\|\omega(t)\|_q < \infty$. *Then, for all* δ *such that* $q \le \delta \le 2q$ *and* $\delta \le p$, *we have* $\sup_{0<t}\|\omega(t)\|_\delta < \infty$.

Proof. Note first that $\omega(t) \in L^\eta$ for all $\eta \in [1,p]$, which is verified by Hölder's inequality. We have

$$\frac{d}{dt}\|\omega(t)\|_\delta^\delta = 2\mu(\delta-1)\|\omega(t)\|_p\|\omega(t)\|_\delta^\delta - 2\pi\nu(\delta-1)\delta\int_0^\infty \omega^{\delta-2}\omega_r^2 r dr.$$

We then use the Gagliardo-Nirenberg theorem for $g = \omega^{\delta/2}$ with $\alpha = 2, \beta = 2q/\delta, \gamma = 2, s = 1 - q/\delta$ to obtain

$$\|\omega\|_\delta \le c\left(\int_0^\infty \omega^{\delta-2}\omega_r^2 r dr\right)^{(\delta-q)/(\delta^2)},$$

where the boundedness of $\|\omega(t)\|_q$ is used. Similarly we have

$$\|\omega\|_p \le c\left(\int_0^\infty \omega^{\delta-2}\omega_r^2 r dr\right)^{(p-q)/(p\delta)}$$

by choosing $\alpha = 2p/\delta, \beta = 2q/\delta, \gamma = 2, s = (p-q)/p$. Combining these two inequalities, we obtain

$$\|\omega\|_p\|\omega\|_\delta^\eta \le c\int_0^\infty \omega^{\delta-2}\omega_r^2 r dr,$$

where

$$\eta = \frac{\delta^2}{\delta - q}\left(1 - \frac{1}{\delta}\left(1 - \frac{q}{p}\right)\right).$$

Since $\eta > \delta$, we have the boundedness of $\|\omega\|_\delta$.

□

Making repeated use of this lemma, we see that $\omega(t)$ is bounded in $L^1 \cap L^p$. We therefore have proved the following

Theorem 3 *Assume that $\omega(0,\cdot) \in L^1 \cap L^p$. Then, for all $1 \le p < \infty$, the solution of* (5) *exists for all time and is bounded in $L^1 \cap L^p$.*

6 Solutions for vortex sheet

Let us consider (9) again. If we look for a solution of the following form:

$$\omega(t,x) = (T-t)^{-\alpha}\phi(x/\sqrt{T-t}),$$

then it turns out that $\alpha = 1 + (2p)^{-1}$, and

$$-\alpha\phi(\xi) - \frac{1}{2}\xi\phi'(\xi) = \mu\|\phi\|_p\left(\phi + \xi\phi'(\xi)\right) + \nu\phi''(\xi). \tag{21}$$

where $\xi = x/\sqrt{T-t}$. No positive function satisfy this equation for any $1 \le p \le \infty$. Therefore, we can expect global existence of the solutions in the vortex sheet models.

In fact, the Gagliardo-Nirenberg theorem also holds true in one dimension and we obtain, in almost the same way, the following theorem.

Theorem 4 *For any $\omega_0 \in L^1 \cap L^p$, the solution of the equation* (9) *exists globally in time. The same conclusion holds true for* (10).

7 Conclusion

Existence and/or blow-up of solutions to some model equations are considered. Although we rely on hypotheses about the relation between the strain-rate and the vorticity, it should be noted that the solutions nevertheless represent exact solutions of the Navier-Stokes equations. Solutions of the vortex sheet models (9) and (10) exist globally in time for any choice of p including $p = \infty$. Vortex tube model (8) has blow-up solutions for all $p \in (1, \infty]$. On the other hand, for the model (5), blow-up exists if $p = \infty$ but not if $1 \leq p < \infty$.

We have derived the same conclusion on the blow-up asymptotics as in Moffatt [8] but, while he assumes the blow-up strain-rate $\gamma(t) = C/(T - t)$, we have the same conclusion from autonomous systems (5) and (8).

Because the model we have considered here are based upon some assumptions on the choice of strain-rates, the results on the presence or absence of blow-up do not necessarily carry over to the general Navier-Stokes equations. However, it is interesting to us that there is a remarkable difference between vortex tube and vortex sheet solutions within the identical framework, *i.e.* under the same hypotheses. This may suggest that geometrical structure of vortices substantially influence the regularity property, which seems to comply with known theories, see [1].

Many important questions are left unanswered. For instance, stability of the steady-states and asymptotic behavior of the global solutions need further study. Similarity solutions have been sought only in positive solutions. We do not know whether equation (12) with $\alpha = 1 + \frac{1}{p} > 1$, or (21) with $\alpha = 1 + \frac{1}{2p}$ may possess a nontrivial solutions with changing sign.

We have not considered the general, non-axisymmetric solutions, which obey

$$\omega_t + (-\gamma_1 x + u)\omega_x + (-\gamma_2 y + v)\omega_y - (\gamma_1 + \gamma_2)\omega = \nu \triangle \omega,$$

supplemented by

$$\begin{aligned} u(t,x,y) &= \frac{-1}{2\pi} \int_{\mathbf{R}^2} \frac{y-\eta}{(x-\xi)^2 + (y-\eta)^2} \omega(t,\xi,\eta) d\xi d\eta, \\ v(t,x,y) &= \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{x-\xi}{(x-\xi)^2 + (y-\eta)^2} \omega(t,\xi,\eta) d\xi d\eta., \end{aligned}$$

and

$$\gamma_1 = F_1(\omega), \qquad \gamma_2 = F_2(\omega).$$

Certainly more computations than in the present paper are necessary for studying this system. The case where γ_1 and γ_2 are constant were considered in [6, 9]. But the general cases seem to be difficult to analyze.

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