# A singular perturbation problem arising in Oseen's spiral flows

H. Ikeda*, M. Mimura† and H. Okamoto‡

**Abstract**

Oseen's spiral flows for viscous incompressible fluid are considered. Their limiting behavior as the Reynolds number tends to infinity is rigorously analyzed and the width of the interior layer is proved to be of $O(R^{-1/2})$, where $R$ is the Reynolds number.

## 1 Introduction

Oseen' spiral flows ( [6] ) are classical in fluid mechanics: they are exact solutions of the Navier-Stokes equations in the plane and are generalization of the point vortex and the sink ( or the source ) ( see, e.g., Berker [2] ).

Oseen reduced the Navier-Stokes equations into a nonlinear ordinary differential equation of second order, which he solved in terms of the elliptic functions. It is noted in [5] that the flows can be better understood if we consider the problem in the framework of the bifurcation theory. Also, it is shown that the flows have an interesting limiting behavior, which seems to have escaped from researchers' attention, as the Reynolds number tends to infinity.

Our purpose in the present paper is to provide a solider base for analysis of such limiting behavior. Our method consists of numerical experiments and singular perturbation technique.

In order to explain Oseen's flow, let us consider the stationary Navier-Stokes equations in the plane:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\triangle\mathbf{u} - \frac{1}{\rho}\nabla p, \qquad \operatorname{div}\mathbf{u} = 0,$$

*Dept. of Math, Fac. of Sci., Toyama University, Toyama, 930-8555, Japan. Partialy supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan, # 11640107

†Dept. of Math., Fac. of Sci., Hiroshima University,Higashi-Hiroshima, 739-8526 Japan. Partialy supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan, # 11214101

‡Res. Inst. for Math. Sci., Kyoto University, Kyoto 606-8502 Japan. email: okamoto@kurims.kyoto-u.ac.jp, partialy supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture of Japan, # 11304005

where $\mathbf{u} = \mathbf{u}(x, y)$ is the velocity vector field, $p$ is the pressure. $\nu > 0$ is a constant called the viscosity. $\rho > 0$ is a positive constant representing the mass density of the fluid. We consider steady-states which are given by a stream function $\psi$. Namely $\mathbf{u} = (\psi_y, -\psi_x)$. Our standing assumption, which is the same as the one assumed by Oseen [6], is that the stream function $\psi$ is of the following form: $\psi = f(\phi) + C\chi$, where $C$ is a constant, $f$ is a function of one variable, and we define $\phi$ and $\chi$ by

$$\phi = B \log r + \theta, \qquad \chi = \log r - B\theta.$$

Here, $B$ is another constant: $r$ and $\theta$ are polar coordinates. We require that $\psi$ is smooth except at the origin, that the gradient of $\psi$ is a single-valued function of $\theta$, and that the Navier-Stokes equations are satisfied in $\mathbf{R}^2 \setminus \{0\}$.

The total flux of the fluid along the circle of radius $r$ about origin is

$$\int_0^{2\pi} u_r(r, \theta) r d\theta, \tag{1}$$

where $u_r$ denotes the $r$-component of the velocity. It is easily verified that this is independent of $r$. Accordingly we define the Reynolds number $R$ by

$$R = \frac{1}{\nu} \int_0^{2\pi} u_r r d\theta. \tag{2}$$

As is proved in [5], every requirement is met if we have a function $U$ which is $2\pi$-periodic and satisfies

$$\int_0^{2\pi} U(\phi) d\phi = 0 \tag{3}$$

and

$$U'' + AU = U^2 - \frac{1}{2\pi} \int_0^{2\pi} U(\phi)^2 d\phi, \tag{4}$$

where $A = (4 + R/\pi)/(1 + B^2)$.

In fact, it is enough to put

$$\frac{\psi}{\nu(1 + B^2)} = \frac{R\phi}{2\pi(1 + B^2)} + \frac{4B \log r}{1 + B^2} - \int_0^{\phi} U. \tag{5}$$

See, [5]. The vorticity is given by

$$\frac{-\triangle \psi}{\nu(1 + B^2)} = \frac{1 + B^2}{r^2} U'(\phi). \tag{6}$$

Streamline pattern of a solution is shown in Figure 1. The solution represents a sink or a source if $B = 0$, On the other hand, with a suitable rescaling as $B \to \infty$, it becomes a point vortex. In this sense, Oseen's spiral flows are generalizations of sink, source, and point vortex.

It is verified in [5] that nontrivial solutions of the boundary value problem (3) and (4) with the periodic boundary condition bifurcates at $A = 1$ from the trivial solution

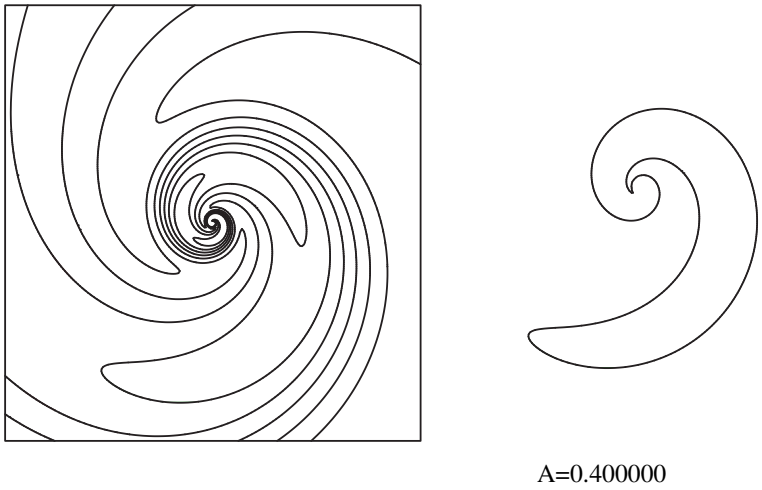


Figure 1: The streamlines of the flow corresponding to the case where $A = 0.4, R = 0$. All the streamlines are closed and similar to each other. One of them are depicted in the right hand side.

$U \equiv 0$. The same paper shows numerically that the bifurcating solutions exist for all $A$ such that $-\infty < A < 1$.

In order to prove the existence and singular behavior of the solutions as $A \to -\infty$, we rescale the unknowns in the following way. We define $\epsilon$ and $u$ by $A = -\epsilon^{-2}$ and $u = -\epsilon^2 U$, respectively. Then the equation becomes

$$\epsilon^2 u_{\phi\phi} - u + u^2 - \frac{1}{2\pi}\int_0^{2\pi} u(\phi)^2 d\phi = 0. \tag{7}$$

# 2 Heuristic argument

We computed by a spectral method the solutions of (4), or equivalently those of (7). Since details of the numerical method is explained in [5], we explain only the numerical results. We remark that the boundary value problem is equivariant with respect to translation, i.e., if $u(\phi)$ is a solution then $u(\phi+\alpha)$ with any constant $\alpha$ is also a solution. In order to eliminate this redundancy, we consider only those solutions which are even functions.

Our computation ( see Figure 2 ) suggests that $u$ is bounded as $\epsilon \to 0$, that $u$ is of the form $u(\phi) = w(\phi/\epsilon)$ near the origin with a suitable function $w$, and that $u$ is almost constant outside the small region containing the origin. Suppose that this is true and that $u$ is of $O(\epsilon)$ outside a small neighborhood of $\phi = 0$. Then $w$ can be defined as $w(x) = u(\epsilon x)$. We now let $\epsilon$ tends to zero. Then we easily see that the limit of $w$, which is again denoted by $w$, satisfies

$$w_{xx} - w + w^2 = 0 \qquad (-\infty < x < +\infty) \tag{8}$$

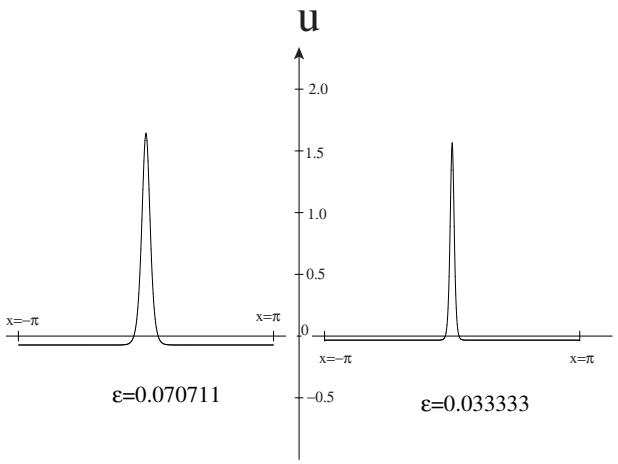


Figure 2: The profiles of $u$, the solution of (7) with periodic boundary condition. $\epsilon^{-2} = 200.0$ and $\epsilon^{-2} = 900.0$

and

$$w(\pm\infty) = 0.$$

The solution can be computed explicitly and we obtain

$$w(x) = \frac{3}{2}\mathrm{sech}^2\left(\frac{x}{2}\right). \tag{9}$$

Since $u$ is nearly a small constant in the region away from the origin, both $u_{\phi\phi}$ and $u^2$ is negligible in (7). Hence we obtain in the region

$$u \sim -\frac{1}{2\pi}\int_0^{2\pi} u(\phi)^2 d\phi \sim -\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} w(y)^2 dy = -\frac{3\epsilon}{\pi}. \tag{10}$$

Therefore our heuristic argument shows that as $\epsilon \to 0$

$$u \sim \begin{cases} -\frac{3}{\pi}\epsilon & \text{away from the origin,} \\ \frac{3}{2}\mathrm{sech}^2\left(\frac{\phi}{2\epsilon}\right) & \text{near the origin.} \end{cases} \tag{11}$$

Such asymptotic behavior implies that the vorticity is large only in a small part of the plane and is very small elsewhere. See Figure 3, where the graph of $-r^2\triangle\psi$ is plotted.

# 3 Asymptotic analysis

Our goal is to prove the existence of a family of solutions of (7) with the periodic boundary condition, which exhibit the singular asymptotics in Figure 2 as $\epsilon \to 0$.

We will use the idea in [3], where a nonlinear boundary value problem of the following form is considered:

$$\begin{aligned} &\epsilon^2 u_{yy} + f(u, y) = 0 \qquad (-1 < y < 1), \\ &u_y(-1) = u_y(1) = 0. \end{aligned}$$

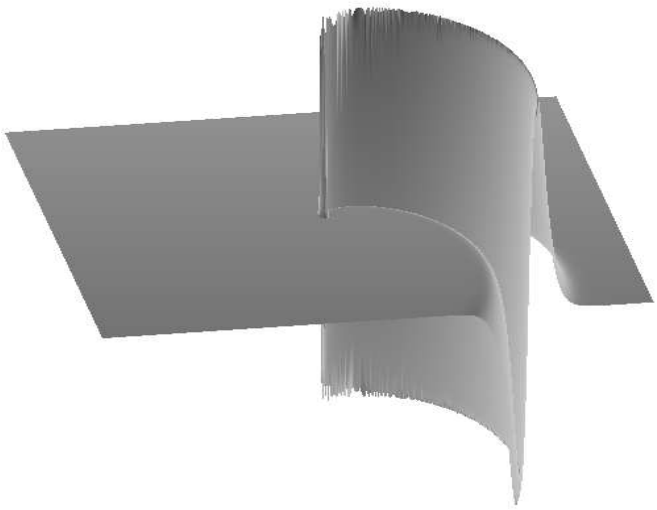

Figure 3: The profiles of $-r^2\triangle\psi$. $\epsilon^{-2} = 900.0$, $B = 1$. The vorticity is large in a small neighborhood of the curve $\log r + \theta = 0$ and is negligibly small elsewhere.

There, the existence of a family of solutions $u(y;\epsilon)$ such that $\lim_{\epsilon\to 0} u(y;\epsilon) = 0$ for all $-1 < y \leq 1$ and $u$ has a transition layer at $y \approx -1$.

Strictly speaking, our problem does not fall into the category of the problem which was dealt with in [3]. However, its idea can equally be used for our problem. We can prove the existence of the solution and its sharp spike in the following way. Since we consider only those solutions which are even functions, the problem is reduced to the following

$$\epsilon^2 u_{\phi\phi} - u = -u^2 + \frac{1}{\pi}\int_0^\pi u(\phi)^2 d\phi \qquad (0 < \phi < \pi), \tag{12}$$

$$u_\phi(0) = u_\phi(\pi) = 0. \tag{13}$$

Note that any solution of this boundary value problem automatically satisfies $\int_0^\pi u(\phi)d\phi = 0$. We here remark that, if $u(\phi)$ satisfies (12) and (13), then $u(-\phi)$ satisfies, too. In order to deal with the unique solution, we hereafter look for a solution which is monotone decreasing in $0 < \phi < \pi$.

The result in the previous section shows that

$$\int_0^\pi u(\phi)^2 d\phi = O(\epsilon)$$

as $\epsilon \to 0$. Keeping this in mind, we consider the following auxiliary problem:

$$\epsilon^2 u_{\phi\phi} - u = -u^2 + \frac{h\epsilon}{\pi} \qquad (0 < \phi < \pi), \tag{14}$$

$$u_\phi(0) = u_\phi(\pi) = 0, \tag{15}$$

where $h$ is a positive constant. Let $u(\phi;\epsilon,h)$ denote this solution. Then the solution of (12) and (13) is obtained by finding $h$ such that

$$\int_0^\pi u(\phi;\epsilon,h)^2 d\phi = h\epsilon. \tag{16}$$

In order to prove the existence of the solution to (14) and (15) in a constructive manner, we first divide the interval $[0,\pi]$ into two parts $[0,\epsilon\delta(\epsilon)]$ and $[\epsilon\delta(\epsilon),\pi]$. Then we consider the following two boundary value problems:

$$\begin{cases} \epsilon^2 u_{\phi\phi} - u = -u^2 + \frac{h\epsilon}{\pi} & (0 < \phi < \epsilon\delta(\epsilon)), \\ u_\phi(0) = 0, \quad u(\epsilon\delta(\epsilon)) = 5/4, \end{cases} \tag{17}$$

$$\begin{cases} \epsilon^2 u_{\phi\phi} - u = -u^2 + \frac{h\epsilon}{\pi} & (\epsilon\delta(\epsilon) < \phi < \pi), \\ u(\epsilon\delta(\epsilon)) = 5/4, \quad u_\phi(\pi) = 0. \end{cases} \tag{18}$$

where the value $5/4$ can be chosen for a technical reason. See Figure 4. We can choose $\delta(\epsilon)$ so that the solutions of (17) and (18) matches at $\phi = \epsilon\delta(\epsilon)$ in a continuously differentiable manner.

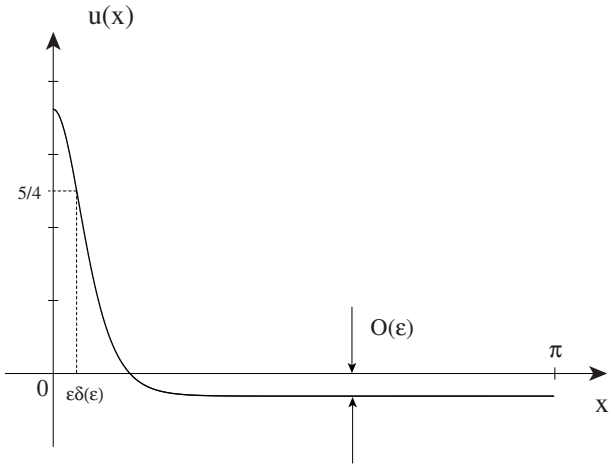


Figure 4: Definition of $\delta(\epsilon)$

This is an idea used in [3]. This program is quite a general one and can be carried out for a variety of nonlinearities. In fact we show that the problem (14) and (15) can be solved in a similar way. In doing so, we solve (14) and (15) directly in terms of Jacobi's elliptic functions, avoiding (17) and (18)

**Theorem 1** *For any* $0 < h < +\infty$ *and* $0 < \epsilon < 1$, *there exists one and only one* $u$ *satisfying* (14), (15), *and* $u_\phi(\phi) < 0 \ \ (0 < \phi < \pi)$.

**Proof**. If a solution exists, then it satisfies

$$\epsilon\frac{du}{d\phi} = -\sqrt{u^2 - \frac{2}{3}u^3 + \frac{2h\epsilon}{\pi}u + c_0}, \tag{19}$$

where $c_0$ is a constant. It is readily seen that the cubic polynomial of $u$ inside the square root sign must have three distinct real roots. ( Otherwise, the solution cannot exist. ) Let $\alpha < \beta < \gamma$ denote the roots. Then, by our assumption, it holds that $\beta = u(\pi)$ and $\gamma = u(0)$. The equation (19) gives us

$$\frac{\phi}{\epsilon} = \sqrt{\frac{3}{2}} \int_u^\gamma \frac{du}{\sqrt{(u-\alpha)(u-\beta)(\gamma-u)}}. \tag{20}$$

Therefore, the solution is uniquely determined if $\alpha, \beta$, and $\gamma$ are uniquely determined.

By the relation of the coefficients of the polynomial and the roots, we have

$$\alpha + \beta + \gamma = \frac{3}{2}, \tag{21}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{3h\epsilon}{\pi}. \tag{22}$$

On the other hand, (20) gives us

$$\frac{\pi}{\epsilon} = \sqrt{\frac{3}{2}} \int_\beta^\gamma \frac{du}{\sqrt{(u-\alpha)(u-\beta)(\gamma-u)}}. \tag{23}$$

We now show that these three equations (21), (22), and (23) determines $\alpha, \beta$, and $\gamma$ uniquely.

Let us define $0 < k < 1$ by

$$k^2 = \frac{\gamma - \beta}{\gamma - \alpha}. \tag{24}$$

Then (23) can be rewritten as follows:

$$\begin{aligned} \frac{\pi}{\epsilon} \frac{\sqrt{\gamma-\alpha}}{\sqrt{6}} &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \\ &= K, \end{aligned}$$

where $K$ is the complete elliptic integral of the first kind, with $k$ being its modulus. We will soon show that $k$ is determined as a function of $\epsilon$ and that $\epsilon \to 0$ corresponds to $k \to 1$. In view of this, it is more convenient to define $\mu = 1 - k^2$ and

$$K = K(\mu) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + \mu \sin^2\theta}}.$$

Note also that (20) is equivalently written as

$$u(\phi) = \gamma - (\gamma - \beta)\mathrm{sn}^2\left(\sqrt{\frac{\gamma-\alpha}{6}}\frac{\phi}{\epsilon}\right), \tag{25}$$

where sn is Jacobi's elliptic function with the modulus $k$. See, for instance,

We next eliminate $\beta$ from (24), which is rewritten as $(1-\mu)(\gamma-\alpha)=\gamma-\beta$, and (21) to obtain

$$(2-\mu)\alpha+(1+\mu)\gamma=\frac{3}{2}.$$

This and $\gamma-\alpha=6\epsilon^2K^2/\pi^2$ gives us

$$\alpha=\frac{1}{2}-\frac{2(1+\mu)\epsilon^2K^2}{\pi^2},\qquad \gamma=\frac{1}{2}+\frac{2(2-\mu)\epsilon^2K^2}{\pi^2}.$$

For notational convenience, we put

$$G=\frac{\epsilon^2K^2}{\pi^2}.$$

Then $\alpha$ and $\gamma$ are represented as

$$\alpha=\frac{1}{2}-2(1+\mu)G,\qquad \gamma=\frac{1}{2}+2(2-\mu)G. \tag{26}$$

On the other hand, (22) is rewritten, with the aid of (23), as

$$\left(\alpha-\frac{1}{2}\right)^2+\left(\alpha-\frac{1}{2}\right)\left(\gamma-\frac{1}{2}\right)+\left(\gamma-\frac{1}{2}\right)^2=\frac{3h\epsilon}{\pi}+\frac{3}{4}.$$

Substituting (26) into this equation, we obtain

$$(1-\mu+\mu^2)G^2=\left(\frac{1}{16}+\frac{h\epsilon}{4\pi}\right),$$

or equivalently

$$(1-\mu+\mu^2)K^4=\left(\frac{\pi^4}{16}+\frac{\pi^3h\epsilon}{4}\right)\epsilon^{-4}.$$

It is elementarily proved that the left hand side of this equation is monotone decreasing in $\mu\in(0,1)$ and tends to $\infty$ and $\pi^4/16$ as $\mu\downarrow 0$ and $\mu\uparrow 1$, respectively. Therefore $\mu$ is determined uniquely for all $\epsilon\in(0,1)$ and $h>0$. On the other hand, $K(\mu)$ has the following asymptotic behavior:

$$K(\mu)\sim\frac{1}{2}\log(16/\mu))\qquad(\mu\to 0).$$

( See [1], page 591, formula 17.3.26, [4], or Art. 22.737 of [7] ). Summing up these observations, we see that $\mu$ can be defined as a smooth function of $\epsilon\in[0,1)$ and that

$$\mu=O(\epsilon^n)\quad\text{as }\epsilon\to 0$$

for any positive integer $n$. Note finally that $G$ is a smooth function of $\epsilon$ and satisfies

$$G=\frac{1}{4}\left(1+\frac{2h\epsilon}{\pi}+O(\epsilon^2)\right)\qquad\text{as }\epsilon\to 0.$$

We now have

$$u(\phi)=\frac{1}{2}+2(2-\mu)G-6(1-\mu)G\mathrm{sn}^2\left(\frac{K}{\pi}\phi\right). \tag{27}$$

□

**Theorem 2** *There exists one and only one solution $u$ of* (12) *and* (13) *satisfying* $u_\phi <$ *in* $0 < \phi < \pi$.

**Proof**. The solution is obtained if we can find $h$ such that

$$h\epsilon = \int_0^\pi u(\phi)^2 d\phi. \tag{28}$$

By the well-known properties of elliptic integrals ( see, for instance, page 94 of [4] ), we have

$$\begin{aligned} \int_0^\pi \mathrm{sn}^2\left(\frac{K}{\pi}\phi\right) d\phi &= \frac{\pi}{K}\int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{\cos^2\theta + \mu\sin^2\theta}} d\theta \\ &= \frac{\pi(K-E)}{(1-\mu)K}, \end{aligned}$$

where

$$E = \int_0^{\pi/2} \sqrt{\cos^2\theta + \mu\sin^2\theta}\, d\theta$$

is the complete elliptic integral of the second kind. Similarly we have

$$\begin{aligned} \int_0^\pi \mathrm{sn}^4\left(\frac{K}{\pi}\phi\right) d\phi &= \frac{\pi}{K}\int_0^{\pi/2} \frac{\sin^4\theta}{\sqrt{\cos^2\theta + \mu\sin^2\theta}} d\theta, \\ &= \frac{\pi\left[(3-\mu)K + 2(\mu-2)E\right]}{3(1-\mu)^2 K}. \end{aligned}$$

Then the equation (28) is written as follows:

$$h\epsilon = \pi\left(\frac{1}{4} - 2(1+\mu)G + 4(1-\mu+\mu^2)G^2\right) + 6E\sqrt{G}\epsilon. \tag{29}$$

In order to show that this equation defines $h$ as a function of $\epsilon$, We need some properties, which we set as a lemma:

**Lemma 1** $\mu = O(\epsilon^n)$ *as* $\epsilon \to 0$ *for any positive integer* $n$. $G = \frac{1}{4} + \frac{h\epsilon}{2\pi} + O(\epsilon^2)$ *as* $\epsilon \to 0$. $E = 1 + O(\mu^{1/2})$ *as* $\mu \to 0$.

The last relation follows from the elementary inequality

$$1 \le E \le \int_0^{\pi/2} (\cos\theta + \sqrt{\mu}\sin\theta)\, d\theta = 1 + \sqrt{\mu}.$$

It actually has a better estimate near $\mu \approx 0$ ( see [4] ) but this inequality is sufficient for our purpose.

We now define

$$F(h,\epsilon) = h - \frac{\pi}{\epsilon}\left(\frac{1}{4} - 2(1+\mu)G + 4(1-\mu+\mu^2)G^2\right) - 6E\sqrt{G}.$$

By Lemma 1, we have

$$\begin{aligned} F(h,\epsilon) &= h - \frac{\pi}{\epsilon}\left((\frac{1}{2}-2G)^2 + O(\epsilon^3)\right) - 6(1+O(\epsilon^2))\left(\frac{1}{2} + \frac{h\epsilon}{2\pi} + O(\epsilon^2)\right) \\ &= h - \frac{h^2\epsilon}{\pi} - 3\left(1 + \frac{h\epsilon}{\pi}\right) + O(\epsilon^2) \end{aligned}$$

as $\epsilon \to 0$. This equation and the implicit function theorem guarantees the existence of $h = h(\epsilon)$ for sufficiently small $\epsilon$.

Finally the asymptotic formula (11) is verified. Putting $x = K\phi/\pi$, we write the formula (27) as

$$u(x) = \frac{1}{2} + 2(2-\mu)G - 6(1-\mu)G\mathrm{sn}^2(x).$$

Since $\lim_{\mu\to 0}\mathrm{sn}(x) = \tanh(x)$, we obtain

$$\lim_{\mu\to 0} u(x) = \frac{3}{2} - \frac{3}{2}\tanh^2(x) = \frac{3}{2}\mathrm{sech}^2(x).$$

Noting that $x = K\phi/\pi \approx \phi/(2\epsilon)$, we obtain the second asymptotic formula of (27). With a little more calculation, we can deduce from (27) that

$$u(\phi) \sim -\frac{3\epsilon}{\pi} \qquad (\ \epsilon \to 0\ )$$

for all $\phi \in (0, \pi]$.

□

**Acknowledgment**. We are grateful to Professor T. Kawai and Professor Y. Takei for their useful advice on the elliptic integrals.

# References

[1] M. Abramowitz and I. A. Stegun ( eds. ), Handbook of Mathematical Functions, Dover (1970).

[2] R. Berker, Intégration des équations du mouvement d'un fluide visqueux incompressible, Handbuch der Physik, vol VIII/2 (1963), pages 1–384.

[3] H. Ikeda, Stability characteristics of transition layer solutions, J. Dynam. Diff. Eqns., vol. 5 (1993), pp. 625–671.

[4] D. F. Lawden, Elliptic Functions and Applications, Springer Verlag, (1989).

[5] H. Okamoto, Localization of singularities in inviscid limit — numerical examples, in Proceedings of Navier-Stokes equations: theory and numerical methods, ed. R. Salvi, Longman, Pitman Research Notes in Mathematics Series vol. 388 (1998), pp. 220–236.

[6] C. W. Oseen, Exacte Lösungen der hydrodynamischen Differentialgleichungen. I. Ark. Mat. Astronom. Fysik, vol. 20 No. 14 (1928), pp. 1-24., II. (ibid), No. 22 (1928), pp. 1–9.

[7] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge Univ. Press, Fourth Ed. (1927)