Carleman Estimates for the Lamé System with Stress Boundary Condition

By

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Abstract

In this paper, for functions without compact supports, we established Carleman estimates for the two-dimensional non-stationary Lamé system with the stress boundary condition.

§1. Introduction and the Main Carleman Estimates

In this paper, for functions without compact supports, we establish Carleman estimates for the two-dimensional non-stationary Lamé system with stress boundary condition:

\begin{equation}
\begin{aligned}
P(x, D)u &\equiv (P_1(x, D)u, P_2(x, D)u)^T \\
&= \rho(\tilde{x})\frac{\partial^2 u}{\partial x_0^2} - \mu(\tilde{x})\Delta u - (\mu(\tilde{x}) + \lambda(\tilde{x}))\nabla_\tilde{x}\text{div} u \\
&\quad - (\text{div} u)\nabla_\tilde{x}\lambda(\tilde{x}) - (\nabla_\tilde{x}u + (\nabla_\tilde{x}u)^T)\nabla_\tilde{x}\mu(\tilde{x}) = f \quad \text{in} \ Q = (0, T) \times \Omega,
\end{aligned}
\end{equation}


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\[
\begin{aligned}
\mathcal{B}(x, D)u & \equiv \left( \sum_{j=1}^{2} n_j \sigma_{j1}, \sum_{j=1}^{2} n_j \sigma_{j2} \right)^T = g \quad \text{on } (0, T) \times \partial \Omega, \\
u(T, \tilde{x}) = \frac{\partial u}{\partial x_0}(T, \tilde{x}) & = \frac{\partial u}{\partial x_0}(0, \tilde{x}) = 0,
\end{aligned}
\]

where \( u = (u_1, u_2)^T, \ f = (f_1, f_2)^T \) are the vector functions, \( u^T \) denotes the transpose of the vector \( u \), \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with \( \partial \Omega \in C^3, x = (x_0, \tilde{x}), \tilde{x} = (x_1, x_2) \) and \( (n_1, n_2)^T \) is the unit outward normal vector to \( \partial \Omega \),

\[ \sigma_{jk} = \lambda(\tilde{x}) \delta_{jk} \text{div } u + \mu(\tilde{x}) \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right). \]

The boundary condition in (1.2) describes the surface stress. In (1.1), the coefficients \( \rho, \mu, \lambda \in C^2(\bar{\Omega}) \) are assumed to satisfy

\[
\rho(\tilde{x}) > 0, \quad \mu(\tilde{x}) > 0, \quad \mu(\tilde{x}) + \lambda(\tilde{x}) > 0, \quad \forall \tilde{x} \in \bar{\Omega}, \quad \lambda(\tilde{x}) \neq 0, \quad \forall \tilde{x} \in \partial \Omega.
\]

Physically \( \lambda \) and \( \mu \) are the Lamé coefficients of the isotropic medium occupying the domain \( \Omega \), and \( \rho \) is the density. A Carleman estimate is an inequality for solutions to a partial differential equation with weighted \( L^2 \)-norm and is a strong tool for proving the uniqueness for Cauchy problems or the unique continuation of partial differential equations with non-analytic coefficients. Moreover Carleman estimates have been applied successfully to estimation of energy of solutions (e.g., \( \text{[KK]} \)) and to inverse problems of determining coefficients by boundary observations (e.g., \( \text{[BuK]}, \text{[K]} \) as initiating works). As a pioneering work, we refer to Carleman \( \text{[Ca]} \) which derived a Carleman estimate and used it to prove the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since \( \text{[Ca]} \), the theory of Carleman estimates has been studied extensively. We refer to Hörmander \( \text{[Hö]} \) in the case where the symbol of a partial differential equation is isotropic and functions under consideration have compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). Later Carleman estimates for functions with compact supports have been obtained for partial differential operators with anisotropic symbols by Isakov (\( \text{[Is]} \)). For general results in the case of functions without compact supports, see \( \text{[Ta]} \) and for hyperbolic equations, see \( \text{[Im]} \). Our main task of establishing a Carleman estimate for (1.1)-(1.2) is difficult twofold: Firstly, in (1.1) the highest order derivatives are coupled and secondly (1.2) contains a boundary condition of the non-Dirichlet type. **First difficulty:** As for Carleman estimates for strongly coupled systems, there are not many works. In fact,
all the above-mentioned works discuss single partial differential equations. As long as the unique continuation is concerned, to our best knowledge, the most general result for such systems of partial differential equations is Calderón’s uniqueness theorem (see e.g., [E], [Zui]). However, the non-stationary Lamé system does not satisfy all the conditions of that theorem. More precisely, the eigenvalues of the matrix associated with the principal symbol of the Lamé system change the multiplicities and at some points of cotangent bundle, they are not smooth, which break the assumptions in the known Caledrón’s uniqueness theorem. On the other hand, for proving the unique continuation, the Lamé system can be decoupled (modulo low order terms) for example by introducing a new function div $u$ and applying to the new system the technique developed for the scalar partial differential equations (see e.g., [EINT]). This method may produce a Carleman estimate for the Lamé system, but the displacement function $u$ is required to have a compact support, so that the method does not work for (1.1) and (1.2) if $u$ does not have a compact support. In [IY1] and [IY3], we have established Carleman estimates for the Dirichlet case where the stress boundary condition in (1.2) is replaced by $u = g$ on $(0, T) \times \partial \Omega$. It is known that there are two types of the interior waves for the Lamé system: the longitudinal wave with the velocity $\sqrt{\frac{\lambda + 2\mu}{\rho}}$ and the transverse wave with the velocity $\sqrt{\frac{\mu}{\rho}}$. Thus a weight function in the Carleman estimate is assumed to be pseudoconvex with respect to the two symbols (see Condition 1.1). Second difficulty: The essential difference between the stress boundary condition and the Dirichlet boundary condition which was studied by the authors in [IY3] and [IY4], is that the stress boundary condition requires us to deal with the new phenomena - the Rayleigh boundary waves. In order to treat the boundary waves, we have to additionally assume that a weight function is strictly pseudoconvex with respect to the pseudodifferential operator whose principal symbol is given by the Lopatsinsiki determinant (see Condition 1.2). Furthermore, from the practical point of view (e.g., in view of the seismology), the stress boundary condition is very important and well describes the reality such as the surface wave, so that the associated inverse problems and energy estimation are highly requested to be studied. Under Conditions 1.1 and 1.2, we state our main results - the Carleman estimates (Theorem 1.1 and Corollaries 1.1 and 1.2). Among applications of the Carleman estimates obtained in this paper, we mention the sharp unique continuation/conditional stability results for the Cauchy problem for (1.1), the exact controllability of the Lamé system with stress boundary conditions by means of controls in a subdomain or on a subboundary, and an inverse problem of determining the Lamé coefficients and
the density by measurements in a subdomain. For the inverse problems, the method in [BuK] and [K] can be validated by means of our Carleman estimates. Thanks to our Carleman estimate for functions without compact supports, we can establish the exact controllability and the stability over the whole domain \( \Omega \) in the inverse problem with controls or measurements in a subdomain satisfying a related geometric optics condition (e.g., [BLR]). Those are longstanding open problems in spite of the physical significance. However we will postpone such applications to our forthcoming papers and we exclusively consider Carleman estimates in the two-dimensional spatial case. The higher dimensional case is more difficult. Really, as is shown in [Y], in the case where the spatial dimension is greater than two, the Lopatinskii determinant equals zero at some case is more difficult. Really, as is shown in [Y], in the case where the spatial dimension is greater than two, the Lopatinskii determinant equals zero at some point. Among related papers, we refer to Bellassoued [B1]–[B3], Dehman and Robbiano [DR], and Imanuvilov and Yamamoto [IY2], where Carleman estimates in the two-dimensional spatial case. The higher dimensional space, the unique continuation for the stationary Lamé system. Also see Weck [W] for the unique continuation for the stationary Lamé system.

Throughout this paper, we use:

**Notations.** \( i = \sqrt{-1}, \bar{z} \): the complex conjugate of \( z \in \mathbb{C}, \bar{e}_1 = (1, 0), \bar{e}_2 = (0, 1), \bar{n} = (n_1, n_2), x = (x_0, x_1, x_2) = (x_0, \bar{x}), x_1 = (x_1, x_2), y = (y_0, y_1, y_2), y' = (y_0, y_1), \xi = (\xi_0, \xi_1, \xi_2), \xi' = (\xi_0, \xi_1), \partial_y \phi = \phi_y = \frac{\partial \phi}{\partial y}, \partial_x \phi = \phi_x = \frac{\partial \phi}{\partial x}, \phi_{x_1 x_2} = \partial_{x_1, x_2} \phi = \partial_{x_1} \partial_{x_2} \phi, \nabla = (\partial_{x_0}, \partial_{x_1}, \partial_{x_2}) \) or \( \nabla = (\partial_{y_0}, \partial_{y_1}, \partial_{y_2}) \) if there is no fear of confusion (Otherwise we will add the subscript \( x \) or \( y \).) \( \nabla_{\bar{x}} = (\partial_{x_1}, \partial_{x_2}) \), \( \text{div } u = \partial_{x_1} u_1 + \partial_{x_2} u_2 \) for \( u = (u_1, u_2)^T \), \( D_{y_j} = \frac{\partial}{\partial y_j} + is\partial_y \phi, D' = (D_{y_0}, D_{y_1}, D_{y_2}), \nabla_{y'} = (\partial_{y_0}, \partial_{y_1}), D_{y_j} = \frac{1}{i} \frac{\partial}{\partial y_j}, \alpha = (\alpha_0, \alpha_1, \alpha_2), \alpha_j \in \mathbb{N} \cup \{0\}, \partial_x^\alpha = \partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}, \zeta = (s, \xi_0, \xi_1), S^2 \) the two dimensional sphere: \( S^2 = \{ \zeta; |\zeta| = 1 \} \). For a domain \( Q \) in the \( x \)-space, \( H^{m,s}(Q) \) is the Sobolev space of scalar-valued functions equipped with the norm

\[
\|u\|_{H^{m,s}(Q)} = \left( \sum_{|\alpha| \leq m} s^{2m-2|\alpha|}\|\partial_x^\alpha u\|^2_{L^2(Q)} \right)^{\frac{1}{2}},
\]

\( H^{m,s}(Q) = H^{m,s}(Q) \times \cdots \times H^{m,s}(Q) \) is the corresponding space of vector-valued functions \( u \). Also we use the space

\[
W_q^m(Q) = \{ u; D^\alpha u \in L^q(Q), |\alpha| \leq m \}, \quad \|u\|_{W_q^m(Q)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^q(Q)}.
\]

For a domain \( \Omega \) in the \( \bar{x} \)-space, we will similarly define the Sobolev spaces \( H^{1,s}(\Omega) \) and \( H^{1,s}(\Omega) \). Let \([A, B] = AB - BA\), and let \( \epsilon(\delta) \) be a nonnegative function such that \( \epsilon(\delta) \to +0 \) as \( \delta \to +0 \). By \( \mathcal{O}(\delta_1) \), we denote the conic
neighbourhood of the point \( \zeta^* \) with \( |\zeta^*| = 1 \): \( \mathcal{O}(\delta_1) = \left\{ \zeta; \left| \frac{x}{\rho} - \zeta^* \right| \leq \delta_1 \right\}, \)

\( B_{\delta}(y^*) = \{ y; |y - y^*| < \delta \} \) is the ball centered at \( y^* \) with radius \( \delta \). \( \mathcal{L}(X_1, X_2) \) is the space of linear continuous operators from a normed space \( X_1 \) to a normed space \( X_2 \), \( E_k \) is the \( k \times k \) unit matrix.

Our main purpose is to establish Carleman estimates for system (1.1)–(1.2) for \( u \) having non-compact supports. Let \( \omega \subset \Omega \) be an arbitrarily fixed open set which is not necessarily connected. Denote by \( \tilde{n} \) and \( \tilde{t} \), the outward unit normal vector and the unit counterclockwise oriented tangential vector on \( \partial u \) and we set

\[
\begin{cases}
(i) \\
(ii)
\end{cases}
\]

\[
\psi.
\]

We formulate that as:\n
**Condition 1.1.** There exists a function \( \psi \in C^2(\overline{Q}) \) such that

\[
|\nabla_x \psi(x)| \neq 0 \quad \text{for} \quad x \in Q \setminus \Omega, \quad \text{and (i), (ii) and (1.6) hold:}
\]

(i)

\[
\{p_k, \{p_k, \psi\}\}(x, \xi) > 0, \quad \forall k \in \{1, 2\}
\]

if \( \xi \in \mathbb{R}^3 \setminus \{0\} \) and \( x \in Q \setminus \Omega \) satisfy \( p_k(x, \xi) = \langle \nabla_x p_k, \nabla_x \psi \rangle = 0 \).

(ii)

\[
\frac{1}{2\iota \delta}\{p_k(x, \xi - \iota s \nabla_x \psi(x)), p_k(x, \xi + \iota s \nabla_x \psi(x))\} > 0, \quad \forall k \in \{1, 2\}
\]

if \( \xi \in \mathbb{R}^3 \setminus \{0\}, \; s > 0 \) and \( x \in Q \setminus \Omega \) satisfy \( p_k(x, \xi + \iota s \nabla_x \psi(x)) = \langle \nabla_x p_k(x, \xi + \iota s \nabla_x \psi(x)), \nabla_x \psi \rangle = 0 \).

On the lateral boundary we assume

\[
(1.6)
\]

\[
\begin{aligned}
&\sqrt{\mu} \frac{\partial \psi_{\nu_0}}{\partial \nu} \leq \frac{\mu}{\sqrt{\lambda + 2\mu}} \left| \frac{\partial \psi}{\partial \nu} \right| + \frac{\sqrt{\lambda + 2\mu}}{\sqrt{\lambda + 2\mu}} \left| \frac{\partial \psi}{\partial \nu} \right|, \quad \forall x \in [0, T] \times (\partial \Omega \setminus \partial \omega), \\
&p_1(x, \nabla \psi) < 0, \quad \forall x \in [0, T] \times (\partial \Omega \setminus \partial \omega), \\
&\frac{\partial \psi}{\partial \nu} < 0, \quad \frac{\partial \psi}{\partial \nu} \neq 0 \quad \text{on} \ [0, T] \times (\partial \Omega \setminus \partial \omega).
\end{aligned}
\]

Let \( \psi \) satisfy Condition 1.1. We introduce the function \( \phi(x) \) by

\[
\begin{equation}
(1.7)
\phi(x) = e^{\tau \phi(x)}, \quad \tau > 1,
\end{equation}
\]

where the parameter \( \tau \) will be fixed below. In order to deal with surface waves, we additionally need Condition 1.2 on the function \( \psi \). We formulate that assumptions below as (1.23), and for the statement, we need to introduce some
boundary differential operators by means of a new local coordinate. For an arbitrarily fixed point \((x_1^0, x_2^0) \in \partial \Omega\), we set \(\tilde{x}_1 = x_1 - x_1^0\) and \(\tilde{x}_2 = x_2 - x_2^0\). We consider (1.1) and (1.2) in the new coordinates \((\tilde{x}_1, \tilde{x}_2)\). Since (1.1) and (1.2) are invariant with respect to the translation by the constant vector \((x_1^0, x_2^0)\), we use the same notations \(x_1, x_2\) instead of \(\tilde{x}_1, \tilde{x}_2\). Therefore we may assume that \((0,0) \in \partial \Omega\) and that locally near \((0,0)\), the boundary \(\partial \Omega\) is given by an equation \(x_2 - \ell(x_1) = 0\), where \(\ell = \ell(x_1)\) is a \(C^3\)-function. Moreover, since the function \(\tilde{u} = O\mathbf{u}(x_0, O^{-1}\tilde{x})\) satisfies system (1.1) and (1.2) with \(\tilde{f} = O\mathbf{f}(x_0, O^{-1}\tilde{x})\) for any orthogonal matrix \(O\), we may assume that

\[
\ell'(0) \equiv \frac{d\ell}{dx_1}(0) = 0.
\]

We make the change of variables \(y = (y_0, y_1, y_2) = Y(x) \equiv (x_0, x_1, x_2 - \ell(x_1))\). Then we reduce equations (1.1) to

\[
P_1(y, D)\mathbf{u} = \rho \frac{\partial^2 u_1}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1) \frac{\partial^2 u_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 u_1}{\partial y_2^2} \right\}
\]

\[
+ \mu \ell''(y_1) \frac{\partial u_1}{\partial y_2} - (\lambda + \mu) \frac{\partial}{\partial y_1} \left( \text{div} \mathbf{u} - \frac{\partial u_1}{\partial y_1} \ell' \right) + (\lambda + \mu) \frac{\partial}{\partial y_2} \left( \text{div} \mathbf{u} - \frac{\partial u_1}{\partial y_2} \ell' \right)
\]

\[+ K_1(y, D)\mathbf{u} = f_1 \quad \text{in} \; G,
\]

(1.8)

\[
P_2(y, D)\mathbf{u} = \rho \frac{\partial^2 u_2}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1) \frac{\partial^2 u_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2) \frac{\partial^2 u_2}{\partial y_2^2} \right\}
\]

\[
+ \mu \ell''(y_1) \frac{\partial u_2}{\partial y_2} - (\lambda + \mu) \frac{\partial}{\partial y_1} \left( \text{div} \mathbf{u} - \frac{\partial u_2}{\partial y_1} \ell' \right) + K_2(y, D)\mathbf{u} = f_2 \quad \text{in} \; G,
\]

(1.9)

where we set \(G = \{y; \ y_2 \geq 0, y \in Y((0, T) \times B_r(0,0))\}\) with some \(r > 0\), and we keep the same notations \(P_1, P_2, \mathbf{u}, \mathbf{f}\) after the change of variables, and \(K_1(y, D)\) are first order differential operators with \(C^1\)-coefficients.

We set \(P(y, D) = (P_1(y, D), P_2(y, D))\). In the new coordinates, the stress boundary condition (1.2) has the form

\[
n_1(\tilde{x}) \left( \lambda(\tilde{x}) \left( \frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2} (\ell') + \frac{\partial u_2}{\partial y_2} \right) + 2\mu(\tilde{x}) \left( \frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2} (\ell') \right) \right)
\]

\[
+ n_2(\tilde{x}) \mu(\tilde{x}) \left( \frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_2} (\ell') + \frac{\partial u_1}{\partial y_2} \right) = g_1,
\]

(1.10)
\[ n_1(\tilde{x})\mu(\tilde{x}) \left\{ \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_2} (-\ell') \right\} + n_2(\tilde{x}) \left\{ \lambda \left( \frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2} (-\ell') + \frac{\partial u_2}{\partial y_2} + 2\mu(\tilde{x}) \frac{\partial u_2}{\partial y_2} \right) \right\} = g_2. \]

Here we use the same notations \( n_1, n_2 \) after the change of the variables. We can solve system (1.10) and (1.11) with respect to \( \left( \frac{\partial u_1}{\partial y_1}, \frac{\partial u_2}{\partial y_2} \right) \) in the form:

\[ (\frac{\partial u_1}{\partial y_1}, \frac{\partial u_2}{\partial y_2}) = A(y_1) \left( \frac{\partial u_1}{\partial y_1}, \frac{\partial u_2}{\partial y_2} \right) + \tilde{A}(y_1)g, \quad A(0) = \left( \begin{array}{cc} 0 & -1 \\ \frac{\lambda}{\nu_2} & 0 \end{array} \right), \quad y \in \partial \mathcal{G}, \]

and \( \tilde{A}(y_1) \) is a \( C^2 \) matrix-valued function. By \( A_1 \) and \( \tilde{A}_1 \), we denote the first rows of the matrices \( A \) and \( \tilde{A} \) respectively, and the second by \( A_2 \) and \( \tilde{A}_2 \):

\[ A_j = (a_{j1}, a_{j2}) \quad \text{and} \quad \tilde{A}_j = (\tilde{a}_{j1}, \tilde{a}_{j2}), \quad j = 1, 2. \]

System (1.1) can be decoupled (up to lower order terms) if we consider as a new unknown functions \( \text{rot} \mathbf{u} \) and \( \text{div} \mathbf{u} \). The great advantage of dealing with \( \text{rot} \mathbf{u} \), \( \text{div} \mathbf{u} \) instead of \( \mathbf{u} \) is that the divergence and the rotation solve the scalar second order wave equations for which the theory of Carleman estimates—the main machinery used in this paper—is well developed.

Below we need a formulae for \( \text{rot} \mathbf{u} \) and \( \text{div} \mathbf{u} \) in new coordinates. After the change of variables, the functions \( z_1 \equiv \text{rot} \mathbf{u} = \partial_1 u_2 - \partial_2 u_1 \) and \( z_2 \equiv \text{div} \mathbf{u} \) have the form

\[ z_1(y) = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2} \ell'(y_1) - \frac{\partial u_1}{\partial y_1}, \quad z_2(y) = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2} \ell'(y_1). \]

Using (1.12), we can transform these functions on the boundary as follows:

\[ (\text{rot} \mathbf{u})(y) = z_1(y) = \frac{\partial u_2}{\partial y_1} - \ell'(y_1) A_2(y_1) \frac{\partial u}{\partial y_1} - A_1(y_1) \frac{\partial u}{\partial y_2} - \ell'(y_1) \tilde{A}_2(y_1) g - \tilde{A}_1(y_1) g \equiv b_{11}(y_1, D') u_1 + b_{12}(y_1, D') u_2 + \tilde{C}_1(y_1) g, \quad y \in \partial \mathcal{G}, \]

where

\[ b_{11}(y_1, \xi) = i(-\ell'(y_1)a_{21}(y_1) - a_{11}(y_1)) \xi_1, \]

\[ b_{12}(y_1, \xi) = i(1 - \ell'(y_1)a_{22}(y_1) - a_{12}(y_1)) \xi_1. \]

For the function \( z_2(y) \), we have

\[ (\text{div} \mathbf{u})(y) = z_2(y) = \frac{\partial u_1}{\partial y_1} + A_2(y_1) \frac{\partial u}{\partial y_1} - \ell'(y_1) A_1(y_1) \frac{\partial u}{\partial y_1} + \tilde{A}_2(y_1) g - \tilde{A}_1(y_1) g \ell'(y_1) \equiv b_{21}(y_1, D') u_1 + b_{22}(y_1, D') u_2 + \tilde{C}_2(y_1) g, \quad y \in \partial \mathcal{G}, \]
where
\begin{equation}
\begin{aligned}
b_{21}(y_1, \xi) &= i(\xi_1 + a_{21}(y_1)\xi_1 - \ell(y_1) a_{11}(y_1)\xi_1), \\
b_{22}(y_1, \xi) &= i(a_{22}(y_1)\xi_1 - a_{12}(y_1)\xi_1\ell(y_1)),
\end{aligned}
\end{equation}
and \( \bar{C}_j \) are \( C^2 \) matrix-valued functions. Denote
\begin{equation}
\begin{aligned}
b_1(y_1, D') &= (b_{11}(y_1, D'), b_{12}(y_1, D')), \\
b_2(y_1, D') &= (b_{21}(y_1, D'), b_{22}(y_1, D')),
\end{aligned}
\end{equation}
by \( \Gamma \)
\begin{equation}
\begin{aligned}
p_\beta(y, s, \xi_0, \xi_1) &= -\rho(\xi_0 + i\partial_\rho \phi)^2 \\
+ \beta [(\xi_1 + i\partial_{y_1} \phi)^2 - 2\ell'(\xi_1 + i\partial_{y_1} \phi)(\xi_2 + i\partial_{y_2} \phi) + (\xi_2 + i\partial_{y_2} \phi)^2]|G|, \\
|G| &= 1 + (\ell'(y_1))^2, \beta \in \{\mu, \lambda + 2\mu\} \text{ and } s \text{ is a positive parameter.}
\end{aligned}
\end{equation}
The roots of polynomial \( p_\beta \) with respect to the variable \( \xi_2 \) are
\begin{equation}
\begin{aligned}
\Gamma_\beta^\pm (y, s, \xi_0, \xi_1) &= -i\partial_{y_2} \phi + \alpha_\beta^\pm (y, s, \xi_0, \xi_1), \\
\alpha_\beta^\pm (y, s, \xi_0, \xi_1) &= (\xi_1 + i\partial_{y_1} \phi)\ell'(y_1) \pm \sqrt{r_\beta(y, s, \xi_0, \xi_1)}, \\
r_\beta(y, s, \xi_0, \xi_1) &= \frac{(\rho(\xi_0 + i\partial_\rho \phi)^2 - \beta(\xi_1 + i\partial_{y_1} \phi)^2)|G| + \beta(\xi_1 + i\partial_{y_1} \phi)^2(\ell')^2}{\beta |G|^2}.
\end{aligned}
\end{equation}
Henceforth, fix \( \zeta^* \in \mathbb{R}^3 \) such that \( |\zeta^*| = 1 \) arbitrarily, and set \( y^* = (y_0, 0, 0) \) and \( \gamma = (y^*, \zeta^*) \). Suppose that \( |r_\beta(\gamma)| \geq 2\delta > 0 \). In [IY3], it was shown that there exists \( \delta_0(\delta) > 0 \) such that for all \( \delta, \delta_1 \in (0, \delta_0) \), there exists a constant \( C_1 > 0 \) such that for one of the roots of the polynomial (1.17), which we denote by \( \Gamma_\beta^-, \) we have
\begin{equation}
-\text{Im} \Gamma_\beta^- (y, s, \xi_0, \xi_1) \geq sC_1, \quad \forall y \in B_\delta(y_0, 0, 0), \quad (s, \xi_0, \xi_1) \in \mathcal{O}(\delta_1), \quad |\zeta| \geq 1.
\end{equation}
Set
\begin{equation}
\mathcal{B}(y', s, D') = \begin{pmatrix} B_{11}(y', s, D') & B_{12}(y', s, D') \\ B_{21}(y', s, D') & B_{22}(y', s, D') \end{pmatrix}, \quad y \in \partial \mathcal{G},
\end{equation}
where
\begin{equation}
\begin{aligned}
B_{11}(y', s, D') &= -\rho D_{y_1}^2 + \mu i\alpha_\beta^+(y', 0, s, D')b_{11}(y_1, D') \\
&\quad - (\lambda + 2\mu)(iD_{y_1} - \ell'(y_1)i\alpha_{\lambda+2\mu}(y', 0, s, D'))b_{21}(y_1, D'),
\end{aligned}
\end{equation}
Next we make the change of variables in the above equations. Observing that
\[ B_{12}(y', s, D') = -(\lambda + 2\mu)\{iD_{y_1} - \ell'(y_1)i\alpha_{\lambda+2\mu}^+(y', 0, s, D')\}\mathbf{b}_{22}(y_1, D'), \]
\[ + \mu \alpha_{\mu}^+(y', 0, s, D')\mathbf{b}_{12}(y_1, D'), \]
\[ B_{21}(y', s, D') = -(\lambda + 2\mu)i\alpha_{\lambda+2\mu}^+(y', 0, s, D')\mathbf{b}_{21}(y_1, D'), \]
\[ - \mu (iD_{y_1} - \ell'(y_1)i\alpha_{\mu}^+(y', 0, s, D'))\mathbf{b}_{11}(y_1, D'). \]

\[ B_{22}(y', s, D') = -\rho D_{y_1}^2 - (\lambda + 2\mu)i\alpha_{\lambda+2\mu}^+(y', 0, s, D')\mathbf{b}_{12}(y_1, D'), \]
\[ - \mu (iD_{y_1} - \ell'(y_1)i\alpha_{\mu}^+(y', 0, s, D'))\mathbf{b}_{11}(y_1, D'). \]

Remark 1. For readers’ convenience we derive the boundary operator \( B(y', s, D'). \) We rewrite equations (1.1) on the boundary in the form
\[ \rho \frac{\partial^2 u_1}{\partial y_0^2} + \mu \frac{\partial}{\partial x_2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) - (\lambda + 2\mu) \frac{\partial}{\partial x_1} \text{div} u + l.o.t. = f_1 \]
\[ \rho \frac{\partial^2 u_2}{\partial y_0^2} - \mu \frac{\partial}{\partial x_1} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) - (\lambda + 2\mu) \frac{\partial}{\partial x_2} \text{div} u + l.o.t. = f_2. \]

Next we make the change of variables in the above equations. Observing that
\( \frac{\partial}{\partial x_1} \rightarrow \frac{\partial}{\partial y_1} - \ell'(y_2) \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_2} \rightarrow \frac{\partial}{\partial y_2} \) we obtain
\[ \rho \frac{\partial^2 u_1}{\partial y_0^2} + \mu \frac{\partial}{\partial y_2} \left( \frac{\partial z_1}{\partial y_1} - \ell' \frac{\partial z_2}{\partial y_2} \right) + l.o.t. = f_1 \]
and
\[ \rho \frac{\partial^2 u_2}{\partial y_0^2} + \mu \left( \frac{\partial z_2}{\partial y_1} - \ell' \frac{\partial z_1}{\partial y_2} \right) - (\lambda + 2\mu) \frac{\partial}{\partial y_2} + l.o.t. = f_2. \]

By (1.13) and (1.15), we have
\[ \rho \frac{\partial^2 u_1}{\partial y_0^2} + \mu \frac{\partial}{\partial y_2} \left( \frac{\partial}{\partial y_1} (b_{21}(y_1, D')u_1 + b_{22}(y_1, D')u_2) - \ell' \frac{\partial z_2}{\partial y_2} \right) + l.o.t. = \tilde{f}_1 \]
and
\[ \rho \frac{\partial^2 u_2}{\partial y_0^2} + \mu \left( \frac{\partial}{\partial y_1} (b_{11}(y_1, D')u_1 + b_{12}(y_1, D')u_2) - \ell' \frac{\partial z_1}{\partial y_2} \right) - (\lambda + 2\mu) \frac{\partial}{\partial y_2} + l.o.t. = \tilde{f}_2. \]

Setting \( v = u \varepsilon^{\alpha}, w = z \varepsilon^{\alpha} \) we obtain
\[ -\rho D_{y_0}^2 v_1 + \mu D_{y_2} w_1 - i(\lambda + 2\mu)(D_{y_1} (b_{21}(y_1, D')v_1 + b_{22}(y_1, D')v_2) + l.o.t. = F_1 \]
\[ \rho D_{y_0}^2 w_2 + \mu (D_{y_1} (b_{11}(y_1, D')u_1 + b_{12}(y_1, D')u_2) - \ell' D_{y_2} w_2) + l.o.t. = F_2. \]
and
\[-\rho D^2 \psi_2 + \mu (D_{y_1}(b_{11}(y_1, D'))v_1 + b_{12}(y_1, D')v_2) - \ell' D_{y_2} w_1 \]
\[-(\lambda + 2\mu)D_{y_2} w_2 + l.o.t. = F_2.\]

Later, provided that symbols $\alpha_\mu^\pm$ and $\alpha_{\lambda + 2\mu}^\pm$ are smooth at a small conic neighbourhood, we will be able to prove that the functions $D_{y_2} w_1 - \alpha_\mu^\pm(y', 0, s, D')w_1$ and $D_{y_2} w_2 - \alpha_{\lambda + 2\mu}(y', 0, s, D')w_2$ are bounded in terms of the right hand side of (1.24). Thus substituting in the above equations instead of $D_{y_2} w_1$ and $D_{y_2} w_2$ functions $\alpha_\mu^\pm(y', 0, s, D')(b_{11}(y_1, D')v_1 + b_{12}(y_1, D')v_2)$ and $\alpha_{\lambda + 2\mu}(y', 0, s, D')(b_{21}(y_1, D')v_1 + b_{22}(y_1, D')v_2)$, we obtain the operator $B(y', s, D').$

Now we formulate a condition which allows us to observe the surface waves. For this purpose, we use the operator $B$ which was introduced in the local coordinates. For an arbitrary point $x^0 \equiv (x^0_0, x^0_1, x^0_2) \in [0, T] \times (\partial \Omega \setminus \partial \omega)$, we rotate and translate $\Omega$ such that after the rotation and the translation, the normal vector to the boundary at $x^0$ is $(0, 0, -1)$. Then by $\gamma(x)$, we denote the transform involved with the rotation and the translation. Now we are ready to state the condition:

**Condition 1.2.** Let $x \in [0, T] \times (\partial \Omega \setminus \partial \omega)$ be an arbitrary point and $y = \gamma(x)$. We assume that

\begin{equation}
\operatorname{Im} \mathbb{I}_j \sum_{j=0}^{1} \frac{\partial \det B(y', s, \xi_0, \xi_1)}{\partial y_j} \frac{\partial \det B(y', s, \xi_0, \xi_1)}{\partial \xi_j} > 0
\end{equation}

for any $(y, s, \xi_0, \xi_1) \in \{(y, s, \xi_0, \xi_1) \in \partial G \times S^2; \det B(y', s, \xi_0, \xi_1) = 0, s > 0, y_0 \in (0, T), \operatorname{Im} \Gamma_{\beta}^\dagger(y', 0, s, \xi_0, \xi_1)/s \geq 0, \forall \beta \in \{\mu, \lambda + 2\mu\}, \xi_0 \neq 0\}.$

Now, under Conditions 1.1 and 1.2, we are ready to state our Carleman estimates:

**Theorem 1.1.** We assume (1.3), Conditions 1.1 and 1.2. Let $f \in H^1(Q), g \in H^2(\partial Q)$ and let the function $\phi$ be given by (1.7). Then there exists $\hat{\tau} > 0$ such that for any $\tau > \hat{\tau}$, we can choose $s_0(\tau) > 0$ such that for any solution $u \in H^1(Q) \cap L^2(0, T); H^2(\Omega)$ to problem (1.1)–(1.2), the following
estimate holds true:

\[ (1.24) \]

\[
\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} |\partial_\alpha u|^2 e^{2s\phi} dx + s \left\| \left( u e^{\phi} \frac{\partial u}{\partial n} e^{\phi} \right) \right\|_{H^\frac{3}{2}(\partial Q) \times H^\frac{1}{2}(\partial Q)}^2 
\leq C \left( ||f e^{\phi}||_{H^1(Q)}^2 + ||g e^{\phi}||_{H^\frac{3}{2}(\partial Q)}^2 + \int_{Q^c} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} |\partial_\alpha u|^2 e^{2s\phi} dx \right),
\]

\[ \forall s \geq s_0(\tau), \]

where the constant \( C = C(\tau) > 0 \) is independent of \( s \).

Remark 2. In (1.3), the final condition is relaxed as

\[ \lambda(\tilde{x}) \neq 0, \quad \forall \tilde{x} \in (\tilde{\Omega} \setminus \partial \omega). \]

Assume in addition that

\[ (1.25) \]

\[ \partial_{\omega_0} \phi(0, \cdot) > 0 \quad \text{and} \quad \partial_{\omega_0} \phi(T, \cdot) < 0 \quad \text{on} \quad \overline{\Omega}. \]

Then we can formulate Carleman estimates in the situations when the right hand side of equation (1.1) belongs to the spaces \( L^2(Q) \) or \( H^{-1}(Q) \).

Corollary 1.1. We assume (1.3), (1.25), Conditions 1.1 and 1.2. Let \( f \in L^2(Q) \), \( g = 0 \) and let the function \( \phi \) be given by (1.7). Then there exists \( \tilde{\tau} > 0 \) such that for any \( \tau > \tilde{\tau} \), we can choose \( s_0(\tau) > 0 \) such that for any solution \( u \in H^1(Q) \) to problem (1.1)–(1.2), the following estimate holds true:

\[ ||u e^{\phi}||_{H^{-1}(Q)} \leq C(||f e^{\phi}||_{L^2(Q)} + ||u e^{\phi}||_{H^{-1}(Q)}), \quad \forall s \geq s_0(\tau). \]

Here \( C = C(\tau) > 0 \) is independent of \( s \).

Corollary 1.2. We assume (1.3), (1.25), Conditions 1.1 and 1.2. Let \( f = f_{-1} + \sum_{j=0}^{2} \partial_{\omega_j} f_j \) where \( f_0, f_1, f_2 \in L^2(Q), f_{-1} \in H^{-1}(Q), \) supp\( f_{-1} \subset Q, \) and let the function \( \phi \) be given by (1.7). Then there exists \( \tilde{\tau} > 0 \) such that for any \( \tau > \tilde{\tau} \), we can choose \( s_0(\tau) > 0 \) such that for any solution \( u \in H^1(Q) \) to problem (1.1)–(1.2), the following estimate holds true:

\[ ||u e^{\phi}||_{L^2(Q)} \leq C \left( ||f_{-1} e^{\phi}||_{H^{-1}(Q)} + \sum_{j=0}^{2} ||f_j e^{\phi}||_{L^2(Q)} + ||u e^{\phi}||_{L^2(Q)} \right), \]

\[ \forall s \geq s_0(\tau). \]

Here \( C = C(\tau) > 0 \) is independent of \( s \).
Similarly to Theorems 2.2 and 2.3 in [IY3], we can derive Corollaries 1.1 and 1.2 from Theorem 1.1, and we omit the arguments. The rest of this section is devoted to describing a sufficient condition for inequality (1.23) in Condition 1.2 which is convenient for the applications to inverse problems, etc.

For any fixed \( \tilde{x} \in \partial \Omega \), we define a cubic polynomial in \( t \) by

\[
H(t) = t^3 - t^2 \left( \frac{8 \mu}{\rho} \right) (\tilde{x}) + t \left( \frac{24 \mu^2}{\rho^2} - \frac{16 \mu^3}{\rho^2(\lambda + 2\mu)} \right) (\tilde{x}) - \left( \frac{16 \mu^3(\lambda + \mu)}{\rho^2(\lambda + 2\mu)} \right) (\tilde{x}).
\]

Then we can directly verify that \( H'(t) > 0 \) if \( t \leq 0 \), \( H(0) < 0 \), \( H \left( \frac{\mu}{\rho} (\tilde{x}) \right) = \tilde{x}^3(\tilde{x}) > 0 \) and \( H'' \left( \frac{\mu}{\rho} (\tilde{x}) \right) = 0 \). Therefore we can prove that \( H(t) = 0 \) possesses a unique simple root \( t \) in the interval \( \left( 0, \left( \frac{\mu}{\rho} (\tilde{x}) \right) \right) \) for any \( \tilde{x} \in \partial \Omega \), and by \( C = C(\tilde{x}) \) we denote this root. Moreover if there exists another real root, then it is greater than \( \left( \frac{\mu}{\rho} \right) (\tilde{x}) \).

**Remark 3.** By means of the Cardano formula, we can compute \( C = C(\tilde{x}) \) explicitly. We set

\[
\tilde{a}_1 = -\frac{8 \mu}{\rho} (\tilde{x}), \quad \tilde{a}_2 = \frac{24 \mu^2}{\rho^2} - \frac{16 \mu^3}{\rho^2(\lambda + 2\mu)} (\tilde{x}),
\]

\[
\tilde{a}_3 = -\frac{16 \mu^3(\lambda + \mu)}{\rho^2(\lambda + 2\mu)} (\tilde{x}).
\]

That is, \( H(t) = t^3 + \tilde{a}_1 t^2 + \tilde{a}_2 t + \tilde{a}_3 \). Moreover we put

\[
\tilde{b}_1 = \frac{\tilde{a}_1^3}{27} - \frac{\tilde{a}_1 \tilde{a}_2}{6} + \frac{\tilde{a}_3}{2}, \quad \tilde{b}_2 = \frac{1}{9} (3\tilde{a}_2 - \tilde{a}_1^2),
\]

\[
\tilde{b}_3 = \tilde{b}_1^2 + \tilde{b}_2^3, \quad \tilde{b}_4 = \text{sign} (\tilde{b}_1) |\tilde{b}_2|^{\frac{1}{2}}.
\]

Then we have:

\[
C = -\frac{\tilde{a}_1}{3} - 2\tilde{b}_4 \cosh \left( \frac{\theta}{3} \right)
\]

if \( \tilde{b}_1 > 0 \) and \( \tilde{b}_2 < 0 \), where \( \theta \) solves the equation: \( \cosh \theta = \frac{\tilde{b}_2}{\tilde{b}_1} \).

\[
C = -\frac{\tilde{a}_1}{3} - 2\tilde{b}_4 \sinh \left( \frac{\theta}{3} \right)
\]

if \( \tilde{b}_2 > 0 \), where \( \theta \) solves the equation: \( \sinh \theta = \frac{\tilde{b}_2}{\tilde{b}_1} \). If \( \tilde{b}_2 < 0 \) and \( \tilde{b}_3 \leq 0 \), then we define \( C \) by the one of the three zeros of the polynomial \( H \) which belongs
to the interval $[0, \mu/\rho(\bar{x})]$: $t_1 = -\frac{\alpha}{3} - 2\bar{b}_4 \cos (\frac{\theta}{4})$, $t_2 = -\frac{a}{4} + 2\bar{b}_4 \cos (\frac{\theta}{4} + \frac{a}{4})$, $t_3 = -\frac{a}{3} + 2\bar{b}_4 \cos (\frac{\pi}{3} + \frac{a}{4})$, where $\theta$ solves the equation: $\cos \theta = \frac{\bar{b}_4}{b_4}$. In terms of $C(\bar{x})$, we can state one sufficient condition:

**Proposition 1.1.** Let $\psi \in C^2(\overline{Q})$, and

$$\partial_{x_\mu}\psi(x) \pm \sqrt{C(\bar{x})} \frac{\partial \psi}{\partial \ell}(x) \neq 0$$

for any $x \in [0, T] \times (\partial \Omega \setminus \partial \omega)$. Then there exists $\tau_0 > 0$ such that Condition 1.2 holds for $\phi = e^{\tau \psi}$ if $\tau > \tau_0$.

Proof of Proposition 1.1. For this, it suffices to prove: Let $\psi \in C^2(\overline{Q})$ satisfy $\partial_{x_\mu}\psi \neq 0$ on $Q$, and for $(x_1^0, x_2^0) \in \partial \Omega \setminus \partial \omega$, let the local coordinate $\bar{y} = (y_1, y_2)$ be introduced by the local representation $x_2 = \ell(x_1)$ of $\partial \Omega$. We assume

$$\partial_{y_\mu}\psi(y^*) \pm \sqrt{C(y^*)} \partial_{y_\mu}\psi(y^*) \neq 0$$

for any $(x_1^0, x_2^0) \in \partial \Omega \setminus \partial \omega$ and $y_0 \in (0, T)$. Then there exists $\tau_0 > 0$ such that Condition 1.2 holds for the function $\phi = e^{\tau \psi}$ if $\tau > \tau_0$.

We recall that $y^* = (y_0, 0)$. The principal symbol of the operator $B$ at the point $y^*$ is

$$B(y^*, \zeta') = \begin{pmatrix} -\rho(0)\bar{c}_0^2 + 2\mu(0)\bar{c}_1^2 & -2\mu(0)\alpha_\mu^+(y^*, \zeta)\bar{c}_1 \\ 2\mu(0)\alpha_\mu^+(y^*, \zeta)\bar{c}_1 & -\rho(0)\bar{c}_0^2 + 2\mu(0)\bar{c}_1^2 \end{pmatrix},$$

where $\zeta = (s, \xi_0, \xi_1) \in S^2$ and $\bar{c}_j = \xi_j + i s \phi_{y_0}(y^*)$. Obviously

$$\det B(y^*, \zeta) = \rho^2(0) \left(-\bar{c}_0^2 + 2\frac{\mu(0)}{\rho}(0)\bar{c}_1^2\right)^2 + 4\mu^2(0)\alpha_\mu^+(y^*, \zeta)\alpha_\mu^+(y^*, \zeta)\bar{c}_1^2.$$

We study the structure of the set

$$\Psi = \left\{ \zeta \in \mathbb{R}^3 \setminus \{0\} : \det B(y^*, \zeta) = 0, \quad \Im \frac{\Gamma_\mu^+(y^*, \zeta)}{s} \geq 0, \quad \Im \frac{\Gamma_{\lambda + 2\mu}^+(y^*, \zeta)}{s} \geq 0 \right\}.$$

We have

**Lemma 1.1.** Let (1.3) hold true and let $\partial_{y_\mu}\psi(y^*) \neq 0$. Then

$$\Psi \subset \Psi_1 \cup \Psi_2, \quad \dist(\Psi_1, \Psi_2) > 0,$$
where $\Psi_1 = \{ \zeta = (s, \xi_0, \zeta_1) \in S^2; \xi_0 + is\phi_{y_0}(y^*) = 0 \}$ and $\Psi_2 = \{ \zeta \in S^2; \xi_0 + is\phi_{y_0}(y^*) = \pm \sqrt{c(0)(\xi_1 + is\phi_{y_1}(y^*))}, \text{Im} \frac{\Gamma_2^+(y^*, \zeta)}{\xi} \geq 0, \text{Im} \frac{\Gamma_2^+(y^*, \zeta)}{\xi} \geq 0 \}$. 

Proof. We can directly see from the definition that $\text{dist} (\Psi(1.6) (0)) = 0$ and we have $\phi(1.6)$. Since $\tau \leq \tau_0$ and $y$ are real, we can consider only the two cases: $\xi_0 = 0$ or $\xi_0 \neq 0$. Really, $\phi_{y_0}(y^*) = 0$, which are distinct from $\Psi(1.6)$, we have only the two cases: $\xi_0 = 0$ or $\xi_0 \neq 0$. Then we have $t_2, t_3 > \frac{\mu(0)}{\rho(0)}$ if they are real. Therefore, noting that $\frac{\Gamma_1^+(y^*, \zeta)}{\xi} = \frac{\mu(0)}{\rho(0)} \frac{\xi_1}{\xi} H(t)$ with $\xi_1 = t\xi_1^2$, we have only to prove that

(1.30) $\det B(y^*, \zeta) \neq 0$

if $(\zeta_0 + is\phi_{y_0}(y^*))^2 = t_j(\zeta_1 + is\phi_{y_1}(y^*))^2$, $\forall j \in \{2, 3\}$.

Moreover we have only the two cases: $t_2, t_3 \in \mathbb{R}$ or $t_2, t_3 \in \mathbb{C} \setminus \mathbb{R}$. First we consider the case of $t_2, t_3 \in \mathbb{R}$. Really $(\zeta_0^2 - \frac{\mu}{\rho}(0)\zeta_1^2)^2 = \zeta_1^2 \left( t_j - \frac{2\mu}{\rho}(0) \right)^2$ and

\[
\alpha_{\beta}(y^*, \zeta) = \sqrt{\zeta_1^2(t_j(\rho/\mu)(0) - 1) = \text{sign}(\phi_{y_1}(y^*)) \zeta_1 \sqrt{t_j(\rho/\mu)(0) - 1},}
\]

where we used the fact that $t_j(\rho/\mu)(0) - 1 > 0$ and a part of the assumption (1.6) which guarantee that $\phi_{y_1}(y^*) \neq 0$. If $t_j(\rho(0))/(\lambda + 2\mu)(0) - 1 > 0$, then

\[
\alpha_{\lambda + 2\mu}(y^*, \zeta) = \text{sign}(\phi_{y_1}(y^*)) \zeta_1 \sqrt{t_j(\rho(0)/(\lambda + 2\mu)(0) - 1}
\]

and we have

\[
\det B(y^*, \zeta) = \zeta_1^4 \left\{ \rho^2(0) \left( t_j - \frac{2\mu}{\rho}(0) \right)^2 + 4\mu^2(0) \sqrt{t_j(\rho/\mu)(0) - 1} \sqrt{t_j(\rho(0)/(\lambda + 2\mu)(0) - 1) \neq 0.}
\]
If \( t_j \rho(0)/(\lambda + 2\mu)(0) - 1 < 0 \), then \( \alpha_j^\pm_{\lambda+2\mu}(y^*, \zeta) = i \text{sign}(\xi_1) \xi_4 \sqrt{-t_j \rho(0)/(\lambda + 2\mu)(0) + 1} \) and

\[
\det B(y^*, \zeta) = \zeta_4^2 \left\{ \rho^2(0) \left( t_j - \frac{2\rho}{\rho(0)} \right)^2 + 4i\mu^2(0)\text{sign}(\xi_1)\text{sign}(\phi_{y^*}(y^*))\sqrt{t_j \rho(0)/\rho(0) - 1}\sqrt{-t_j \rho(0)/(\lambda + 2\mu)(0) + 1} \right\} \neq 0.
\]

Note that if \( \xi_1 = 0 \) then \( \text{Im} \frac{\Gamma_{y^*}}{\zeta_1}(y^*, \zeta) < 0 \). Next we will consider the case of \( t_j \not\in \mathbb{R}, j = 2, 3 \). We set \( \rho_0 = \rho(0) \). Henceforth in place of \( H(t) \) defined by (1.26), we consider

\[
H(t) = t^3 - \frac{8b}{\rho_0}t^2 + t \left( \frac{24b^2}{\rho_0^2} - \frac{16b^3}{\rho_0^3(a + 2b)} \right) - \frac{16b^3(a + b)}{\rho_0^3(a + 2b)}
\]

for all \((a, b) \in \mathbb{R}^2\) such that \( a, b > 0 \). We note that if we set \( a = \lambda(0) \) and \( b = \mu(0) \), then this coincides with \( H(t) \) by (1.26). Moreover, without fear of confusion, by \( t_2 = t_2(a, b) \) and \( t_3 = t_3(a, b) \), we may denote the roots which are not in the interval \([0, \frac{1}{\rho_0}]\) of \( H(t) \) defined by (1.31). We set \( z = (z_0, z_1) \in \mathbb{C}^2 \) and for factorizing \( \mathcal{F}(\tilde{\zeta_0^2, \zeta_1^2}) \) with \( \lambda(0) = a, \mu(0) = b \) and \( \rho(0) = \rho_0 \), we introduce two functions

\[
\mathcal{H}^\pm(z) = \left( \frac{z_0^2}{\rho_0} - 2 - \frac{b}{\rho_0} \right)^2 \pm 4\frac{b^2}{\rho_0^2}z_1^2 \sigma_2^\pm(z) \sigma_2^\pm(z),
\]

and \( \sigma_2^\pm(z) = \sqrt{\frac{b^2}{\rho_0^2}z_0^2 - z_1^2} \) with \( \beta \in \{a + 2b, b\} \). Henceforth we set

\[
\text{Dom} \sigma_2^\pm = \left\{ z = (z_0, z_1); \frac{\rho_0}{\beta}z_0^2 - z_1^2 \not\in \mathbb{R}_+ \right\}.
\]

For \( z \in \text{Dom} \sigma_2^\pm \), we take the complex root \( \sqrt{\frac{b^2}{\rho_0^2}z_0^2 - z_1^2} \) in such a way that \( \text{Im} \sigma_2^\pm(z) > 0 \). We set \( \text{Dom} \mathcal{H}^+ = \text{Dom} \mathcal{H}^- = \text{Dom} \sigma_2^+ \cap \text{Dom} \sigma_2^+ \). In order to prove Lemma 1.1 it suffices to show that

\[
(1.32) \quad \mathcal{H}^-(z^{(j)}(\lambda(0), \mu(0), z_1)) = 0, \quad \forall j \in \{2, 3\}
\]

if

\[
z^{(j)}(a, b, z_1) \equiv (\pm \sqrt{t_j(a, b)z_1, z_1}) \in \text{Dom} \mathcal{H}^+ \quad \text{for} \ j \in \{2, 3\}, \ z_1 \in \mathbb{C}.
\]
In fact, if (1.32) will be proved, then we have (1.30) in the following manner: Assume contrarily that \( \det B(y^*, \zeta) = 0 \). Then, in terms of (1.28), we obtain 
\[
\tilde{\zeta}_0^2 = \frac{2w(0)}{\rho_0} \tilde{\zeta}_1^2,
\]
which contradicts that \( \tilde{\zeta}_0^2 = t_j \tilde{\zeta}_1^2, \; j = 2, 3 \) where \( t_j \in \mathbb{C} \setminus \mathbb{R} \).

**Proof of (1.32).** Let \( \delta_2 > 0 \) be an arbitrary but fixed number. We introduce the sets 
\[
\Pi_1 = \{(a, b); \; a, b > \delta_2, \; \text{there exists} \; z_1 \in \mathbb{C} \; \text{with} \; |z_1| = 1 \; \text{such that} \; \mathcal{H}^+(z^{(j)}(a, b, z_1)) = 0 \; \text{for} \; j = 2 \; \text{or} \; j = 3\}
\]
and \[
\Pi_2 = \{(a, b); \; a, b > \delta_2, \; \mathcal{H}^+(z^{(j)}(a, b, z_1)) \neq 0 \; \text{for any} \; z_1 \in \mathbb{C} \; \text{with} \; |z_1| = 1 \; \text{and any} \; j \in \{2, 3\} \; \text{such that} \; z^{(j)}(a, b, z_1) \in \Dom \mathcal{H}^+\}.
\]
In order to prove (1.32) it suffices to prove that \( \Pi_1 = \emptyset \) because we have either \( (a, b) \in \Pi_1 \) or \( (a, b) \in \Pi_2 \). Let \((\tilde{a}, \tilde{b}) \in \Pi_2 \). Such a point exists because there exist \( a_0, b_0 \) such that \( t_j(a_0, b_0) \in \mathbb{R} \) for \( j = 2, 3 \) and then we have already shown that \( \det B(y^*, \zeta^*) \neq 0 \), so that \( (a_0, b_0) \in \Pi_2 \) in terms of (1.28) and the definition of \( \mathcal{H}^+ \). Assume contrarily that the set \( \Pi_1 \neq \emptyset \). Then \( \dist ((\tilde{a}, \tilde{b}), \Pi_1) > 0 \). There exist sequences \( \{(a_n, b_n)\}_{n=1}^\infty \subset \Pi_1 \) and \( \{z_{1,n}\}_{n=1}^\infty \subset \mathbb{C} \) such that \( |z_{1,n}| = 1 \), \( \lim_{n \to \infty} (a_n, b_n) = (\tilde{a}, \tilde{b}) \), \( \lim_{n \to \infty} z_{1,n} = \tilde{z}_1 \), 
\[
\dist ((\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b})) = \dist ((\tilde{a}, \tilde{b}), \Pi_1)
\]
and \( \mathcal{H}^+(z^{(j)}_{1,n}) = 0 \), where we set 
\[
\begin{align*}
z^{(j)}_{1,n} &= (\pm \sqrt{t_{j1}}(a_n, b_n)z_{1,n}, z_{1,n}), \\
&\text{for some} \; j_1 \in \{2, 3\} \; \text{and some} \; z_{1,n} \in \mathbb{C} \; \text{with} \; |z_{1,n}| = 1.
\end{align*}
\]
Let us show that there exists \( \tilde{z}_1 \in \mathbb{C} \) such that \( |\tilde{z}_1| = 1 \) and \( \tilde{z} \equiv (\pm \sqrt{t_{j1}}(\tilde{a}, \tilde{b})\tilde{z}_1, \; \tilde{z}_1) \in \Dom \mathcal{H}^+ \). Really if \( \tilde{z} \equiv (\pm \sqrt{t_{j1}}(\tilde{a}, \tilde{b})\tilde{z}_1, \; \tilde{z}_1) \in \Dom \mathcal{H}^+ \), then we can take \( \tilde{z} = \tilde{z} \). On the other hand, if \( \tilde{z} \not\in \Dom \mathcal{H}^+ \), then 
\[
\tilde{z} \not\in \Dom \alpha^+_b \quad \text{or} \quad \tilde{z} \not\in \Dom \alpha^+_{\alpha^+_b + 2k}.
\]
Let us assume for example that 
\[
\tilde{z} \not\in \Dom \alpha^+_{\alpha^+_b}.
\]
Then \( \sqrt{t_{j_1}(\tilde{a}, \tilde{b})/\tilde{b}} - 1 = e^{i\theta} \) and \( \sqrt{t_{j_1}(\tilde{a}, \tilde{b})rho/\tilde{a} + 2\tilde{b}} - 1 = r_0e^{i\theta_0} \). Since the imaginary part of \( t_{j_1} \) is not zero, we have \( \hat{\theta}_0 \neq \hat{\theta} \mod 2\pi \). Then either

\[
\mathcal{H}^+ (\tilde{z}) = 0 \quad \text{for} \beta \in \{\tilde{b}, \tilde{a} + 2\tilde{b}\} \quad \text{with} \quad \tilde{z} = (\pm \sqrt{t_{j_1}(\tilde{a}, \tilde{b})z_1, \tilde{z}_1}), \quad \tilde{z}_1 = e^{\pm i(\hat{\theta} + \hat{\theta}_0)},
\]
or

\[
\mathcal{H}^+(\tilde{z}) = 0 \quad \text{for} \beta \in \{\tilde{b}, \tilde{a} + 2\tilde{b}\} \quad \text{with} \quad \tilde{z} = (\pm \sqrt{t_{j_1}(\tilde{a}, \tilde{b})z_1, \tilde{z}_1}), \quad \tilde{z}_1 = e^{-\frac{i}{\pi}(\hat{\theta} + \hat{\theta}_0) + i\pi}.
\]

On the other hand, we have

(1.33) \( [(\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b})] \equiv \{ t(\tilde{a}, \tilde{b}) + (1-t)(\tilde{a}, \tilde{b}); 0 < t \leq 1 \} \subset \Pi_2 \).

In fact, noting that \( \{(a, b); a, b > \delta_2\} \) is convex and is contained in \( \Pi_1 \cup \Pi_2 \), we see that \( [(\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b})] \subset \Pi_1 \cup \Pi_2 \). Assume contrarily that there exists \( (a^*, b^*) \in \Pi_1 \) such that \( (a^*, b^*) \) is in the open segment \( ((\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b})) \). Then \( \text{dist} ((\tilde{a}, \tilde{b}), (a^*, b^*)) < \text{dist} ((\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b}) = \text{dist} ((\tilde{a}, \tilde{b}), \Pi_1) \), which is a contradiction. Thus we have proved that \( [(\tilde{a}, \tilde{b}), (\tilde{a}, \tilde{b})] \subset \Pi_2 \). We set \( (a_\varepsilon, b_\varepsilon) = \varepsilon(\tilde{a}, \tilde{b}) + (1-\varepsilon)(\hat{a}, \hat{b}) \). Then, for sufficiently small \( \varepsilon > 0 \), we have

\[
\tilde{z}_\varepsilon = (\pm \sqrt{t_j(a_\varepsilon, b_\varepsilon)z_1, \tilde{z}_1}) \in \text{Dom}\mathcal{H}^+.
\]

Then, by (1.33), we have \( \lim_{\varepsilon \to 0} \mathcal{H}^-(\tilde{z}_\varepsilon) = \mathcal{H}^-(\tilde{z}) = 0 \). Moreover by the choice of \( \tilde{z} \), we have \( \mathcal{H}^+(\tilde{z}) = 0 \). Hence \( \mathcal{H}^-(\tilde{z}) = 0 \). This implies that \( t_{j_1}(\hat{a}, \hat{b}) = \frac{2\tilde{b}}{\rho_0} \) but this is impossible because the left hand side is not real and the right hand side is real. Thus we have a contradiction. Thus the proof of Lemma 1.1 is complete.

Now we proceed to

**Completion of Proof of Proposition 1.1.** Let \( C(0) \in [0, (\mu/\rho)(y^*)] \) be the zero of the polynomial \( H \). By Lemma 1.1, for any \( \zeta \in S^2 \), the set of all the possible solutions to the equation

\[
\det B(y^*, \zeta) = 0, \quad \text{Im} \frac{1}{\rho} \frac{\Gamma^j}{\rho}(y^*, \zeta) \geq 0, \quad \forall \beta \in \{\mu, \lambda + 2\mu\}, \quad \xi_0 \neq 0
\]

is given by the formula

\[
\xi_0 + i\phi_{\rho_0}(y^*) \pm \sqrt{C(0)}(\xi_1 + i\phi_{\rho_1}(y^*)) = 0.
\]
Let \((\xi^*_1, s^*) \in S^1\) be an arbitrary but fixed point. Let \(z_0^* = \xi_0^* + is^*\phi_{y_0}(y^*)\) with some \(\xi_0^* \neq 0\) and \(z_1^* = \xi_1^* + is^*\phi_{y_1}(y^*)\) satisfy \(z_0^* = \pm \sqrt[3]{z_1^*}\) and \(\text{Im} \frac{\partial}{\partial s}\psi_{y_0}(y^*)\). Consider the following function \(J(y', z_0, z_1) \equiv \det \mathcal{B}(y', s, \xi_0, \xi_1), z_0 = \xi_0 + is\phi_{y_0}(y^*), z_1 = \xi_1 + is\phi_{y_1}(y^*)\). Applying the implicit function theorem (keeping in mind that \(J_{y_0}(0, z_0^*, z_1^*) \neq 0\), we see that there exists a function \(q(y', z_1)\) which is defined in a neighbourhood of \((y^*, z_1^*)\) and analytic in \(z_1\) such that \((y', q(y', z_1), z_1)\) is a solution to the equation \(J(y', z_0, z_1) = 0\). Note that

\[
q(y^*, z_1) = \pm \sqrt[3]{C(0)}z_1. 
\]

Set \(r(y', s, \xi_0, \xi_1) = \xi_0 + is\phi_{y_0}(y) - q(y', \xi_1 + is\phi_{y_1}(y))\). Since

\[
\text{det} \mathcal{B}(y', s, \xi_0, \xi_1) = r(y', s, \xi_0, \xi_1) \times \frac{\text{det} \mathcal{B}(y', s, \xi_0, \xi_1)}{r(y', s, \xi_0, \xi_1)}
\]

Applying the implicit function theorem, we have that

\[
\text{Im} \frac{1}{s} \sum_{k=0}^{1} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k} > 0. 
\]

Computing the left hand side of this inequality, we obtain

\[
\sum_{k=0}^{1} \text{Im} \frac{1}{s} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k} \Big|_{(y^*, s, \xi_0, \xi_1)} 
\]

\[
= \text{Im} \frac{1}{s} \left[ is\phi_{y_0y_0}(y^*) - q_{y_0}(y_0, 0, \xi_1 + is\phi_{y_1}(y^*)) \pm \sqrt{C(0)}is\phi_{y_0y_1}(y^*) \right. 
\]

\[
\left. + \sqrt{C(0)}(is\phi_{y_0y_1}(y^*) - q_{y_1}(y_0, 0, \xi_1 + is\phi_{y_1}(y^*)) \mp \sqrt{C(0)}is\phi_{y_0y_1}(y^*)) \right] 
\]

\[
= \phi_{y_0y_0}(y^*) \pm \sqrt{C(0)}\phi_{y_0y_1}(y^*) + \mathcal{C}(0)\phi_{y_1y_1}(y^*) 
\]

\[
- \text{Im} \frac{1}{s} \left[ q_{y_0}(y_0, 0, \xi_1 + is\phi_{y_1}(y^*)) \pm \sqrt{C(0)}\phi_{y_0y_1}(y^*) \right] 
\]

By the implicit function theorem, \(q_{y_0}(y_0, 0, z_1^*) = \frac{J_{y_0}(0, z_0^*, z_1^*)}{J_{y_1}(0, z_0^*, z_1^*)} = m_k(\xi_1^* + is^*\phi_{y_1}(y^*)), k = 0, 1\), where a number \(m_k\) depends on the sign in formula (1.34). Using this formula we obtain

\[
\sum_{k=0}^{1} \text{Im} \frac{1}{s} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k} \frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k} \Big|_{(y^*, s, \xi_0, \xi_1)} 
\]

\[
= \phi(r(y^*) \pm \sqrt{C(0)}\psi_{y_1}(y^*))^2 + \psi_{y_0y_0}(y^*) 
\]

\[
\pm \sqrt{C(0)}\psi_{y_0y_1}(y^*) + \mathcal{C}(0)\psi_{y_1y_1}(y^*) - (m_0\psi_{y_0}(y^*) \pm \sqrt{C(0)}m_1\psi_{y_1}(y^*)). 
\]
Obviously under condition (1.27), for all sufficiently large \( \tau > 0 \), inequality (1.23) holds true at the point \( y^* \). The proof of Proposition 1.1 is completed. 

§2. Proof of Theorem 1.1 (the beginning)

We shall prove Theorem 1.1 in several steps. Our proof is based on de-coupling of the Lamé system into the scalar acoustic equations for \( \text{rot} \, u \) and \( \text{div} \, u \). Then we apply the standard procedure (e.g., [Hö]) to these equations for obtaining a Carleman estimate. Finally we analyze the boundary integrals, which appear in the previous steps by means of the microlocalization technique.

First we show that it suffices to consider the case when displacement has a support in a ball of a small radius (Lemma 2.1). The strategic goal is a priori estimates for the traces of displacement and its normal derivative on the boundary. For overcoming the difficulties caused by non-compact supports of \( u \), we shall argue microlocally and the argument is lengthy but based on standard source books ([Ku], [T1], [T2]). The estimates for \( u \) and \( \partial_n u \) are of two types. Outside of the set \( \Psi_2 \) (see Lemma 1.1) in the cotangent bundle we have estimate (3.1) while in a neighbourhood of this set we have a weaker estimates (5.2), (5.3).

In order to prove estimate (3.1) we shall separate several cases corresponding to that the roots \( r_\mu(\gamma) \) and \( r_{\lambda+2\mu}(\gamma) \) defined by (1.20) are zero or non-zero. In Section 3 we consider the case \( r_\mu(\gamma) = 0 \) and Section 4 is devoted to the case \( r_{\lambda+2\mu}(\gamma) = 0 \). Finally in Section 5 we consider the remaining case \( r_\mu(\gamma) \neq 0 \) and \( r_{\lambda+2\mu}(\gamma) \neq 0 \) and obtain estimate (1.24).

First we show that it suffices to consider only the case where the support of \( u \) is located in a small neighbourhood of an arbitrary point \( y^* = (y^*_0, 0, 0) \in Q \).

Lemma 2.1. Under the conditions of Theorem 1.1 it suffices to prove (1.24) under the assumption that

\[
(2.1) \quad \text{supp} \, u \subset B_\delta(y^*),
\]

where \( \delta > 0 \) is an arbitrary small number and \( y^* \) is an arbitrary point in \( Q \).

Proof of Lemma 2.1. Let us consider the finite covering of \( \overline{Q} \) by balls \( B_\delta(y^*_0) \). Let \( e \in C_0^\infty(B_{2\delta}(0)) \) be a non-negative function such that \( e|_{B_{\frac{3}{4}\delta}(0)} = 1 \) and \( e(x) < 1 \) for all \( x \in B_{2\delta}(0) \setminus B_{\frac{3}{4}\delta}(0) \) and let \( \tilde{e} \in C_0^\infty(B_{2\delta}(0)) \) be a non-negative function such that \( \tilde{e}|_{B_{\frac{3}{2}\delta}(0)} = 1 \) and \( \tilde{e}(x) < 1 \) for all \( x \in B_{2\delta}(0) \setminus B_{\frac{3}{2}\delta}(0) \).
We set \( e_j(x) = \epsilon(x - y_j^*) \) and \( \tilde{e}_j(x) = \tilde{\epsilon}(x - y_j^*) \). For the function \( e_j u \) we have the following boundary condition

\[
(2.2) \quad B(x, D)e_j u = -[e_j, B]u + e_j g.
\]

Let \( \psi_j(x) = \psi(x) + \epsilon(\tilde{\epsilon}_j(x) - 1) \), \( \phi_j(x) = e^{r\psi_j(x)} \) and \( \epsilon \in (0, 1) \). The function \( \psi_j \) satisfies Condition 1.1 and Condition 1.2 for all sufficiently small \( \epsilon \).

We set

\[
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\]

Let \( \psi \) be such that

\[
\|\psi\|_{H^{\frac{1}{2}+\epsilon}(\partial Q)} \leq C_1 \|\psi\|_{H^{\frac{1}{2}-\epsilon}(\partial Q)} + \int_{Q} \sum_{|\alpha| = 0}^{2} s^{4-2|\alpha|}|\partial_x^s u|^{\frac{1}{2}} e^{2\epsilon\phi} dx.
\]

Note that

\[
\|e_j, \partial Q\|_{H^{\frac{1}{2}+\epsilon}(\partial Q)} \leq \|e_j, \partial Q\|_{H^{\frac{1}{2}-\epsilon}(\partial Q)} + \int_{Q} \sum_{|\alpha| = 0}^{2} s^{4-2|\alpha|}|\partial_x^s u|^{\frac{1}{2}} e^{2\epsilon\phi} dx,
\]

for all \( \alpha \geq 0 \).

Hence the fourth term on the right hand side of (2.3) can be absorbed into the left hand side, so that we obtain (1.24). Thus the proof of Lemma 2.1 is completed.
Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$. We can directly prove that the functions $\text{rot } u \equiv \partial_x u_2 - \partial_y u_1$ and $\text{div } u \equiv \partial_x u_1 + \partial_y u_2$ satisfy the equations

\begin{equation}
\begin{aligned}
\partial_x^2 \text{rot } u - \mu \Delta \text{rot } u &= m_1 \quad \text{in } Q, \\
\partial_y^2 \text{div } u - (\lambda + 2\mu) \Delta \text{div } u &= m_2 \quad \text{in } Q,
\end{aligned}
\end{equation}

where $m_1 = M_1(x, D)\text{rot } u + M_2(x, D)\text{div } u + M_1(x, D)u + \text{rot } f,$ $m_2 = M_3(x, D)\text{rot } u + M_4(x, D)\text{div } u + M_2(x, D)u + \text{div } f.$

Here and henceforth $M_1(x, D), M_2(x, D), M_j(x, D), j = 1, 2, 3, 4, K_j(y, D),$ $K_j(y, D),$ etc. denote first order differential operators with $L^\infty$-coefficients, if they are not specified.

If $y^* \in Q$, then we take $\delta > 0$ sufficiently small and we can assume $B_\delta(y^*) \cap \partial Q = \emptyset$. In that case, directly by means of (2.4), we have (1.24). Therefore, thanks to Lemma 2.1, we have to concentrate on the case $y^* \in \partial Q$. Henceforth without loss of generality, we assume that

$$y^* = (y_0^*, 0, 0), \quad y^{*'} = (y_0^*, 0).$$

Below we will apply to equations (2.4) a Carleman estimate for the scalar equations. Since the functions $\text{div } u$ and $\text{rot } u$ do not have compact supports, some boundary integrals in this estimates will appear. In order to estimate these boundary integrals, it is convenient to use a weight function $\varphi$ such that $\varphi|_{\partial \Omega} = \phi|_{\partial \Omega}$ and $\varphi(x) < \phi(x)$ for all $x$ in a neighborhood of $\partial Q$. We construct such a function $\varphi$ locally near the boundary $\partial \Omega$:

\begin{equation}
\varphi(x) = e^{\tau \tilde{\varphi}(x)}, \quad \tilde{\psi}(x) = \psi(x) - \frac{1}{\sqrt{N}} \ell_1(x) + N \ell_2(x),
\end{equation}

where $N > 0$ is a large positive parameter and $\ell_1 \in C^2(\bar{\Omega})$ satisfies

$$\ell_1(x') > 0 \quad \forall x' \in \Omega, \quad \ell_1|_{\partial \Omega} = 0, \quad \nabla \ell_1|_{\partial \Omega} \neq 0.$$ 

Denote $\Omega_{\frac{1}{N}} = \{x' \in \Omega; 0 < \text{dist}(x', \partial \Omega) < \frac{1}{N}\}$. Obviously for any fixed $\tilde{\tau} > 0$, there exists $N_0 > 0$ such that

$$\varphi(x) < \phi(x), \quad \forall x \in [0, T] \times \Omega_{\frac{1}{N^2}}, \quad N \in (N_0, \infty).$$

Our goal is to prove an analogue of (1.24) for the weight function $\varphi$ instead of $\phi$. We make the additional assumption

$$\text{supp } u \subset B_\delta(y^*) \cap \mathcal{G} = \mathbb{R}^2 \times [0, 1/N^2].$$
Below in order to simplify the notations we denote the functions \( \psi(x(y)), \phi(x(y)), \varphi(x(y)) \) as \( \psi(y), \phi(y), \varphi(y) \). Therefore

\[
\phi(y) = \varphi(y) \quad \text{on } \partial \mathcal{G}
\]

and for some \( N > N_1 \) we have

\[
\varphi(y) < \phi(y) \quad \forall y \in \mathcal{G}.
\]

We need the following proposition:

**Proposition 2.1.** Let (2.7) holds true. There exist \( \hat{\tau} > 1 \) and \( N_1 > 1 \) such that for any \( \tau > \hat{\tau} \), there exists \( s_0(\tau, N) > 0 \) such that for any function \( u \in H^2(\Omega) \) satisfying (1.8) and (1.9), we have

\[
N \| e^{s\varphi} \|_{H^{1,\gamma}(\Omega)}^2 \leq C_4 \left( N \| e^{s\varphi} \|_{L^2(\Omega)}^2 + s \| (\text{rot } u, \text{div } u) e^{s\varphi} \|_{H^{1,\gamma}(\Omega)}^2 \right) + N \left\| \left( u \frac{\partial u}{\partial n} \right) e^{s\varphi} \right\|_{H^{1,\gamma}(\partial \Omega)}^2,
\]

\( s \geq s_0(\tau, N), \quad N \geq N_1 \)

and \( C_4 \) is independent of \( N, s \).

We give the proof of this proposition in Appendix I. Thanks to Lemma 2.1 we can work with the variable \( y \) instead of \( x \). By (1.8) and (1.9) on the boundary \( \partial \mathcal{G} \), we have

\[
\frac{\partial^2 u_1}{\partial y_0^2} - \mu \left( \frac{\partial^2 u_1}{\partial y_1^2} - 2\ell' (y_1) \frac{\partial^2 u_1}{\partial y_1 \partial y_2} \right) + \mu \ell'' (y_1) \frac{\partial u_1}{\partial y_2} - (\lambda + \mu) \frac{\partial^2 u_1}{\partial y_1 \partial y_2} \ell (y_1)
\]

\[
= f_1 + \mu (1 + |\ell'|^2) \frac{\partial^2 u_1}{\partial y_2^2} - (\lambda + \mu) \ell + (\lambda + \mu) \frac{\partial^2 u_1}{\partial y_2^2} |\ell'|^2 - \tilde{K}_3(y, D) u
\]

and

\[
\frac{\partial^2 u_2}{\partial y_0^2} - \mu \left( \frac{\partial^2 u_2}{\partial y_1^2} - 2\ell' (y_1) \frac{\partial^2 u_2}{\partial y_1 \partial y_2} \right) + \mu \ell'' (y_1) \frac{\partial u_2}{\partial y_2} - (\lambda + \mu) \frac{\partial^2 u_2}{\partial y_1 \partial y_2} \ell (y_1)
\]

\[
= f_2 + \mu (1 + |\ell'|^2) \frac{\partial^2 u_2}{\partial y_2^2} + (\lambda + \mu) \left( \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2} \ell \right) - \tilde{K}_4(y, D) u.
\]

Here and below by \( K_j(y, D) \) we denote a general first order differential operator with \( C^1 \) coefficient. By (1.12) we know that

\[
\frac{\partial u_1}{\partial y_2} = \left( A_1(y_1), \frac{\partial u}{\partial y_1} \right) + (\tilde{A}_1(y_1), g), \quad \frac{\partial u_2}{\partial y_2} = \left( A_2(y_1), \frac{\partial u}{\partial y_1} \right) + (\tilde{A}_2(y_1), g).
\]
Hence
\begin{align}
\frac{\partial^2 u_1}{\partial y_2 \partial y_1} &= \left( A_1(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_1(y_1), \frac{\partial u}{\partial y_1} \right) + \frac{\partial}{\partial y_1}(\tilde{A}_1(y_1), g), \\
\frac{\partial^2 u_2}{\partial y_2 \partial y_1} &= \left( A_2(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_2(y_1), \frac{\partial u}{\partial y_1} \right) + \frac{\partial}{\partial y_1}(\tilde{A}_2(y_1), g).
\end{align}

Using these equations we may transform (2.8) and (2.9) to
\begin{align}
B_1(y', D')u &= \frac{\partial^2 u_1}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1) \left( \left( A_1(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_1(y_1), \frac{\partial u}{\partial y_1} \right) \right) \right\} \\
&\quad + \mu \ell''(y_1) \left( A_1(y_1), \frac{\partial u}{\partial y_1} \right) - (\lambda + \mu) \frac{\partial^2 u_1}{\partial y_1^2} \\
&\quad - (\lambda + \mu) \left\{ \left( A_2(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_2(y_1), \frac{\partial u}{\partial y_1} \right) \right\} \ell'(y_1) \\
&\quad + (\lambda + \mu) \ell''(y_1) \left\{ \left( A_1(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_1(y_1), \frac{\partial u}{\partial y_1} \right) \right\} \\
&\equiv f_1 + \mu(1 + |\ell'|^2) \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}(\lambda + \mu)\ell' \\
&\quad + (\lambda + \mu) \frac{\partial^2 u_1}{\partial y_2^2}|\ell'|^2 + \tilde{K}_5(y, D')u + \mathcal{R}_1(y, D)g
\end{align}

and
\begin{align}
B_2(y', D')u &= \frac{\partial^2 u_2}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1) \left( \left( A_2(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_2(y_1), \frac{\partial u}{\partial y_1} \right) \right) \right\} \\
&\quad - \mu \ell''(y_1) \left( A_2(y_1), \frac{\partial u}{\partial y_1} \right) - (\lambda + \mu) \left\{ \left( A_1(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A'_1(y_1), \frac{\partial u}{\partial y_1} \right) \right\} \\
&\equiv f_2 + \mu(1 + |\ell'|^2) \frac{\partial^2 u_2}{\partial y_2^2} + (\lambda + \mu) \left( \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell' \right) \\
&\quad + \tilde{K}_6(y, D')u + \mathcal{R}_2(y, D)g.
\end{align}

Here \( \mathcal{R}_1(y, D), \mathcal{R}_2(y, D) \) are first order differential operators with \( C^1 \) coeffi-
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(2.17) \[ P_{\mu}(y, D)z_2 = D_0^2 z_2 - (\lambda + 2\mu)(D_2^2 z_2 - 2\ell'(y_1)D_1 D_2 z_2) + (1 + |\ell'(y_1)|^2)D_2^2 z_2 - (\lambda + 2\mu)\ell''(y_1)D_2 z_2 = m_2. \]

After the change of the coordinates, we use the same letters \( m_1, m_2 \) as in (2.4).

We consider a finite covering of the unit sphere \( S^2 \equiv \{(s, \xi_0, \xi_1); s^2 + \xi_0^2 + \xi_1^2 = 1\} \). That is, \( S^2 \subset \bigcup_{\nu=1}^{K(\delta_1)} \{(s, \xi_0, \xi_1) \in S^2; |\zeta - \zeta_\nu^*| < \delta_1\} \) where \( \zeta_\nu^* \in S^2 \), and by \( \{\chi_\nu(\zeta)\}_{1 \leq \nu \leq K(\delta_1)} \) we denote the corresponding partition of unity: \( \sum_{\nu=1}^{K(\delta_1)} \chi_\nu(\zeta) = 1 \) for any \( \zeta \in S^2 \) and \( \text{supp } \chi_\nu \subset \{\zeta \in S^2; |\zeta - \zeta_\nu^*| < \delta_1\} \). Henceforth we extend \( \chi_\nu \) to the set \( \{\zeta; |\zeta| \neq 1\} \) as the homogeneous function of the order zero in \( C^\infty(\mathbb{R}^3 \setminus \{0\}) \) such that

\[ \text{supp } \chi_\nu \subset O(\delta_1) \equiv \left\{ \zeta; \left| \frac{\zeta}{|\zeta|} - \zeta_\nu^* \right| < \delta_1 \right\}. \]
Moreover we set

\[ P_{\mu,s}(y, s, D) = P_{\mu}(y, D) = D^2_{y_0} - \mu(D^2_{y_2} - 2\ell'(y_1)D_{y_1}D_{y_2} + (1 + |\ell'(y_1)|^2)D^2_{y_2}) - \mu i\ell''(y_1)D_{y_2} \]

\[ P_{\lambda+2\mu,s}(y, s, D) = P_{\lambda+2\mu}(y, D) = D^2_{y_0} - (\lambda + 2\mu)(D^2_{y_2} - 2\ell'(y_1)D_{y_1}D_{y_2} + (1 + |\ell'(y_1)|^2)D^2_{y_2}) - (\lambda + 2\mu) i\ell''(y_1)D_{y_2}. \]

Under some condition, we can factor the operator \( P_{\beta,s} \) as a product of two pseudodifferential operators.

**Proposition 2.2.** Let \( \beta \in \{\mu, \lambda + 2\mu\} \) and \( |r_{\beta}(y, \zeta)| \geq \hat{\delta} > 0 \) for all \( (y, \zeta) \in B_\delta(y^*) \times O(2\delta_1) \). Then we can factor the operator \( P_{\beta,s} \) into the product of two pseudodifferential operators:

\[ P_{\beta,s} \chi_{\nu}(s, D')V = \beta|G|(D_{y_2} - \Gamma^-_{\beta}(y, s, D'))(D_{y_2} - \Gamma^+_{\beta}(y, s, D'))\chi_{\nu}(s, D')V + T_{\beta}V, \]

where \( \text{supp} V \subset B_\delta(y^*) \cap \mathcal{G} \) and

\[ T_{\beta} \in \mathcal{L}(L^2(0, 1; H^{1,s}(\mathbb{R}^3)); L^2(0, 1; L^2(\mathbb{R}^3))). \]

Let us consider the equation

\[ (D_{y_2} - \Gamma^-_{\beta}(y, s, D'))\chi_{\nu}(s, D')V = q, \quad V|_{y_2=\frac{1}{N}} = 0, \quad \text{supp} V \subset B_\delta(y^*) \cap \mathcal{G}. \]

For solutions of this problem we have an a priori estimate:

**Proposition 2.3.** Let \( \beta \in \{\mu, \lambda + 2\mu\} \) and \( |r_{\beta}(y, \zeta)| \geq \hat{\delta} > 0 \) for all \( (y, \zeta) \in B_\delta(y^*) \times O(2\delta_1) \). Then there exists a constant \( C_5 > 0 \) independent of \( N \) such that

\[ \|\sqrt{s}\chi_{\nu}(s, D')V|_{y_2=0}\|_{L^2(\mathbb{R}^3)} \leq C_5\|q\|_{L^2(\mathcal{G})}. \]

The proofs of Propositions 2.2 and 2.3 can be found for example in [IY3]. Next we consider the equation

\[ (D_{y_2} - \Gamma^+_{\beta}(y, s, D'))\chi_{\nu}(s, D')w = g, \quad w|_{y_2=1/N^2} = 0, \quad \text{supp} w \subset B_\delta(y^*) \cap \mathcal{G}. \]

We have
Proposition 2.4. Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y, \zeta)| \geq \delta > 0$ for all $(y, \zeta) \in B_\delta(y^*) \times O(\delta_1)$, $s^* \neq 0$, or $\text{Im} \, \alpha_\beta^\gamma(\gamma) \neq 0$ and $\chi_\nu \subset O(\delta_1)$. Then for sufficiently small $\delta, \delta_1$ there exists a constant $C_6 > 0$ independent of $N$ such that

$$
\|\chi_\nu(s, D')w\|_{H^{1,\gamma}(G)} \leq C_6 \left( \frac{1}{\sqrt{s}} \|g\|_{H^{1,\gamma}(G)} + s^2 \|\chi_\nu w(\cdot, 0)\|_{H^{1,\gamma}(\partial G)} \right).
$$

For the proof of this proposition, we can use the exactly same arguments as in [IP], and we give it in Appendix II for completeness.

Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $\tilde{w} = \tilde{w}(y)$ satisfy a scalar second order hyperbolic equation

$$
P_{\beta,s}(y, s, D)\tilde{w} = q \quad \text{in } \mathcal{G}, \quad \frac{\partial \tilde{w}}{\partial y_0}|_{y_2 = 1/N^2} = \tilde{w}|_{y_2 = 1/N^2} = 0, \quad \text{supp } \tilde{w} \subset B_\delta(y^*).
$$

Next we remind main facts related to the Carleman estimates with boundary for the operator $P_{\beta,s}(y, s, D)$. Set $L_{+,\beta}(y, s, D) = \frac{P_{\beta,s}(y, s, D) + P_{\beta,s}^*(y, s, D)}{2}$ and $L_{-,\beta}(y, s, D) = \frac{P_{\beta,s}(y, s, D) - P_{\beta,s}^*(y, s, D)}{2}$, where $P_{\beta,s}^*$ is the formally adjoint operator to $P_{\beta,s}$. One can easily check that the principal part of the operator $L_{-,\beta}$ is given by formula

$$
L_{-,\beta}(y, s, D)\tilde{w} = -2s^2 \varphi \varphi_0 \frac{\partial \tilde{w}}{\partial y_0} + 2s^2 \varphi \varphi_0 \left( \varphi_0 \frac{\partial \tilde{w}}{\partial y_0} + \varphi_0 \frac{\partial \tilde{w}}{\partial y_2} \right) + 2s(1 + (\ell(y_1))^2)\varphi_0 \frac{\partial \tilde{w}}{\partial y_2}
$$

$$
+ |l.o.t.
$$

Obviously $L_{+,\beta}(y, s, D)\tilde{w} + L_{-,\beta}(y, s, D)\tilde{w} = q$. For all $s \in \mathbb{R}^1$ the following equality holds true:

$$
\Sigma_\beta(\tilde{w}) + \|L_{-,\beta}(y, s, D)\tilde{w}\|_{L^2(G)}^2 + \|L_{+,\beta}(y, s, D)\tilde{w}\|_{L^2(G)}^2
$$

$$
+ \text{Re} \int_G [(L_{+,\beta}, L_{-,\beta})\tilde{w}, \bar{\tilde{w}}] \, dy = \|q\|_{L^2(G)}^2,
$$

where

$$
\Sigma_\beta(\tilde{w}) = \int_{\mathcal{G}} \tilde{p}_\beta(y, \nabla \varphi, 0, -\tilde{e}_2, -1, 0) (s\tilde{p}_\beta(y, \nabla \tilde{w}, \nabla \tilde{w})
$$

$$
- s^3 \tilde{p}_\beta(y, \nabla \varphi, \nabla \varphi) |\tilde{w}|^2 \, dy_0 \, dy_1 + \text{Re} \int_{\mathcal{G}} \tilde{p}_\beta(y, \nabla \tilde{w}, -\tilde{e}_2) L_{-,\beta}(y, s, D)\tilde{w} \, dy_0 \, dy_1,
$$

for all $(y, \zeta) \in B_\delta(y^*) \times O(\delta_1)$.
and
\[ \hat{p}_\beta(y, \xi, \tilde{\xi}) = \xi_0 \tilde{\xi}_0 - \beta(\xi_1 \tilde{\xi}_1 - \ell'(y_1)(\xi_1 \tilde{\xi}_2 + \xi_2 \tilde{\xi}_1) + (1 + |\ell'(y_1)|^2)\xi_2 \tilde{\xi}_2). \]

We note that by (2.5) \( \phi_{y_k} |_{\partial G} = \phi_{y_k} \) for \( k \in \{0, 1\} \). Therefore on \( \partial G \) the function \( \nabla_y \phi \) is independent of \( N \) and \( |\nabla \phi(y') - \nabla \phi(y')| \leq C_7 \sqrt{N} \) where the constant \( C_7 \) is independent of \( N \). In particular, for all sufficiently large \( N \), we have (1.6) for the function \( \hat{\psi} \).

It is convenient for us to rewrite (2.21) in the form
\[ \Sigma_\beta(\tilde{w}) = \Sigma_\beta^{(1)}(\tilde{w}) + \Sigma_\beta^{(2)}(\tilde{w}), \]
where
\[ \Sigma_\beta^{(1)}(\tilde{w}) = \Re \int_{y_2=0} \Re \beta(y^*) \frac{\partial \tilde{w}}{\partial y_2} \]
\[ \times \left\{(\beta(y^*) \frac{\partial \tilde{w}}{\partial y_1} \varphi_{y_1}(y^*) + \beta(y^*) \frac{\partial \tilde{w}}{\partial y_2} \varphi_{y_2}(y^*) - \frac{\partial \tilde{w}}{\partial y_0} \varphi_{y_0}(y^*))\right\} dy_0 dy_1 \]
\[ + \int_{y_2=0} \Re \beta(y^*) \varphi_{y_2}(y^*) \left\{ \left| \frac{\partial \tilde{w}}{\partial y_0} \right|^2 - \beta(y^*) \left( \left| \frac{\partial \tilde{w}}{\partial y_1} \right|^2 + \left| \frac{\partial \tilde{w}}{\partial y_2} \right|^2 \right) \right\} dy_0 dy_1. \]

Then
\[ (2.23) \quad |\Sigma_\beta^{(2)}(\tilde{w})| \leq \epsilon(\delta)s \left\| \left( \frac{\partial \tilde{w}}{\partial y_2} \right) \right\|_{L^2(\partial G) \times H^{\gamma_1}(\partial G)}, \]
where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \). It is known (see e.g., [Im]) that there exists a parameter \( \tau > 1 \) such that for any \( \tau > \tau \) there exists \( s_0(\tau) \) such that
\[ (2.24) \quad \frac{1}{4} \left| L_{-\beta}(y, s, D) \tilde{w} \right|_{L^2(\partial G)}^2 + \frac{1}{4} \left| L_{+\beta}(y, s, D) \tilde{w} \right|_{L^2(\partial G)}^2 \]
\[ + \Re \left[ L_{+\beta, L_{-\beta}} \tilde{w} \right]_{L^2(\partial G)} \geq C_9 \| \varphi_{y_2} \|_{L^2(\partial G)} \| \partial_{y_2} \tilde{w} \|_{L^2(\partial G)} \quad \forall s \geq s_0(\tau), \]
where \( C_9 > 0 \) is independent of \( s \). Combining (2.21) and (2.24), we arrive at
\[ (2.25) \quad \frac{1}{4} \left| L_{-\beta}(y, s, D) \tilde{w} \right|_{L^2(\partial G)}^2 + \frac{1}{4} \left| L_{+\beta}(y, s, D) \tilde{w} \right|_{L^2(\partial G)}^2 \]
\[ + C_9 \| \varphi_{y_2} \|_{H^{\gamma_1}(\partial G)} + \Sigma_\beta(\tilde{w}) \leq C_0(\| q \|_{L^2(\partial G)} + s \| \tilde{w} \|_{L^2(\partial G)} \| \partial_{y_2} \tilde{w} \|_{L^2(\partial G)} + \| \tilde{w} \|_{H^{\gamma_1}(\partial G)}^2), \quad \forall s \geq s_0(\tau). \]
We finish this section by recalling some results of calculus of pseudodifferential operators with symbols of limited smoothness. Let

$$C^k S_{1,0}^p = \{a(x,s,\xi); |\partial_x^\alpha \partial_\xi^\beta a(x,s,\xi)| \leq c_{\alpha,\beta}(1 + |(\xi,s)|^{p-|\beta|}), \quad |\alpha| \leq k\}$$

for some compact set $K \subset \Omega$ we have $a(x,s,\xi) = 0$ for $x \notin K$.

For every symbol $a(x,s,\xi) \in C^k S_{1,0}^p$, we introduce the operator $a(x,s,D)u = \int_{\mathbb{R}^n} a(x,s,\xi) \hat{u}(\xi)e^{i(x,\xi)}d\xi$. It is known (see [T2]) that $a(x,s,D) \in \mathcal{L}(H^m_{0,+p,s}(\Omega), H^{m,s}(\mathbb{R}))$ for $m \in (-k,k)$. Here $\hat{u}$ is the Fourier transform.

For the operators with nonsmooth symbols, we have the Gårding inequality: Let $p(x,s,\xi) \in C^\ell S_{1,0}^m$, $\ell > m$ and there exists $C > 0$ such that if $\text{Re} \, p(x,s,\xi) \geq C \|\xi,s\|^m$ for all large $|\xi,s|$, then

$$\text{Re}(p(x,s,D)u, u)_{L^2(\Omega)} \leq C_0 \|u\|^2_{H^{m/2,-}(\Omega)} - C_1 \|u\|^2_{L^2(\Omega)}$$

for any $u$ such that supp $u \subset K$.

We say that a symbol $a(x,s,\xi) \in C^k S_{cl}^p$ if $a(x,s,\xi) \in C^k S_{1,0}^p$ and

$$a(x,s,\xi) = \sum_{j \geq 0} a_j(x,s,\xi), \quad \forall |(s,\xi)| \geq 1$$

where $a(x,s,\xi) = \sum_{j \geq 0} a_j(x,s,\xi) \in C^k S_{1,0}^{p-N}$.

The following proposition will be very useful.

**Proposition 2.5** [T2]. Let $A_j(x,s,\xi) \in C^1 S_{cl}^1$, $B(x,s,\xi) \in C^1 S_{cl}^m$. Then

$$A_j(x,s,D)B(x,s,D) = C_j(x,s,D) + R_j \quad j \in \{0, 1\},$$

$$C_j(x,s,\xi) = A_j(x,s,\xi)B(x,s,\xi)$$

and

$$R_0 : H^{\mu+\ell,s} \to H^{\ell+1,s}; \quad R_1 : H^{\mu,s} \to L^2.$$
§3. Case \( r_\mu(\gamma) = 0 \)

In this section we treat the case where \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \) and \( r_\mu(\gamma) = 0 \) for \( \gamma = (y^*, \zeta^*) \in \partial G \times S^2 \). Throughout this paper, we use the following notations:

\[
v = u e^{s \varphi}, \quad w = z e^{s \varphi},
\]

\[z(y) = (z_1(y), z_2(y)) = (\text{rot } u, \text{div } u)(x), \quad u = (u_1, u_2), \quad v = (v_1, v_2).\]

Henceforth \( \hat{v}(s, \xi', y_2) \) is the Fourier transform of \( v(s, y_0, y_1, y_2) \) with respect to \( y_0, y_1 \), and we set \( w_\nu \equiv (w_{1, \nu}, w_{2, \nu}) = \chi_\nu(s, D') w \). This section is devoted to the proof of the following lemma.

**Lemma 3.1.** Let \( \gamma = (y^*, \zeta^*) \in \partial G \times S^2 \) be a point such that \( r_\mu(\gamma) = 0 \) and \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \). Then for all sufficiently small \( \delta_1 > 0 \), we have

\[
N \sum_{|\alpha| = 0}^2 s^{4-2|\alpha|} \| \partial^\alpha y v_\nu \|_{L^2(\partial G)}^2 + s \left\| \left( \nu_\nu, \frac{\partial v_\nu}{\partial y_2}, \frac{\partial^2 v_\nu}{\partial y_2^2} \right) \right\|_{H^{1, r}((\partial G) \times H^{1, r}((\partial G) \times L^2(\partial G))}
\leq C(\|v\|_{H^{1, r}(G)}^2 + ||fe^{s \varphi}\|_{H^{1, r}(G)}^2 + s\|xe^{s \varphi}\|_{H^{1, r}(G)}^2),
\]

where \( C \) is independent of \( s, N \).

**Proof.** There exists a constant \( C_1 > 0 \) such that

\[
|\xi_0^2 - s^2 \varphi_{y_0}^2(y^*) - \mu(0)\xi_1^2 + \mu(0)s^2 \varphi_{y_1}^2(y^*)|
\leq C_1 \delta_1 (|\xi_0|^2 + |\xi_1|^2 + s^2), \quad \forall \xi \in \mathcal{O}(\delta_1).
\]

We recall that by (2.21)–(2.25) there exist constants \( C_2, C_3 > 0 \) such that

\[
C_2 s\|w_{1, \nu}\|_{H^{1, r}(G)}^2 + \Sigma^{(1)}_{\mu}
\leq C_3 \|P_{\mu, s} w_{1, \nu}\|_{L^2(G)}^2 + \epsilon(\delta)s \left\| \left( \frac{\partial w_{1, \nu}}{\partial y_2}, w_{1, \nu} \right) \right\|_{L^2(\partial G) \times H^{1, r}(\partial G)},
\]

where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \). Note that \( \Sigma^{(1)}_{\mu} \) can be written in the form

\[
\Sigma^{(1)}_{\mu}(w_{1, \nu}) = \int_{\partial G} \left( s \mu^2(0) \varphi_{y_2}(y^*) \left| \frac{\partial w_{1, \nu}}{\partial y_2} \right|^2 + s^3 \mu^2(0) \varphi_{y_2}^3(y^*) |w_{1, \nu}|^2 \right) d\Sigma
\]

\[
+ \text{Re} \int_{\partial G} 2s \mu(0) \frac{\partial w_{1, \nu}}{\partial y_2} \left( \mu(0) \varphi_{y_1}(y^*) \frac{\partial w_{1, \nu}}{\partial y_1} - \varphi_{y_0}(y^*) \frac{\partial w_{1, \nu}}{\partial y_0} \right) d\Sigma
\]

\[
+ \int_{\partial G} s \mu(0) \varphi_{y_2}(y^*) (\xi_0^2 - \mu(0)\xi_1^2 - s^2 \varphi_{y_0}^2(y^*) + s^2 \mu(0) \varphi_{y_0}^2(y^*)) |\hat{w}_{1, \nu}|^2 d\Sigma
\]

\[\equiv J_1 + J_2 + J_3.\]
Note that $r_{\lambda+2\mu}(\gamma) \neq 0$. Really if $r_{\lambda+2\mu}(\gamma) = 0$ then 
\[ \rho_{(0)}(\lambda+2\mu,0)(\xi_0^* + is^*\varphi_{y_0}(y^*))^2 - (\xi_1^* + is^*\varphi_{y_1}(y^*))^2 = 0. \]
On the other hand since $r_{\mu}(\gamma) = 0$ we have 
\[ \rho_{(0)}(\xi_0^* + is^*\varphi_{y_0}(y^*))^2 - (\xi_1^* + is^*\varphi_{y_1}(y^*))^2 = 0. \]
Hence $\xi_0^* + is^*\varphi_{y_0}(y^*) = 0$ and $\xi_1^* + is^*\varphi_{y_1}(y^*) = 0$. Since $|\xi^*| = 1$, these equalities imply 
\[ \varphi_{y_0}(y^*) = 0, \quad \varphi_{y_1}(y^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \quad s^* = 1. \]
This contradicts assumption (1.6) which states $\frac{\partial \psi}{\partial t} \neq 0$. Therefore $r_{\lambda+2\mu}(\gamma) \neq 0$ and factorization (2.18) holds true. We set $V_{\lambda+2\mu}^+ = (D_{y_2} - \Gamma_{\lambda+2\mu}(y, s, D'))w_{2,\nu}$. Then 
\[ P_{\lambda+2\mu, s}(y, s, D)w_{2,\nu} = (\lambda + 2\mu)G((D_{y_2} - \Gamma_{\lambda+2\mu}(y, s, D'))V_{\lambda+2\mu}^+ + T_{\lambda+2\mu}w_{2,\nu}, \]
where $T_{\lambda+2\mu} \in \mathcal{L}(H^{1,\alpha}(\mathcal{G}), L^2(\mathcal{G}))$. Therefore Proposition 2.3 immediately yields 
\[ \sqrt{s}||D_{y_2} - \Gamma_{\lambda+2\mu}^+(y, s, D')||w_{2,\nu}|_{y_2=0}||_{L^2(\partial\mathcal{G})} \leq C_4(||P_{\lambda+2\mu, s}w_{2,\nu}||_{L^2(\mathcal{G})} + ||w||_{H^{1,\alpha}(\mathcal{G})}). \]
Now we have to estimate $\Sigma_\mu^{(1)}$. First we note that 
\[ \frac{\partial z_1}{\partial y_2}|_{y_2=0} = \frac{\partial^2 u_2}{\partial y_1 \partial y_2} - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2} \ell'(y_1) \]
\[ = \left( A_2(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A_1^*(y_1), \frac{\partial u}{\partial y_1} \right) + \left( \tilde{A}_2(y_1), g \right) - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2} \ell'(y_1), \]
\[ \frac{\partial z_2}{\partial y_2}|_{y_2=0} = \frac{\partial^2 u_1}{\partial y_1 \partial y_2} + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2} \ell'(y_1) \]
\[ = \left( A_1(y_1), \frac{\partial^2 u}{\partial y_1^2} \right) + \left( A_1^*(y_1), \frac{\partial u}{\partial y_1} \right) + \left( \tilde{A}_1(y_1), g \right) + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2} \ell'(y_1). \]
We may rewrite (3.6) and (3.7) as 
\[ e^{s^2 \left( \frac{\partial}{\partial y_2} \right)} = -A(y_1)D_{y_1}^2 v - iA'(y_1)D_{y_1} v + \mathcal{H}(y', s, D')(ge^{s^2}) - I(y_1)D_{y_1}^2 v. \]
where we used the notations 
\[ A(y_1) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad I(y_1) = \begin{pmatrix} -1 - \ell'(y_1) \\ -\ell'(y_1) & 1 \end{pmatrix}. \]
\[ A(0) = \begin{pmatrix} -\frac{\Lambda}{\lambda + \gamma}(0) & 0 \\ 0 & -1 \end{pmatrix}, \]

\(a_{ij}\) are the elements of the matrix \(A\) introduced in (1.12) and \(\Phi(y', s, D')\) stands for some first order differential operator. Therefore

\[ (3.8) \quad I^{-1}(y_1)e^{s\varphi} \left( \frac{\partial_2}{\partial y_2} \right) = -I^{-1}(y_1)A(y_1)D^2_{y_1}v - iI^{-1}(y_1)A'(y_1)D_{y_1}v + I^{-1}(y_1)\tilde{\Phi}(y', s, D')(ge^{s\varphi}) - D^2_{y_2}v. \]

Using the definition of the operator \(\Gamma^+_{\lambda+2\mu}(y, s, D')\), we have

\[ (3.9) \quad \chi_\nu(s, D') \left( \frac{\partial_2}{\partial y_2} e^{s\varphi} \right) = \chi_\nu(s, D')iD_{y_2}w_2 = i\lambda_{y_2}w_{2,\nu} + i[\chi_\nu, D_{y_2}]z_2 \]

\[ = iV^+_{\lambda+2\mu}(y', 0) - [\chi_\nu, s\varphi_{y_2}]w_2 + i\alpha^+_{\lambda+2\mu}(y', 0, s, D')(b_{21}(y_1, D_{y_1})v_{1,\nu} + b_{22}(y_1, D_{y_1})v_{2,\nu} + [\chi_\nu, b_2(y, D)]v + \chi_\nu(\tilde{C}_2(y_1)ge^{s\varphi})). \]

Here we recall that \(b_{21}\) and \(b_{22}\) are defined by (1.16). Applying to the both sides of (3.8) the operator \(\chi_\nu\) and substituting (3.9) into (3.8), we can obtain

\[ (3.10) \quad D^2_{y_2}v_\nu = \bar{f} - iI^{-1}(y_1)D_{y_2}w_{1,\nu}e_1 + R(y', s, D')v_\nu, \]

where we recall that \(e_1 = (1, 0)\), and we set

\[ \bar{f} = -\chi_\nu I^{-1} \]

\[ \times \left( iV^+_{\lambda+2\mu}(y', 0) + i\alpha^+_{\lambda+2\mu}(y', 0, s, D')\chi_\nu(\tilde{C}_2(y_1)ge^{s\varphi}) - [\chi_\nu, s\varphi_{y_2}]w_2 + [\chi_\nu, b_2]v \right) \]

\[ + [\chi_\nu, -I^{-1}(y_1)A(y_1)D^2_{y_2}w - iI^{-1}(y_1)A'(y_1)D_{y_1}v + \chi_\nu(I^{-1}(y_1)\tilde{R}(y', s, D')(ge^{s\varphi})) - i[\chi_\nu, I^{-1}(y_1)D_{y_2}]w - [\chi_\nu, D^2_{y_2}]v, \]

\[ R(y', s, D')v_\nu = I^{-1}(y_1) \]

\[ \times \left( -i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{21}(y_1, D_{y_1}) - i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{22}(y_1, D_{y_1}) \right) v_\nu \]

\[ - I^{-1}(y_1)A(y_1)D^2_{y_1}v_\nu - I^{-1}(y_1)A'(y_1)D_{y_1}v_\nu, \]

\[ w_\nu = (w_{1,\nu}, w_{2,\nu}), \quad v_\nu = (v_{1,\nu}, v_{2,\nu}), \quad v_\nu = \chi_\nu(s, D')v. \]
By (2.14) and (3.10), we obtain

\[ \chi_\nu(s, D') \tilde{B}(y', D') \nu = \chi_\nu(s, D')(Q^{-1}e^{s\varphi} + \chi_\nu(s, D')(Q^{-1}e^{s\varphi}(\mathcal{R}_1(y, D), \mathcal{R}_2(y, D)))g \]

\[ - [\chi_\nu, D^2] y - \tilde{I} + iI^{-1}(y_1)D_{y_1}w_{1,\nu} \tilde{e}_1 - R(y', s, D')\nu_y \ y \in \partial G. \]

Next we note that

\[ \det (\tilde{B}(y^{*'}, \xi^{*'} + is^*\nabla'\varphi(y^{*'})) + R(y^{*'}, s^*, \xi^{*})) = \frac{\mu(0)}{(\lambda + 2\mu)(0)}(\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^4. \]

Really by (2.15)

\[ \tilde{B}(y^{*'}, \xi^{*'} + is^*\nabla'\varphi(y^{*'})) = \left( -\frac{(\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2 + (\mu(0) + 2\mu + \frac{\lambda(\lambda + 2\mu)}{2})(0)(\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2}{\mu(0)} \frac{0}{0} \frac{0}{(\lambda + 2\mu)(0)} \right). \]

Since \( r_\mu(\gamma) = 0 \) we have \((\xi_0^* + is^*\varphi_{y_0}(y^{*'}))^2 = \mu(0)(\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2\) which implies

\[ \tilde{B}(y^{*'}, \xi^{*'} + is^*\nabla'\varphi(y^{*'})) = (\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2 \left( \frac{2(\lambda + 2\mu)}{\lambda + 2\mu} \frac{0}{(\lambda + 2\mu)} \right) \]

and

\[ \alpha_{\lambda + 2\mu}(\gamma) = isign(\xi_1^*) \sqrt{\frac{\lambda + 2\mu}{\lambda + 2\mu} \frac{0}{(\lambda + 2\mu)}}(\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2. \]

Moreover by (1.16)

\[ \left( -is\alpha_{\lambda + 2\mu}^+(y^*, s^*, \xi^{*})b_{11}(y^*, \xi^* + is^*\nabla'\varphi(y^{*})) - is\alpha_{\lambda + 2\mu}^+(y^*, s^*, \xi^{*})b_{22}(y^*, \xi^* + is^*\nabla'\varphi(y^{*})) \right) \]

\[ = \left( isign(\xi_1^*) \sqrt{\frac{\lambda + 2\mu}{\lambda + 2\mu}} \frac{0}{\lambda + 2\mu} \frac{0}{\lambda + 2\mu} (\xi_1^* + is^*\varphi_{y_1}(y^{*'}))^2 \right) \frac{0}{0} \]

and

\[ I^{-1}(0)A(0) = \left( \frac{\lambda + 2\mu}{\lambda + 2\mu} \frac{0}{0} \frac{0}{-1} \right). \]

Thus
\[(3.14) \quad R(y^*, \zeta^*) = \begin{pmatrix} \frac{-\lambda_{m_n}^0(0)}{\alpha + 2\mu(0)} & \frac{-\lambda_{m_n}^0(0)}{\alpha + 2\mu(0)} \\ \frac{\alpha + 2\mu(0)}{\alpha + 2\mu(0)} & \frac{\alpha + 2\mu(0)}{\alpha + 2\mu(0)} \end{pmatrix} \left( \xi^* + is^*\varphi_{y_1}(y^*) \right)^2. \]

In terms of (3.13) and (3.14), we easily obtain (3.12). Set

\[S(y', s, \xi_0, \xi_1) = \tilde{B}(y', \xi' + is\nabla'\varphi(y)) + R(y', s, \xi').\]

Let \(S(y', s, D')\) be the corresponding pseudodifferential operator. Then

\[(3.15) \quad S(y', s, D')v + [\chi_{y'}, \tilde{B}(y, D)]v = \chi_{y'}(Q^{-1}e^{s\varphi} - [\chi_{y'}, D_{y_2}]v + \chi_{y'}(Q^{-1}e^{s\varphi}(\mathfrak{R}_1(y, D), \mathfrak{R}_2(y, D))g) - \tilde{f} + i\tau^{-1}(y_1)D_{y_2}w_{1,\nu}\tilde{e}_1.\]

Since we can directly verify that \(\det S(y', s^*, \xi'^*) \neq 0\), we have

\[(3.16) \quad \epsilon \left\| \left( \frac{\partial^2 v_{y'}}{\partial y_{2}^2} \right) \right\|^2_{H^{2,\infty}(\partial \Omega) \times H^{1,\infty}(\partial \Omega) \times L^2(\partial \Omega)} \leq C_5(s\|f e^{s\varphi}\|_{L^2(\partial \Omega)} + s\|g e^{s\varphi}\|_{H^{1,\infty}(\partial \Omega)} + s\|v\|_{H^{1,\infty}(\partial \Omega)}) + J_1 + s\|V^+_{y_1}\|_{L^2(\partial \Omega)}).\]

Note by (3.2) that for any \(\epsilon > 0\) there exists \(\delta_2(\epsilon) > 0\) such that

\[(3.17) \quad J_3 \leq \epsilon s \left\| \left( \frac{\partial^2 v_{y'}}{\partial y_{2}^2} \right) \right\|^2_{H^{2,\infty}(\partial \Omega) \times H^{1,\infty}(\partial \Omega) \times L^2(\partial \Omega)} + C_6(s\|g e^{s\varphi}\|_{H^{1,\infty}(\partial \Omega)} + \|v\|_{H^{1,\infty}(\partial \Omega)}), \quad \forall \zeta \in \mathcal{O}(\delta_2(\epsilon)).\]

In order to estimate the term \(J_2\), we consider two cases. First we assume that \(s^* \neq 0\). Then by (1.18)–(1.20) and \(r_\mu(\gamma) = 0\), for given \(\epsilon > 0\), there exists \(\delta_0\) such that if \(\delta \in (0, \delta_0)\), then

\[(3.18) \quad |\mu(0)\xi_1\varphi_{y_1}(y^*) - \xi_0\varphi_{y_1}(y^*)| \leq \epsilon|\zeta|, \quad \forall \zeta \in \mathcal{O}(\delta).\]

By this inequality, we obtain

\[(3.19) \quad J_2 \leq \epsilon s \left\| \left( \frac{\partial^2 v_{y'}}{\partial y_{2}^2} \right) \right\|^2_{H^{2,\infty}(\partial \Omega) \times H^{1,\infty}(\partial \Omega) \times L^2(\partial \Omega)} + C_7(s\|g e^{s\varphi}\|_{H^{1,\infty}(\partial \Omega)} + \|v\|_{H^{1,\infty}(\partial \Omega)}).\]

Second let us assume that \(s^* = 0\). We solve equation (3.15) with respect to the variable \(v_{y'}\):
where $S(y', s, D')^{-1}$ is the bounded operator from $L^2(\partial G)$ into $H^{2,\alpha}(\partial G)$ such that the principal symbol of the operator $S(y', s, D')^{-1}$ is the inverse matrix to the matrix $S(y', s, \xi_0, \xi_1)$ and $T \in \mathcal{L}(H^{1,\alpha}(\partial G), H^{2,\alpha}(\partial G))$. Therefore on $\partial G$ we have

\begin{equation}
(3.21)
\mu(0)\varphi_{y_1}(y^*) \frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(y^*) \frac{\partial w_{1,\nu}}{\partial y_0} = i(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})b_1(y_1, D') \times \{S(y', s, D')^{-1}(-[\chi_{y^*}, \tilde{B}(y, D)]\nu) \\
+ \chi_{y^*}(Q^{-1}fe^{s\nu}) - [\chi_{y^*}, D^2_{y_2}]\nu + \chi_{y^*}(Q^{-1}(\mathfrak{R}_1(y, D), \mathfrak{R}_2(y, D))\tilde{g}e^{s\nu}) \\
+ iI^{-1}(y_1)D_{y_2}w_{1,\nu} + T\} \\
+ i(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})\{[\chi_{y^*}, b_1]\nu - \chi_{y^*}(\tilde{e}_1\tilde{g}e^{s\nu})\}.
\end{equation}

By $\mathcal{M}$ we denote the pseudodifferential operator with the symbol

\begin{equation}
\mathcal{M}(y', s, \xi') = (\varphi_{y_0}(y^*)\xi_0 - \mu(0)\varphi_{y_1}(y^*)\xi_1)b_1(y', \xi' + i\delta \nabla'\varphi)S(y', s, \xi')^{-1}I^{-1}(y_1)\tilde{e}_1.
\end{equation}

Since $b_{11}(y^*, \xi') = 0$ and $b_{12}(y^*, \xi') = 2i\xi_1$, we have $\text{Re} \mathcal{M}(y^*, s, \xi_0^*, \xi_1^*) = 0$. Therefore, by Gårding’s inequality, we see

\begin{equation}
(3.22)
\text{Re} \int_{\partial G} \frac{\partial w_{1,\nu}}{\partial y_2} (\mu(0)\varphi_{y_1}(y^*)\partial_{y_1} - \varphi_{y_0}(y^*)\partial_{y_0})b_1(y', D')S(y', s, D')^{-1}iI^{-1}(y_1)\tilde{e}_1 \frac{\partial w_{1,\nu}}{\partial y_2} d\Sigma \\
\geq -\epsilon \left\| \left( \frac{\partial \nu_{y_1}}{\partial y_2}, \frac{\partial^2 \nu_{y_1}}{\partial y_2^2} \right) \right\|_{H^{2,\alpha}(\partial G) \times H^{1,\alpha}(\partial G) \times L^2(\partial G)}^2.
\end{equation}
On the other hand

\[s ||(\mu(0)\varphi_{y_1}(y^*)\partial_{y_1} - \varphi_{y_0}(y^*)\partial_{y_0})b_1(y', D')\times\{S(y', s, D')^{-1}(-\varphi_{y_1}(y^*)\partial_{y_1} + \tilde{\varphi}_{y_0}(Q^{-1}fe^{s\varphi})) + Tv\}\]

+ \chi_{\nu}(Q^{-1}(\mathfrak{R}_1(y, D), \mathfrak{R}_2(y, D))ge^{s\varphi}) + \bar{\tilde{\varphi}} + \chi_{\nu}(Q^{-1}fe^{s\varphi})) + TV\}

\[\leq s ||S(y', s, D')^{-1}(-\varphi_{y_1}(y^*)\partial_{y_1} + \tilde{\varphi}_{y_0}(Q^{-1}fe^{s\varphi})) + TV\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

\[\leq s ||(\mu(0)\varphi_{y_1}(y^*)D_{y_1} - \varphi_{y_0}(y^*)D_{y_0})([\chi_{\nu}, b_1][v - \chi_{\nu}(\tilde{\varphi}_{y_1}ge^{s\varphi})])\|_{L^2(\partial \Omega)}^2\]

Here we recall that \(\tilde{C}_{1}\) is defined in (1.13). Inequalities (3.22) and (3.23) imply

\[J_2 \leq \left\| \left( \frac{\partial \varphi_{y_1}}{\partial y_1}, \frac{\partial^2 \varphi_{y_1}}{\partial y_1^2} \right) \right\|_{H^{2, \nu}(\partial \Omega) \times H^{1, \nu}(\partial \Omega) \times L^2(\Omega)}^2

+ C_9(s\|v\|_{H^{1, \nu}(\partial \Omega)}^2 + ||fe^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)}^2 + s||ge^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)}^2).

By (3.5), (3.16), (3.17), (3.19) and (3.24), there exist constants \(C_{10} > 0\) and \(C_{11} > 0\) such that

\[\Sigma^{(1)}_{\mu}(w_{1, \nu}) \geq C_{10}^8 \left\| \left( \frac{\partial \varphi_{y_1}}{\partial y_1}, \frac{\partial^2 \varphi_{y_1}}{\partial y_1^2} \right) \right\|_{H^{2, \nu}(\partial \Omega) \times H^{1, \nu}(\partial \Omega) \times L^2(\Omega)}^2

- C_{11}(s\|v\|_{H^{1, \nu}(\partial \Omega)}^2 + ||fe^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)}^2 + s||ge^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)}^2 + ||w||_{H^{1, \nu}(\partial \Omega)}^2).

By (3.3) and (3.25), we obtain

\[\sqrt{s}\|w_{1, \nu}\|_{H^{1, \nu}(\partial \Omega)} + \sqrt{s} \left\| \left( \frac{\partial \varphi_{y_1}}{\partial y_1}, \frac{\partial^2 \varphi_{y_1}}{\partial y_1^2} \right) \right\|_{H^{2, \nu}(\partial \Omega) \times H^{1, \nu}(\partial \Omega) \times L^2(\Omega)}^2

\leq C_{12}(\sqrt{s}\|v\|_{H^{1, \nu}(\partial \Omega)} + ||fe^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)} + \sqrt{s}||ge^{s\varphi}\|_{H^{1, \nu}(\partial \Omega)} + ||w||_{H^{1, \nu}(\partial \Omega)}^2).\]
Note that
\[(3.28) \sqrt{s}\left\| w_{2,\nu} \right\|_{H^{1,\nu}(\partial \Omega) \times L^2(\partial \Omega)} \leq C_{13} \left( \left\| \left( v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \right) \right\|_{H^{2,\nu}(\partial \Omega) \times H^{1,\nu}(\partial \Omega) \times L^2(\partial \Omega)} + \sqrt{s} \| \psi \|_{H^{1,\nu}(\partial \Omega)} \right) \]
\leq C_{14} \sqrt{s} \| \psi \|_{H^{1,\nu}(\partial \Omega)} + \| f_{\nu} e^{s^2}\|_{H^{1,\nu}(\partial \Omega)} + \sqrt{s} \| g e^{s^2}\|_{H^{1,\nu}(\partial \Omega)} + \| w \|_{H^{1,\nu}(\partial \Omega)}.

By (3.27), (2.24) and (2.25) with \( \beta = \lambda + 2\mu \), we have
\[(3.28) \sqrt{s} \| w_{2,\nu} \|_{H^{1,\nu}(\partial \Omega)} \leq C_{15} \sqrt{s} \| \psi \|_{H^{1,\nu}(\partial \Omega)} + \| f_{\nu} e^{s^2}\|_{H^{1,\nu}(\partial \Omega)} + \sqrt{s} \| g e^{s^2}\|_{H^{1,\nu}(\partial \Omega)} + \| w \|_{H^{1,\nu}(\partial \Omega)}.

Therefore combining (3.26) and (3.28), we obtain
\[
\sqrt{s} \| w_{2,\nu} \|_{H^{1,\nu}(\partial \Omega)} + \sqrt{s} \| \left( v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \right) \|_{H^{2,\nu}(\partial \Omega) \times H^{1,\nu}(\partial \Omega) \times L^2(\partial \Omega)} \leq C_{16} \sqrt{s} \| \psi \|_{H^{1,\nu}(\partial \Omega)} + \| f_{\nu} e^{s^2}\|_{H^{1,\nu}(\partial \Omega)} + \sqrt{s} \| g e^{s^2}\|_{H^{1,\nu}(\partial \Omega)},
\forall s \geq s_0(r, N), N \geq N_1.
\]
This inequality and Proposition 2.1 imply (3.1). Thus the proof of Lemma 3.1 is completed.

\[\Box\]

\section{Case \( r_{\lambda+2\mu}(\gamma) = 0 \)}

In this section, we will prove

\textbf{Lemma 4.1.} Let \( \gamma = (y^*, \xi^*) \in \partial \Omega \times S^2 \) be a point such that \( r_{\lambda+2\mu}(\gamma) = 0 \) and \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \). Then for all sufficiently small \( \delta_1 > 0 \), estimate (3.1) holds true.

\textbf{Proof.} By (1.19) and (1.20), there exist \( \delta_0 > 0 \) and \( C_1 > 0 \) such that for all \( \delta_1 \in (0, \delta_0) \) we have
\[(4.1) \xi^2 \leq C_1 (\xi^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1).
\]
We note that if \( r_\mu(\gamma) = 0 \), then \( \xi^2 = \xi^1 = 0 \), \( s^1 = 1 \) and \( \varphi_{g_\mu}(y^*) = \varphi_{g_1}(y^*) = 0 \). By (1.6) this is impossible. Therefore \( r_\mu(\gamma) \neq 0 \) must hold true.

\textbf{Case A.} Assume that \( s^1 = 0 \) and \( \varphi_{g_\mu}(y^*) > \frac{\left| \xi e_{g_\mu}(y^*) - \xi \xi e_\mu(y^*) \right|}{\sqrt{\lambda_\mu(\gamma)[\zeta^2]}} \).

Then by decreasing the parameter \( \delta_1 \), we can assume that for some constant \( C_2 > 0 \)
\[(4.2) \xi^2 + s^2 \leq C_2 \xi^2, \quad \forall \zeta \in \mathcal{O}(\delta_1).
\]
Then there exists a constant $C_3 > 0$ such that

$$- \text{Im} \Gamma_\mu^+(y, \zeta) \geq C_3 |s|, \quad \forall (y, \zeta) \in B_{\delta_1}(y^*) \times \mathcal{O}(\delta_1),$$

provided that $|\delta_1| + |\delta_1|$ is sufficiently small. We set $V_\mu^\pm = (D_{y_2} - \Gamma_\mu^+(y, s, D')) w_{1,\nu}$. Then we can represent $P_{\mu,s}$ as

$$P_{\mu,s}(y, s, D) w_{1,\nu} = \mu|G|(D_{y_2} - \Gamma_\mu^+(y, s, D')) V_\mu^\pm + T_\mu^\pm w_{1,\nu},$$

where $T_\mu^\pm \in \mathcal{L}(H^{1+s}(\mathcal{G}), L^2(\mathcal{G}))$. This decomposition and Proposition 2.3 immediately imply

$$\|\sqrt{s}(D_{y_2} - \Gamma_\mu^+(y, s, D')) w_{1,\nu}|_{y_2=0}\|_{L_2(\partial \mathcal{G})} \leq C_4 (\|P_{\mu,s} w_{1,\nu}\|_{L^2(\mathcal{G})} + \|w\|_{H^{1+s}(\mathcal{G})}).$$

This inequality implies

$$\sqrt{s}\|D_{y_2} w_{1,\nu}\|_{L^2(\partial \mathcal{G})} + \sqrt{\delta}\|w_{1,\nu}\|_{H^{1+s}(\partial \mathcal{G})} \leq C_5 (\|P_{\mu,s} w_{1,\nu}\|_{L^2(\mathcal{G})} + \|w\|_{H^{1+s}(\mathcal{G})}).$$

Next we need the estimate for $(v_\nu, \frac{\partial v_\nu}{\partial y_2}, \frac{\partial^2 v_\nu}{\partial y_2^2})$. We may rewrite equations (2.8) and (2.9) as

$$-D_{y_0}^2 v_1 - (\lambda + 2\mu)(iD_{y_1} - i\ell'(y_1)D_{y_2}) w_2 + \mu iD_{y_2} w_1 + \tilde{K}_7(y, D)v = f_1 e^{s\varphi},$$

$$-D_{y_0}^2 v_2 - (\lambda + 2\mu)(iD_{y_1} - i\ell'(y_1)D_{y_2}) w_1 + \mu iD_{y_2} w_2 + \tilde{K}_8(y, D)v = f_2 e^{s\varphi},$$

where $\tilde{K}_7$ and $\tilde{K}_8$ are first order differential operators. Furthermore, setting

$$q_1 = f_1 e^{s\varphi} - \mu iD_{y_2} w_1 - \tilde{K}_7(y, D)v,$$

$$q_2 = f_2 e^{s\varphi} + \mu iD_{y_2} w_2 - \tilde{K}_8(y, D)v,$$

we rewrite (4.5) and (4.6) as

$$\begin{cases} -D_{y_0}^2 v_1 - (\lambda + 2\mu)(iD_{y_1} - i\ell'(y_1)D_{y_2}) w_2 = q_1, \\ -D_{y_0}^2 v_2 - (\lambda + 2\mu)(iD_{y_1} - i\ell'(y_1)D_{y_2}) w_1 = q_2. \end{cases}$$

Using (4.7) we get rid of the term $D_{y_2} w_2$ in (4.6):

$$-D_{y_0}^2 v_1 - (\lambda + 2\mu)(iD_{y_1} b_{21}(y_1, D_{y_2}) v_1 + b_{22}(y_1, D_{y_1}) v_2) - \ell'(y_1) D_{y_0}^2 v_2 = q_1 + \ell'(y_1) q_2 + \tilde{K}_9(y', D') g.$$

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Using (1.13), we obtain

\begin{equation}
(4.9)
\begin{align*}
D_{y_1}b_{11}(y', D_{y_1})v_1 + D_{y_1}b_{12}(y', D_{y_1})v_2 = D_{y_1}w_1|_{y_2=0} + \tilde{K}_{10}(y', D')(ge^{s^*}y).
\end{align*}
\end{equation}

Here and henceforth \(\tilde{K}_9\) and \(\tilde{K}_{10}\) are first order differential operators. Set

\begin{equation}
(4.10)
\begin{align*}
\mathcal{K}(y', D') &= 
\begin{pmatrix}
-D_{y_0}^2 - (\lambda + 2\mu) \xi D_{y_1}b_{21}(y', D_{y_1}) & -(\lambda + 2\mu) \xi D_{y_1}b_{22}(y', D_{y_1}) - \ell'(y_1) D_{y_1}^2 \\
D_{y_1}b_{11}(y', D_{y_1}) & D_{y_1}b_{12}(y', D_{y_1})
\end{pmatrix}
\end{align*}
\end{equation}

By (4.8) and (4.9), we have

\begin{equation}
\mathcal{K}(y', D')v = m,
\end{equation}

where \(m = (q_1 + \ell'(y_1)q_2 + \tilde{K}_9(y', D')g, D_{y_1}w_1|_{y_2=0} + \tilde{K}_{10}(y', D')(ge^{s^*}y))\). Therefore

\begin{equation}
\mathcal{K}(y', D')v_{\nu} = \chi_{\nu}(s, D')m - [\chi_{\nu}(s, D'), \mathcal{K}]v,
\end{equation}

and since \(\det \mathcal{K}(0, \zeta^*) \neq 0\) if \(\lambda(0) \neq 0\), we have

\begin{equation}
v_{\nu} = \mathcal{K}^{-1}(y', D')(\chi_{\nu}(s, D')m - [\chi_{\nu}(s, D'), \mathcal{K}]v) + T(y', s, D')v_{\nu},
\end{equation}

where \(T \in \mathcal{L}(H^{1,s}(\mathbb{R}^2), H^{2,s}(\mathbb{R}^2))\). Hence

\begin{equation}
(4.10)
\begin{align*}
\sqrt{s} \left\| \begin{pmatrix} v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \end{pmatrix} \right\|_{H^{2,s}((\partial \Omega))^2 \times L^2(\partial \Omega)} &
\leq C_0 \left( \sqrt{s} \| v \|_{H^{1,s}(\partial \Omega)} + \| fe^{s^*}y \|_{H^{1,s}(\partial \Omega)} + \sqrt{s} \| ge^{s^*}y \|_{H^{1,s}(\partial \Omega)} 
+ \sqrt{s} \left\| \begin{pmatrix} \frac{\partial \xi_{1,\nu}}{\partial y_2}, w_{1,\nu} \end{pmatrix} \right\|_{L^2(\partial \Omega) \times H^{1,s}(\partial \Omega)} \right).
\end{align*}
\end{equation}

By (2.25) with \(\beta = \lambda + 2\mu\) and \(\beta = 2\mu\), we obtain

\begin{equation}
(4.11)
\begin{align*}
s \| w_{\nu} \|_{H^{1,s}(\partial \Omega)}^2 + s \left( \begin{pmatrix} v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \end{pmatrix} \right) &
\leq C_7 (s \| v \|_{H^{1,s}(\partial \Omega)}^2 + \| w \|_{H^{1,s}(\partial \Omega)}^2 + \| fe^{s^*}y \|_{H^{1,s}(\partial \Omega)}^2 + \| ge^{s^*}y \|_{H^{1,s}(\partial \Omega)}^2).
\end{align*}
\end{equation}

Applying Proposition 2.1 we obtain (3.1).
Next we need to consider two more cases. Using the definition of the operator $\Gamma_{\lambda+2\mu}(y, s, D')$, we have

\begin{equation}
\chi_{\nu}(s, D') \left( \frac{\partial_2}{\partial y_2} e^{s\varphi} \right) = i V^{\nu +}_{\mu}(y', 0) - [\chi_{\nu}, s\varphi_{y_2}] w_1 \\
+ i\alpha^{\nu}_{\mu}(y', 0, s, D') (b_{11}(y', D_{y_1}) v_{1,\nu} + b_{12}(y', D_{y_1}) v_{2,\nu} \\
+ [\chi_{\nu}, b_{1}] v + \chi_{\nu} (\tilde{C}(y_1) g e^{s\varphi})).
\end{equation}

Here we recall that the operators $b_{21}$ and $b_{22}$ are defined by (1.16). Substituting (4.12) into (3.8), we can obtain

\begin{equation}
D_{y_2}^2 v_{\nu} = \tilde{F}_0 - i I^{-1}(y_1) D_{y_2} w_{2,\nu} \tilde{e}_2 + \tilde{R}(y', s, D') v_{\nu},
\end{equation}

where we recall that $\tilde{e}_2 = (0, 1)$, and we set

\[ \tilde{F}_0 = \]

\[ \chi_{\nu} I^{-1}(y_1) \left( i V^{\nu +}_{\mu}(y', 0) + i\alpha^{\nu}_{\mu}(y', 0, s, D') \tilde{C}(y_1) g e^{s\varphi} - [\chi_{\nu}, s\varphi_{y_2}] w_1 + [\chi_{\nu}, b_{1}] v \right) \]

\[ + [\chi_{\nu}, I^{-1}A(y_1) D_{y_1}^2 - I^{-1}(y_1) A(y_1) D_{y_1}] v \]

\[ + \chi_{\nu} (I^{-1}(y_1) \tilde{R}(y', s, D) g e^{s\varphi} - i[\chi_{\nu}, I^{-1}D_{y_2}] w_1 \tilde{e}_1 - [\chi_{\nu}, D_{y_2}^2] v, \]

and

\[ \tilde{R}(y', s, D') v_{\nu} = I^{-1}(y_1) \left( -i\alpha^{\nu}_{\mu}(y', 0, s, D') b_{11}(y', D_{y_1}) - i\alpha^{\nu}_{\mu}(y', 0, s, D') b_{12}(y', D_{y_1}) \right) v_{\nu} \]

\[ - I^{-1}(y_1) A(y_1) D_{y_1}^2 v_{\nu} - I^{-1}(y_1) A(y_1) D_{y_1} v_{\nu}. \]

By (2.14) and (4.13), we obtain

\begin{equation}
\chi_{\nu}(s, D') \tilde{B}(y', D') v = \chi_{\nu}(s, D') (Q^{-1} f e^{s\varphi}) \\
+ \chi_{\nu} (Q^{-1}(\mathcal{R}_{1}(y, D), \mathcal{R}_{2}(y, D)) g e^{s\varphi}) \\
- [\chi_{\nu}, D_{y_2}^2] v - (\tilde{F}_0 - i I^{-1}(y_1) D_{y_2} w_{2,\nu} \tilde{e}_2 + \tilde{R}(y', s, D') v_{\nu}).
\end{equation}

Next we note that for the principal symbol of the operator $\Pi(y', s, D') = \tilde{B}(y', s, D') + \tilde{R}(y', s, D')$ we have

\begin{equation}
\Pi(\gamma) = \Pi(\gamma) = \left( \frac{-\lambda}{\mu}(0)(\xi_1^*)^2 - 2 \sqrt{\frac{\lambda+\mu}{\mu}(0)\xi_1^*|\xi_1|} \right),
\end{equation}
where \( I = \text{sign}(\xi_0^* \varphi_{y_0}(y^*) - \mu(0)\xi_1^* \varphi_{y_1}(y^*)) \). Really, since \( b_{11}(y^*, \xi_1^* + i s^* \varphi_{y_1}(y^*)) = 0 \) and \( b_{12}(y^*, \xi_1^* + i s^* \varphi_{y_1}(y^*)) = 2i \) we have

\[
-I^{-1}(y^*)i(\alpha_\mu(\gamma)w_1) = \begin{pmatrix}
0
-2\sqrt{\frac{\lambda_\mu(0)}{\mu}}(0)I
\end{pmatrix} \xi_1^* \xi_1^*
\]

and

\[
(-I^{-1}(0)A(0)\xi_1^2 + \tilde{B}(\gamma)) = (\xi_1^2)^2 \begin{pmatrix}
\frac{\lambda}{\lambda + 2\mu}(0) & 0 \\
0 & 1
\end{pmatrix} + \begin{pmatrix}
\frac{\lambda(\lambda + \mu)}{(\lambda + 2\mu)^2}(0) & 0 \\
0 & -2(\lambda + \mu)(0)
\end{pmatrix}
\]

\[
= (\xi_1^2)^2 \begin{pmatrix}
\frac{\lambda}{\mu}(0) & 0 \\
0 & \frac{-\lambda}{\lambda + 2\mu}(0)
\end{pmatrix}
\]

By (4.16) and (4.17), formula (4.15) follows immediately.

**Case B.** Assume that \( s^* = 0 \) and

\[
\varphi_{y_2}(y^*) \leq \frac{|\frac{1}{\mu(0)} \xi_0^* \varphi_{y_0}(y^*) - \xi_1^* \varphi_{y_1}(y^*)|}{\sqrt{\frac{\lambda_\mu(0)}{\mu}}(0)\xi_1^*}.
\]

Note that in this case we still have (4.2). Then, since \( s^* = 0 \), we see that \( \text{Re} r_\mu(y^*, \zeta^*) > 0 \). Hence

\[
\Gamma^+_{\mu}(y^*, \zeta^*) = I|\text{Re} r_\mu(y^*, \zeta^*)|, \quad I = \text{sign}(\xi_0^* \varphi_{y_0}(y^*) - \mu(0)\xi_1^* \varphi_{y_1}(y^*)).\]

Therefore

\[
\Gamma^+_{\mu}(y^*, \zeta^*)(\mu(0)\varphi_{y_2}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*) = \sqrt{\text{Re} r_\mu(y^*, \zeta^*)}\left((\mu(0)\varphi_{y_2}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*)\right) < 0.
\]

We note that by (4.15)

\[
w_{1, \nu} = R_1(y', s, D')\frac{\partial w_2}{\partial y_2} + q_3, \quad w_{2, \nu} = R_2(y', s, D')\frac{\partial w_2}{\partial y_2} + q_4,
\]

where

\[
R_1(\gamma) = -\frac{2i\lambda(0)(0)}{(\lambda + 2\mu)^2(0)}(y^*), \quad R_2(\gamma) = \frac{2i\lambda(0)I}{(\lambda + 2\mu)(0)|\xi_1^*|}(y^*)
\]
and

\begin{equation}
\sqrt{s} \| (q_1, q_2) \|_{H^1(\partial G)} \leq C \sqrt{s} \| v \|_{H^1(\partial G)} + \| w \|_{H^1(\partial G)} + \| \mathbf{f} e^{\sigma \mathbf{r}} \|_{H^1(\partial G)} \| e^{\sigma \mathbf{r}} \|_{H^1(\partial G)}.
\end{equation}

Consider two cases. First let

\[ (\varphi_{y_1}(y^*) \xi^*_1) \Gamma^*_\mu (y^*, \zeta^*) > 0. \]

This inequality and (4.18) yield that \( |\xi^*_1 \varphi_{y_0}(y^*)| > |\xi^*_1 \varphi_{y_1}(y^*)| \). If \( |\xi^*_1 \varphi_{y_0}(y^*)| > 0 \), then \( \Gamma^*_\mu (y^*, \zeta^*) = |\sqrt{r_\mu(\gamma)}| \) and \( |\xi^*_1 \varphi_{y_1}(y^*)| > 0 \). Hence by (1.6)

\[ \varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*) \xi^*_1 - \frac{\varphi_{y_0}(y^*) \xi^*_0}{\mu(0)}}{\sqrt{\lambda + \mu(0)}} |\xi^*_1| = \frac{-\varphi_{y_1}(y^*) \xi^*_1 + \frac{\varphi_{y_0}(y^*) \xi^*_0}{\mu(0)}}{\sqrt{\lambda + \mu(0)}} |\xi^*_1|. \]

This contradicts (4.18). If \( |\xi^*_1 \varphi_{y_0}(y^*)| < 0 \), then \( \Gamma^*_\mu (y^*, \zeta^*) = -|\sqrt{r_\mu(\gamma)}| \) and \( \xi^*_1 \varphi_{y_1}(y^*) < 0 \). Therefore

\[ \varphi_{y_2}(y^*) > \frac{|\varphi_{y_1}(y^*) \xi^*_1 - \frac{\varphi_{y_0}(y^*) \xi^*_0}{\mu(0)}}{\sqrt{\lambda + \mu(0)}} |\xi^*_1| = \frac{-\varphi_{y_1}(y^*) \xi^*_1 + \frac{\varphi_{y_0}(y^*) \xi^*_0}{\mu(0)}}{\sqrt{\lambda + \mu(0)}} |\xi^*_1|. \]

By (1.6) this again contradicts (4.18). As the second case, one has to consider \( (\varphi_{y_1}(y^*) \xi^*_1) \Gamma^*_\mu (y^*, \zeta^*) < 0 \). Similarly to (3.4), we note that \( \Sigma_{\lambda+2\mu}^{(1)} \) can be written in the form

\begin{equation}
\Sigma_{\lambda+2\mu}^{(1)}(w_{2,\nu})
\end{equation}

\[ = \int_{\partial G} s(\lambda + 2\mu)^2(0) \varphi_{y_2}(y^*) \left( \frac{\partial w_{2,\nu}}{\partial y_2} \right)^2 + s^3(\lambda + 2\mu)^2(0) \varphi_{y_2}(y^*) w_{2,\nu}^2 \right) d\Sigma \]

\begin{equation}
+ \operatorname{Re} \int_{\partial G} 2s(\lambda + 2\mu)(0) \frac{\partial w_{2,\nu}}{\partial y_2} \left( (\lambda + 2\mu)(0) \varphi_{y_1}(y^*) \frac{\partial w_{2,\nu}^2}{\partial y_1} - \varphi_{y_0}(y^*) \frac{\partial w_{2,\nu}}{\partial y_0} \right) d\Sigma \]

\begin{equation}
+ \int_{\partial G} s(\lambda + 2\mu)(0) \varphi_{y_2}(y^*) \left( \xi^*_0 - (\lambda + 2\mu)(0) \xi^*_1 - s^2 \varphi_{y_0}(y^*) \\
+ s^2(\lambda + 2\mu)(0) \varphi_{y_2}(y^*) \right) |\hat{w}_{2,\nu}|^2 d\Sigma \]

\[ = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3. \]

Using (4.20) and (4.22), we can transform \( \tilde{J}_2 \) as

\begin{equation}
\tilde{J}_2 = \operatorname{Re} \int_{\partial G} 2s(\lambda + 2\mu)(0) \frac{\partial w_{2,\nu}}{\partial y_2} (i) \times \{ (\lambda + 2\mu)(0) \varphi_{y_1}(y^*) D_{y_1} - \varphi_{y_0}(y^*) D_{y_0} \} R_2(y, s, D^* \frac{\partial w_{2,\nu}}{\partial y_2} d\Sigma + I_3, \}
\end{equation}
where

\[ |I_3| \leq \epsilon s \left\| \left( \frac{\partial \nu}{\partial y_2}, \frac{\partial^2 \nu}{\partial y_2^2} \right) \right\|_{L^2(\partial G) \times H^{1,+}(\partial G) \times L^2(\partial G)}^2 + C_9 \left( ||fe^{s\varphi}||^2_{H^{1,+}(\partial G)} + s||ge^{s\varphi}||^2_{H^{1,+}(\partial G)} \right). \]

Note that for the symbol of the operator we have

\[ (4.25) \quad (-i) \{ (\lambda + 2\mu)(0)\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^* \} R_2(\gamma)^* \]
\[ = (-i) \{ (\lambda + 2\mu)(0)\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^* \} \frac{-2i\lambda(0)I}{(\lambda + 2\mu)(0)|\xi_1^*|} \]
\[ = - \{ (\lambda + 2\mu)(0)\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^* \} \frac{2\lambda(0)I}{(\lambda + 2\mu)(0)|\xi_1^*|} \]
\[ = - (\lambda + \mu)(0)(\varphi_{y_1}(y^*)\xi_1^* - \varphi_{y_0}(y^*)\xi_0^*) \Gamma^{(0)}_y (y^*, \xi_1^*) > 0. \]

By (4.25), (4.24) and Gårding’s inequality, we obtain

\[ (4.26) \quad \tilde{J}_2 \geq C_{10}s \left\| \left( \frac{\partial u_{1,\nu}}{\partial y_2}, u_{1,\nu}, w_{2,\nu} \right) \right\|_{L^2(\partial G) \times H^{1,+}(\partial G) \times L^2(\partial G)}^2 \]
\[ - C_{11}(||fe^{s\varphi}||^2_{H^{1,+}(\partial G)} + ||v||^2_{H^{1,+}(\partial G)}) \]
\[ - \epsilon s \left\| \left( \nu_{\nu}, \frac{\partial \nu}{\partial y_2}, \frac{\partial^2 \nu}{\partial y_2^2} \right) \right\|_{H^{1,+}(\partial G) \times H^{1,+}(\partial G) \times L^2(\partial G)}^2. \]

Now we estimate \( \tilde{J}_3. \) By (1.18)–(1.20) there exists a constant \( C_{12} > 0 \) such that

\[ (4.27) \quad |\xi_0^2 - s^2\varphi_{y_0}(y^*) - (\lambda + 2\mu)(0)\xi_1^2 + s^2(\lambda + 2\mu)(0)\varphi_{y_1}(y^*)| \]
\[ \leq C_{12}\delta_1 (\xi_0^2 + \xi_1^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1), \]

which yields

\[ (4.28) \quad |\tilde{J}_3| \leq C_{13}\epsilon(\delta_1) \left\| \left( \frac{\partial u_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|_{L^2(\partial G) \times H^{1,+}(\partial G)}^2, \]

where \( \epsilon(\delta_1) \to 0 \) as \( \delta_1 \to 0. \) By (4.26) and (4.28), we can choose constants \( C_{14}, C_{15} > 0 \) such that

\[ \Sigma_{\lambda + 2\mu}(u_{2,\nu}) \geq C_{14}s \left\| \left( \nu_{\nu}, \frac{\partial \nu}{\partial y_2}, \frac{\partial^2 \nu}{\partial y_2^2} \right) \right\|_{H^{1,+}(\partial G) \times H^{1,+}(\partial G) \times L^2(\partial G)}^2 \]
\[ - C_{15}(||fe^{s\varphi}||^2_{H^{1,+}(\partial G)} + s||ge^{s\varphi}||^2_{H^{1,+}(\partial G)} + ||v||^2_{H^{1,+}(\partial G)})]. \]
This inequality immediately implies (4.11). Applying Proposition 2.1 we obtain (3.1).

**Case C.** Assume that \( s^* \neq 0 \). If \( \delta_1 > 0 \) is small enough, then there exists a constant \( C_{16} > 0 \) such that

\[
|\xi_0 \varphi_{y_1}(y^*) - (\lambda + 2\mu)(0)\xi_1 \varphi_{y_1}(y^*)|^2 \leq \delta_1^2 C_{16}(\xi_1^2 + s^2).
\]

By (2.24) and (2.25), there exist constants \( C_{17} > 0 \) and \( C_{18} > 0 \) such that

\[
\Sigma^{(1)}_{\lambda + 2\mu}(w_{2,\nu}) + C_{17} s\|w_{2,\nu}\|_{H^{11}(\bar{G})}^2 \leq C_{18}(\|P_{\lambda + 2\mu, s}w_2\|_{L^2(\bar{G})}^2 + \|v\|_{H^{21}(\bar{G})}^2)
\]

\[
+ \epsilon \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \|_{L^2(\partial \bar{G}) \times H^{11}(\partial \bar{G})},
\]

where \( \epsilon(\delta) \to 0 \) as \( \delta \to +0 \). By (4.27) and (4.29), we have

\[
|\tilde{J}_2 + \tilde{J}_3| \leq C_{19} \epsilon(\delta_1) \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \|_{L^2(\partial \bar{G}) \times H^{11}(\partial \bar{G})},
\]

where \( \epsilon(\delta_1) \to 0 \) as \( \delta_1 \to 0 \). By (4.31) we obtain from (4.23) that there exists a constant \( C_{20} > 0 \) such that

\[
\Sigma^{(1)}_{\lambda + 2\mu}(w_{2,\nu}) \geq C_{20} \int_{\partial \Sigma} \left\{ s(\lambda + 2\mu)^2(0)\varphi_{y_2}(y^*) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2
\]

\[
+ s^3(\lambda + 2\mu)^2(0)\varphi_{y_2}^3 (y^*)|w_{2,\nu}|^2 \right\} d\Sigma
\]

\[
- \epsilon \left( \frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \|_{L^2(\partial \bar{G}) \times H^{11}(\partial \bar{G})}.
\]

In terms of (1.9), we can represent

\[
D_{y_0}^2 v_{2,\nu} - 2\mu D_{y_1}^2 v_{2,\nu} + T_5 v_{\nu} + T_6 v + T_7 (f e^{\phi}, g e^{\psi})
\]

\[
= (\lambda + 2\mu)D_{y_2} w_{2,\nu} + \chi_\nu(s, D')(f_2 e^{\psi}), \quad y \in \partial G,
\]

where we have estimates:

\[
\|T_5 v_{\nu}\|_{L^2(\partial \bar{G})} \leq \epsilon(\delta)\|v_{\nu}\|_{H^{21}(\partial \bar{G})}, \quad \|T_6 v\|_{L^2(\partial \bar{G})} \leq C_{21}\|v\|_{H^{11}(\partial \bar{G})},
\]

\[
\|T_7 (f e^{\phi}, g e^{\psi})\|_{L^2(\partial \bar{G})} \leq C_{21}\|(f e^{\phi}, g e^{\psi})\|_{L^2(\partial \bar{G}) \times H^{11}(\partial \bar{G})},
\]
where \( \epsilon(\delta) \to +0 \) as \( \delta \to +0 \). On the other hand, by \( r_{\lambda+2\mu}(\gamma) = 0 \), we have

\[
(\xi_0^* + i\varphi_{y_0}(y^*)s^*)^2 - (\lambda + 2\mu)(\xi_1^* + i\varphi_{y_1}(y^*)s^*)^2 = 0,
\]

that is,

\[
(\xi_0^* + i\varphi_{y_0}(y^*)s^*)^2 - 2\mu(0)(\xi_1^* + i\varphi_{y_1}(y^*)s^*)^2 = \lambda(0)(\xi_1^* + i\varphi_{y_1}(y^*)s^*)^2 \neq 0
\]

by the final condition in (1.3). Therefore, from (4.1) and (4.33) we obtain

\[
\|\sqrt{s}v_{2,\nu}\|_{H^{2,\nu}(\Omega)} \leq C_{22} \left\{ \left\| \sqrt{s} \frac{\partial v_{2,\nu}}{\partial y_2} \right\|_{L^2(\Omega)} + \|\sqrt{s}v\|_{H^{1,\nu}(\Omega)} + \|\sqrt{s}f e^{\delta\phi}\|_{L^2(\Omega)} \right\}
+ \epsilon\sqrt{s}\left( \|v_{\nu,\nu}\| + \|v^{\delta\nu}\|_{L^2(\Omega)} \right)
+ \epsilon(\delta, \delta_1)\sqrt{s}\left( \|v_{\nu,\nu}\| + \|v^{\delta\nu}\|_{L^2(\Omega)} \right).
\]

Thanks to (1.12), (1.15) and the fact that \( \xi_1^* + is^\nu \varphi_{y_1}(y^*) \neq 0 \), we have

\[
\sqrt{s}\|v_{2,\nu}\|_{H^{2,\nu}(\Omega)} \leq C_{23} \left\{ \left\| \sqrt{s} \frac{\partial v_{2,\nu}}{\partial y_2} \right\|_{L^2(\Omega)} + \|\sqrt{s}v\|_{H^{1,\nu}(\Omega)} + \|\sqrt{s}f e^{\delta\phi}\|_{H^{1,\nu}(\Omega)} + \sqrt{s}\|\sqrt{s}g\|_{H^{1,\nu}(\Omega)} \right\}
+ \epsilon(\delta, \delta_1)\sqrt{s}\left( \|v_{\nu,\nu}\| + \|v^{\delta\nu}\|_{L^2(\Omega)} \right).
\]

Consequently (4.34), (4.35), (1.12), (1.8) and (1.9) imply

\[
\sqrt{s}\left( \|v_{\nu,\nu}\| + \|v^{\delta\nu}\|_{L^2(\Omega)} \right) \leq C_{24} \left\{ \left\| \sqrt{s} \frac{\partial v_{1,\nu}}{\partial y_2} \right\|_{L^2(\Omega)} \right\}
+ \sqrt{s}\|v\|_{H^{1,\nu}(\Omega)} + \|e^{\delta\phi}\|_{H^{1,\nu}(\Omega)} + \sqrt{s}\|\sqrt{s}g\|_{H^{1,\nu}(\Omega)} \right\}.
\]

By (4.32) and (4.36)

\[
\sum_{\lambda+2\mu}(w_{1,\nu}) \geq C_{25}s \left( \|v_{\nu,\nu}\| + \|v^{\delta\nu}\|_{L^2(\Omega)} \right) - C_{25}(s\|v\|_{H^{1,\nu}(\Omega)} + \|e^{\delta\phi}\|_{H^{1,\nu}(\Omega)} + s\|\sqrt{s}g\|_{H^{1,\nu}(\Omega)}).}
\]
Hence, from this inequality and (2.25) with \( \beta = \lambda + 2\mu \), we obtain
\[
(4.37) \quad \sqrt{s} \left\| \left( \psi_{v,\gamma}, \frac{\partial \psi_{v,\gamma}}{\partial y_2}, \frac{\partial^2 \psi_{v,\gamma}}{\partial y_2^2} \right) \right\|_{H^{2,\gamma}(\partial G) \times H^{1,\gamma}(\partial G) \times L^2(\partial G)} + \sqrt{s} \| w_{2,v} \|_{H^{1,\gamma}(G)} \\
\leq C_{26}(\sqrt{s}) \| v \|_{H^{1,\gamma}(G)} + \| fe^{\psi \varphi} \|_{H^{1,\gamma}(G)} + \sqrt{s} \| ge^{\psi \varphi} \|_{H^{1,\gamma}(G)}.
\]
By (4.37) and (2.25) with \( \beta = \mu \), we have
\[
\sqrt{s} \left\| \left( \psi_{v,\gamma}, \frac{\partial \psi_{v,\gamma}}{\partial y_2}, \frac{\partial^2 \psi_{v,\gamma}}{\partial y_2^2} \right) \right\|_{H^{2,\gamma}(\partial G) \times H^{1,\gamma}(\partial G) \times L^2(\partial G)} + \sqrt{s} \| w_{2,v} \|_{H^{1,\gamma}(G)} \\
\leq C_{27}(\sqrt{s}) \| v \|_{H^{1,\gamma}(G)} + \| fe^{\psi \varphi} \|_{H^{1,\gamma}(G)} + \sqrt{s} \| ge^{\psi \varphi} \|_{H^{1,\gamma}(G)}.
\]
Finally using Proposition 2.1, we obtain (3.1). The proof of Lemma 4.1 is completed.

\[\square\]

\section{Case \( r_\mu(\gamma) \neq 0 \) and \( r_{\lambda+2\mu}(\gamma) \neq 0 \)}

In this section, we consider the case where
\[
(5.1) \quad |r_\mu(y,\zeta^\ast)| \neq 0 \quad \text{and} \quad |r_{\lambda+2\mu}(y^\ast,\zeta^\ast)| \neq 0.
\]
Denote
\[
H^{2,\gamma}(\partial G) = \{ u(y_0, y_1) \in L^2(\mathbb{R}^2) ; \| u \|_{H^{2,\gamma}(\partial G)}^2 = \int_{\mathbb{R}^2} (1 + s^3 + |\xi_0|^3 + |\xi_1|^3)|\hat{u}|^2 d\xi_0 d\xi_1 < \infty \},
\]
where we set \( \hat{u}(\xi_0, \xi_1) = \int_{\mathbb{R}^2} u(y_0, y_1) e^{-i(\xi_0 y_0 + \xi_1 y_1)} dy_0 dy_1 \). We have

\textbf{Lemma 5.1.} \quad Let (5.1) hold at \( \gamma \equiv (y^\ast, \zeta^\ast) \) and let \( \text{supp} \chi_\nu \subset \mathcal{O}(\delta_1) \) where \( \delta_1 \) is a sufficiently small positive number. If \( \zeta^\ast \in \Psi_2 \), then we have
\[
(5.2) \quad \| v_{\nu} \|_{H^2(\partial G)} \leq C_1 \left\{ \frac{1}{s^{1/2}} \left( \| \psi_{\nu,\gamma} \|_{H^1(\partial G)} + \| v \|_{H^1(\partial G)} \right) \\
+ s^{1/2} \| ge^{\psi \varphi} \|_{H^1(\partial G)} + \frac{1}{s^{1/2}} \| v \|_{H^2(\partial G)} \right\}
\]
and
\[
(5.3) \quad \sqrt{s} \left\| \left( \psi_{v,\gamma}, \frac{\partial \psi_{v,\gamma}}{\partial y_2}, \frac{\partial^2 \psi_{v,\gamma}}{\partial y_2^2} \right) \right\|_{H^{2,\gamma}(\partial G) \times H^{1,\gamma}(\partial G) \times L^2(\partial G)} \leq C_2(\sqrt{s}) \| v \|_{H^1(\partial G)} + \| w \|_{H^1(\partial G)} \\
+ \| fe^{\psi \varphi} \|_{H^1(\partial G)} + \| ge^{\psi \varphi} \|_{H^1(\partial G)}.
\]
If \( \zeta^\ast \notin \Psi_2 \), then estimate (3.1) hold true.
We recall that the sets $\Psi$, $\Psi_1$ and $\Psi_2$ are defined by (1.29) and Lemma 1.1.

Proof. Thanks to (5.1) and Proposition 2.2, decomposition (2.18) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Therefore we have

\begin{equation}
(D_{y_2} - \Gamma_{\mu}^-(y,s,D'))w_{1,|y_2=0} = V_{\mu}^+(\cdot,0),
\end{equation}

(5.4)

\begin{equation}
(D_{y_2} - \Gamma_{\lambda+2\mu}^-(y,s,D'))w_{2,|y_2=0} = V_{\lambda+2\mu}^+(\cdot,0).
\end{equation}

(5.5)

By Proposition 2.3 we have the a priori estimate:

\begin{equation}
\sqrt{\nu} V_{\mu}^+(\cdot,0)_{\mathcal{L}_2(\partial\mathcal{G})} + \sqrt{\nu} V_{\lambda+2\mu}^+(\cdot,0)_{\mathcal{L}_2(\partial\mathcal{G})} \\
\leq C_\delta(\|P_{\lambda+2\mu,s}w_{2}\|_{\mathcal{L}_2(\mathcal{G})} + \|\mu_{\mathcal{G}} w_{1}\|_{\mathcal{L}_2(\mathcal{G})} + \|w\|_{H^{1,\nu}(\mathcal{G})}).
\end{equation}

(5.6)

System (1.1) can be written in the following form:

\begin{align*}
\frac{\partial^2 u_1}{\partial x_0^2} + \mu \frac{\partial \text{rot } u}{\partial x_2} - (\lambda + 2\mu) \frac{\partial \text{div } u}{\partial x_1} &= m_1, \\
\frac{\partial^2 u_2}{\partial x_0^2} - \mu \frac{\partial \text{rot } u}{\partial x_1} - (\lambda + 2\mu) \frac{\partial \text{div } u}{\partial x_2} &= m_2,
\end{align*}

where $m = (m_1, m_2) = (\nabla_x u + (\nabla_x u)^T)\nabla_x \mu(x) + f$. In $y$- coordinates these equations can be written as:

\begin{align*}
\frac{\partial^2 u_1}{\partial y_0^2} + \mu \frac{\partial z_1}{\partial y_2} - (\lambda + 2\mu) \left( \frac{\partial}{\partial y_1} - \ell' \frac{\partial}{\partial y_2} \right) z_2 &= m_1, \\
\frac{\partial^2 u_2}{\partial y_0^2} - \mu \left( \frac{\partial}{\partial y_1} - \ell' \frac{\partial}{\partial y_2} \right) z_1 - (\lambda + 2\mu) \frac{\partial z_2}{\partial y_2} &= m_2.
\end{align*}

Here we still keep the same notations for $u$ and $m$ in the new coordinates. The above equations can be written in the form:

\begin{align*}
-D_{y_0}^2 v_1 + \mu \text{D}_{y_2} w_{1} - (\lambda + 2\mu) i(D_{y_1} - \ell' D_{y_2}) w_{2} &= m_1 e^{s\varphi}, \\
-D_{y_0}^2 v_2 - i\mu (D_{y_1} - \ell' D_{y_2}) w_{1} - (\lambda + 2\mu) iD_{y_2} w_{2} &= m_2 e^{s\varphi}.
\end{align*}

Applying to these equations the operator $\chi_{\nu}(s,D')$ we obtain

\begin{align*}
-D_{y_0}^2 v_{1,\nu} + \mu iD_{y_2} w_{1,\nu} - (\lambda + 2\mu) i(D_{y_1} - \ell' D_{y_2}) w_{2,\nu} &= h_1, \\
-D_{y_0}^2 v_{2,\nu} - i\mu (D_{y_1} - \ell' D_{y_2}) w_{1,\nu} - (\lambda + 2\mu) iD_{y_2} w_{2,\nu} &= h_2.
\end{align*}

(5.7) \quad (5.8)
Here \( h_1 = \chi_v(s, D') (m_1 e^{s \nu}) + [x_v, D_{h1}^\nu] v_1 + [\mu D_{h2}, \chi_v] v_1 - [(\lambda + 2\mu)i(D_{y_1} - \ell' D_{y_2}), \chi_v] w_1, h_2 = \chi_v(s, D') (m_2 e^{s \nu}) + [x_v, D_{h2}^\nu] v_2 - i[\mu(D_{y_1} - \ell' D_{y_2}), \chi_v] w_1 - [(\lambda + 2\mu)iD_{h2}, \chi_v] w_2.\) Finally using (5.4) and (5.5) we arrive at the system

\[
(5.9) \quad -D_{h}^2 v_{1,\nu} + \mu i\alpha_{\mu}^+(y, s, D') w_{1,\nu} - (\lambda + 2\mu)i(D_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y, s, D')) w_{2,\nu} = h_1 - i\mu V^+_{\mu} - i(\lambda + 2\mu)\ell' V^+_{\lambda+2\mu} \quad \text{on } \mathcal{G},
\]

\[
(5.10) \quad -D_{h}^2 v_{2,\nu} - \mu i(D_{y_1} - \ell'\alpha_{\mu}^+(y, s, D')) w_{1,\nu} - (\lambda + 2\mu)i(\alpha_{\lambda+2\mu}^+(y, s, D')) w_{2,\nu} = h_2 - i\mu\ell' V^+_{\mu} - (\lambda + 2\mu)\mu V^+_{\lambda+2\mu} \quad \text{on } \mathcal{G}.
\]

We set \( \tilde{h_1} = h_1 - i\mu V^+_{\mu} - i(\lambda + 2\mu)\ell' V^+_{\lambda+2\mu}, \tilde{h_2} = h_2 - i\mu\ell' V^+_{\mu} - (\lambda + 2\mu)iV^+_{\lambda+2\mu}. \) By (1.13)–(1.16) we have

\[
(5.11) \quad -D_{h}^2 v_{1,\nu} + \mu i\alpha_{\mu}^+(y, s, D') b_{11}(y', D') v_{1,\nu} - (\lambda + 2\mu)i(D_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y, s, D')) b_{21}(y', D') v_{1,\nu} + \mu i\alpha_{\mu}^+(y, s, D') b_{21}(y', D') v_{2,\nu} - (\lambda + 2\mu)i(D_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y, s, D')) b_{22}(y', D') v_{2,\nu} = q_1 \quad \text{on } \mathcal{G},
\]

\[
(5.12) \quad -D_{h}^2 v_{2,\nu} - \mu i(D_{y_1} - \ell'\alpha_{\mu}^+(y, s, D')) b_{11}(y', D') v_{1,\nu} - (\lambda + 2\mu)i(\alpha_{\lambda+2\mu}^+(y, s, D')) b_{21}(y', D') v_{1,\nu} - i\mu(D_{y_1} - \ell'\alpha_{\mu}^+(y, s, D')) b_{22}(y', D') v_{2,\nu} - (\lambda + 2\mu)i(\alpha_{\lambda+2\mu}^+(y, s, D')) b_{22}(y', D') v_{2,\nu} = q_2 \quad \text{on } \mathcal{G},
\]

with

\[
q_1 = \tilde{h_1} + \mu i\alpha_{\mu}^+(y, s, D') ([x_v, b_1(y', D)] C_1(y) e^{s \nu}) - (\lambda + 2\mu)i(D_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y, s, D')) ([x_v, b_2(y', D)] C_2(y) e^{s \nu}),
\]

\[
q_2 = \tilde{h_2} - i\mu(D_{y_1} - \ell'\alpha_{\mu}^+(y, s, D')) ([x_v, b_1(y', D)] C_1(y) e^{s \nu}) - (\lambda + 2\mu)i(\alpha_{\lambda+2\mu}^+(y, s, D')) ([x_v, b_2(y', D)] C_2(y) e^{s \nu}).
\]

By (5.11), (5.12) and (1.22) we have :

\[
(5.13) \quad B(y', s, D') v_{\nu} = q, \quad y' \in \partial \mathcal{G},
\]

where we recall that the operator \( B(y', s, D') \) is defined by (1.22) and we set

\[
(5.14) \quad q = T_1(g e^{s \nu}) + T_2(f(y', 0) e^{s \nu}) + T_3 v + G(y')(V^+_{\mu}(\cdot, 0), V^+_{\lambda+2\mu}(\cdot, 0)),
\]
$T_1, T_3 \in \mathcal{L}(H^{1+s}(\partial\mathcal{G}), L^2(\partial\mathcal{G})), T_2 \in \mathcal{L}(L^2(\partial\mathcal{G}), L^2(\partial\mathcal{G}))$ and $G(y')$ is a $C^1$ matrix-valued function.

Now we consider the following three cases.

**Case A.** Let $\det \mathcal{B}(\gamma) \neq 0$ and $\zeta^* \notin \Psi$. In that case, there exists a parametrix of the operator $\mathcal{B}(y', s, D')$ which we denote by $\mathcal{B}^{-1}(y', s, D')$, and we have

$$v_\nu = \mathcal{B}^{-1}(y', s, D') q + K v_\nu,$$

where

$$K \in \mathcal{L}(H^{1+s}(\partial\mathcal{G}), H^{2-s}(\partial\mathcal{G})) \quad \text{and} \quad \mathcal{B}^{-1}(y', s, D') \in \mathcal{L}(L^2(\partial\mathcal{G}), H^{2-s}(\partial\mathcal{G})).$$

Then

$$\|v_\nu\|_{H^{2-s}(\partial\mathcal{G})} \leq C_6 (\|q\|_{L^2(\partial\mathcal{G})} + \|v\|_{H^{1-s}(\partial\mathcal{G})}).$$

This estimate and (2.10) imply

$$\left\| \frac{\partial v_\nu}{\partial y_2} \right\|_{H^{1-s}(\partial\mathcal{G})} \leq C_6 (\|q\|_{L^2(\partial\mathcal{G})} + \|v\|_{H^{1-s}(\partial\mathcal{G})} + \|e^{s\phi}\|_{H^{1-s}(\partial\mathcal{G})}).$$

Next (2.14) and (5.16), (5.17) give us

$$\left\| \frac{\partial^2 v_\nu}{\partial y_2^2} \right\|_{L^2(\partial\mathcal{G})} \leq C_6 (\|q\|_{L^2(\partial\mathcal{G})} + \|v\|_{H^{1-s}(\partial\mathcal{G})} + \|e^{s\phi}\|_{H^{1-s}(\partial\mathcal{G})} + \|f e^{s\phi}\|_{L^2(\partial\mathcal{G})}).$$

Finally taking into account

$$\|q\|_{L^2(\partial\mathcal{G})} \leq C_7 (\|f e^{s\phi}\|_{L^2(\partial\mathcal{G})} + \|e^{s\phi}\|_{H^{1-s}(\partial\mathcal{G})} + \|v\|_{H^{1-s}(\partial\mathcal{G})})$$

we obtain from (5.16)–(5.18)

$$|\Sigma_\mu(w_{1, \nu})| + |\Sigma_{\lambda+2\mu}(w_{2, \nu})| \leq C_8 (\|f e^{s\phi}\|^2_{H^{1-s}(\partial\mathcal{G})} + s \|v\|^2_{H^{1-s}(\partial\mathcal{G})} + s \|e^{s\phi}\|^2_{H^{1-s}(\partial\mathcal{G})}).$$

Inequalities (2.25) and (5.19) yield

$$s \|w_{1, \nu}\|^2_{H^{1-s}(\partial\mathcal{G})} + s \left( \left\| v_\nu \left( \frac{\partial v_\nu}{\partial y_2}, \frac{\partial^2 v_\nu}{\partial y_2^2} \right) \right\|_{H^{1-s}(\partial\mathcal{G})} \times L^2(\partial\mathcal{G}) \right)^2 \leq C_9 (\|f e^{s\phi}\|^2_{H^{1-s}(\partial\mathcal{G})} + s \|v\|^2_{H^{1-s}(\partial\mathcal{G})} + s \|e^{s\phi}\|^2_{H^{1-s}(\partial\mathcal{G})}).$$
By (5.20) and Proposition 2.1 we obtain (3.1).

If det $B(\gamma) = 0$ and $\zeta^* \notin \Psi$ then either $\text{Im} \frac{\Gamma^+}{s}(\gamma) < 0$ or $\text{Im} \frac{\Gamma^{1+2\nu}}{s}(\gamma) < 0$.

In the first case: $\text{Im} \frac{\Gamma^+}{s}(\gamma) < 0$, we have

$$\sqrt{s}\|w_{1,\nu}\|_{H^{1,\tau}(\partial G)} + \sqrt{s}\|(b_{11} + b_{12})(y_1, s, D')v_\nu\|_{H^{1,\tau}(\partial G)}$$

$$\leq C_{10}(\|P_{\lambda + 2\mu, s}w_2\|_{L^2(G)} + \|P_{\mu, s}w_1\|_{L^2(G)} + \|w\|_{H^{1,\tau}(G)}).$$

Next we consider the operator $\Phi(y', s, D')$ given by

$$\Phi(y', s, D')v_\nu = (-D_{y_0}^2 v_2, \nu - i\mu(D_{y_1} - \ell' \alpha^+_\mu(y, s, D'))b_2(y', D)v_\nu, \Lambda(s, D')b_1(y', D)v_\nu) = (q_5, q_6),$$

where $\Lambda$ is the pseudodifferential operator with the symbol $\sqrt{1 + s^2 + \xi_0' + \xi_1'}$.

By (5.21) and (5.14), we have

$$\sqrt{s}\|\Phi(\gamma)\|_{H^{1,\tau}(\partial G)}\leq C_{11}(\|P_{\lambda + 2\mu, s}w_2\|_{L^2(G)} + \|P_{\mu, s}w_1\|_{L^2(G)} + \|w\|_{H^{1,\tau}(G)}$$

$$+ \sqrt{s}(\|f e^{s\phi}\|_{L^2(\partial G)} + \|g e^{s\phi}\|_{H^{1,\tau}(\partial G)} + \|v\|_{H^{1,\tau}(\partial G))).$$

Next we observe that since the principal symbol of the operator $\Phi(\gamma) \neq 0$ we have

$$\text{Im} \frac{\Gamma^{1+2\nu}}{s}(\gamma) < 0,$$
Next we observe that since the principal symbol of the operator $\hat{\Phi}(\gamma) \neq 0$ we have
\begin{equation}
\|v\|_{L^2(\partial\Omega)} \leq C_{15}(\|q\|_{L^2(\partial\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)}).
\end{equation}

Then repeating the arguments (5.17)–(5.20), we obtain (3.1).

Now we have consider the case where $\zeta^* \in \Psi$. In order to treat this case, we will use the Calderon method. First we introduce the new variables $U = (U_1, U_2, U_3, U_4)$, where $(U_1, U_2) = \Lambda(s, D')v, (U_3, U_4) = D_{y_2}v$. Then problem (1.8)–(1.9) can be written in the form:
\begin{equation}
D_{y_2}U = M(y, s, D')U + F \quad \text{in } \mathbb{R}^2 \times [0, 1], \quad B(y', s, D')U(y)|_{y_2=0} = ge^{s\varphi},
\end{equation}

where $F = (0, P(y, D)v)$ and we set $D = (D_{y_0}, D_{y_1}, D_{y_2})$. $\tilde{D}_{y_2} = \frac{\partial}{\partial y_2} + is\varphi_{y_2}$ and $M(y, s, D')$ is the matrix pseudodifferential operator whose principal symbol $M_1(y, \zeta)$ is given by formula (see [Y]):
\begin{equation}
M_1(y, \zeta) = \begin{pmatrix}
0 & \Lambda_1 E_2 \\
\Lambda^{-1} M_21 \Lambda^{-1} & \Lambda^{-1} M_22
\end{pmatrix} - is\varphi_{y_2} E_4,
\end{equation}

where $\Lambda_1 = |\zeta|$, $M_{21}(y, \zeta) = (\Lambda_1 - is\varphi_{y_0})^2 - (\Lambda_1 + is\varphi_{y_1})^2)E_2 - (\lambda + \mu)\tilde{\theta}^T \tilde{\theta}$, $M_{22}(y, \zeta) = -(\lambda + \mu)(\tilde{\theta}^T G + G^T \tilde{\theta}) - 2\mu\tilde{\theta} G^T E_2$, $\Lambda = \Lambda(y) = (\lambda + \mu)G^T G + \mu(G)^2 E_2$, $G(y) = (\lambda' (y_1), 1)$, $\tilde{\theta} = (\xi_1 + is\varphi_{y_1}, 0)$. Hence and henceforth, $\tilde{\theta}^T$ denotes the transpose of the vector $\tilde{\theta}$. For the stress boundary conditions, we have
\begin{equation}
B(y', s, D')U = (B_1(y', s, D'), B_2(y', s, D'))U = ge^{s\varphi},
\end{equation}

where
\begin{equation}
B_1(y', s, \zeta') = \frac{\lambda G^T \tilde{\theta} + \mu \tilde{\theta}^T G + \mu \tilde{\theta} G^T E_2}{\sqrt{s^2 + \xi_1^2 + \xi_0^2}}, \quad B_2(y', s, D') = \Lambda(y', 0).
\end{equation}

**Case B.** Let $\det B(\gamma) = 0$ and $\zeta^* \in \Psi_2$. We introduce the matrix symbol $K(y', s, \zeta, \xi_1)$ by formula
\begin{equation}
K(y', \zeta) = \frac{1}{1 + s^2 + \xi_0^2 + \xi_1^2} \begin{pmatrix}
B_{22}(y', \zeta) & -B_{12}(y', \zeta) \\
-B_{21}(y', \zeta) & B_{11}(y', \zeta)
\end{pmatrix}.
\end{equation}

Applying the pseudodifferential operator $K(y', s, D')$ to equation (5.13), we have
\begin{equation}
K(y', s, D')B(y', s, D')v = K(y', s, D')q.
\end{equation}
The principal symbol of the operator $K(y', s, D')B(y', s, D')$ is given by the formula $K(y', \zeta)B(y', \zeta) = \det B(y', \zeta)E_4/|\zeta|^2$. Note that if $\zeta^* \in \Psi_2$, then $\alpha_3^2(\gamma) = (\frac{\mu(0)}{\beta}(0) - 1)(\xi^*_1 + i\sigma^*\varphi(y^*))^2 \not\in \mathbb{R}_+$ for $\beta \in \{\mu(0), (\lambda + 2\mu)(0)\}$. Therefore $\text{Im } \alpha_3^2(\gamma) < 0$ and $\text{Im } \alpha_{\lambda+2\mu}(\gamma) < 0$. Hence we may rewrite estimate (5.6) in the form

$$
(5.30) \quad \left\| V^+_\mu(\cdot, 0) \right\|_{H^{\frac{1}{4}}(\partial G)} + \left\| V^+_{\lambda+2\mu}(\cdot, 0) \right\|_{H^{\frac{1}{4}}(\partial G)} \leq C_{16}(\left\| P_{\lambda+2\mu}w_2 \right\|_{L^2(G)} + \left\| P_{\mu}w_1 \right\|_{L^2(G)} + \left\| w \right\|_{H^{1}(G)}).
$$

Thanks to (5.14), (5.30) and Condition 1.2, applying Theorem 1.1 (Chapter 8) from [E] we have estimate (5.3).

Now we need to show that estimate (5.2) holds true. By $\zeta^* \in \Psi_2$, the matrix $M_1(\gamma)$ has four distinct eigenvalues given by (1.18)–(1.20). Following [T1], in terms of the change of variables $W = S^{-1}(y, s, D')U$, we can transform system (5.27) to

$$
(5.31) \quad D_{\zeta}W = \tilde{M}(y, s, D')W + T(y, s, D')W + \tilde{F},
$$

where

$$
(5.32) \quad \left\| \tilde{F} \right\|_{L^2(\mathbb{R}^4; H^{1/4}(\partial G))} \leq C_{17}(\left\| P(y, D)v \right\|_{H^{1/4}(G)} + \left\| v \right\|_{L^2(\mathbb{R}^4; H^{1/4}(\partial G))}).
$$

Here the matrix $\tilde{M}$ has the form

$$
\tilde{M}(y, \zeta) = \begin{pmatrix}
M^+_{\lambda+2\mu}(y, \zeta) & 0 \\
0 & M^-_{\lambda+2\mu}(y, \zeta)
\end{pmatrix}, \quad M_{\pm}(y, \zeta) = \begin{pmatrix}
\Gamma^+_{\lambda+2\mu}(y, \zeta) & 0 \\
0 & \Gamma^-_{\mu}(y, \zeta)
\end{pmatrix},
$$

and the operator $T(y, s, D') \in L^\infty(0, 1; \mathcal{L}(L^2(G), L^2(G)))$. We represent the symbol $S$ in the form $S = (s^+_1, s^+_2, s^-_1, s^-_2)$. Here $\Lambda_1 = \{(s, \zeta_0, \zeta_1) |$,

$$
\begin{array}{c}
s^+_1 = ((\theta + \alpha^+_3+2\mu)G\Lambda^{-1}_1, \alpha^+_3+2\mu(\theta + \alpha^+_3+2\mu)G\Lambda^{-2}_1), \\
s^+_2 = (\alpha^+_3\zeta_1 + is\varphi_{\gamma_1} - \alpha^+_3\zeta_1', -\zeta_1 + is\varphi_{\gamma_1} - \alpha^+_3\zeta_1')\Lambda^{-2}_1, \\
a^+_\mu\Lambda^{-1}_1(\alpha^+_\mu(\zeta_1 + is\varphi_{\gamma_1} - \alpha^+_\mu\zeta_1')\Lambda^{-2}_1, -(\zeta_1 + is\varphi_{\gamma_1} - \alpha^+_\mu\zeta_1')\Lambda^{-2}_1),
\end{array}
$$

are the eigenvectors of the matrix $M_1(y, \zeta)$, $\zeta \in S^2$, which corresponds to the eigenvalues $\Gamma^+_{\lambda+2\mu}$ and $\Gamma^-_{\mu}$. Now using the standard arguments (see e.g., §4 of Chapter 7 in [Ku]), we can estimate the last two components of $W$ as follows

$$
(5.33) \quad \left\| (W_3, W_4) \right\|_{H^{1/4}(G)} \leq C_{18}(\left\| P(y, D)v \right\|_{L^2(G)} + \left\| v \right\|_{H^{1/4}(G)}),
$$

where the constant $C_{18}$ is independent of $N$. Now we need the estimate for the first two components of vector $W$. Henceforth we set $j(\beta) = 2$ if $\beta = \mu$ and $j(\beta) = 1$ if $\beta = \lambda + 2\mu$. 

There are the following two possibilities (i) and (ii): (i) \( \text{Im} \Gamma^+_{\bar{\beta}}(\gamma) > 0 \) for any \( \beta \in \{\mu, \lambda + 2\mu\} \).

Then, by the same argument (see e.g., [Ku], pp. 241-247), we have

\[
(5.34) \quad \|W_{j(\beta)}\|_{H^{1,1}(\mathcal{G})} \leq C_{19}(\|P(y, D)v\|_{L^2(\mathcal{G})} + \|W_{j(\beta)}(\cdot, 0)\|_{H^{1/2,1}(\partial \mathcal{G})} + \|v\|_{H^{1,1}(\mathcal{G})}).
\]

Combining this inequality with a priori estimate (5.3), we have

\[
(5.35) \quad \|W_{j(\beta)}\|_{H^{1,1}(\mathcal{G})} \leq C_{20}(\|P(y, D)v\|_{L^2(\mathcal{G})} + \|v\|_{H^{1,1}(\mathcal{G})} + \|ge^{s\varphi}\|_{H^{1,1}(\mathcal{G})} + \|v\|_{H^{1,1}(\mathcal{G})} + 1 + \mathcal{C}_s(\|w\|_{H^{1,1}(\mathcal{G})} + \|fe^{s\varphi}\|_{H^{1,1}(\mathcal{G})} + \|v\|_{H^{1,1}(\mathcal{G})}).
\]

(ii) There exists \( \beta \in \{\mu, \lambda + 2\mu\} \) such that \( \text{Im} \Gamma^+_{\bar{\beta}}(\gamma) = 0 \).

Applying Proposition 2.4, we obtain

\[
\|W_{j(\beta)}\|_{H^{1,1}(\mathcal{G})} \leq C_{21}\left(\frac{1}{\sqrt{s}}(\|P(y, D)v\|_{H^{1,1}(\mathcal{G})} + \|v\|_{H^{2,1}(\mathcal{G})}) + s^{\frac{1}{2}}\|W_{j(\beta)}(\cdot, 0)\|_{H^{1,1}(\mathcal{G})}\right).
\]

Combining this inequality with a priori estimate (5.3), we have

\[
(5.36) \quad \|W_{j(\beta)}\|_{H^{1,1}(\mathcal{G})} \leq C_{22}\left\{\frac{1}{\sqrt{s}}(\|P(y, D)v\|_{H^{1,1}(\mathcal{G})} + \|v\|_{H^{2,1}(\mathcal{G})}) + s^{\frac{1}{2}}\|ge^{s\varphi}\|_{H^{1,1}(\mathcal{G})} + s^{\frac{1}{2}}(\|v\|_{H^{1,1}(\mathcal{G})} + \|w\|_{H^{1,1}(\mathcal{G})} + \|fe^{s\varphi}\|_{H^{1,1}(\mathcal{G})})\right\}.
\]

In view of (5.33), (5.34) and (5.36), we obtain (5.2).

**Case C.** det \( B(\gamma) = 0 \) and \( \zeta^* \in \Psi_1 \). In that case we have

\[
(5.37) \quad \xi_0^* = 0, \quad s^*\varphi_{y_0}(y^*) = 0.
\]

Then we can assume that

\[
(5.38) \quad \text{Im} \Gamma^+_{\bar{\beta}}(\gamma) = \text{Im} \Gamma^+_{\lambda + 2\mu}(\gamma) \geq 0.
\]

In fact, if

\[
(5.39) \quad \text{Im} \Gamma^+_{\bar{\beta}}(\gamma) = \text{Im} \Gamma^+_{\lambda + 2\mu}(\gamma) < 0,
\]
then the situation is simple because we have the decomposition

\[
P_{\lambda}(y, s, D)w_{j(\beta)}(\nu) = \beta(G)[D_{y_2} - \Gamma_{\beta}(y, s, D')]V_{\beta}^\pm + T_{\beta}^\pm w_{j(\beta)},\]

where \( T_{\beta}^\pm \in \mathcal{L}(H^{1,\ast}(G), L^2(G)) \), \( \beta \in \{\mu, \lambda + 2\mu\} \), \( j(\beta) = 1 \) for \( \beta = \mu \) and \( j(\beta) = 2 \) for \( \beta = \lambda + 2\mu \). This decomposition, (5.39) and Proposition 2.3 imply

\[
||\sqrt{s}(D_{y_2} - \Gamma_{\beta}(y, s, D'))w_{j(\beta),\nu}|_{y_2=0}||_{L^2(\partial G)} \\
\leq C_{24}(||P_{\beta,\nu}w_{j(\beta)}||_{L^2(G)} + ||v||_{H^{2,\ast}(G)}).
\]

Obviously

\[
-V_{\mu}^+(\cdot, 0) + V_{\mu}^-(\cdot, 0) = (\alpha_{\mu}^+(y', 0, s, D') \\
-\alpha_{\mu}^-(y', 0, s, D')((b_{11}(y', D_{y_1}) + b_{12}(y', D_{y_1}))\nu + \tilde{C}_1(y_1)\text{ge}^{s\phi}) \quad \text{on } \partial G
\]

and

\[
-V_{\lambda+2\mu}^+(\cdot, 0) + V_{\lambda+2\mu}^-(\cdot, 0) = (\alpha_{\lambda+2\mu}^+(y', 0, s, D') \\
-\alpha_{\lambda+2\mu}^-(y', 0, s, D')((b_{21}(y', D_{y_1}) + b_{22}(y', D_{y_1}))\nu + \tilde{C}_2(y_1)\text{ge}^{s\phi}) \quad \text{on } \partial G.
\]

Since \( \alpha_{\beta}^+(y^*, \zeta^*) - \alpha_{\beta}^-(y^*, \zeta^*) = 2\sqrt{r_{\mu}(y^*, \zeta^*)} \neq 0 \) and the determinant of the matrix

\[
\begin{pmatrix}
2\sqrt{r_{\mu}(y^*, \zeta^*)}b_{11}(y^*, \xi^* + is^\ast \varphi_{y_1}(y^*)) \\
2\sqrt{r_{\mu}(y^*, \zeta^*)}b_{21}(y^*, \xi^* + is^* \varphi_{y_1}(y^*)) \\
2\sqrt{r_{\mu}(y^*, \zeta^*)}b_{12}(y^*, \xi^* + is^* \varphi_{y_1}(y^*)) \\
2\sqrt{r_{\mu}(y^*, \zeta^*)}b_{22}(y^*, \xi^* + is^* \varphi_{y_1}(y^*))
\end{pmatrix}
\]

is not equal to zero, by (5.38) and Gårding’s inequality, we obtain

\[
\left(\sqrt{s}\left\|\begin{matrix}
\nu_{\beta} \\
\frac{\partial \nu_{\beta}}{\partial y_2} \\
\frac{\partial^2 \nu_{\beta}}{\partial y_2^2}
\end{matrix}\right\|_{H^{2,\ast}(\partial G) \times H^{1,\ast}(\partial G) \times L^2(\partial G)}
\leq C_{24}(||f\text{ge}^{s\phi}||_{H^{1,\ast}(G)} + \sqrt{s}||\text{ge}^{s\phi}||_{H^{1,\ast}(\partial G)} + ||v||_{H^{2,\ast}(G)}).
\]

In terms of (5.41) and (2.25), we obtain (3.1).

The matrix \( M_1(\gamma) \) has only two eigenvalues given by (1.18)–(1.20). Moreover it is known that the Jordan form of the matrix \( M_1(\gamma) \) has two Jordan blocks of the form:

\[
M^\pm = \begin{pmatrix}
\Gamma_{\beta}^{\pm}(\gamma) & 1 \\
0 & \Gamma_{\beta}^{\pm}(\gamma)
\end{pmatrix}.
\]

Following [T1], in terms of the change of variables \( W = S^{-1}(y, s, D')U \), we can transform system (5.27) to the form

\[
D_{y_2}W = \tilde{M}(y, s, D')W + T(y, s, D')W + \tilde{F},
\]
are given explicitly by
\[ (5.43) \]
\[ \tilde{B}(y', s, D')W = ge^{s\phi}, \]
where
\[ \|F\|_{L^2(\mathbb{R}^1; \mathcal{H}^{1,\gamma}(\partial G))} \leq C_{25} \|P(y, D)v\|_{\mathcal{H}^{1,\gamma}(G)} + \|v\|_{L^2(\mathbb{R}^1; \mathcal{H}^{1,\gamma}(\partial G))} \]
and the principal symbol of the operator \( \tilde{B} \) is defined by formula
\[ (5.44) \]
\[ \tilde{B}(y', s, \xi') = B(y', s, \xi')S(y', 0, s, \xi'). \]

Here the matrix \( \tilde{M} \) has the form
\[ \tilde{M}(y, \zeta) = \begin{pmatrix} M_+(y, \zeta) & 0 \\ 0 & M_-(y, \zeta) \end{pmatrix}, \]
\[ M_\pm(y, \zeta) = \begin{pmatrix} \Gamma^{\pm}_{\lambda+2\mu}(y, \zeta) & m^{\pm}_{1,2}(y, \zeta) \\ 0 & \Gamma^{\pm\mu}_{\lambda}(y, \zeta) \end{pmatrix}, \]
and the operator \( T(y, s, D') \in L^\infty(0, 1; \mathcal{L}(\mathcal{H}^{1,s}(G), \mathcal{H}^{1,s}(G))) \). We describe the construction of the pseudodifferential operator \( S \). We write the symbol \( S \) in the form \( S(y, s, \zeta') = (s_1^+ s_2^+, s_1^-, s_2^-) \). Here \( s_1^\pm = ((\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu} G_2)^{-1} \alpha^{\pm}_{\lambda+2\mu} (\vec{\theta} + \alpha^{\pm}_{\lambda+2\mu} G_2)^{-1} \) are the eigenvectors of the matrix \( M_1(y, \zeta) \) on the sphere \( \zeta \in S^2 \) which corresponds to the eigenvalue \( \Gamma^\pm_{\lambda+2\mu}(y, s, \zeta') \) and the vectors \( s_2^\pm \) are given by
\[ s_2^\pm(y, s, \zeta') = E_{\pm} s^\pm, \quad E_{\pm} = \frac{1}{2\pi i} \int_{C^\pm} (z - M_1(y, \zeta))^{-1} dz, \]
where \( C^\pm \) are small circles oriented counterclockwise and centered at \( \Gamma^\pm_{\mu}(\gamma) \), and \( s^\pm \) solves the equation \( M_1(\gamma) s^\pm - \Gamma^\pm_{\mu}(\gamma) s^\pm = s^\pm(\gamma) \). By \( (5.38) \) the circles \( C^\pm \) can be taken such that the disks bounded by these circles do not intersect. Note that the vectors \( s_2^\pm \in C^2(B_\delta \times \mathcal{O}_{\delta_1}) \) are homogeneous functions of the order zero in \( (s, \xi_0, \xi_1) \). Now, similarly to \( (5.34) \), we can estimate the last two components of \( W \) as follows
\[ (5.45) \]
\[ \|W_3, W_4\|_{\mathcal{H}^{1,\gamma}(\partial G)} \leq C_{26} \|P(y, D)v\|_{\mathcal{H}^{1,\gamma}(G)} + \|v\|_{\mathcal{H}^{1,\gamma}(G)}, \]
where the constant \( C_{26} \) is independent of \( N \). Now we need to estimate the first two components of the vector \( W \) on \( \partial G_1 \). We can decompose the boundary operator \( \tilde{B}(y', s, D') = (\tilde{B}^+(y', s, D'), \tilde{B}^-(y', s, D')) \) such that
\[ (5.46) \]
\[ \tilde{B}^+(y', s, D')(W_1, W_2) = -\tilde{B}^-(y', s, D')(W_3, W_4) + ge^{s\phi}, \]
where \( \tilde{B}^+(y', \zeta) = (\tilde{B}_1(y', \zeta), \Lambda(y', 0))S_+(y', 0, \zeta) \), \( \tilde{B}_1 \) is the principal symbol of the boundary operator \( \tilde{B} \) and \( S_+ = (s_1^+, s_2^+) \). At the point \( \gamma \) the vectors \( s_1, s_2 \) are given explicitly by
\[ \tilde{\eta} = (\xi_1^+ i s^+ \varphi_{y_1}(y^+), i \text{sign}(\xi_1^+) (\xi_2^+ i s^\varphi_{y_1}(y^+)) \)
\[ s_1^+(\gamma) = \left( \frac{\text{sign}(\xi_1^*)(\xi_1^* + is^* \varphi_{y_1}(y^*))}{\sqrt{(\xi_1^*)^2 + (s^*)^2}} \right), \]

\[ \zeta = -\frac{(\lambda + 3\mu)(0)}{2\sqrt{(\xi_1^*)^2 + (s^*)^2}(\lambda + \mu)(0)}(\text{sign}(\xi_1^*), 1), \]

\[ s_2^+(\gamma) = \left( \frac{1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}(\text{sign}(\xi_1^*)(\xi_1^* + is^* \varphi_{y_1}(y^*))\zeta + \eta) \right). \]

Therefore

(5.47)

\[ \det \mathcal{B}^+(\gamma) = \begin{pmatrix} 2\mu(0)(\xi_1^* + is^* \varphi_{y_1}(y^*))^2 & \mu(0)(\xi_1^* + is^* \varphi_{y_1}(y^*)) \\ 2\mu(0)(\xi_1^* + is^* \varphi_{y_1}(y^*))^2 & -i\mu(\lambda + 2\mu)(0)(\xi_1^* + is^* \varphi_{y_1}(y^*)) \end{pmatrix} \]

\[ = (\xi_1^* + is^* \varphi_{y_1}(y^*))^3 \mu(0) \frac{2\lambda + 4\mu}{\lambda + \mu} (0) \neq 0. \]

By (5.44)–(5.47) and Gårding’s inequality, we obtain

(5.48)

\[ \sqrt{s} \left\| \left( v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \right) \right\| \| H^{1,1}(\partial \Omega) \times H^{1,1}(\partial \Omega) \times L^2(\partial \Omega) \]

\[ \leq C_{2T}(\| P(y, D)v \| H^{1,1}(\Omega) + \| v \| H^{2,1}(\Omega)). \]

This inequality implies (5.19). Then from (2.25) and Proposition 2.1 we obtain (3.1).

Now we will proceed to

**Completion of the proof of Theorem 1.1.** Microlocally we obtain two types of estimates: If \( \zeta^* \in \Psi_2 \), then we have estimate (5.2), while if \( \zeta^* \notin \Psi_2 \), then we have estimate (3.1). By \( \mathcal{O}\Psi_2(\delta_2) \), we denote the \( \delta_2 \)-neighbourhood of the set \( \Psi_2 \) in \( S^2 \). We take the parameter \( \delta_2 \) sufficiently small. From the covering of the set \( \mathcal{O}\Psi_2(\frac{\delta_2}{2}) \) by balls with radius \( 4\delta_2 \), we take the finite subcovering \( \{ B_{4\delta_2}(\zeta_j) \}_{j \in \mathcal{T}_1}, \zeta_j \in \Psi_2 \). Let \( \{ \chi_{\nu} \}_{\nu \in \mathcal{T}_1} \) be a partition of unity associated with this subcovering. For the set \( S^2 \setminus \mathcal{O}\Psi_2(3\delta_2) \), we take the finite covering by balls with radius \( \delta_2 \). Let \( \{ \chi_{\nu} \}_{\nu \in \mathcal{T}_2} \) be a partition of unity associated with this subcovering. We extend the functions \( \chi_{\nu} \) as a homogeneous functions of order zero to a function in \( C^\infty(\mathbb{R}^3) \). Since \( \Psi_2 \subset \mathcal{O}\Psi_2(\frac{\delta_2}{2}) \cup S^2 \setminus \mathcal{O}\Psi_2(3\delta_2) \),
it follows from (5.2) and (3.1) that

\begin{equation}
\|v\|_{H^{1/2}({\mathcal G})} \leq C_{28} \sum_{\nu \in \mathcal{T}_1 \cup \mathcal{T}_2} \|\chi_{\nu} v\|_{H^{1/2}({\mathcal G})} \leq \frac{1}{\sqrt{N}} \left( \frac{1}{s^3} \right)^{1/2} \|P(y, D)v\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})} \right)
\end{equation}

\begin{equation}
\sup_{|\alpha| = 0} s^{2-|\alpha|} \|\partial^\alpha_y v\|_{L^2({\mathcal G})} \right), \; \forall N \geq \tilde{N}, s \geq s_0(\tilde{N}).
\end{equation}

Fixing the parameters \(\tilde{N}\) and \(s_0(N)\) sufficiently large, we obtain

\begin{equation}
\|v\|_{H^{1/2}({\mathcal G})} \leq C_{29}(\|P(y, D)v\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})}), \; \forall s \geq s_0.
\end{equation}

Combination of (5.50) with estimates (5.3) and (3.1), yields

\begin{equation}
\sqrt{s} \left\| \left( v_{\nu}, \frac{\partial v_{\nu}}{\partial y_2}, \frac{\partial^2 v_{\nu}}{\partial y_2^2} \right) \right\|_{H^{1/2}({\partial \mathcal G}) \times H^{1/2}({\partial \mathcal G}) \times L^2({\partial \mathcal G})} \leq C_{30}(\|P(y, D)v\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})}), \; \forall \nu \in \mathcal{T}_2
\end{equation}

and

\begin{equation}
\sqrt{s} \|v_{\nu}\|_{H^{1/2}({\partial \mathcal G})} \leq C_{31}(\|P(y, D)v\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})}), \; \forall \nu \in \mathcal{T}_1
\end{equation}

Estimates (5.51) and (5.52) yield

\begin{equation}
\sqrt{s} \|ue^{s \phi}\|_{H^{1/2}({\partial \mathcal G})} \leq C_{32}(\|P(y, D)v\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})}),
\end{equation}

where we used \(\phi = \varphi\) on \(\partial \Omega\). We note that estimate (5.53) is obtained under additional assumption (2.7). Now we will get rid of (2.7). For this, we consider the function \(\theta u\) instead of the function \(u\), where \(\theta\) is a smooth cut-off function such that \(\theta|\Omega = 1\) and \(\theta|_{\Omega \setminus \{\partial \Omega\}} = 0\). Then it suffices to modify (5.53) and prove

\begin{equation}
\sqrt{s} \|ue^{s \phi}\|_{H^{1/2}({\partial \mathcal G})} \leq C_{33} \left( \|fe^{s \phi}\|_{H^{1/2}({\mathcal G})} + \|g e^{s \phi}\|_{H^{1/2}({\partial \mathcal G})} + \frac{1}{\sqrt{s}} \|ue^{s \phi}\|_{H^{1/2}({\mathcal G})} \right), \; \forall s \geq s_0,
\end{equation}

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where we used the fact that \( \varphi(x) \leq \phi(x) \) in \((0, T) \times (\Omega \setminus \Omega_\varepsilon) \) (see (2.6)).

Next by Lemma 5.1 we note that the analogues to estimates (5.3) and (5.2) hold true for the weight function \( \phi \) instead of \( \varphi \):

\[
(5.55) \quad \sqrt{s}||\chi_w(ue^{s\delta})||_{H^{2,s}(\Omega)} + \sqrt{s}||\chi_w(ue^{s\delta})||_{H^{1,s}(\partial\Omega)} \\
\leq C_{34}(||ue^{s\delta}||_{H^{2,s}(\Omega)} + ||fe^{s\delta}||_{H^{1,s}(\Omega)} + ||ge^{s\delta}||_{H^{2,s}(\partial\Omega)}), \quad \forall \nu \in \mathcal{T}_1.
\]

Let \( \chi_{-1} \) and \( \chi_{-2} \) be \( C^\infty \)-functions on the sphere \( S^2 \) such that \( \chi_{-1} \in C^\infty_c(O\Psi_2(\frac{1}{2}B_2)), \chi_{-2} \in C^\infty_c(S^2 \setminus O\Psi_2(3\delta_2)) \) and \( \chi_{-1}O\Psi_2(\frac{1}{2}B_2) = 1, \chi_{-2}|S^2 \setminus O\Psi_2(3\delta_2) = 1 \). Hence we have \( |\chi_{-1}(s,\xi)| + |\chi_{-2}(s,\xi')| \geq 1 \) for all \( \xi \in S^2 \). We extend the functions \( \chi_{-1} \) and \( \chi_{-2} \) as homogeneous functions of order zero to functions in \( C^\infty(\mathbb{R}^3) \). By (1.8) and (1.9), we have

\[
(5.56) \quad A^{-1}P(y,D)\chi_{-1}v + [\chi_{-1},A^{-1}P(y,D)]v = \chi_{-1}A^{-1}fe^{s\delta}.
\]

Note that we can estimate

\[
(5.57) \quad ||[\chi_{-1},A^{-1}P(y,D)]v||_{H^{1,s}(\Omega)} \leq C_{35}(||ue^{s\delta}||_{H^{2,s}(\Omega)} + ||fe^{s\delta}||_{H^{1,s}(\Omega)}),
\]

Hence applying estimate (5.55) and (5.57) to (5.56), we obtain.

\[
(5.58) \quad \sqrt{s}||\chi_{-1}(s,D')(ue^{s\delta})||_{H^{2,s}(\Omega)} \\
\leq C_{36}(||ue^{s\delta}||_{H^{2,s}(\Omega)} + ||fe^{s\delta}||_{H^{1,s}(\Omega)} + ||ge^{s\delta}||_{H^{2,s}(\partial\Omega)}).
\]

Setting \( \tilde{P}_{\beta,s}(y,s,D) = \frac{1}{\beta(1+|\ell'(y_1)|^2)}P_{\beta,s}(y,s,D) \), we have

\[
\tilde{P}_{\mu,s}(y,s,D)\chi_{-2}(s,D')w_1 + [\chi_{-2},\tilde{P}_{\mu,s}]w_1 = \chi_{-2} \left( \frac{1}{\mu(1+|\ell'(y_1)|^2)}q_1 \right)
\]

and

\[
\tilde{P}_{\lambda+2\mu,s}(y,s,D)\chi_{-2}(s,D')w_2 + [\chi_{-2},\tilde{P}_{\lambda+2\mu,s}]w_2 = \chi_{-2} \left( \frac{1}{(\lambda+2\mu)(1+|\ell'(y_1)|^2)}q_2 \right).
\]

Note that

\[
||\Sigma_{\mu}(\chi_{-2}(s,D')w_1) + \Sigma_{\lambda+2\mu}(\chi_{-2}(s,D')w_2)|| \\
\leq C_{37}||\chi_{-2}(s,D')\left( \frac{\partial w}{\partial y_2}, w \right)||_{L^2(\partial\Omega)}H^{1,s}(\partial\Omega).
\]
Combining this inequality with (2.25) we have

\[
\sqrt{s} \| \chi_2(s, D') w \|_{H^{1.5}(G)} \leq C_{38} \left\{ \| f e^{s \phi} \|_{H^{1.5}(G)} + \| g e^{s \phi} \|_{H^{2.5}(\partial G)} + \| \chi_2 - \tilde{P}_{\mu, s} \|_{L^2(G)} + \| \chi_2 - \tilde{P}_{\lambda + 2\mu, s} \|_{L^2(G)} \right\}.
\]

Thanks to the estimates

\[
\| \chi_2 - \tilde{P}_{\mu, s} \|_{L^2(G)} + \| \chi_2 - \tilde{P}_{\lambda + 2\mu, s} \|_{L^2(G)} \leq C_{39} \left( \| f e^{s \phi} \|_{H^{1.5}(G)} + \| w \|_{H^{1.5}(G)} \right)
\]

and

\[
\| \chi_2(s, D') \left( \frac{\partial w}{\partial y_2} \right) \|_{L^2(\partial G) \times H^{1.5}(\partial G)} \leq C_{40} \left\{ \frac{1}{\sqrt{s}} \| f e^{s \phi} \|_{H^{1.5}(G)} + \| w e^{s \phi} \|_{H^{1.5}(\partial G)} + \| \chi_2(s, D')(w e^{s \phi}) \|_{H^{2.5}(\partial G)} \right\},
\]

we obtain

\[
\sqrt{s} \| \chi_2(s, D') w \|_{H^{1.5}(G)} \leq C_{41} \left( \| f e^{s \phi} \|_{H^{1.5}(G)} + \| g e^{s \phi} \|_{H^{2.5}(\partial G)} + \| (1 - \chi_2(s, D')) w \|_{H^{1.5}(G)} \right) + \sqrt{s} \left( \| w e^{s \phi} \|_{H^{1.5}(\partial G)} + \| \chi_2(s, D') w e^{s \phi} \|_{H^{2.5}(\partial G)} \right).
\]

Estimating the last term on the right hand side of (5.60), we obtain

\[
\sqrt{s} \| \chi_2(s, D') w \|_{H^{1.5}(G)} \leq C_{42} \left( \| f e^{s \phi} \|_{H^{1.5}(G)} + \| g e^{s \phi} \|_{H^{2.5}(\partial G)} + \| (1 - \chi_2(s, D')) w \|_{H^{1.5}(G)} \right).
\]

Since \( \chi_1 \text{supp}(1 - \chi_2) = 1 \), by (5.55) we have

\[
\| (1 - \chi_2(s, D')) w \|_{H^{1.5}(G)} \leq \| \chi_1(s, D') w \|_{H^{1.5}(G)} \leq C_{44} \left( \| f e^{s \phi} \|_{H^{1.5}(G)} + \| g e^{s \phi} \|_{H^{2.5}(\partial G)} + \| w e^{s \phi} \|_{H^{2.5}(\partial G)} \right).
\]

Hence, using this estimate and (5.58), we obtain from (5.61)

\[
\sqrt{s} \| w \|_{H^{1.5}(G)} \leq C_{44} \left( \| f e^{s \phi} \|_{H^{1.5}(G)} + \| g e^{s \phi} \|_{H^{2.5}(\partial G)} + \frac{1}{\sqrt{s}} \| w e^{s \phi} \|_{H^{2.5}(\partial G)} \right).
\]
We can rewrite equation (1.1) as

$$\rho(\tilde{x}) \frac{\partial^2 u}{\partial x_0^2} - \mu(\tilde{x}) \Delta u$$

$$= (\mu(\tilde{x}) + \lambda(\tilde{x})) \nabla_\tilde{x} \text{div } u + (\text{div } u)\nabla_\tilde{x} \lambda(\tilde{x}) + (\nabla_\tilde{x} u + (\nabla_\tilde{x} u)^T)\nabla_\tilde{x} \mu(\tilde{x}) + f.$$ 

Applying to each component of this system the Carleman estimate for hyperbolic equations we have

$$s\|ue^{s\phi}\|_{H^{1.5}(G)} \leq C_{45} \sqrt{s} \|fe^{s\phi}\|_{L^2(G)}$$

$$+ \sqrt{s} \|ue^{s\phi}\|_{H^{1.5}(\tilde{\partial} G)} + \sqrt{s} \|e^{s\phi}\|_{H^{1.5}(\tilde{\partial} G)} + s \|ue^{s\phi}\|_{H^{1.5}(\partial G)}.$$ 

Estimating the term with \text{div } u on the right hand side of this inequality by (5.62), we obtain

$$s\|ue^{s\phi}\|_{H^{1.5}(G)} \leq C_{46} (\|fe^{s\phi}\|_{H^{1.5}(G)} + \|ge^{s\phi}\|_{H^{1.5}(\tilde{\partial} G)} + s \|ue^{s\phi}\|_{H^{1.5}(\partial G)}).$$ 

We set rot^* v = \(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\). Then we have

$$-\Delta_\tilde{x} u \equiv -\Delta u = \text{rot}^* \text{rot } u - \nabla_\tilde{x} (\text{div } u).$$ 

Therefore we obtain

$$s\|\partial^2_{x_0}(ue^{s\phi})\|_{L^2(G)} \leq C_{47} \|ue^{s\phi}\|_{L^2(0,T; H^{1.5}(\partial G))} + s \|ue^{s\phi}\|_{H^{1.5}(G)}.$$ 

In view of equations (1.8) and (1.9), we can estimate \(\partial^2_{x_0}(ue^{s\phi})\):

$$\|\partial^2_{x_0}(ue^{s\phi})\|_{L^2(G)} \leq C_{48} \sum_{|\alpha|=0, \alpha=(0,0,0)}^2 s^{2-|\alpha|} \|\partial^\alpha(ue^{s\phi})\|_{L^2(G)} + \|fe^{s\phi}\|_{L^2(\tilde{\partial} G)}.$$ 

Hence (5.54) and (5.63)–(5.65) yield

$$\|ue^{s\phi}\|_{H^{1.5}(G)} \leq C_{49} \left\{ \|fe^{s\phi}\|_{H^{1.5}(G)} + \|ge^{s\phi}\|_{H^{1.5}(\tilde{\partial} G)} + \frac{1}{\sqrt{s}} \|ue^{s\phi}\|_{H^{1.5}(\partial G)} \right\}.$$ 

Estimate (5.66) implies (1.24). Thus the proof of Theorem 1.1 is completed. ☐
Appendix I. Proof of Proposition 2.1

Let \( x_0 \in [0, T] \) be arbitrary fixed.

First we choose \( N_0 > 0 \) sufficiently large such that

\[
\nabla \tilde{v}(x) \neq 0, \quad \forall \tilde{x} \in \Omega \frac{1}{\sqrt{T}}, \ x_0 \in [0, T].
\]

The existence of such \( N_0 \) follows from condition (1.6).

Let \( \text{rot}^* v = \left( \frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right) \). Using the well-known formula \( \text{rot} \text{rot} = -\Delta \tilde{v} + \nabla \text{div} \), we obtain

\[
\Delta \tilde{v} u = -\text{rot}^* (\text{rot} u) + \nabla \tilde{v} (\text{div} u) \quad \text{in} \ \Omega \frac{1}{\sqrt{T}}.
\]

The function \( v = u e^{\sqrt{r}} \) satisfies equations

\[
L_1 v + L_2 v = q_s \quad \text{in} \ \Omega \frac{1}{\sqrt{T}}, \quad v|_{\partial \Omega} = 0,
\]

where \( L_1 v = -\Delta \tilde{v} v - s^2 |\nabla \tilde{v}|^2 v, \ L_2 v = 2s \sum_{k=1}^{2} v_{x_k} \varphi_{x_k} + s(\Delta \tilde{v}) v \) and \( q_s = (\text{rot}^* \text{rot} u - \nabla \tilde{v} (\text{div} u)) e^{\sqrt{r}} \). Taking the \( L^2 \) norms of the right and the left hand sides of the first equation in (2), we obtain

\[
\|L_1 v\|_{L^2}^2 + \|L_2 v\|_{L^2}^2 + 2(L_1 v, L_2 v)_{L^2} = \|q_s\|_{L^2}^2.
\]

After integrations, we will arrive at the formula:

\[
(L_1 v, L_2 v)_{L^2} = \int_{\Omega \frac{1}{\sqrt{T}}} \left\{ \sum_{k,j=1}^{2} v_{x_j} v_{x_k} \varphi_{x_j x_k} + s^3 |\nabla \tilde{v}|^2 |\nabla \tilde{v}|^2 - s \sum_{j=1}^{2} \frac{\partial^2 \Delta \tilde{v}}{\partial x_j^2} |v|^2 \right\} d\tilde{x} - \int_{\partial \Omega} \left( \frac{\partial v}{\partial n} L_2 v \right) d\sigma + s \int_{\partial \Omega} \left( \frac{\partial v}{\partial n} \right)^2 (\nabla \tilde{v}, \tilde{n}) d\sigma + s \int_{\partial \Omega} (\nabla \tilde{v}, \tilde{n}) s^3 |\nabla \tilde{v}|^2 |v|^2 d\sigma.
\]

Denote \( \psi_1(x) = \psi(x) - \tilde{\ell}_1(x) \). Then

\[
\text{div} (|\nabla \tilde{v}|^2 |\nabla \tilde{v}|^2 - |\nabla \tilde{v}|^2 \Delta \tilde{v}) = 2 \sum_{k,j=1}^{2} \varphi_{x_k} \varphi_{x_j} \varphi_{x_k x_j}
\]

\[
= 2s^3 \sum_{k,j=1}^{2} \tau^4 (\partial_{x_k} \psi_1 + 2N \ell_1 \partial_{x_k} \ell_1)^2 (\partial_{x_j} \psi_1 + 2N \ell_1 \partial_{x_j} \ell_1)^2 + \tau^3 (\partial_{x_k} \psi_1 + 2N \ell_1 \partial_{x_k} \ell_1) (\partial_{x_j} \psi_1 + 2N \ell_1 \partial_{x_j} \ell_1) (\partial_{x_k} \ell_1)^2 + 2N (\partial_{x_k} \ell_1) \partial_{x_j} \ell_1 + 2N \ell_1 (\partial_{x_k} \ell_1) \ell_1.
\]
Since \((\nabla_x \psi_1, \nabla_x \ell_1) > 0\) on \(\partial \Omega\), there exists a constant \(C_1 > 0\) independent of \(N, \tau, s\), such that

\[
\text{div}(|\nabla_x \varphi|^2 \nabla_x \varphi) - |\nabla_x \varphi|^2 \Delta_x \varphi \geq 2\varphi^3 \tau^4 |\nabla_x \psi_1|^4 + C_1 N \tau^3 \varphi^3 + \varphi^2 O(\tau^3).
\]

On the other hand,

\[
\sum_{k,j=1}^{2} v_{x_j} v_{x_k} \varphi_{x_j x_k} = \tau^2 (\nabla_x \varphi, \nabla_x \varphi)^2 \varphi
\]

\[
+ 2\tau \sum_{k,j=1}^{2} v_{x_j} v_{x_k} (\psi_{x_j x_k} + N \ell_k \partial_{x_j x_k} \ell_1) \varphi - \frac{\tau \varphi}{\sqrt{N}} \sum_{j,k=1}^{2} v_{x_j} v_{x_k} \partial_{x_j x_k} \ell_1
\]

\[
+ N \tau (\nabla_x \varphi, \nabla_x \ell_1)^2 \varphi.
\]

Note that there exists a constant \(C_2 > 0\), independent of \(N\), such that

\[
\|N \ell_1 \partial_{x_j x_k} \ell_1\|_{C^0 \left(\frac{\Omega}{\sqrt{N}}\right)} \leq \frac{C_2}{N}
\]

By (3)–(6) we obtain

\[
\|L_1 \mathbf{v}\|_{L^2 \left(\frac{\Omega}{\sqrt{N}}\right)} + \|L_2 \mathbf{v}\|_{L^2 \left(\frac{\Omega}{\sqrt{N}}\right)} + \int_{\Omega \left(\frac{\Omega}{\sqrt{N}}\right)} (2\varphi^3 \tau^4 |\nabla_x \psi_1|^4 + C_3 N \tau^3 \varphi^3)|\mathbf{v}|^2 d\bar{x}
\]

\[
- \sigma C_4 \int_{\Omega \left(\frac{\Omega}{\sqrt{N}}\right)} \varphi |\nabla_x \mathbf{v}|^2 d\bar{x}
\]

\[
\leq \|q_s\|_{L^2 \left(\frac{\Omega}{\sqrt{N}}\right)}^2 + C_5 \left( s \|\mathbf{v}\|_{H^{1, r}(\partial \Omega)} + s \left\| \frac{\partial \varphi}{\partial n} \right\|_{L^2(\partial \Omega)}^2 \right).
\]

Multiplying the first equation in (2) by \(s N \varphi \mathbf{v}\) and integrating by parts, we obtain

\[
\int_{\Omega \left(\frac{\Omega}{\sqrt{N}}\right)} \left\{ s N \nabla_x \varphi |\mathbf{v}|^2 + s^2 N (\Delta_x \varphi) \varphi |\mathbf{v}|^2 - s^3 \varphi^3 |\nabla_x \varphi|^2 |\mathbf{v}|^2 - \frac{s N}{2} \text{div} \varphi |\mathbf{v}|^2 \right\} d\bar{x}
\]

\[
+ \int_{\partial \Omega} \left\{ - \left( \frac{\partial \mathbf{v}}{\partial n}, s N \varphi \mathbf{v} \right) + \left( s^2 \varphi N + \frac{s N}{2} \right) (\nabla_x \varphi, \vec{n}) |\mathbf{v}|^2 \right\} d\sigma
\]

\[
= \int_{\Omega \left(\frac{\Omega}{\sqrt{N}}\right)} q_s s N \varphi \mathbf{v} d\bar{x}.
\]
Next we note that
\[ \Delta \tilde{\varphi} = (|\nabla \tilde{\varphi}|^2)^2 + \tau \Delta \tilde{\varphi} \psi_1 + 2\tau N |\nabla \tilde{\varphi}|^2 + 2\tau N \Delta \tilde{\varphi} \psi_1 \geq C_6 \tau N \varphi. \]
This inequality and (8) imply
\[
\int_{\Omega_N} \left( sN|\nabla \tilde{\varphi}|^2 + \frac{1}{2}s^2 N(\Delta \tilde{\varphi})\varphi|v|^2 - s^3 \varphi^2|\nabla \tilde{\varphi}|^2|v|^2 \right) \, d\tilde{x}
\leq C_6 \|q_s\|^2_{L^2\left(\Omega, \frac{1}{\tau}\right)} + C_6 \left( s\|v\|^2_{H^{1.5}(\partial \Omega)} + s \left\| \frac{\partial v}{\partial n} \right\|^2_{L^2(\partial \Omega)} \right).
\]
In view of (7) and (9), we obtain
\[
\|L_1 v\|^2_{L^2\left(\Omega, \frac{1}{\tau}\right)} + \|L_2 v\|^2_{L^2\left(\Omega, \frac{1}{\tau}\right)} + \int_{\Omega_N} \left( \frac{1}{2}s^3 \tau^4 |\nabla \tilde{\varphi}|^4 \right) |v|^2 \, d\tilde{x} + sN \int_{\Omega_N} \varphi|\nabla \tilde{\varphi}|^2 |v|^2 \, d\tilde{x}
\leq C_7 \|q_s\|^2_{L^2\left(\Omega, \frac{1}{\tau}\right)} + C_8 \left( s\|v\|^2_{H^{1.5}(\partial \Omega)} + s \left\| \frac{\partial v}{\partial n} \right\|^2_{L^2(\partial \Omega)} \right).
\]
Let \( v = \tilde{v}_1 + \tilde{v}_2 \) where
- \( \Delta \tilde{v}_1 = L_1 v \) in \( \Omega, \tau \), \( \tilde{v}_1|_{\partial \Omega, \tau} = v \),
- \( \Delta \tilde{v}_2 = s^2|\nabla \tilde{\varphi}|^2 v \) in \( \Omega, \tau \), \( \tilde{v}_2|_{\partial \Omega, \tau} = 0 \).

By standard a priori estimates for the Laplace operator, we have
\[
\|\tilde{v}_1\|_{H^2\left(\Omega, \frac{1}{\tau}\right)} \leq C_9 (\|L_1 v\|_{L^2\left(\Omega, \frac{1}{\tau}\right)} + \|v\|_{H^{2.5}(\partial \Omega)}),
\]
\[
\sqrt{\frac{N}{s}} \|\tilde{v}_2\|_{H^2\left(\Omega, \frac{1}{\tau}\right)} \leq C_{10} \sqrt{N} s \|\nabla \tilde{\varphi}|^2 v\|_{L^2\left(\Omega, \frac{1}{\tau}\right)},
\]
where \( C_9, C_{10} > 0 \) are independent of \( N \). Taking \( s_0(\tau, N) \geq N \), from (9)–(12) we obtain
\[
N \sum_{|\alpha| = 0, \alpha \neq (0,0,0)} 2^{s-2|\alpha|} \left\| (\partial^\alpha u) e^{s\varphi} \right\|^2_{L^2(Q)} \leq C_{11} \left\{ \left\| (\text{div } u) e^{s\varphi} \right\|^2_{L^2(Q)} + s^2 \|v\|^2_{H^{1.5}(\partial Q)} + s^2 \left\| \frac{\partial v}{\partial n} \right\|^2_{L^2(\partial Q)} + N\|v\|^2_{L^2(0,T; H^{3/2}(\partial \Omega))} \right\}.
\]
By (1.8) and (1.9), we estimate the norm of $\partial^2_{x_0} u$:

\begin{equation}
N \|(\partial^2_{x_0} u)e^{s^2}\|_{L^2(Q)}^2 \leq C_{12} \left\{ \|(\text{div } u)e^{s^2}\|_{L^2(Q)}^2 + s^2 \|v\|^2_{H^{1/2}(\partial Q)} + s^2 \left| \frac{\partial v}{\partial n} \right|^2_{L^2(\partial Q)} + N\|v\|^2_{L^2(0,T;H^{1/2}H(\partial Q))} + N\|v\|^2_{L^2(Q)} \right\}.
\end{equation}

Finally we note

\begin{equation}
N s^2\|(\partial_{x_0} u)e^{s^2}\|_{L^2(Q)}^2 \leq C_{13}(N \|(\partial^2_{x_0} u)e^{s^2}\|_{L^2(Q)}^2 + s^4 N \|u e^{s^2}\|_{L^2(Q)}^2)
\end{equation}

and

\begin{equation}
N \|\partial_{x_k} v\|_{L^2(Q)}^2
\leq C_{14} \left( N\|\partial_{x_0} v\|_{L^2(Q)}^2 + \|\partial_{x_k} v\|_{L^2(Q)}^2 + \left\| \left( v, \frac{\partial v}{\partial n} \right) \right\|^2_{H^{1/2}((\partial Q) \times H^{1/2}((\partial Q))} \right).
\end{equation}

Thus the proof of Proposition 2.1 is completed. \hfill \Box

\section*{Appendix II. Proof of Proposition 2.4}

Let us consider the following problem

\begin{equation}
L^* p = \left( -\frac{\partial}{\partial y_2} - \tilde{\Gamma}^+_{ij}(y,s,D') \right) p = i\chi_0 w \quad \text{in } G,
\end{equation}

where $\beta \in \{\mu, \lambda + 2\mu\}$ and $\tilde{\Gamma}^+_{ij}$ is the operator which is formally adjoint to $\tilde{\Gamma}^+_{ij}$. We have

**Lemma 1.** There exist constants $C_{1} > 0$ and $s_0 > 0$ such that for every $s \geq s_0$, there exists $p$ satisfying (1) and

\begin{equation}
s \int_{G} |p|^2 dy + \sqrt{s} \int_{G \setminus \{y_2 = 0\}} |p(y',0)|^2 dy' \leq C_1 \int_{G} |\chi_0 w|^2 dy.
\end{equation}

**Proof of Lemma 1.** For $\epsilon > 0$, let us consider the functional:

\begin{equation}
J_{\epsilon}(p) = \frac{1}{2} \|p\|_{L^2(G)}^2 + \frac{1}{2\epsilon} \left| \frac{\partial p}{\partial y_2} + \tilde{\Gamma}_{ij}(y,s,D') p + i\chi_0 w \right|_{L^2(G)}^2.
\end{equation}

Notice that there exists $p$ such that $J_{\epsilon}(p)$ is finite and, for example, we can set $p = 0$. We consider the minimization problem

\begin{equation}
\min_{p \in \mathcal{U}} J_{\epsilon}(p),
\end{equation}

where

\begin{equation}
\mathcal{U} = \left\{ p \mid \left| \frac{\partial p}{\partial y_2} + \tilde{\Gamma}_{ij}(y,s,D') p + i\chi_0 w \right|_{L^2(\Omega)}^2 \leq s \int_{\Omega} |p|^2 dx \right\}.
\end{equation}

\begin{equation}
\mathcal{U} = \left\{ p \mid \left| \frac{\partial p}{\partial y_2} + \tilde{\Gamma}_{ij}(y,s,D') p + i\chi_0 w \right|_{L^2(\Omega)}^2 \leq s \int_{\Omega} |p|^2 dx \right\}.
\end{equation}
where
\begin{equation}
U = \left\{ p \in L^2(\mathcal{G}) : \frac{\partial p}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p + i\chi_v w \in L^2(\mathcal{G}) \right\}.
\end{equation}

There exists a minimizing sequence \( \{p_n\}_{n=1}^\infty \) such that \( p_n \in U \) and
\begin{equation}
\lim_{n \to \infty} J_\epsilon(p_n) = \inf_{p \in U} J_\epsilon(p).
\end{equation}

Then the sequences \( \{p_n\} \) and \( \{\frac{\partial p_n}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_n + \chi_v w\} \) are bounded in \( L^2(\mathcal{G}) \). Therefore \( \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_n \) is bounded in \( L^2 \left( 0, 1; \frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2) \right) \) and \( \frac{\partial p_n}{\partial y_2} \), \( n \in \mathbb{N} \) are bounded in \( L^2 \left( 0, 1; \frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2) \right) \). Consequently we can extract a subsequence, still denoted by \( \{p_n\}_{n=1}^\infty \) such that
\begin{equation}
p_n \rightharpoonup p_\epsilon \ \text{weakly in} \ L^2(\mathcal{G}),
\end{equation}
\begin{equation}
\frac{\partial p_n}{\partial y_2} \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} \ \text{weakly in} \ L^2 \left( 0, \frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2) \right),
\end{equation}
\begin{equation}
\frac{\partial p_n}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_n + i\chi_v w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_\epsilon + i\chi_v w
\end{equation}
weakly in \( L^2 \left( 0, \frac{1}{N^2}; H^{-1}(\mathbb{R}^2) \right) \).

On the other hand, as \( \left\| \frac{\partial p_n}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_n + i\chi_v w \right\|_{L^2(\mathcal{G})} \) remains bounded, we have
\begin{equation}
\frac{\partial p_n}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_n + i\chi_v w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_\epsilon + i\chi_v w
\end{equation}
weakly in \( L^2(\mathcal{G}) \). Then \( p_\epsilon \) is a minimizer of \( J_\epsilon \), that is, \( p_\epsilon \in U \) and
\begin{equation}
J_\epsilon(p_\epsilon) = \inf_{p \in U} J_\epsilon(p).
\end{equation}

Writing the first order optimality conditions, we have for every \( r \in H^1(\mathcal{G}) \):
\begin{equation}
(J'_\epsilon(p_\epsilon), r) = 0.
\end{equation}
Let us define \( q_\epsilon \) by
\begin{equation}
q_\epsilon = \frac{1}{\epsilon} \left( \frac{\partial p_\epsilon}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')p_\epsilon + i\chi_v w \right).
\end{equation}
In view of (9), for every \( r \in H^1(\mathcal{G}) \), we see:
\begin{equation}
\int_{\mathcal{G}} p_\epsilon r dy + \int_{\mathcal{G}} q_\epsilon \left( \frac{\partial r}{\partial y_2} + \tilde{\Gamma}_\beta^{+,*}(y, s, D')r \right) dy = 0.
\end{equation}
Then \( q_e \) satisfies the following problem

\[
\frac{\partial q_e}{\partial y_2} - \tilde{\Gamma}_\beta^+(y, s, D') q_e = p_e \quad \text{in} \quad \mathcal{G},
\]

\[
q_e(y', 0) = 0, \quad q_e \left(y', \frac{1}{N^2}\right) = 0, \quad \forall y' \in \mathbb{R}^2.
\]

Denote \( L_1 = \frac{1}{2}(-\tilde{\Gamma}_\beta^+ - \tilde{\Gamma}_\beta^{+*, -}) \) and \( L_2 = \frac{\partial}{\partial y_2} + \frac{1}{2}(\tilde{\Gamma}_\beta^{+*, -} - \tilde{\Gamma}_\beta^+) \). Then we can rewrite (12) and (13) as follows.

\[
L q_e \equiv (L_1 + L_2) q_e = p_e \quad \text{in} \quad \mathcal{G}, \quad q_e(y', 0) = 0, \quad q_e \left(y', \frac{1}{N^2}\right) = 0, \quad y' \in \mathbb{R}^2.
\]

There exist constants \( C_2 > 0 \) and \( s_0 > 0 \) such that

\[
\|Lq_e\|_{L^2(G)}^2 + \|L_2 q_e\|_{L^2(G)}^2 + sN \int_{\mathcal{G}} |q_e|^2 dy \leq C_2 \|p_e\|_{L^2(G)}^2, \quad \forall s \geq s_0.
\]

Notice that \( L_1 q_e \in L^2(G) \) implies \( q_e \in L^2 \left(0, \frac{1}{N^2}; H^1(\mathbb{R}^2)\right)\), which implies \( \frac{\partial q_e}{\partial y_2} \in L^2(G) \) from (12). Now it follows from definition (10) of \( q_e \) that \( p_e \) satisfies

\[
\frac{\partial p_e}{\partial y_2} + \tilde{\Gamma}_\beta^{+*, -}(y, s, D') p_e = \epsilon q_e - i\chi_w w,
\]

which can be written as

\[
(L_2 - L_1)p_e = \frac{\partial p_e}{\partial y_2} + \tilde{\Gamma}_\beta^{+*, -}(y, s, D') p_e = \epsilon q_e - i\chi_w w.
\]

Multiplying (17) by \( q_e \) in \( L^2(G) \) and using the boundary conditions for \( q_e \), we obtain

\[
- \int_{\mathcal{G}} p_e (L_1 + L_2) q_e dy = \epsilon \int_{\mathcal{G}} |q_e|^2 dy - \int_{\mathcal{G}} i\chi_w w q_e dy,
\]

so that

\[
\int_{\mathcal{G}} |p_e|^2 dy + \epsilon \int_{\mathcal{G}} |q_e|^2 dy = \int_{\mathcal{G}} i\chi_w w q_e dy
\]

and

\[
\int_{\mathcal{G}} |p_e|^2 dy \leq \left( \int_{\mathcal{G}} |\chi_w w|^2 dy \right)^{\frac{1}{2}} \left( \int_{\mathcal{G}} |q_e|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq \frac{C_3}{s^{\frac{1}{2}}} \left( \int_{\mathcal{G}} |\chi_w w|^2 dy \right)^{\frac{1}{2}} \left( \int_{\mathcal{G}} |p_e|^2 dy \right)^{\frac{1}{2}}.
\]
Therefore we obtain the first estimate on \( p_e \)

\[
(20) \quad s \int_{\mathcal{G}} |p_e|^2 dy \leq c_1 \int_{\mathcal{G}} |\chi_{\nu} w|^2 dy.
\]

Let us now notice from (12) and (13) that we have

\[
p_e(y', 0) = \frac{\partial q_e}{\partial y_2}(y', 0), \quad y' \in \mathbb{R}^2.
\]

Let \( \theta = \theta(y_2) \in C^\infty[0, 1/2] \) such that \( 0 \leq \theta \leq 1, \theta(0) = 1 \) and \( \theta(1/2) = 0 \). We have

\[
(21) \quad (L_1 + L_2)(\theta q_e) = \theta (L_1 + L_2)(q_e) + \frac{\partial \theta}{\partial y_2} q_e = \theta p_e + \frac{\partial \theta}{\partial y_2} q_e \text{ in } \mathcal{G},
\]

\[
(22) \quad (\theta q_e)(y', 0) = 0, \quad (\theta q_e) \left( y', \frac{1}{N^2} \right) = 0, \quad \frac{\partial (\theta q_e)}{\partial y_2} \left( y', \frac{1}{N^2} \right) = 0, \quad y' \in \mathbb{R}^2.
\]

Now we apply the operator \((L_2 - L_1)\) to the first equation, so that

\[
(23) \quad (L_2 - L_1)(L_2 + L_1)(\theta q_e) = (L_2 - L_1)(\theta p_e) + (L_2 - L_1) \left( \frac{\partial \theta}{\partial y_2} q_e \right)
\]

\[
= \theta (L_2 - L_1)(p_e) + \frac{\partial \theta}{\partial y_2} p_e + \frac{\partial \theta}{\partial y_2} (L_2 - L_1)q_e + \frac{\partial^2 \theta}{\partial y_2^2} q_e.
\]

Then we have

\[
(24) \quad L_2^2(\theta q_e) - L_1^2(\theta q_e) + [L_2, L_1](\theta q_e)
\]

\[
= \epsilon \theta q_e - i \theta \chi_{\nu} w + \frac{\partial \theta}{\partial y_2} p_e + \frac{\partial \theta}{\partial y_2} (L_2 - L_1)q_e + \frac{\partial^2 \theta}{\partial y_2^2} q_e.
\]

We now take the scalar product in \( L^2(\mathcal{G}) \) of this equation with \( L_2(\theta q_e) \) and take the real part. Henceforth we give the computations of the successive terms. We can calculate the first term as follows.

\[
(25) \quad (L_2^2(\theta q_e), L_2(\theta q_e))_{L^2(\mathcal{G})} = \left( \frac{\partial}{\partial y_2} (L_2(\theta q_e)), L_2(\theta q_e) \right)_{L^2(\mathcal{G})}
\]

\[
- \frac{1}{2} \| [\Gamma^+(\beta)(y, s, D') - \Gamma^+_{\beta, \gamma}(y, s, D')] L_2(\theta q_e), L_2(\theta q_e) \|_{L^2(\mathcal{G})}^2
\]

\[
= - \| L_2(\theta q_e)(\cdot, 0) \|_{L^2(\mathbb{R}^2)}^2 - (L_2(\theta q_e), L_2^2(\theta q_e))_{L^2(\mathcal{G})}
\]

\[
= - \left\| \frac{\partial q_e}{\partial y_2}(\cdot, 0) \right\|_{L^2(\mathbb{R}^2)}^2 - (L_2(\theta q_e), L_2^2(\theta q_e))_{L^2(\mathcal{G})}.
\]
Therefore we have
\[(26) \quad \text{Re} (L_2^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} = -\frac{1}{2} \left\| \frac{\partial q_c}{\partial y_2}(\cdot, 0) \right\|_{L^2(\mathbb{R}^2)}^2.\]

Now for the second term
\[(27) \quad 2\text{Re} (L_2^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} = (L_2^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} + (L_2(\theta q_c), L_2^2(\theta q_c))_{L^2(\mathbb{G})}
\quad = -(L_2 L_1^2(\theta q_c), (\theta q_c))_{L^2(\mathbb{G})} + (L_1^2 L_2(\theta q_c), (\theta q_c))_{L^2(\mathbb{G})}
\quad = (L_1 [L_1, L_2](\theta q_c), (\theta q_c))_{L^2(\mathbb{G})} + ([L_1, L_2] L_1(\theta q_c), (\theta q_c))_{L^2(\mathbb{G})}
\quad = 2\text{Re} ([L_1, L_2](\theta q_c), L_1(\theta q_c))_{L^2(\mathbb{G})}.
\]

Here we notice that \(([L_1, L_2]u, v)_{L^2(\mathbb{G})} = (u, [L_1, L_2]v)_{L^2(\mathbb{G})}\).

We have already seen that
\[(28) \quad [L_1, L_2] = (N + 1)K(s),\]
where \(K \in C([0, 1]; \mathcal{L}(H^{1,\alpha}(\mathbb{R}^2), L^2(\mathbb{R}^2)))\) is some operator. Therefore
\[(29) \quad \left| \text{Re} (L_1^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} \right| \leq \|\theta q_c\|_{L^2(0, \mathbb{T}; H^{1,\alpha}(\mathbb{R}^2))} \|L_1(\theta q_c)\|_{L^2(\mathbb{G})}.
\]

We already know from (15) that
\[(30) \quad \|L_1(\theta q_c)\|_{L^2(\mathbb{G})} \leq \|L_1 q_c\|_{L^2(\mathbb{G})} \leq C_3 \|q_c\|_{L^2(\mathbb{G})}
\]
and from the definition of \(L_1\)
\[(31) \quad \|\theta q_c\|_{L^2(0, \mathbb{T}; H^{1,\alpha}(\mathbb{R}^2))} \leq \|q_c\|_{L^2(0, \mathbb{T}; H^{1,\alpha}(\mathbb{R}^2))} \leq C_6 (\|L_1 q_c\|_{L^2(\mathbb{G})} + \|q_c\|_{L^2(\mathbb{G})}).\]

Using again (15), we obtain
\[(32) \quad \left| \text{Re} (L_1^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} \right| \leq C_7 \sqrt{s} \|p_c\|_{L^2(\mathbb{G})}^2 + C_8 \|p_c\|_{L^2(\mathbb{G})},
\]
so that we have
\[(33) \quad \left| \text{Re} (L_1^2(\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} \right| \leq C_9 \sqrt{s} \|p_c\|_{L^2(\mathbb{G})}^2
\]
for \(s \geq s_0\).

Concerning the third term, we have
\[(34) \quad \left| \text{Re} ([L_2, L_1](\theta q_c), L_2(\theta q_c))_{L^2(\mathbb{G})} \right| \leq \|[L_2, L_1](\theta q_c)\|_{L^2(\mathbb{G})} \|L_2(\theta q_c)\|_{L^2(\mathbb{G})}.
\]
Using the form of \([L_2, L_1]\) we obtain

\[
(35) \quad \left| \text{Re} \left([L_2, L_1]([\theta q_e], L_2(\theta q_e))_{L^2(G)} \right) \right| \\
\leq C_9 N \|\theta q_e\|_{L^2(0, \frac{1}{N^2}; L^1(\mathbb{R}^2))} \|L_2(\theta q_e)\|_{L^2(G)}
\]

and, since \(L_2(\theta q_e) = \theta L_2 q_e + \frac{\partial \theta}{\partial y_2} q_e\), from (15) and (31) we have

\[
(36) \quad \left| \text{Re} \left([L_2, L_1]([\theta q_e], L_2(\theta q_e))_{L^2(G)} \right) \right| \leq C_{10} \sqrt{s} \|p_e\|_{L^2(G)}^2 + C_{11} N^3 \|p_e\|_{L^2(G)}^2
\]

for \(s \geq s_0\) and \(s_0 \geq N^8 + 1\).

On the other hand, for the right hand side of (24), we have

\[
\left| \text{Re} \left( \frac{e^{i\theta q_e - \theta i\chi \nu w} + \frac{\partial \theta}{\partial y_2} p_e + \frac{\partial \theta}{\partial y_2} (L_2 - L_1) q_e + \frac{\partial^2 \theta}{\partial y_2^2} q_e, L_2(\theta q_e) \right) \right|_{L^2(G)} \\
\leq C_{12} \|p_e\|_{L^2(G)} \|p_e\|_{L^2(G)} + C_{12} \|\chi \nu w\|_{L^2(G)} \|p_e\|_{L^2(G)} + C_{12} \|p_e\|_{L^2(G)}^2.
\]

Therefore we obtain

\[
(37) \quad \left| \text{Re} \left( \frac{e^{i\theta q_e - \theta i\chi \nu w} + \frac{\partial \theta}{\partial y_2} p_e + \frac{\partial \theta}{\partial y_2} (L_2 - L_1) q_e + \frac{\partial^2 \theta}{\partial y_2^2} q_e, L_2(\theta q_e) \right) \right|_{L^2(G)} \\
\leq C_{13} \|p_e\|_{L^2(G)}^3 + C_{13} \|\chi \nu w\|_{L^2(G)} \|p_e\|_{L^2(G)}.
\]

Putting together (28), (33), (36) and (37), we obtain the following estimate:

\[
(38) \quad \left\| \frac{\partial q_e}{\partial y_2} (\cdot, 0) \right\|_{L^2([G \cap (y_2 = 0)])}^2 \leq C_{14} \sqrt{s} \|p_e\|_{L^2(G)} + C_{14} \|\chi \nu w\|_{L^2(G)} \|p_e\|_{L^2(G)}.
\]

Now combining (15), (20) and (38), we can easily obtain

\[
(39) \quad s \int_G |p_e|^2 dy + \sqrt{s} \int_{\mathbb{R}^2} |p_e(y', 0)|^2 dy' \leq C_{16} \int_G |\chi \nu w|^2 dy,
\]

which is estimate (2) for \(p_e\).

Now \(p_e\) and \(\frac{\partial p_e}{\partial y_2} + \Gamma^{\nu_{s,D}}_y(y, s, D') p_e\) are bounded in \(L^2(G)\) uniformly in \(\epsilon\).

After extraction of a subsequence (still denoted by \(p_e\)), we can assume that

\[
(40) \quad p_e \rightharpoonup p \quad \text{weakly in } L^2(G), \\
\frac{\partial p_e}{\partial y_2} \rightharpoonup \frac{\partial p}{\partial y_2} \quad \text{weakly in } L^2 \left(0, \frac{1}{N^2}; H^{-1}(\mathbb{R}^2)\right),
\]

for
so that
\[ p_\kappa(\cdot, 0) \to p(\cdot, 0) \text{ weakly in } H^{\frac{1}{2}}(\mathbb{R}^2). \]
Since \( p_\kappa(\cdot, 0) \) remains bounded in \( L^2(\mathbb{R}^2) \), we also have \( p_\kappa(\cdot, 0) \to p(\cdot, 0) \) weakly in \( L^2(G \cap \{y_2 = 0\}) \). By (15) and (40), we easily see that \( p \) satisfies
\[ L^* p = \left( -\frac{\partial}{\partial y_2} - \tilde{\Gamma}^+, s, D' \right) p = i\chi_\nu w \text{ in } G, \]
which is (1), and from (39) we see that
\[ s \int_G |p|^2 dy + \sqrt{s} \int_{G \cap \{y_2 = 0\}} |p(y', 0)|^2 dy' \leq C_{16} \int_G |\chi_\nu w|^2 dy, \]
which is (2). The proof of Lemma 1 is now completed.

We take the scalar product of equation (1) and the function \( \chi_\nu w \) in \( L^2(G) \)
\[ \|\chi_\nu w\|_{L^2(G)}^2 = (g, p)_{L^2(G)} + (p(\cdot, 0), \chi_\nu w(\cdot, 0))_{L^2(\mathbb{R}^2)}. \]
Applying estimate (2) to this equality, we have
\[ \|\chi_\nu w\|_{L^2(G)}^2 \leq C_{17} \left( \frac{1}{\sqrt{s}} \|g\|_{L^2(G)} \|\chi_\nu w\|_{L^2(G)} + \frac{1}{s^4} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \|\chi_\nu w\|_{L^2(G)} \right). \]
Therefore
\[ (41) \quad \sqrt{s}\|\chi_\nu w\|_{L^2(G)} \leq C_{17} (\|g\|_{L^2(G)} + s^{\frac{1}{4}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)}), \quad \forall s \geq s_0. \]
If \( s^* \neq 0 \), then estimate (41) imply (2.20) immediately. If \( \text{Im } \alpha^*(\gamma) \neq 0 \), then we obtain
\[ \|\chi_\nu w\|_{L^2(0, \frac{1}{s^*}, H^{1, -\cdot}(\mathbb{R}^2))} \leq C_{18} \left( \frac{1}{\sqrt{s}} \|g\|_{H^{1, -\cdot}(G)} + s^{\frac{1}{4}} \|\chi_\nu w(\cdot, 0)\|_{H^{1, -\cdot}(\mathbb{R}^2)} \right). \]
Thus the proof of Proposition 2.4 is finished.

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Carleman Estimate for Lamé System