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The Orbibundle Miyaoka-Yau-Sakai Inequality and an Effective Bogomolov-McQuillan Theorem

Dedicated to Professor Heisuke Hironaka on his 77th Birthday

By

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Abstract

Let X be a minimal projective surface of general type defined over the complex numbers and let $C \subset X$ be an irreducible curve of geometric genus g. Given a rational number $\alpha \in [0,1]$, we construct an orbibundle $\tilde{\mathcal{E}}_\alpha$ associated with the pair (X, C) and establish the Miyaoka-Yau-Sakai inequality for $\tilde{\mathcal{E}}_\alpha$. By varying the parameter α in the inequality, we derive several geometric consequences involving the "canonical degree" CK_X of C. Specifically we prove the following two results. (1) If K_X^2 is greater than the topological Euler number $c_2(X)$, then CK_X is uniformly bounded from above by a function of the invariants g, K_X^2 and $c_2(X)$ (an effective version of a theorem of Bogomolov-McQuillan). (2) If C is nonsingular, then $CK_X \leq 3g - 3 + o(g)$ when g is large compared to $K_X^2, c_2(X)$ (an affirmative answer to a conjecture of McQuillan).

§1. Introduction

In 1977, F. Bogomolov [1] showed that, given a pair (g, X) of a non-negative integer g and a minimal complex projective surface X of general type with $K_X^2 > c_2(X)$, the irreducible curves of genus g on X form a bounded family. In particular, such X contains only finitely many rational/elliptic curves.

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Bogomolov's proof (see also the expository article [3]) involves beautiful ideas, eventually leading to M. McQuillan's partial solution [8] of the Green-Griffiths conjecture [4] concerning the algebraicity and finiteness of entire holomorphic curves (*i.e.,* non-constant holomorphic images of $\mathbf{C}$) on a surface of general type.

Unfortunately, the result of Bogomolov-McQuillan is not effective. For instance, when the ambient surface X deforms in an analytic family, their theory cannot rule out the possibility that the number of rational/elliptic curves therein tends to infinity. In this note we clear up this problem by proving that the *canonical degree* of an irreducible curve of genus g on X is bounded from above by a function of g, K_X^2 and $c_2(X)$:

Theorem 1.1 (Uniform bound of the canonical degree). *Let X be a minimal projective surface of general type defined over $\mathbf{C}$ and let $C \subset X$ be an irreducible curve of geometric genus g. If $K^2 = K_X^2 > c_2 = c_2(X)$, then the canonical degree $CK = CK_X$ of C is bounded by $a(g-1)+b$, where a and b are functions of K^2 and c_2. When C is not a smooth rational curve, we can choose the functions as follows*:

$$a = \frac{2K^2 + \sqrt{2\left(K^2\right)\left(3c_2 - K^2\right)}}{K^2 - c_2},$$
$$b = \frac{\left(K^2\right)\left(3c_2 - K^2\right) + c_2\sqrt{2\left(K^2\right)\left(3c_2 - K^2\right)}}{2\left(K^2 - c_2\right)}.$$

Corollary 1.2 (Uniformity of the number of rational/elliptic curves). *Let X be a minimal projective surface of general type over $\mathbf{C}$. If $K_X^2 > c_2(X)$, then the number of irreducible curves of genus ≤ 1 on X is bounded by a function of K_X^2 and $c_2(X)$.*

Theorem 1.1 is a direct consequence of the following

Theorem 1.3 (Orbibundle Miyaoka-Yau-Sakai inequality for the pair (X, C)). *Let X be a surface of non-negative Kodaira dimension* (*i.e., X is not a ruled surface*) *and let C be an irreducible curve of genus g on it.*

(i) *If α is a real number with $0 \leq \alpha \leq 1$, then the inequality*

$$\frac{\alpha^2}{2}\left(C^2 + 3CK_X - 6g + 6\right) - 2\alpha(CK_X - 3g + 3) + 3c_2(X) - K_X^2 \geq 0$$

holds.

(ii) *Assume that C is not isomorphic to $\mathbf{P}^1$ and that $CK_X > 3g-3$. Then*

$$2(CK_X - 3g + 3)^2 - (3c_2(X) - K_X^2)(C^2 + 3CK_X - 6g + 6) \leq 0.$$

(iii) *Assume that K_X is nef and $K_X^2 > 0$ (i.e., X is a minimal surface of general type). Put $x = CK_X/K_X^2$, $\sigma = c_2(X)/K_X^2$, $\gamma = (g-1)/K_X^2$, $y^2 = -(C - xK_X)^2/K_X^2$* (we have $x \geq 0$, $\sigma \geq 1/3$, $\gamma \geq -1$, $y^2 \geq 0$ by the minimality of X, the Miyaoka-Yau inequality [10], the definition of the genus and the Hodge index theorem). *If $C \not\simeq \mathbf{P}^1$ and $x > 3\gamma$, then we have the inequality*

$$(1) \qquad (\sigma - 1)x^2 + (4\gamma + 3\sigma - 1)x - 2\gamma(3\gamma + 3\sigma - 1) \geq \left(\sigma - \frac{1}{3}\right) y^2.$$

When $\sigma < 1$ (i.e., $c_2(X) < K_X^2$), the left-hand side $P(x)$ of (1) is a quadratic polynomial in x with negative leading coefficient, and hence x is bounded from above by the larger root $R_+(\sigma,\gamma)$ of $P(x)$ or by 3γ:

$$(2) \qquad x \leq \max\{3\gamma, R_+(\sigma,\gamma)\} = \frac{2 + \sqrt{6\sigma - 2}}{1 - \sigma}\gamma + o(\gamma).$$

Theorem 1.3 says something new about nonsingular curves as well. If C is nonsingular with $C^2 + CK_X = 2g-2$, then the assertion (ii) yields an inequality, which seems to have been unknown to date (see Remarks D and F below):

Corollary 1.4. *Let X be a surface with nef canonical divisor. If C is a nonsingular curve of genus $g \geq 1$ on X, then*

$$(3) \qquad CK \leq 3g - 3 + \frac{\sqrt{3c_2 - K^2}\sqrt{4g - 4 + 3c_2 - K^2}}{2} + \frac{3c_2 - K^2}{2},$$

where K and c_2 are the abbreviations of K_X and $c_2(X)$. Thus, when g is very large, the canonical degree CK of a non-singular curve C of genus g on X is bounded roughly by $3g-3$.

Remarks. **A.** Corollary 1.4 is true also for $g = 0$ with minor modifications. A smooth rational curve C on a minimal non-ruled surface X has negative self-intersection $-n \leq -2$ and can be contracted to a cyclic quotient singularity. Then [11] Theorem 1.1 tells us that

$$CK = n - 2 \leq \left\lfloor 3c_2 - K^2 - 4 - \frac{1}{n} \right\rfloor = 3c_2 - K^2 - 5.$$

B. There is an essential difference between our approach and the original method of Bogomolov-McQuillan.

The basic idea of Bogomolov-McQuillan is to look at (a) the projective bundle $\pi\colon \mathbf{P}(\Omega^1_X) \to X$ with the tautological divisor $\mathbf{1}$, (b) an effective divisor $F \in |n\mathbf{1} - m\pi^* K_X|$, and (c) the rational map $\sigma\colon C \to \mathbf{P}(\Omega^1_X)$ which is induced by the natural homomorphism $\Omega_X|_C \to \omega_C$ defined on the non-singular locus of C. Unless $\sigma(C) \subset F$ (*i.e.*, C is a leaf of the multi-valued foliation induced by F), the intersection number $\sigma(C)F$ is non-negative, whence follows the inequality $mCK_X \leq n(2g-2)$. Extra analytic techniques (Jouanolou's theorem [5] in Bogomolov's paper and Nevanlinna theory for foliations in McQuillan's) take care of the exceptional case $\sigma(C) \subset F$. The hypothesis $K_X^2 > c_2(X)$ appears as a guarantee of the existence of some $F \in |n\mathbf{1} - m\pi^* K_X|$ for $n \gg m > 0$.

Our proof of Theorem 1.3 does not rely on the existence of an effective member $F \in |n\mathbf{1} - m\pi^* K_X|$ as above, but depends on old inequalities of [10], [11]. What is new in the present note is more or less of technical nature: the systematic use of *orbibundles* (although [11] already used the notion implicitly), along with an elementary reduction process ("*G-nef reduction*") for vector bundles of special type (Lemma 2.3). Theorem 1.3 is exactly the Miyaoka-Yau-Sakai inequality [11] applied to a family of orbibundles $\widetilde{\mathcal{E}}_\alpha$ with parameter $\alpha \in [0,1] \cap \mathbf{Q}$. The precise construction of $\widetilde{\mathcal{E}}_\alpha$ via G-nef reduction process will be elaborated in Section 3 below.

C. In connection with S. Kobayashi's complex hyperbolic geometry and P. Vojta's value distribution theory, S. Lang [6] conjectured that the union of all the entire holomorphic curves (rational/elliptic curves, for example) on a variety of general type should be contained in a certain proper algebraic closed subset. L. Caporaso, J. Harris and B. Mazur [2] pointed out that Lang's conjecture in arbitrary dimension would entail Corollary 1.2 (uniform finiteness of rational/elliptic curves on surfaces) even when the assumption $K^2 > c_2$ is dropped (their argument does not give any *explicit* bound, though). It is yet to be seen whether the assumption is really necessary or redundant. It might be any way noteworthy that Bogomolov-McQuillan's proof and ours alike require the very same condition $K^2 > c_2$ in quite different contexts.

D. When the curve C is nonsingular of genus g on a non-ruled surface X, the Miyaoka-Yau-Sakai inequality [13], [11] for the open surface $X \setminus C$ reads

$$CK_X \leq 4g - 4 + 3c_2(X) - K_X^2,$$

regardless of the ratio $\sigma = c_2(X)/K_X^2$ (our Corollary 1.4 improves this inequality for $g \geq 2$). The smoothness hypothesis on C was marginally relaxed in [7]. Without the assumption $K^2 > c_2$, we still have a similar bound for CK_X provided C contains neither ordinary double points nor ordinary triple points. Strangely, complicated singularities of high multiplicity do no harm to estimating canonical degrees. Curves with many ordinary double points are technically the most difficult to deal with.

E. Consider a sufficiently ample divisor H on a surface X. In the complete linear system $|H|$ of dimension $(H^2 - HK_X + 2\chi(X, \mathcal{O}_X) - 2)/2$, the curves with m ordinary double points form a locally closed subset of codimension $\leq m$. Hence we have a crude estimate of the minimum g_H of the geometric genera of the members of $|H|$:

$$g_H \leq HK_X - \frac{K_X^2 + c_2(X)}{12} + O(1).$$

To put it another way, the supremum

$$\mathfrak{d}(X, g) = \sup\{CK_X \mid C \subset X, g(C) = g\} \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$$

of the canonical degrees of curves of genus g is expected to satisfy

$$\mathfrak{d}(X, g) \geq g + \frac{K_X^2 + c_2(X)}{12} + O(1). \tag{4}$$

This lower bound, depending only on the genus and the topology of X, will be very crude; it may well happen that a curve of low genus and high degree suddenly emerges as the complex structure of X varies. Such possibilities taken into account, our uniform estimates (1) and (2) would be fairly nice.

A good example is the extremal case $\sigma = 1/3$. In this situation, the left-hand side of the inequality (1) is $(2/3)(x - 3\gamma)^2$ so that Theorem 1.3 boils down to the simple linear estimate $CK_X \leq 3g - 3$, which happens to be identical with the estimate (3) stated exclusively for nonsingular curves.

F. The referee informed the authour of an extremely interesting preprint [9] of McQuillan. The linear estimate of Theorem 1.1 was already proved there through the combination of the complete classification of foliated surfaces with his previous result [8]. However, his method does not produce explicit descriptions of the coefficients a, b, apart from its dependence on a very hard classification theorem. In the same preprint, an asymptotic behaviour

$$K_X C \leq (3 + \varepsilon)(g - 1) + o(g)$$

of the canonical degree of smooth curves is stated as a conjecture based on certain properties of normalized holomorphic sectional curvature. Theorem 1.4 confirms his conjecture with well controlled correction terms.

G. In the subsequent proof, we do not use the irreducibility of C. Theorem 1.3 (i) stays true for reduced reducible curve $C = \sum C_i$ if we define the genus $g = g(C)$ by

$$g(C) - 1 = \sum (g(C_i) - 1).$$

Conventions. In this note we work in the category of complex algebraic varieties (or, more generally, varieties defined over an algebraically closed field of characteristic zero). All the surfaces will be non-singular and projective unless otherwise mentioned. Chern classes of coherent sheaves (or of elements of the Grothendieck group $\mathrm{K}(\cdot)$) are regarded as elements in the real Betti cohomology ring $H^*(\cdot, \mathbf{R})$. By the (geometric) genus of an irreducible curve, we mean the genus of its normalization. Effective divisors are often identified with closed subschemes of pure codimension one via the correspondence $A \mapsto \operatorname{Spec}(\mathcal{O}/\mathcal{O}(-A))$.

§2. G-Nef Reductions of Certain Vector Bundles

In this section, we formulate a couple of technical but elementary results, which allow us to modify vector bundles of certain type into much simpler ones, without changing their second Chern classes.

Let Z be a surface. A finite sum $G = \sum G_i$ of irreducible curves $G_i \subset Z$ is said to be a *negative definite cycle* if the intersection matrix $(G_i G_j)_{ij}$ is negative definite. From the definition, it follows that the components G_i of a negative definite cycle G must be mutually distinct, *i.e.*, G is reduced. Given a surjective morphism $f \colon Z \to Y$ onto another surface, the f-exceptional locus (the union of the curves which f contracts to points) is a typical example of negative definite cycles.

Proposition 2.1 (The Zariski decomposition with support in a negative definite cycle). *Let $G = \sum G_i \subset Z$ be a negative definite cycle and let A be an effective $\mathbf{Q}$-divisor on Z. Then there exists a unique decomposition $A = P + N$ into $\mathbf{Q}$-divisors which satisfies the following four conditions*:

(a) *Both P and N are effective: $P \geq 0$, $N \geq 0$.*

(b) *N is supported by a subset of G, i.e., $N = \sum \nu_i G_i$, $\nu_i \geq 0$.*

(c) *P is nef on G, i.e., $PG_i \geq 0$ for every i.*

(d) *P and N are mutually orthogonal, i.e., $PN = 0$ (hence $A^2 = P^2 + N^2$ and, in view of* (c)*, P is numerically trivial on N, i.e., $PG_i = 0$ for each $G_i \subset \operatorname{supp} N$).*

Furthermore, P is the largest effective $\mathbf{Q}$-divisor $\leq A$ that is nef on G:

(e) *If a $\mathbf{Q}$-divisor B with $0 \leq B \leq A$ is nef on G, then $B \leq P$.*

Definition. The unique decomposition $A = P + N$ as above is said to be the *Zariski decomposition with support in G*. We call the $\mathbf{Q}$-divisors P and N the *G-nef part* and the *G-negative part* of A, respectively.

This definition generalizes classical Zariski decompositions [15]. Indeed, the classical decomposition $A = \overline{P} + \overline{N}$ is the decomposition with support in $\overline{N}$. Among the Zariski decompositions of a fixed divisor A with support in various negative cycles, the classical one is characterized as the decomposition which has the largest negative part and the smallest positive part (see Corollary 2.2 (iii) below). In order to avoid unnecessary confusions, we hereafter call the classical Zariski decomposition the *absolute* Zariski decomposition.

Proposition 2.1 is proved in exactly the same manner (essentially an exercise of linear algebra) as in the case of absolute Zariski decompositions [15], [12].

Corollary 2.2. (i) *Take two effective $\mathbf{Q}$-divisors A, A' and let $A = P + N$, $A' = P' + Q'$ be the decompositions into the G-nef parts and the G-negative parts. If $A \leq A'$, then $P \leq P'$.*

(ii) *Let G and $\widehat{G}$ be two negative definite cycles and let $A = P + N = \widehat{P} + \widehat{N}$ be the Zariski decompositions with supports in G and $\widehat{G}$. If $G \leq \widehat{G}$, then $N \leq \widehat{N}$, $P \geq \widehat{P}$. Furthermore, we have the inequalities between self-intersection numbers: $0 \geq N^2 \geq \widehat{N}^2$, $P^2 \leq \widehat{P}^2$.*

(iii) *Let $A = P + N$ be the Zariski decomposition with support in G and let $A = \overline{P} + \overline{N}$ denote the absolute Zariski decomposition. Then we have $N \leq \overline{N}$, $P \geq \overline{P}$, $0 \geq N^2 \geq \overline{N}^2$, $P^2 \leq \overline{P}^2$.*

Proof. (i) is a direct consequence of the property (e) of P'. Let us prove (ii). By definition, $\widehat{P}$ is nef on $\widehat{G}$ and hence on G, so that $\widehat{P} \leq P$ and $\widehat{N} \geq N$. In particular, supp $\widehat{N} \geq$ supp N. Let $\widehat{V}$ [resp. V] denote the subspace of $H^2(Z, \mathbf{R})$ generated by the curves in $\operatorname{supp} \widehat{N}$ [resp. $\operatorname{supp} N$]. Then $\widehat{P}$ [resp. P]

sits in the orthogonal complement $\widehat{V}^\perp$ [resp. $V^\perp$], while $R = P - \widehat{P} = \widehat{N} - N \in V^\perp \cap \widehat{V}$. Hence $P = \widehat{P} + R \in \widehat{V}^\perp \oplus (V^\perp \cap \widehat{V})$ is an orthogonal decomposition in $V^\perp$ and so is $\widehat{N} = N + R \in V \oplus (V^\perp \cap \widehat{V})$ in $\widehat{V}$. Noting that $R^2 \leq 0$, we get (ii). The proof of (iii) is quite similar to that of (ii). $\square$

Take a surjective morphsm f: $Z \to Y$ between nonsingular projective surfaces and denote by $G \subset Z$ the f-exceptional locus. In what follows we assume that G is a divisor of simple normal crossings (this condition is automatic when f is birational). Let $\Delta \subset Z$ be an effective, reduced, normal crossing divisor which contains G.

Pick up an effective reduced divisor Γ on Y such that $G \subset f^{-1}(\Gamma) \subset \Delta$. The image $f(G)$ of G in Y is a finite subset of Γ and therefore we can find an affine open neighbourhood $U \subset Y$ of $f(G)$ on which Γ is defined by a single equation φ. Since the inverse image $f^{-1}(\Gamma)$ is supported by a normal crossing divisor $\subset \Delta$, the pull back $f^*\varphi$ of its defining equation is of the form $(\text{unit})z_1^a z_2^b$, where $a \geq 0, b \geq 0, a + b > 0$ and z_1, z_2 are local coordinates of Z around a point $q \in G \subset f^{-1}(\Gamma)$. Thus the logarithmic 1-form

$$f^*\mathrm{d}\log\varphi = a\frac{\mathrm{d}z_1}{z_1} + b\frac{\mathrm{d}z_2}{z_2} + (\text{regular 1-form}), \qquad (a, b) \neq (0, 0)$$

is a nowhere-vanishing section of $\Omega^1_Z(\log\Delta)|_{f^{-1}(U)}$ near G.

Lemma 2.3. *Let $Y, Z, f, G, \Gamma, \Delta, d\log\varphi$ be as above. Assume that a vector bundle $\mathcal{E}$ of rank two on Z satisfies the following four conditions*:

(a) $\mathcal{E} \subset \Omega^1_Z(\log\Delta)$.

(b) *The determinant divisor $D = \det\mathcal{E}$ is effective* (as a $\mathbf{Q}$-divisor).

(c) *The G-nef part P and the G-negative part N of D are integral divisors.*

(d) *$\mathcal{E}$ contains $f^*\mathrm{d}\log\varphi \in \Omega^1_Z(\log\Delta)$ on a certain neighborhood V of G.*

Then, after shrinking V to a smaller neighborhood if necessary, we have an exact sequence $0 \to \mathcal{O}_V \to \mathcal{E}|_V \to \mathcal{O}_V(\det\mathcal{E}) \to 0$ and there is a standard procedure to construct a new vector bundle $\widetilde{\mathcal{E}}$ on Z which satisfies the following three conditions:

i) $\widetilde{\mathcal{E}} \subset \mathcal{E}$ globally on Z and $\widetilde{\mathcal{E}} = \mathcal{E}$ outside G;

ii) $\det\widetilde{\mathcal{E}} = P$;

iii) $c_2(\widetilde{\mathcal{E}}) = c_2(\mathcal{E})$.

Proof. The logarithmic 1-form $f^* d \log \varphi$ gives a nowhere vanishing global section of $\mathcal{E}|_V$ and hence an injection $\mathcal{O}_V \to \mathcal{E}|_V$, of which the cokernel is locally free and isomorphic to $\det \mathcal{E}|_V$.

We regard N as the subscheme determined by the ideal $\mathcal{O}_Z(-N)$. Consider the composite of the natural projections $\mathcal{E} \to \mathcal{E}|_V \to \mathcal{O}_V(\det \mathcal{E}) \to \mathcal{O}_N(\det \mathcal{E})$ and define $\widetilde{\mathcal{E}}$ to be the kernel of this composite map. We have the equality $c(\widetilde{\mathcal{E}})c(\mathcal{O}_N(\det \mathcal{E})) = c(\mathcal{E})$ in $H^*(Z, \mathbf{R})$. Since $\det \mathcal{E} = P + N$, P being numerically trivial on N, we get

$$c(\mathcal{O}_N(\det \mathcal{E})) = c(\mathcal{O}_N(N)) = c(\mathcal{O}_Z(N))\, c(\mathcal{O}_Z)^{-1} = 1 + N.$$

Hence

$$\begin{aligned} P + N &= \det \mathcal{E} = c_1(\mathcal{E}) = c_1(\widetilde{\mathcal{E}}) + N = \det \widetilde{\mathcal{E}} + N, \\ c_2(\mathcal{E}) &= c_2(\widetilde{\mathcal{E}}) + c_1(\widetilde{\mathcal{E}})N = c_2(\widetilde{\mathcal{E}}) + PN = c_2(\widetilde{\mathcal{E}}). \end{aligned} \qquad \square$$

Definition. The new vector bundle $\widetilde{\mathcal{E}} \subset \mathcal{E}$ obtained in Lemma 2.3 is called the *G-nef reduction* of $\mathcal{E}$.

§3. Proof of Theorem 1.3

In this section, we construct an orbibundle $\mathcal{E}_\alpha$ defined in terms of the triple (α, X, C), where $\alpha \in [0,1] \cap \mathbf{Q}$, X is a surface of non-negative Kodaira dimension and C is an irreducible curve on it. Theorem 1.1 follows from the Miyaoka-Yau-Sakai inequality applied to the nef reduction $\widetilde{\mathcal{E}}_\alpha$ of $\mathcal{E}_\alpha$.

Throughout the section, we use the following symbols:

α	a parameter $\in [0,1] \cap \mathbf{Q}$,
X	a surface with effective K_X (as a $\mathbf{Q}$-divisor),
C	an irreducible curve on X,
e	the topological Euler number $c_2 = c_2(X)$ of X,
g	geometric genus of C,
s	the number of the singular points of C,
$\mu : Y \to X$	the blowing up at the s singular points of C,
$E_1, \ldots, E_s$	the exceptional curves on Y,
Γ	the exceptional locus $E_1 + \cdots + E_s$ of μ.

Since $\mu^{-1}(C)$ may not be a divisor of simple normal crossings, we choose a log-resolution π: $(\widetilde{Y}, \overline{C}) \to \left(Y, \mu^{-1}(C)\right)$. Namely,

(a) $\pi : \widetilde{Y} \to Y$ is the composite $\widetilde{Y} = Y_r \xrightarrow{\pi_r} Y_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_1} Y_0 = Y$, where π_i is the blowing up at a singular point of $\pi_{i-1}^{-1} \cdots \pi_1^{-1} \mu^{-1}(C)$;

(b) $\overline{C}$, the inverse image $\pi^{-1}\mu^{-1}(C)$ with reduced structure, is a divisor of simple normal crossings. (In particular, the strict transform $\tilde{C} \subset \tilde{Y}$ of $C \subset X$ is nonsingular. $\overline{C}$ is a sum of $\tilde{C}$ and a disjoint union of s connected trees of $\mathbf{P}^1$'s.)

We introduce further symbols associated with this log resolution:

- $E_{s+i} \subset Y_i$ is the (-1)-curve produced by $\pi_i : Y_i \to Y_{i-1} \quad (i = 1, \dots, r)$.

- $\overline{E}_i \in \mathrm{Div}(\widetilde{Y})$ is the total transform of $E_i \quad (i = 1, \dots, s, s+1, \dots, s+r)$, and $\overline{E}$ is the sum $\overline{E}_1 + \cdots + \overline{E}_{s+r}$ (hence $K_{\widetilde{Y}} = \pi^*\mu^*K_X + \overline{E}$).

- $F_1, \dots, F_s$ [resp. $G_1, \dots, G_r$] are the strict transforms on $\widetilde{Y}$ of $E_1, \dots, E_s$ [resp. of $E_{s+1}, \dots, E_{s+r}$], and $F = F_1 + \cdots + F_s \quad$ (the strict transform of $\Gamma = E_1 + \cdots + E_s \subset Y$).

Thus the exceptional locus G of π is $G_1 + \cdots + G_r$, while the inverse image $\pi^{-1}\mu^{-1}(C)$ is $\overline{C} = \widetilde{C} + F + G$.

For a given parameter $\alpha \in [0,1] \cap \mathbf{Q}$, we define the *orbibundle* $\mathcal{E}_\alpha$ to be the kernel of the homomorphism $\rho\colon \Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_{(1-\alpha)\widetilde{C}}$ induced by the natural residue map $\Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_X/\mathcal{O}_X(-\widetilde{C})$. To be more precise, choose any branched Galois covering $f\colon Z \to \widetilde{Y}$ such that $A_\alpha = (1-\alpha)f^*\widetilde{C} \leq f^*\widetilde{C}$ is an integral divisor. Then let $\mathcal{E}_\alpha$ denote $\mathrm{Ker}\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C}) \to \mathcal{O}_Z/\mathcal{O}_Z(-A_\alpha)\right)$, which is a well defined vector bundle on Z with equivariant Galois action.[1] Its total Chern class $c(\mathcal{E}_\alpha)$ is computed by

$$\begin{aligned} c(\mathcal{E}_\alpha) &= c\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C})\right) c\left(\mathcal{O}_{A_\alpha}\right)^{-1} \\ &= c\left(f^*\Omega^1_{\widetilde{Y}}(\log \overline{C})\right) \left(1 - (1-\alpha)f^*\widetilde{C}\right). \end{aligned}$$

Denoting by d the mapping degree $[\mathbf{C}(Z) : \mathbf{C}(\widetilde{Y})] = [\mathbf{C}(Z) : \mathbf{C}(X)]$ of f, we have:

[1] The pair of Z and the Galois action is called an orbifold structure of $\widetilde{Y}$, while an orbibundle on $\widetilde{Y}$ is by definition a vector bundle on Z together with an equivariant Galois action. Rigorously speaking, our choice of the covering $f\colon Z \to \widetilde{Y}$ depends on the (denominator of the) rational number α and we should better write $f_\alpha\colon Z_\alpha \to \widetilde{Y}$; however, our subsequent argument is not at all affected by the choice of coverings and we may always replace Z by its branched Galois cover of sufficiently large degree. In this sense, the most natural framework for our purpose will be the limit orbifold structure of $\widetilde{Y}$, *i.e.*, the projective limit $\varprojlim Z$ of branched Galois covers of $\tilde{Y}$.

$$\begin{aligned} c_1(\mathcal{E}_\alpha) &= f^*\left(K_{\widetilde{Y}} + F + G + \alpha\widetilde{C}\right) = f^*\left(\pi^*\mu^*K_X + \overline{E} + F + G + \alpha\widetilde{C}\right), \\ c_2(\mathcal{E}_\alpha) &= c_2\left(f^*\Omega^1_{\widetilde{Y}}(\log(F+G))\right) + c_2\left(\mathcal{O}_{\alpha f^*\widetilde{C}}(-(1-\alpha)f^*\widetilde{C})\right) \\ &\quad + c_1\left(f^*\Omega^1_{\widetilde{Y}}(\log(F+G))\right) c_1\left(\mathcal{O}_{\alpha f^*\widetilde{C}}(-(1-\alpha)f^*\widetilde{C})\right) \\ &= d\left(e - s + \alpha(K_{\widetilde{Y}} + F + G)\widetilde{C} + \alpha\widetilde{C}^2\right) \\ &= d\left(e - s + \alpha(2g-2) + \alpha\widetilde{C}(F+G)\right) \in H^4(Z, \mathbf{Z}) \simeq \mathbf{Z}. \end{aligned}$$

Put $D_\alpha = \pi^*\mu^*K_X + \overline{E} + F + G + \alpha\widetilde{C}$. The $\mathbf{Q}$-divisor D_α and the second Chern class of $\mathcal{E}_\alpha$ can be computed in terms of resolution data. Write

$$\widetilde{C} = \pi^*C - \sum_{i=1}^{s+r} m_i\overline{E}_i, \qquad F + G = \sum_{i=1}^{s}\overline{E}_i - \sum_{i=s+1}^{s+r}\varepsilon_i\overline{E}_i,$$

where

$$\begin{cases} m_i \geq 2 & \text{for } i = 1, \ldots, s \\ m_i \geq 1, \varepsilon_i \geq 0 & \text{for } i = s+1, \ldots, s+r. \end{cases}$$

Then

$$\begin{aligned} D_\alpha &= \pi^*\mu^*(K_X + \alpha C) + \sum_{i=1}^{s}(2 - \alpha m_i)\overline{E}_i + \sum_{j=s+1}^{s+r}(1 - \varepsilon_j - \alpha m_j)\overline{E}_j, \\ \frac{c_2(\mathcal{E}_\alpha)}{d} &= \frac{c_2(\widetilde{\mathcal{E}}_\alpha)}{d} = e + \alpha(2g-2) + \sum_{i=1}^{s}(\alpha m_i - 1) - \sum_{i=s+1}^{r}\alpha\varepsilon_i m_i. \end{aligned}$$

As the expression above shows, the effective $\mathbf{Q}$-divisor D_α and hence $c_1(\mathcal{E}_\alpha)$ are in general not nef on G or on $f^{-1}(G)$. Let $D_\alpha = P_\alpha + N_\alpha$ be the Zariski decomposition into the G-nef part and the G-negative part (thanks to the uniqueness property, the Zariski decomposition of $c_1(\tilde{\mathcal{E}}_\alpha)$ with support in $f^{-1}(G)$ is given by $f^*D_\alpha = f^*P_\alpha + f^*N_\alpha$). We write

$$N_\alpha = \sum_{j=s+1}^{s+r} b_j\overline{E}_j,$$

$b_j = b_j(\alpha)$ being a rational number. If $j = s+1, \ldots, s+r$, then the effective divisor $\overline{E}_j$ is supported by a subset of G so that $0 \leq P_\alpha\overline{E}_j = (D_\alpha - N_\alpha)\overline{E}_j = -1 + \varepsilon_j + \alpha m_j + b_i$, thus proving

Lemma 3.1. *$b_j \geq 1 - \varepsilon_j - \alpha m_j$ for $j = s+1, \ldots, s+r$, and therefore*

$$-N_\alpha^2 \geq \sum_{j=s+1}^{s+r}\left(\max\{1 - \varepsilon_j - \alpha m_j, 0\}\right)^2. \tag{5}$$

Noting the inclusion relations $(\pi f)^*\Omega^1_Y(\log \Gamma) \subset \mathcal{E}_\alpha \subset \Omega^1_Z(\log f^{-1}(\overline{C}))$, we can apply Lemma 2.3 to $\mathcal{E} = \mathcal{E}_\alpha$, with minor modifications (we change f, G, Δ, P, N *etc.* to $\pi \circ f, f^{-1}(G), f^{-1}(\overline{C}), f^*P_\alpha, f^*N_\alpha$ *etc.* and, if necessary, we replace Z by its suitable ramified cover in order to make the pull-backs f^*P_α, f^*N_α integral divisors). Let $\widetilde{\mathcal{E}}_\alpha$ be the $f^{-1}(G)$-nef reduction of $\mathcal{E}_\alpha$ with $c_1(\widetilde{\mathcal{E}}_\alpha) = f^*P_\alpha$, $c_2(\widetilde{\mathcal{E}}_\alpha) = c_2(\mathcal{E}_\alpha)$.

Proposition 3.2. *We have the formula*:

$$\begin{aligned}
&\frac{3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha)}{d} = \frac{3c_2(\mathcal{E}_\alpha) - c_1^2(\mathcal{E}_\alpha) + (f^*N_\alpha)^2}{d} \\
&\quad = 3e - K_X^2 - 2\alpha(CK - 3g + 3) - \alpha^2 C^2 \\
&\qquad + \sum_{i=1}^{s} \left(1 - \alpha m_i + \alpha^2 m_i^2\right) + \sum_{j=s+1}^{t} \left(-3\alpha\varepsilon_j m_j + (1 - \varepsilon_j - \alpha m_j)^2 - b_j^2\right).
\end{aligned}$$

The proof is immediate by simple calculation.

In this formula, the right-hand side is a sum of three terms: the first is independent of the singularity of C, the second involves certain data coming from the first s blowups and the third is concerned with further infinitely near singularities of C. As for the third term, we observe

Lemma 3.3. *For $j = s+1, \ldots, s+r$, we have the estimate*

$$-3\alpha\varepsilon_j m_j + (1 - \varepsilon_j - \alpha m_j)^2 - b_j^2 \leq \frac{3\alpha^2 m_j(m_j - 1)}{2}. \tag{6}$$

Proof. If $1 - \varepsilon_j - \alpha m_j > 0$, then the assertion follows from the two inequalities $b_j \geq 1 - \varepsilon_j - \alpha m_j$ and $\varepsilon_j \geq 0$. Assume that $1 - \varepsilon_j \leq \alpha m_j$. Then

$$\begin{aligned}
-3\alpha\varepsilon_j m_j + (1 - \varepsilon_j - \alpha m_j)^2 &= \alpha^2 m_j^2 - \alpha(2 + \varepsilon_j)m_j + (1 - \varepsilon_j)^2 \\
&\leq \alpha^2 m_j^2 - \alpha(2 + \varepsilon_j)m_j + \alpha(1 - \varepsilon_j)m_j = \alpha^2 m_j^2 - \alpha(1 + 2\varepsilon_j)m_j \\
&\leq \alpha^2 m_j^2 - \alpha m_j \leq \alpha^2 m_j^2 - \alpha^2 m_j \leq (3/2)\alpha^2 m_j(m_j - 1). \qquad \square
\end{aligned}$$

The G-nef divisor P_α may not be nef on $F + G$. Let $P_\alpha = \widehat{P}_\alpha + \widehat{N}_\alpha$ [resp. $P_\alpha = \overline{P}_\alpha + \overline{N}_\alpha$] be the Zariski decomposition with support in $F + G$ [resp. the absolute Zariski decomposition].

Lemma 3.4 (*Cf.* Lemma 3.1, Equation (5)). *In the notation above*,

$$\overline{N}_\alpha^2 \leq \widehat{N}_\alpha^2 \leq -\sum_{i=1}^{s} \left(\max\{2 - \alpha m_i, 0\}\right)^2.$$

Proof. We showed that $\overline{N}_\alpha^2 \leq \widehat{N}_\alpha^2$ in Corollary 2.2. Put $\widehat{N}_\alpha = \sum_{i=1}^{s+r} \widehat{b}_i \overline{E}_i$. Then the coefficient $2 - \alpha m_i - \widehat{b}_i$ of $\overline{E_i}$ in $\widehat{P}_\alpha$ ($i \leq s$) is non-positive by the same reason as in Lemma 3.1. Therefore $|\widehat{b}_i| \geq \max\{2 - \alpha m_i, 0\}$ for $i \leq s$, whence follows $\widehat{N}_\alpha^2 = -\sum_{i=1}^{s+r} b_i^2 \leq -\sum_{i=1}^{s} b_i^2 \leq -\sum_{i=1}^{s} (\max\{2 - \alpha m_i, 0\})^2$. □

Proposition 3.5. *We have*

$$\begin{aligned}
&\frac{1}{d}\left(3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha) + \frac{(f^*\overline{N}_\alpha)^2}{4}\right) \\
&\quad \leq 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) - \alpha^2 C^2 + \frac{3\alpha^2}{2}\sum_{i=1}^{s+r} m_i(m_i - 1) \\
&\quad = 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) + \frac{\alpha^2}{2}\left(C^2 + 3CK_X - 6g + 6\right).
\end{aligned}$$

Proof. The second equality readily follows from the adjunction formula $C^2 + CK_X - \sum m_i(m_i - 1) = 2g - 2$ for $\widetilde{C} \subset \widetilde{Y}$. In view of Proposition 3.2, Lemma 3.3 and Lemma 3.4, the first inequality reduces to the estimate

$$4\left(1 - \alpha m_i + \alpha^2 m_i\right) - (\max\{2 - \alpha m_i, 0\})^2 \leq 6\alpha^2 m_i(m_i - 1)$$

for $i = 1, \dots, s$. In case $\alpha m_i \leq 2$, the left-hand side is $3\alpha^2 m_i^2$, which satisfies $\leq 6\alpha^2 m_i(m_i - 1)$ because $m_i \geq 2$. If $\alpha m_i \geq 2$, then

$$\begin{aligned}
&4 - 4\alpha m_i + 4\alpha^2 m_i^2 - 6\alpha^2 m_i(m_i - 1) \\
&\quad \leq 2\alpha m_i - 4\alpha m_i + 4\alpha^2 m_i^2 - 6\alpha^2 m_i(m_i - 1) \\
&\quad = 2\alpha m_i(3\alpha - 1 - \alpha m_i) \leq 2\alpha m_i(2 - \alpha m_i) \leq 0.
\end{aligned}$$

□

Proof of Theorem 1.3. Recall that $\widetilde{\mathcal{E}}_\alpha \subset f^*\Omega_{\widetilde{Y}}(\log \overline{C})$ and that $\det \widetilde{\mathcal{E}}_\alpha \geq f^*\pi^*\mu^* K_X$ is effective (as a **Q**-divisor). Then in view of Proposition 3.5, the assertion (i) readily follows from the Miyaoka-Yau-Sakai inequality

$$3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha) + (1/4)(f^*\overline{N}_\alpha)^2 \geq 0$$

([11, Theorem 1.1]). When C is not isomorphic to $\mathbf{P}^1$, we have $C^2 + CK_X \geq 0$. If, in addition, $CK_X > 3g - 3$, then $C^2 + 3CK_X - 6g - 6 \geq 2(CK_X - 3g - 3) > 0$, so that the quadratic polynomial

$$Q(\alpha) = \frac{\alpha^2}{2}\left(C^2 + 3CK_X - 6g + 6\right) - 2\alpha(CK_X - 3g + 3) + 3c_2(X) - K_X^2$$

in α has positive leading coefficient and takes minimum value at

$$\alpha_0 = \frac{2(CK_X - 3g + 3)}{C^2 + 3CK_X - 6g + 6} \in [0, 1].$$

This means that

$$Q(\alpha_0) = -\frac{2(CK_X - 3g + 3)^2}{C^2 + 3CK_X - 6g + 6} + 3c_2(X) - K_X^2 \geq 0,$$

which amounts to the assertion (ii). The final statement (iii) is a direct consequence of (ii) and the Hodge index theorem $(C^2)(K_X^2) \leq (CK_X)^2$. □

Remark. The right-hand side of the inequality (6) in Lemma 3.3 can be replaced by $\alpha^2 m_j(m_j - 1)$. Also in Lemma 3.5, our estimate is far from being optimal when the multiplicity m_i is three or more. In particular if C has neither double points nor triple points, we can easily deduce the inequality

$$\begin{aligned} 0 &\leq \frac{1}{d}\left(3c_2(\widetilde{\mathcal{E}}_\alpha) - c_1^2(\widetilde{\mathcal{E}}_\alpha) + \frac{(f^*\overline{N}_\alpha)^2}{4}\right) \\ &\leq 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) - \alpha^2 C^2 + \alpha^2 \sum_{i=1}^{s+r} m_i(m_i - 1) \\ &= 3e - K_X^2 - 2\alpha(CK_X - 3g + 3) + \alpha^2 \left(CK_X - 2g + 2\right), \end{aligned}$$

for $\alpha \in [0, 1]$. Putting $\alpha = 1$, we obtain $CK_X \leq 4g - 4 + 3e - K_X^2$, the main result of [7]. However, we can do better. Assume that $CK_X \geq 3g - 3 \geq 0$ and choose α to be $\dfrac{CK_X - 3g - 3}{CK_X - 2g - 2} \in [0, 1]$. Then we get the same estimate of CK_X as in Corollary 1.4.

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