On $\mathbb{Q}$-conic Bundles, II

By

Shigefumi MORI* and Yuri PROKHOROV**

Abstract

A $\mathbb{Q}$-conic bundle germ is a proper morphism from a threefold with only terminal singularities to the germ $(Z \ni o)$ of a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. We obtain the complete classification of $\mathbb{Q}$-conic bundle germs when the base surface germ is singular. This is a generalization of [MP08], which further assumed that the fiber over $o$ is irreducible.

§1. Introduction

This note is a continuation of our previous work [MP08] where we studied the local structure of $\mathbb{Q}$-conic bundles.

(1.1) Definition. A $\mathbb{Q}$-conic bundle is a projective morphism $f: X \rightarrow Z$ from a threefold with only terminal singularities to a surface such that

(i) $f_* \mathcal{O}_X = \mathcal{O}_Z$ and all fibers are one-dimensional,

(ii) $-K_X$ is $f$-ample.

For $f: X \rightarrow Z$ as above and for a point $o \in Z$, we call the analytic germ $(X, f^{-1}(o)_{\text{red}})$ a $\mathbb{Q}$-conic bundle germ.

In [MP08] we completely classified $\mathbb{Q}$-conic bundle germs over a singular base and such that the central fiber is irreducible. For convenience of quotations

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we reproduce briefly the classification. For more detailed explanations we refer to the original paper [MP08].

(1.2) Theorem. Let $f : (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ, where $C$ is irreducible and $(Z, o)$ is singular. Then we are in one of the following cases:

<table>
<thead>
<tr>
<th>Type</th>
<th>No.</th>
<th>singularities</th>
<th>$(Z, o)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>toroidal</td>
<td>(1.2.1)</td>
<td>$\frac{1}{n}(1, a, -a)$ and $\frac{1}{n}(-1, a, -a)$, $\gcd(n, a) = 1$</td>
<td>$A_{n-1}$</td>
</tr>
<tr>
<td>(IA)+(IA)</td>
<td>(1.2.2)</td>
<td>$\frac{1}{n}(a, -1, 1)$ and $\frac{1}{n}(a + 1, 1, -1)$, $n = 2a + 1$</td>
<td>$A_{n-1}$</td>
</tr>
<tr>
<td>(IE)</td>
<td>(1.2.3)</td>
<td>$\frac{1}{n}(5, 1, 3)$</td>
<td>$A_3$</td>
</tr>
<tr>
<td>(ID)</td>
<td>(1.2.4)</td>
<td>$cA/2$ or $cAx/2$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>(IA)</td>
<td>(1.2.5)</td>
<td>$\frac{1}{n}(1, 1, 3)$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>(II)</td>
<td>(1.2.6)</td>
<td>$cAx/4$</td>
<td>$A_1$</td>
</tr>
</tbody>
</table>

In this paper we consider the case where the base surface is singular and the central fiber is reducible. Our main result is the following.

(1.3) Theorem. Let $f : (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ. Assume that $C$ is reducible and the base surface $(Z, o)$ is singular. Then $(Z, o)$ is Du Val of type $A_1$ and $(X, C)$ is the $\mu_2$-quotient of the index-two $\mathbb{Q}$-conic bundle $f' : (X', C') \to (Z', o')$ over a smooth base, where $\mu_2$ acts on $(Z', o')$ freely in codimension one. Moreover, $C'$ has four irreducible components, $\mu_2$ does not fix any of them and $X$ has a unique non-Gorenstein point $P$. Furthermore, $X'$ is given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1, \ldots, y_4} \times \mathbb{C}^2_{u,v}$

$$
\begin{align*}
  y_1^2 - y_3^2 &= \psi_1(y_1, \ldots, y_4; u, v), \\
  y_2^2 - y_3^2 &= \psi_2(y_1, \ldots, y_4; u, v),
\end{align*}
$$

where $\mu_2$ acts as follows:

$$(y_1, y_2, y_3, y_4; u, v) \mapsto (-y_1, -y_2, y_3, -y_4; -u, -v).$$

Here $\psi_i(y_1, \ldots, y_4; u, v)$ are weighted quadratic in $y_1, \ldots, y_4$ with respect to $\text{wt}(y_1, \ldots, y_4) = (1, 1, 1, 2)$ and $\psi_i(y_1, \ldots, y_4; 0, 0) = 0$. The following are the only possibilities:

(1.3.1) $(X, P)$ is a cyclic quotient singularity of type $\frac{1}{4}(1, 1, -1)$ and for any component $C_i \subset C$ germ $(X, C_i)$ is of type $(\text{IA})$. 

(1.3.2) \((X, P)\) is a singularity of type \(cAx/4\) and for any component \(C_i \subset C\) germ \((X, C_i)\) is of type \((\Pi')\).

Conversely, if the quotient \((X, C) = (X', C')/\mu_2\), where \((X', C')\) and the action of \(\mu_2\) are as above, has only terminal singularities, then \((X, C)\) is a conic bundle germ over \(\mathbb{C}^2_{u,v}/\mu_2\) with reducible central fiber \(C\).

(1.4) Corollary (Reid’s general elephant conjecture). Let \(f: (X, C) \to (Z, o)\) be a \(\mathbb{Q}\)-conic bundle germ. Assume that \((Z, o)\) is singular. Then a general member \(F \in |-K_X|\) has only Du Val singularities. Moreover, in cases (1.3.1) and (1.3.2) \(F\) does not contain any component of \(C\) and is of type \(A_3\) and \(D\) respectively.

Below are a series of explicit examples of \(\mathbb{Q}\)-conic bundles as in (1.3).

(1.5) Example. Consider the subvariety \(X' \subset \mathbb{P}(1,1,1,2) \times \mathbb{C}^2\) defined by the following two equations:

\[
\begin{align*}
y_1^2 - y_2^2 + u^{2k+1}y_4 + v^2 y_2^2 &= 0, \\
y_2^2 - y_3^2 + vy_4 &= 0.
\end{align*}
\]

The projection \(f': X' \to \mathbb{C}^2\) is a \(\mathbb{Q}\)-conic bundle of index 2 (cf. [MP08, 12.1.3]). Define the action of \(\mu_2\) on \(X'\) as follows

\[(y_1, y_2, y_3, y_4; u, v) \mapsto (-y_1, -y_2, y_3, -y_4; -u, -v).\]

Then \(X'/\mu_2 \to \mathbb{C}^2/\mu_2\) is a \(\mathbb{Q}\)-conic bundle with a unique non-Gorenstein point \(P\). The point \(P\) is of type (1.3.1) if \(k = 0\) and of type (1.3.2) if \(k \geq 1\).

The basic idea of the proof is to reduce the problem of classifying \(\mathbb{Q}\)-conic bundles \((X, C)\) as in Theorem (1.3) to the case where the central fiber is irreducible by applying the MMP to a \(\mathbb{Q}\)-factorialization \((X^q, C^q)\). Then the resulting \(\mathbb{Q}\)-conic bundle \((\bar{X}, \bar{C})\) belongs to the list (1.2). We trace back from \((\bar{X}, \bar{C})\) to \((X, C)\). It turns out that in many cases the steps of the MMP do not affect the singularities of \((\bar{X}, \bar{C})\). Here we use some results about divisorial contractions and flips (see §2) based on [KM92] and [Kaw96]. Then the base change trick allows us to show that \((X, C)\) is a \(\mu_2\)-quotient of an index-two conic bundle, see §3.

§2. Preliminary Results on Extremal Contractions

(2.1) Let \((E^\sharp, P^\sharp)\) be a Du Val singularity. (We assume that \((E^\sharp, P^\sharp)\) is singular). Assume that \(\mu_m\) acts on \(E^\sharp\) freely outside \(P^\sharp\) and the quotient
Let \((E, P) = (E^\sharp, P^\sharp) / \mu_m\) is also Du Val. Then there is a \(\mu_m\)-equivariant embedding \((E^\sharp, P^\sharp) \subset (\mathbb{C}^3_{x,y,z}, 0)\) such that \(x, y, z\) and the equation of \(E^\sharp\) are semi-invariant. Let \(F^\sharp \subset \mathbb{C}^3\) be the locus of points at which the action of \(\mu_m\) is not free. By our assumption \(F^\sharp\) is a curve. Define the invariant \(\zeta(E^\sharp, P^\sharp, \mu_m)\) as the local intersection number \((E^\sharp \cdot F^\sharp)_0\). According to [Rei87, 4.10] we have only the following cases:

\[
\begin{array}{|c|c|c|}
\hline
m & (E^\sharp, P^\sharp) \rightarrow (E, P) & \zeta(E^\sharp, P^\sharp, \mu_m) \\
\hline
\text{any} & A_{r-1} \rightarrow A_{mr-1} & r \\
4 & A_{2r-2} \rightarrow D_{2r+1} & 2r - 1 \\
2 & A_{2r-1} \rightarrow D_{r+2} & 2 \\
3 & D_4 \rightarrow E_6 & 2 \\
2 & D_{r+1} \rightarrow D_{2r} & r \\
2 & E_6 \rightarrow E_7 & 3 \\
\end{array}
\]

(2.1.1)

(2.1.2) Let \((W, P)\) be a three-dimensional terminal singularity of index \(m > 1\) and let \(E \in | - K_W|\) be a divisor having a Du Val singularity at \(P\). Assume that \((W, P)\) is not a cyclic quotient. Let \(\pi: (W^\sharp, P^\sharp) \rightarrow (W, P)\) be the index-one \(\mu_m\)-cover and let \((W^\sharp, P^\sharp) = \{ \phi = 0 \} \subset \mathbb{C}^4_{x_1, x_2, x_3, x_4}\) be a \(\mu_m\)-equivariant embedding. Let \(E^\sharp := \pi^{-1}(E)\) and \(F^\sharp \subset \mathbb{C}^4\) be the locus of points at which the action of \(\mu_m\) is not free. Since \(\pi\) is free in codimension two, \(F^\sharp\) is a curve. Recall that the local intersection number \((W^\sharp \cdot F^\sharp)_0\) is called the \emph{axial multiplicity} of \((W, P)\) [Mor88, 1a.5]. We denote it by \(\text{am}(W,P)\). By the classification of terminal singularities we may assume that \(F^\sharp\) is the \(x_4\)-axis, and either \(\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 0, 0) \mod m\), or \(m = 4\) and \(\text{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 2, 2) \mod 4\), where \(\gcd(a, m) = 1\). Since \((E^\sharp, P^\sharp)\) is a Du Val singularity, its Zariski tangent space at the origin is three-dimensional. Hence there is a \(\mu_m\)-stable hypersurface \(H^\sharp \subset \mathbb{C}^4\) such that \(E^\sharp = H^\sharp \cap W^\sharp\) and \(H^\sharp\) is smooth.

(2.1.3) Claim. \(F^\sharp \subset H^\sharp\).

\textbf{Proof.} Let \(\psi\) be the \(\mu_m\)-semi-invariant equation of \(H^\sharp\). Then \(\text{wt} \psi \equiv a\). Hence \(\psi\) does not contain terms \(x_4^k\) and so it vanishes on \(F^\sharp\). \(\square\)

(2.1.4) We define the invariant \(\zeta(W, E, P)\) as the local intersection number \((E^\sharp \cdot F^\sharp)_0\) inside \(H^\sharp\). Clearly it coincides with \(\zeta(E^\sharp, P^\sharp, \mu_m)\) defined above.
(2.1.5) **Lemma.** Assume that \((W, P)\) is not a cyclic quotient singularity. The invariant \(\varsigma(W, E, P)\) does not depend on the choice of \(E\) and \(\varsigma(W, E, P) = \am(W, P)\).

**Proof.** Both sides of the equality coincide with the order of vanishing of \(\phi|_{F^\sharp}\).

(2.1.6) **Corollary.** Let \((W, P)\) is a three-dimensional terminal singularity of index \(m > 1\) which is not a cyclic quotient and let \(E \in |-K_{(W,P)}|\) be a member having a Du Val singularity of A-type at \(P\). Then \(E\) is isomorphic to a general member \(E_{\text{gen}} \in |-K_{(W,P)}|\).

**Proof.** By the above lemma we have \(\varsigma(E^\sharp, P^\sharp, \mu_m) = \varsigma(E_{\text{gen}}^\sharp, P_{\text{gen}}^\sharp, \mu_m) = \am(W, P)\). Then the statement follows by the first line in (2.1.1).

(2.2) **Proposition.** Let \(\varphi: (V, \Gamma) \to (W, o)\) be the analytic germ of a divisorial extremal contraction of threefolds with terminal singularities (in particular, \(W\) is Q-Gorenstein) such that the central fiber \(\Gamma := \varphi^{-1}(o)_{\text{red}}\) is one-dimensional and irreducible.

(i) The point \((W, o)\) cannot be of type \(cAx/4\).

(ii) If \((W, o)\) is of type \(cAx/2\), then \((V, \Gamma)\) has a unique non-Gorenstein point which is of type \((II^\vee)\).

(iii) If \((W, o)\) is analytically isomorphic to

\[
\{x_1x_2 + x_3^2 + x_4^{2k} = 0\}/\mu_2(1, 1, 0, 1),
\]

then \((V, \Gamma)\) has a unique non-Gorenstein point \(P\) which is locally imprimitive of index 4 and splitting degree 2. Moreover, \(P \in (V, \Gamma)\) is either of type \((II^\vee)\) or \((IA^\vee)\) and in the second case \((X, P)\) is a cyclic quotient singularity.

**Proof.** For the proof we assume that \((W, o)\) is of type \(cAx/4\), \(cAx/2\), or as in (2.2.1). We will use the classification [KM92, Th. 2.2]. Let \(m\) be the index of \((W, o)\). Then the canonical class \(K_W\) is an \(m\)-torsion element in \(\text{Cl}^{\text{loc}}(W, o)\). Its pull-back \(\varphi^*K_W\) is a well-defined Cartier divisor on \(V \setminus \Gamma\) such that \(m(\varphi^*K_W) \sim 0\). Hence \(\text{Cl}^{\text{loc}}(V, \Gamma)\) contains an \(m\)-torsion element, say \(\xi\). By the classification [KM92, Th. 2.2] and by [Mor88, (1.10)] the group \(\text{Cl}^{\text{loc}}(V, \Gamma)\) can contain a torsion only when \((V, \Gamma)\) is of type \((k1A)\) (with a point of type \((IA^\vee)\), \((II^\vee)\), or \((k2A)\).
Assume that \((V, \Gamma)\) is of type \((k2A)\). Then by [KM92, Th. 2.2] a general member \(D \in | - K_V|\) and its image \(\varphi(D) \in | - K_W|\) have only Du Val singularities. Moreover, \((\varphi(D), o)\) is a singularity of type \(A_\ast\) and so \((W, o)\) is of type \(cA/\ast\). Clearly, the contraction \(\varphi|_D: D \to \varphi(D)\) is crepant. By our assumptions \((W, o)\) is given by (2.2.1). So, \(am(W, o) = 2\). By Corollary (2.1.6) the singularity \((\varphi(D), o)\) is of type \(A_3\). Since \(\varphi_D: D \to \varphi(D)\) is crepant and \(V\) has two singular points, the only possibility is that \(D\) has two singularities of type \(A_1\). But in this case \(V\) is of index two and then by [KM92, Th. 4.7] \(V\) has a unique non-Gorenstein point, a contradiction.

In the remaining cases \((\Pi^\vee)\) and \((k1A)\), \(V\) has a unique non-Gorenstein point \(P\). Then \((V, \Gamma)\) is locally imprimitive at \(P\) and the splitting degree equals \(m\). In particular, the index of \(P\) is \(> m\) [Mor88, Cor. 1.16]. Thus if \((V, \Gamma)\) is of type \((\Pi^\vee)\), then we are in the case (ii) or (iii).

Assume that \((V, \Gamma)\) is of type \((k1A)\). Then by [KM92, Th. 2.2] a general member \(D \in | - K_V|\) does not contain \(\Gamma\), has only Du Val singularity at \(P := \{D \cap \Gamma\}\), and \(\varphi|_D: D \to \varphi(D)\) is an isomorphism. Hence \(\varphi(D) \in | - K_W|\) has a Du Val singularity of type \(A\) at \(o\). In this case, \((W, o)\) cannot be of type \(cA\ast/\ast\). Thus \((W, o)\) is given by (2.2.1). By Corollary (2.1.6) \(D \simeq \varphi(D)\) is of type \(A_3\). Since the index of \((V, P)\) is \(> 2\), \((V, P)\) must be a cyclic quotient singularity \(\frac{1}{4}(1, 1, -1)\). So we are in the case (iii). This proves the proposition.

\section*{(2.3) Proposition}

Let \(\chi: (V, \Gamma) \dasharrow (V^+, \Gamma^+)\) be a flip of threefolds with terminal singularities with irreducible flipping curve \(\Gamma\). Then \((V^+, \Gamma^+)\) contains none of the following configurations of singularities:

\begin{enumerate}
  \item two cyclic quotient singularities \(P_1^+\) and \(P_2^+\) of indices \(m_1\) and \(m_2\) with \(\gcd(m_1, m_2) > 1\) such that \((V^+, \Gamma^+)\) is locally primitive at \(P_1^+\) and \(P_2^+\);
  \item an imprimitive point \(P^+\) of splitting degree \(s > 1\).
\end{enumerate}

\textbf{Proof.} By [KM92, Cor. 13.4] \(\Gamma^+\) is irreducible. Assume that one of the cases (i)-(ii) holds. As in [Mor88, Cor. 1.12] there is a \(d\)-torsion element \(\xi^+ \in \Cl_{\text{can}} V^+\) for some \(d > 1\). Its proper transform \(\xi\) on \(V\) is a \(d\)-torsion element in \(\Cl_{\text{can}} V\). In [KM92] flips are classified into 6 types \((k1A), (k2A), (cD/3), (\Pi A), (IC), (kAD)\) according to a general member of the anti-canonical linear system \(| - K_V|\) [KM92, Th. 2.2]. The group \(\Cl_{\text{can}} V\) can contain a torsion only in cases \((k1A)\) and \((k2A)\) (in all other cases the flipping variety is locally primitive and indices of non-Gorenstein points are coprime, cf. [Mor88, (1.10)])]. The torsion
elements $\xi$ and $\xi^+$ induce the following cyclic $\mu_d$-coverings:

\[(2.3.1)\]

\[
\begin{array}{c}
(V', \Gamma') \lr B (V'^+, \Gamma'^+)
\end{array}
\]

Consider the flipping diagram

\[
\begin{array}{c}
(V, \Gamma) \lr X (V'^+, \Gamma'^+)
\end{array}
\]

By [Mor88, Th. 7.3, 9.10] and [KM92, Th. 2.2], a general member $D \in |-K_V|$ has only Du Val singularities. Since the restriction $\varphi_D : D \to \varphi(D)$ is crepant, the same holds for $\varphi(D) \in |-K_W|$. Further, if we put $D^+ = \chi(D)$, then $D^+ \in |-K_{V^+}|$ and $D^+$ also has only Du Val singularities. Since $K_{V^+}.\Gamma^+ > 0$, $D^+ \supset \Gamma^+$.

\textbf{(2.3.2)} First we consider the case where our flip is of type $(k_1A)$. Then $V$ has a unique non-Gorenstein point $P$ and $P$ is of type $cA/\ast$. In this case $D \cap \Gamma = \{P\}$ and $(\varphi(D), o) \simeq (D, P)$ is of type $A_*$. Since $\text{Cl}^{\text{gr}} V$ has a torsion, $(V, \Gamma)$ is locally imprimitive at $P$.

\textbf{(2.3.3)} Assume that we are in the case (i). We claim that $V^+$ has at least one analytically non-$\mathbb{Q}$-factorial singular point ($\neq P_1^+, P_2^+$). Indeed, since the germ $(V, \Gamma)$ has only one non-Gorenstein point, it is locally imprimitive and in the diagram (2.3.1) $\pi$ is the splitting cover [Mor88, Cor. 1.12]. Here $\Gamma'$ has exactly $d$ components and $V'^+$ is the relative canonical model of $V'$. Since $(V^+, \Gamma^+)$ is locally primitive at $P_1^+$ and $P_2^+$, the curve $\Gamma'^+$ is irreducible. Now the map $\chi'$ can be decomposed as follows

\[
\chi' : V' = V'_0 \to V'_1 \to \ldots \to V'_n \to V'^+,
\]

where every $V'_i \to V'_{i+1}$ is a flip along an irreducible curve and $V'_n \to V'^+$ is a crepant small contraction (cf. [KM92, Proof of 13.5]). Every step $V'_i \to V'_{i+1}$ preserves the number of components of the central fiber. Hence the crepant contraction $V'_n \to V'^+$ is nontrivial and gives us an analytically non-$\mathbb{Q}$-factorial point $Q \in \Gamma^+ \subset V^+$ (because $V'^+ \to V^+$ is étale outside of $P_1^+$ and $P_2^+$). This proves our claim. Thus the divisor $D^+$ has at least three singular points: $P_1^+$,
$P_2^+$, and $Q$. But then $\varphi^+_D: D^+ \to \varphi(D)$ contracts $\Gamma^+$ to a Du Val singularity of type $D_*$ or $E_*$, a contradiction.

(2.3.4) Now we assume that we are in the case (ii). We claim that the log divisor $K_{D^+} + \Gamma^+$ is not plt at $P^+$. Indeed, in the diagram (2.3.1) $\pi^+$ is the splitting cover (see [Mor88, Cor. 1.12.1]). In particular, $\pi^+$ is étale outside $P^+$, $\pi^{+ -1}(P^+)$ is one point, and $\Gamma^{t+}$ has $s > 1$ irreducible components, all of them pass through $\pi^{+ -1}(P^+)$. Let $D^{t+} := \pi^{+ -1}(D^+)$. Since $\Gamma^{t+}$ is singular at $\pi^{+ -1}(P^+)$, the log divisor $K_{D^{t+}} + \Gamma^{t+}$ is not plt at this point. This proves our claim because the restriction $\pi^+: D^{t+} \to D^+$ is étale in codimension one (see, e.g., [Kol92, Cor. 20.4]). Now since the contraction $D^+ \to \varphi(D)$ is crepant, $D^+$ is dominated by the minimal resolution $D^\text{min}$ of $\varphi(D)$: $D^\text{min} \to D^+ \to \varphi(D)$. Since $K_{D^+} + \Gamma^+$ is not plt, the exceptional divisor of $D^\text{min} \to \varphi(D)$ is not a chain of smooth rational curves. Hence $(\varphi(D), o)$ is not a singularity of type $A_*$, a contradiction.

(2.3.5) Finally, we consider the case where our flip is of type (k2A). These flips are described in [Mor02]. We will use notation of [Mor02]. By [Mor02, Th. 4.7] $(V^+, \Gamma^+)$ is locally primitive. Hence we have the case (i). Moreover, $V^+$ has exactly two singular points and they are analytically isomorphic to germs of the following $cA/m_i$ singularities:

$$\{\xi_i \eta_i = G_{k-i}(\xi_i^{m_i}, u^{e(k+2-i)})\}/\mu_{m_i} \subset \mathbb{C}^4_{\xi_i, \eta_i, \zeta_i, u}/\mu_{m_i}, (1, -1, a_i, 0),$$

where $k$, $a_i$ are some positive numbers and $e(j)$ is some function. Hence these points coincide with $P_1^+$ and $P_2^+$. Since $P_i^+ \in \Gamma^+ \subset V^+$ are cyclic quotient singularities, we have $e(k) = e(k+1) = 1$ ($u$ needs to be eliminated). If we put $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2$, then $\delta \geq d$ and by definition [Mor02, Def. 3.2] we have $e(3) = 0$, $e(4) = \delta \alpha_1 \geq d > 1$, $e(5) = (\delta^2 + 1)\alpha_1 + \delta \alpha_2 \geq d > 1$ (see [Mor02, Rem. 3.6]). Thus, $k \geq 6$. On the other hand, by [Mor02, Lemma 3.5, Cor. 3.7] we have $k \leq 5$, a contradiction.

(2.4) Proposition. Let $\varphi: (V, \Gamma) \to (W, o)$ be the germ of a birational crepant contraction of threefolds with terminal singularities, where $\Gamma$ is irreducible.

(i) $(V, \Gamma)$ contains at most two non-Gorenstein points.

(ii) If $(V, \Gamma)$ is imprimitive at some point $P$, then $(W, o)$ cannot be a singularity of type $cA/\ast$. 

Proof. For the proof we assume that $V$ is not Gorenstein. Since $\varphi$ is crepant, the point $(W, o)$ is not Gorenstein. Let $m$ be its index. Let $D \in | - K_W |$ be a general member and let $S := \varphi^{-1}(D)$. Then $S \in | - K_{\varphi(V, \Gamma)} |$ and both $S$ and $D$ have only Du Val singularities. Moreover, the restriction map $\varphi_S: S \to D$ is crepant. Hence $S$ is dominated by the minimal resolution $D_{\min}^{\text{min}}$ of $D$ and obtained from $D_{\min}^{\text{min}}$ by contracting all but one exceptional curves.

First assume that $(V, \Gamma)$ has at least three non-Gorenstein points, say $P$, $Q$, and $R$. By the classification of Du Val singularities $(D, o)$ is a singularity of type $D_*$ or $E_*$ and $S$ is obtained from $D$ by blowing up the exceptional curve corresponding to the central vertex in the Dynkin diagram. In this case exceptional curves on $D_{\min}^{\text{min}}$ over $(S, P)$, $(S, Q)$ and $(S, R)$ form strings and the proper transform of $\Gamma$ is adjacent to the ends of them. This means that the log divisor $K_S + \Gamma$ is plt. The latter implies that the germ $(V, \Gamma)$ is locally primitive (cf. (2.3.4)). Now consider the index-one cover $\pi: (W^*, o^*) \to (W, o)$. It induces the following diagram

$$
\begin{array}{ccc}
(V^*, \Gamma^*) & \xrightarrow{\nu} & (V, \Gamma) \\
\downarrow \varphi^* & & \downarrow \varphi \\
(W^*, o^*) & \xrightarrow{\pi} & (W, o)
\end{array}
$$

Since $(V, \Gamma)$ is locally primitive, $\Gamma^* = \varphi^* o^* \Gamma$ is irreducible. The group $\mu_m$ naturally acts on $\Gamma^* \simeq \mathbb{P}^1$ and has exactly two fixed points. Thus we may assume that $\nu^{-1}(R)$ contains no fixed points. But then $\nu^{-1}(R)$ consists of $m > 1$ non-Gorenstein points of the same index. By [Mor88, Cor. 1.12] there is a torsion element in $\text{Cl}^{\text{sc}}(V^*, \Gamma^*) \simeq \text{Cl}^{\text{sc}}(W^*, o^*)$. This contradicts the fact that $W^* \setminus \{o^*\}$ is simply connected. Thus (i) is proved.

Now assume that $(V, \Gamma)$ contains an imprimitive point $P$. By the proof of (i) $S$ has at most two singular points and the log divisor $K_S + \Gamma$ is not plt at $P$. On the other hand, assume that $(D, o)$ is a point of type $A_*$. Then the exceptional curves of the minimal resolution $D_{\min}^{\text{min}} \to S$ and $\Gamma$ form a chain. Hence $K_S + \Gamma$ is not plt, a contradiction.

(2.5) Proposition (cf. [Mor88, 1.14]). Let $f: (X, C) \to (Z, o)$ be the germ of a contraction from a threefold with only terminal singularities to a surface such that

(i) $-K_X$ is nef and big,

(ii) $C := f^{-1}(o)_{\text{red}}$ is a curve having at least three components,
(iii) each $K_X$-trivial component $C_j \subset C$ contains a non-Gorenstein point.

Then $X$ has index $> 1$ at all singular points of $C$.

Proof. By the Kawamata-Viehweg vanishing theorem we have $R^1f_*\mathcal{O}_X = 0$. Hence $C$ is a union of $\mathbb{P}^1$'s whose configuration is a tree. Let $P \in C$ be a singular point and let $C_i \subset C$ be a component passing through $P$. We have $gr^0_{C_i} \omega \simeq \mathcal{O}(-1)$. Indeed, take a positive integer $m$ such that $mK_X$ is Cartier. Then there is a natural embedding $(gr^0_{C_i} \omega)^{\otimes m} \hookrightarrow \mathcal{O}_{C_i}(mK_X)$. Since $K_X \cdot C_i \leq 0$ we have $\deg gr^0_{C_i} \omega \leq 0$. Moreover, if $K_X \cdot C_i < 0$, then $\deg gr^0_{C_i} \omega < 0$. Assume that $K_X \cdot C_i = 0$ Since $C_i$ contains a non-Gorenstein point, the above embedding is not an isomorphism and so again $\deg gr^0_{C_i} \omega < 0$. On the other hand, $C_i$ is contractible over $Z$. Hence, by the Grauert-Riemenschneider vanishing theorem we have $H^1(gr^0_{C_i} \omega) = 0$. This shows $gr^0_{C_i} \omega \simeq \mathcal{O}(-1)$.

Now let $C_j$ be another component of $C$ passing through $P$. As above, $gr^0_{C_j} \omega \simeq \mathcal{O}(-1)$. Consider the following exact sequence

$$0 \longrightarrow gr^0_{C_i \cup C_j} \omega \longrightarrow gr^0_{C_i} \omega \oplus gr^0_{C_j} \omega \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\text{Supp} \mathcal{F} = P$. Since $C_i \cup C_j \neq C$, $C_i \cup C_j$ is contractible over $Z$ and again by the Grauert-Riemenschneider vanishing theorem $H^1(gr^0_{C_i \cup C_j} \omega) = 0$. This implies $gr^0_{C_i \cup C_j} \omega \simeq gr^0_{C_i} \omega \oplus gr^0_{C_j} \omega$. So $gr^0_{C_i \cup C_j} \omega$ is not locally free at $P$ and this point cannot be Gorenstein. \qed

§3. The Proof of the Main Theorem

In this section we prove Theorem (1.3).

(3.1) Notation. Let $f: (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with reducible central fiber $C$. Then $\rho(X/Z) > 1$. Recall that according to [MP08, Th. 1.2.7] $(Z, o)$ is either smooth or Du Val of type $A$ (see also the construction (3.1.2) below). We assume that $(Z, o)$ is singular of type $A_{n-1}, n \geq 2$.

(3.1.1) Lemma. Notation as above.

(i) If $(X, C)$ has a point $P$ such that either

(a) $P$ is of type $cAx/4$, or

(b) for each component $C_i \subset C$ passing through $P$ the germ $(X, C_i)$ is locally imprimitive at $P$.

Then $P$ is the only non-Gorenstein point on $X$.  

(ii) Conversely, if \( P \) is a unique non-Gorenstein point on \( X \), then all the components \( C_i \subset C \) pass through \( P \) and the germ \((X, C_i)\) is locally imprimitive at \( P \). If furthermore \((X, P)\) is of index 4, then \((X, C)\) is a quotient of an index two \( \mathbb{Q} \)-conic bundle germ \((X', C')\) over a smooth base by \( \mu_2 \), where the action is free in codimension one, \( C' \) has four irreducible components and \( \mu_2 \) does not fix any of them.

**Proof.** Let \( P \in X \) be a point as in (i). For each component \( C_i \subset C \) passing through \( P \) the germ \((X, C_i)\) is an extremal neighborhood and by [KM92, Th. 2.2] \((X, C_i)\) has no non-Gorenstein point other than \( P \). Since each singular point of \( C \) is not Gorenstein [Kol99, Prop. 4.2], [MP08, 4.4.2] and \( C \) is connected, \( P \) is the only non-Gorenstein point on the whole \( X \).

Now assume that \( P \) is the only non-Gorenstein point. Consider the base change [MP08, 2.4]: \((X', C') \to (X, C)\). Here \((X', C')\) is a conic bundle germ over a smooth base and \( X' \to X \) is a \( \mu_n \)-cover étale outside \( P \). Thus \((X, C) = (X', C')/\mu_n\). If \( \mu_n \) fixes a component \( C_i' \subset C \), then there are two \( \mu_n \)-fixed points on \( C_i \) and they give us two non-Gorenstein points on \( X \), a contradiction. So the first assertion of (ii) is proved.

Finally assume that \((X, P)\) is of index 4. Since the index of \((X, P)\) is divisible by \( n \), \( n = 4 \) or 2. If \( n = 4 \), then \( X' \) is Gorenstein. In this case, by [Pro97, Th. 2.4] \( C \) is irreducible, a contradiction. Thus \( n = 2 \) and \((X', C')\) is of index 2. By the above, \( \mu_2 \) does not fix any component of \( C' \). On the other hand, \( C' \) has at most four components [MP08, Th. 12.1]. Hence \( C' \) has exactly four components. This proves the lemma. \( \square \)

(3.1.2) Let \( q: X^q \to X \) be a \( \mathbb{Q} \)-factorialization. (It is possible that \( q \) is the identity map.) Run the MMP over \( Z \): \( X^q = X_0 \to X_{N+1} = \tilde{X} \). Since \( X/Z \) is a rational curve fibration, \( X_{N+1} \) is not a minimal model over \( Z \). Therefore, at the end we get an extremal contraction \( \tilde{f}: \tilde{X} \to \tilde{Z} \) of Fano type over \( Z \). Since the composition \( f^q: X^q \to Z \) has only one-dimensional fibers, \( Z = \tilde{Z} \) and \( X^q \to \tilde{X} \) is a sequence of flips and extremal divisorial contractions that contract a divisor to a curve which is not contained in the fiber over \( o \in Z \). Thus we have the following diagram:
Here each $X_k$ has a morphism $f_k: X_k \to Z$ with connected one-dimensional fibers and $C_k := f_k^{-1}(o)$ is the central fiber (with reduced structure). Since $\rho(\bar{X}/Z) = 1$, $\bar{f}: \bar{X} \to \bar{Z}$ is a $\mathbb{Q}$-conic bundle with irreducible central fiber $\bar{C}$. Since the base $(Z,o)$ is singular, $\bar{X}$ is not Gorenstein. So $\bar{f}$ is classified in [MP08], see also (1.2).

\textbf{(3.1.3) Note} that each component of the central fiber $C_k$ is contractible and the resulting variety is again projective over $Z$ (because it has one-dimensional fibers over $Z$). Hence each component of $C_k$ generates an extremal ray (not necessarily $K$-negative). This implies that all our flipping curves are irreducible and all the divisorial contractions have irreducible fibers. Note also that all the varieties $X_k$ are analytically $\mathbb{Q}$-factorial at each point on $C_k$ (again because $X_k \to Z$ has one-dimensional fibers, cf. [Mor88, Proof of 1.7]).

The following is the key argument in the proof.

\textbf{Proposition.} In the above notation one of the following holds.

\textbf{(3.2.1) There is a component $C^a_0 \subset C^a$ containing two cyclic quotient singularities $P^a$ and $Q^a$ of index $n$. No other components of $C^a$ pass through $P^a$ and $Q^a$.}

\textbf{(3.2.2) There is a point $P^a \in (X^a, C^a)$ of index $m > 1$ which is contained in only one component $C^a_0 \subset C^a$ and such that $(X^a, C^a_0)$ is locally imprimitive at $P^a$. The following are the possibilities for $(n,m)$: $(4, 8)$, $(2, 4)$, and $(2, 2)$.}

\textbf{(3.2.3) There is a point $P^a \in (X^a, C^a)$ which is contained in exactly two components $C^a_0, C^a_1 \subset C^a$ and such that both germs $(X^a, C^a_i)$ are locally imprimitive at $P^a$. The point $(X^a, P^a)$ is of type $cAx/4$ or $\frac{1}{3}(1,1,-1)$. Here $n = 2$. Moreover, there is an $n$-torsion element $\xi^a \in Cl^{sc}(X^a, C^a)$ which is not Cartier at $P^a$ (and at $Q^a$ is the case (3.2.1)).}
Proof. Since \((Z,o)\) is of type \(A_{n-1}\), there is an \(n\)-torsion element \(\eta \in \text{Cl}(Z,o)\). Put \(\xi := f^* \eta, \xi_i := f_i^* \eta, \) and \(\xi^0 := f^{0*} \eta\).

Assume that \((\bar{X},\bar{C})\) is either toroidal of type \((\text{IA})+(\text{IA})\). Let \(P, Q\) be the singular points of \(\bar{X}\). Then \(\bar{\xi}\) is not Cartier at \(P\) and \(Q\). We claim that the map \(\psi: \bar{X} \rightarrow X^q\) is an isomorphism near \(P\) and \(Q\). Indeed, by induction, since \(P, Q\) are cyclic quotient singularities of index \(n\), there is no divisorial contractions over these points by [Kaw96] and by Proposition (2.3) on each step the proper transform of \(\bar{C}\) cannot be a flipped curve. So if we put \(P^q := \psi(\bar{P}), Q^q := \psi(\bar{Q}), \) and \(C^q_0 := \psi(\bar{C})\), we get the case (3.2.1).

Now assume that \((\bar{X},\bar{C})\) is of type \((\text{IE}^\vee), (\text{IA}^\vee), \) or \((\text{II}^\vee)\). Let \(\bar{P}\) be a (unique) non-Gorenstein point. Then \((\bar{X},\bar{P})\) is either a cyclic quotient singularity of type \(cAx/4\) and again \(\bar{\xi}\) is not Cartier at \(\bar{P}\). Moreover, \((\bar{X},\bar{C})\) is locally imprimitive at \(\bar{P}\). As above, there is no divisorial contractions over \(\bar{P}\) by [Kaw96] and Proposition (2.2) and the proper transform of \(\bar{C}\) cannot be a flipped curve by Proposition (2.3). Put \(P^q := \psi(\bar{P}), C^q_0 := \psi(\bar{C})\). We get the case (3.2.2).

Finally consider the case where \((\bar{X},\bar{C})\) is of type \((\text{II}^\vee)\). Then \(n = 2, i.e., (Z,o)\) is of type \(A_1\). Let \(\bar{P}\) be a (unique) non-Gorenstein point. Then \((\bar{X},\bar{C})\) is locally imprimitive at \(\bar{P}\) and \((\bar{X},\bar{P})\) is of type \(cA/2\) or \(cAx/2\). Moreover, in the first case, \((\bar{X},\bar{P})\) is analytically isomorphic to a singularity given by (2.2.1). If there is no divisorial contractions over \(\bar{P}\), we can argue as above and get the case (3.2.2). Otherwise on some step, the map \(\psi_{k+1}: \bar{X} \rightarrow X^q_{k+1}\) is an isomorphism near \(\bar{P}\) and there is a divisorial contraction \(g_k: X_k \rightarrow X_{k+1}\) which blows up a curve passsing through \(P_{k+1} := \psi_{k+1}(\bar{P})\). Let \(C_{k,0} := g_k^{-1}(P_{k+1})\) and let \(C_{k,1}\) be the proper transform of \(\bar{C}\) on \(X_k\). By Proposition (2.2) \(X_k\) has exactly one non-Gorenstein point \(P_k\) on \(C_{k,0}\). Moreover, \(P_k\) is either a cyclic quotient singularity \(\frac{1}{4}(1, 1, -1)\) or of type \(cAx/4\) and \((X_k, C_{k,0})\) is locally imprimitive at \(P_k\) of splitting degree 2. Note that \(\xi_k = g_k^* \xi_{k+1}\) is not Cartier at some point of \(C_{k,0}\). Since \(P_k\) is the only non-Gorenstein point on \(C_{k,0}\), \(\xi_k\) is not Cartier at \(P_k\). Now if \(C_{k,1}\) does not pass through \(P_k\), then as above we get the case (3.2.2). Assume that \(C_{k,0} \cap C_{k,1} = \{P_k\}\).

We claim that \((X_k, C_{k,1})\) is locally imprimitive at \(P_k\). Indeed, \(\xi_k\) defines the double cover \(\pi_k: (X'_k, C'_{k,1}) \rightarrow (X_k, C_k)\) which is étale outside \(\text{Sing} X_k\). Since \(\xi_k\) is not Cartier at \(P_k\), \(\pi_k\) does not split over \(P_k\). Hence, \(C'_{k,1} := \pi_k^{-1}(C_{k,1})\) is connected. On the other hand, since \((\bar{X},\bar{C})\) is locally imprimitive at \(\bar{P}\), the curve \(C'_{k,1}\) is reducible. This means that \(C_{k,1}\) is locally imprimitive at \(P_k\). Finally as above the map \(X_k \rightarrow X^q\) is an isomorphism near \(P_k\). We get case (3.2.3).  

\[\square\]
\textbf{(3.3) Proposition.} Notation as in (3.1). Then \((X, C)\) contains only one non-Gorenstein point \(P\). This point is either a cyclic quotient \(\frac{1}{4}(1,1,-1)\) or of type \(cAx/4\). Moreover, for each component \(C_i \subset C\) the germ \((X, C_i)\) is imprimitive at \(P\) and \((Z, o)\) of type \(A_1\).

\textit{Proof.} By Proposition (3.2) there is a component \(C_0^q \subset C^q\) as in (3.2.1), (3.2.2), or (3.2.3). First assume that \(C_0^q\) is not contracted by \(q: X^q \to X\). Put \(C_0 := q(C_0^q)\). Then \((X, C_0)\) is an extremal neighborhood. In the case (3.2.1) it has two cyclic quotient singularities at \(q(P^q)\) and \(q(Q^q)\) and no other components of \(C\) pass through \(q(P^q)\) and \(q(Q^q)\). On the other hand, \(C \neq C_0\) and intersection points \(C_0 \cap (C - C_0)\) are non-Gorenstein \([Kol99, \text{Prop. 4.2}], [MP08, 4.4.2]\). Thus the extremal neighborhood \((X, C_0)\) has at least three non-Gorenstein points. This contradicts \([Mor88, \text{Th. 6.2}]\). Similarly, in the case (3.2.2), \((X, C_0)\) is locally imprimitive at \(q(P^q)\) and no other components of \(C\) pass through \(q(P^q)\). We get a contradiction by Lemma (3.1.1). Consider the case (3.2.3). If \(C_0^q\) is not contracted by \(q\), then we are done by Lemma (3.1.1). If \(C_1^q\) is contracted by \(q\), then \(q(C_1)\) is a point of type \(cAx/4\) by Proposition (2.4) and because \(P^q\) is of index 4. Then again the assertion follows by Lemma (3.1.1).

From now on we assume that \(q\) contracts \(C_0^q\), i.e., \(K_{X^q} \cdot C_0^q = 0\). In the case (3.2.3) by symmetry and by the above arguments we may assume that \(q\) contracts \(C_1^q\). Consider the decomposition

\[ q: X^q \xrightarrow{\varphi} X^\delta \xrightarrow{\delta} X, \]

where \(\varphi \) contracts all the \(K_{X^q}\)-trivial components of \(C^q\) except for \(C_0^q\). Put \(C^\delta := \varphi(C_0^q)\) and \(C_0^\delta := \varphi(C_0^q)\). Thus \(-K_{X^\delta}\) is nef and big over \(Z\) and \(C_0^\delta\) is the only \(K_{X^\delta}\)-trivial curve on \(X^\delta/Z\). Let \(C^\delta := C^\delta - C_0^\delta\). Then \(C^\delta\) has at least two components. Let \(P := \delta(C_0^\delta)\) and \(R^\delta = C^\delta \cap C_0^\delta\). By Proposition (2.5) \(R^\delta\) is not Gorenstein.

In the case (3.2.1), \(C_0^\delta\) contains at least three non-Gorenstein points: \(R^\delta\), \(P^\delta := \varphi(P^q)\), and \(Q^\delta := \varphi(Q^q)\). This contradicts Proposition (2.4).

In the case (3.2.2), \(P^\delta := \varphi(P^q)\) is a locally imprimitive point of \((X^\delta, C_0^\delta)\). By Proposition (2.4) the singularity \((X, P = \delta(C_0^\delta))\) is not of type \(cA^*/\). If the index of \((X, P)\) is \(\geq 4\), then \((X, P)\) is of type \(cAx/4\) and we can apply Lemma (3.1.1). Thus we assume that \((X, P)\) is of index 2 and \(n = 2\). Let \(C_i \subset C\) be a component passing through \(P\). By \([Mor88, \text{Cor. 1.16}]\) \((X, C_i)\) is primitive at \(P\). Further, \(\xi := f^* \eta = q_* \xi^q\) is an 2-torsion element of \(\mathcal{O}^{\text{ec}}(X, C)\) and is not Cartier at \(P\). This defines a double étale in codimension one cover \((X', C_i') \to (X, C_i)\) which does not splits over \(P\). Hence there is a point \(Q \in (X, C_i)\) of even index.
This contradicts the classification [KM92, Th. 2.2] (cf. [Mor07]).

Consider the case (3.2.3). Then \( P^\delta := \varphi(P^4) \) is a point of index \( \geq 4 \) (because \( \varphi \) is a crepant contraction). Recall that \( \varphi \) contracts \( C_1^4 \) by our assumption. Then by Proposition (2.4) \( (X^\delta, P^\delta) \) is a point of type \( cAx/4 \). As in the proof of Proposition (2.4), let \( D \in | - K_{(X, \delta(P^\delta))} | \) be a general element and let \( S := \delta^{-1}(D) \). Then both \( D \) and \( S \) have only Du Val singularities and the contraction \( \delta_S: S \to D \) is crepant. Since \( (S, P^\delta) \) is not of type \( A_4 \), the germ \( (D, P) \) also cannot be of type \( A_4 \). Hence, \( (X, P) \) is not of type \( cA/4 \) (because its index is \( \geq 4 \)). Then the assertion follows by Lemma (3.1.1).

(3.4) Explicit forms. By Proposition (3.3) and Lemma (3.1.1) \( f: (X, C) \to (Z, o) \) is a quotient of an index-two \( \mathbb{Q} \)-conic bundle \( f': (X', C') \to (Z', o') \) over a smooth base by \( \mu_2 \), where \( \mu_2 \) acts on \( X' \) and \( Z' \) freely in codimension one. By [MP08, Prop. 12.1.10] there is a \( \mu_2 \)-equivariant diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{f} & \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2 \\
\downarrow & & \downarrow p \\
\mathbb{C}^2 & & \\
\end{array}
\]

where the actions of \( \mu_2 \) on \( (\mathbb{C}^2, 0) \simeq (Z', o') \) and \( \mathbb{P}(1, 1, 1, 2) \) are linear. Further, we can make coordinates \( y_1, y_2, y_3, u, v \) in \( \mathbb{P}(1, 1, 1, 2) \) and \( \mathbb{C}^2 \) to be semi-invariant. By [MP08, Th. 12.1] \( X' \) is given by two semi-invariant equations

\[
\begin{align*}
q_1(y_1, y_2, y_3) - \psi_1(y_1, \ldots, y_4; u, v) &= 0, \\
q_2(y_1, y_2, y_3) - \psi_2(y_1, \ldots, y_4; u, v) &= 0,
\end{align*}
\]

where \( \psi_i \) and \( q_i \) are weighted quadratic in \( y_1, \ldots, y_4 \) with respect to \( \text{wt}(y_1, \ldots, y_4) = (1, 1, 1, 2) \) and \( \psi_i(y_1, \ldots, y_4; 0, 0) = 0 \). Since the action of \( \mu_2 \) on \( Z \simeq \mathbb{C}^2 \) is free outside 0, this action is given by \( u \mapsto -u, v \mapsto -v \). Modulo multiplication on \( \pm 1 \) and permutations of \( y_1, y_2, y_3 \), we may assume also that \( y_1 \mapsto -y_1, y_2 \mapsto -y_2, y_3 \mapsto y_3 \). Otherwise all the points of \( \{ y_4 = 0 \} \cap C' \) are fixed by \( \mu_2 \), while \( P \) is the only non-Gorenstein on \( X \).

The central fiber \( C' \) is defined by \( q_1 = q_2 = 0 \). By Lemma (3.1.1) \( C' \) has exactly four components and \( \mu_2 \) does not fix any of them. Thus we may assume that \( C' = \cup C'_i, i = 1, 2, 3, 4 \) and \( \mu_2 \) interchanges \( C'_1 \) and \( C'_2 \) (resp. \( C'_3 \) and \( C'_4 \)). For any two components \( C'_i \neq C'_j \) of \( C' \), there is a linear form \( l_{i,j}(y_1, \ldots, y_3) \) that vanishes along \( C'_i \cup C'_j \). Then quadratic forms \( l_{1,2}l_{3,4}, l_{1,3}l_{2,4}, l_{1,4}l_{2,3} \) vanish along \( C' \). Hence they belong to the pencil \( \lambda_1 q_1 + \lambda_2 q_2 \) and semi-invariant. This
implies that the action of $\mu_2$ on the pencil is trivial. Moreover, we can put $q_1 = l_{1,3}l_{2,4}$ and $q_2 = l_{1,4}l_{2,3}$. In view of the $\mu_2$-action we may assume that $l_{1,3} = y_1 + y_3$, $l_{2,4} = y_1 - y_3$, $l_{1,4} = y_2 + y_3$, $l_{2,3} = y_2 - y_3$ after some linear coordinate change of $y_1, y_2, y_3$.

We claim that $y_4 \mapsto -y_4$. The arguments below are similar to ones in the proof of [MP08, Lemma 12.1.12]. Assume to the contrary that $y_4 \mapsto y_4$. Let $U \subset \mathbb{P}(1,1,1,2)$ be the chart $y_4 \neq 0$. Then $U \simeq \mathbb{C}^3_{z_1,z_2,z_3}/\mu_2(1,1,1)$. Let $X^\sharp$ be the pull-back of $X \cap (U \times \mathbb{C}^2_{u,v})$ on $\mathbb{C}^3_{z_1,z_2,z_3} \times \mathbb{C}^2_{u,v}$, and let $P^\sharp \in X^\sharp$ be the preimage of $P$. Since the induced map $X^\sharp \to X$ is étale in codimension one, $(X^\sharp, P^\sharp) \to (X, P)$ is the index-one cover. Hence $(X^\sharp, P^\sharp) \to (X, P)/\mu_2$ is also the index-one cover of the terminal point $(X, P)/\mu_2$ of index 4 (the last is true because the action of $\mu_2$ is free in codimension one). Hence the morphism is a $\mu_4$-covering by the structure of terminal singularities. However $(X, P)/\mu_2$ is the quotient of $(X^\sharp, P^\sharp)$ by commuting $\mu_2$-actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (-z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have $y_4 \mapsto -y_4$ as claimed. This finishes the proof of Theorem (1.3).

**Proof of Corollary (1.4).** If $C$ is irreducible, the assertion follows by [MP08, Proposition (1.3.7)], so we have to check only cases (1.3.1) and (1.3.2). Thus we assume that $X$ has a unique non-Gorenstein point, say $P$, and $C$ is reducible. For each component $C_i \subset C$, the germ $(X, C_i)$ is an extremal neighborhood with a unique non-Gorenstein point. Let $F \in |-K_{(X,P)}|$ be a general member of the anti-canonical linear system of the germ $(X, P)$. The point $(F, P)$ is Du Val by [Rei87, (6.4), (B)]. Further, $F$ is also a member of $|-K_{(X,C_i)}|$ for each $C_i$, see [Mor88, Theorem (7.3)]. Hence $F \in |-K_X|$. 

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