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Beurling's Theorem and $L^p - L^q$ Morgan's Theorem for Step Two Nilpotent Lie Groups

By

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Abstract

We prove Beurling's theorem and $L^p - L^q$ Morgan's theorem for step two nilpotent Lie groups. These two theorems together imply a group of uncertainty theorems.

§1. Introduction

Roughly speaking the *Uncertainty Principle* says that "A nonzero function f and its Fourier transform $\widehat{f}$ cannot be sharply localized simultaneously". There are several ways of measuring localization of a function and depending on it one can formulate different versions of qualitative uncertainty principle (QUP). The most remarkable result in this genre in recent times is due to Hörmander [13] where decay has been measured in terms of a single integral estimate involving f and $\widehat{f}$.

Theorem 1.1 (Hörmander 1991). *Let $f \in L^2(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(y)|e^{|x||y|}\ dx\ dy < \infty.$$

Then $f = 0$ almost everywhere.

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Hörmander attributes this theorem to A. Beurling. The above theorem of Hörmander was further generalized by Bonami et al [7] which also accommodates the optimal point of this trade-off between the function and its Fourier transform:

Theorem 1.2. *Let $f \in L^2(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\widehat{f}(y)|e^{|x||y|}}{(1+|x|+|y|)^N} dx\ dy < \infty$$

for some $N \geq 0$. Then $f = 0$ almost everywhere whenever $N \leq n$. If $N > n$, then $f(x) = P(x)e^{-a|x|^2}$ where P is a polynomial with $\deg P < \frac{(N-n)}{2}$ and $a > 0$.

Following Hörmander we will refer to the theorem above simply as Beurling's theorem.

This theorem is described as *master theorem* by some authors as theorems of Hardy, Cowling-Price and some versions of Morgan's as well as $L^p - L^q$ Morgan's follow from it. (See Theorem 2.1 for precise statements of these theorems.)

There is some misunderstanding regarding the implication of Beurling's theorem. However it was observed by Bonami et al. ([7]) that Beurling's theorem does not imply Morgan's theorem in its sharpest form. Indeed Beurling's theorem (Theorem 1.2) together with $L^p - L^q$ Morgan's theorem (Theorem 2.1 (v)) can claim to be the master theorem. We can summarize the relations between these theorems on $\mathbb{R}^n$ in the following diagram.

$$\begin{array}{ccc|cc} & \Rightarrow & \text{Hardy's} & \text{Morgan's} \Leftarrow & \\ \text{Beurling's} & & \Uparrow & \Uparrow & L^p - L^q\text{Morgan's} \\ & \Rightarrow & \text{Cowling-Price} & \text{Gelfand-Shilov} \Leftarrow & \end{array}$$

The aim of this paper is to prove analogues of Beurling's theorem and $L^p - L^q$ Morgan's theorem (Theorem 1.2, Theorem 2.1 (case v)) for the step two nilpotent Lie groups. It is clear from the diagram that all other theorems mentioned above follow from these two theorems. Note that the diagram above remains unchanged when $\mathbb{R}^n$ is substituted by the step two nilpotent Lie groups.

For the convenience of the presentation and easy readability we will first deal with the special case of the Heisenberg groups and then extend the argument for general step two nilpotent Lie groups. The organization of the paper is as follows. In Section 2 we prove modified versions of Theorem 1.2 and Theorem 2.1 for $\mathbb{R}^n$ which are important steps towards proving those theorems for the class of groups mentioned above. In section 3 we establish the preliminaries of the Heisenberg group and prove the two theorems for this group. In section 4 we put the required preliminaries for general step two nilpotent Lie groups. Finally in section 5 we prove the analogues of Beurling's and $L^p - L^q$-Morgan's theorems for step two nilpotent groups. We indicate how the other theorems of this genre follow from those two theorems. We also show the necessity and sharpness of the estimates used in the two theorems.

Some of the other theorems, which follow from Beurling's and $L^p - L^q$-Morgan's (Hardy's and Cowling-Price to be more specific) were proved independently on Heisenberg groups or nilpotent Lie groups in recent years by many authors (see [1, 3, 4, 5, 14, 17] etc.). However we may note that these theorems were proved under some restrictions. But as corollaries of the Beurling's and $L^p - L^q$-Morgan's theorem we get exact analogues of these theorems. We include a precise comparison with the earlier results in the last section. For a general survey on uncertainty principles on different groups we refer to [11, 20].

§2. Euclidean Spaces

We can state a group of uncertainty principles in a compact form as follows:

Theorem 2.1. *Let f be a measurable function on $\mathbb{R}$. Suppose for some $a, b > 0$, $p, q \in [1, \infty]$, $\alpha \geq 2$ and $\beta > 0$ with $1/\alpha + 1/\beta = 1$, f satisfies*

$$e^{a|x|^\alpha} f \in L^p(\mathbb{R}) \quad \text{and} \quad e^{b|y|^\beta} \widehat{f} \in L^q(\mathbb{R}).$$

If moreover

$$(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > \left(\sin \frac{\pi}{2}(\beta - 1)\right)^{1/\beta} \tag{2.1}$$

then $f = 0$ almost everywhere.

The case

(i) $\alpha = \beta = 2$ and $p = q = \infty$ is Hardy's theorem.

(ii) $\alpha = \beta = 2$ is Cowling-Price theorem.

(iii) $\alpha > 2$, $p = q = \infty$ is Morgan's theorem.

(iv) $\alpha > 2$ and $p = q = 1$ is Gelfand-Shilov theorem.

(v) $\alpha > 2$, $p, q \in [1, \infty]$ is $L^p - L^q$ Morgan's theorem.

This theorem has ready generalization for $\mathbb{R}^n$ where by $|x|$ we mean the Euclidean norm of x.

It is clear that we have two separate sets of results in the theorem above namely the cases (i) and (ii) where $\alpha = 2$ and cases (iii), (iv), (v) where $\alpha > 2$. Note that for the first set, condition (2.1) reduces to $ab > 1/4$. Back in 1934 Morgan [15] observed that at the *optimal point* of (2.1) these two sets behave differently. To emphasize this we consider cases (i) and (iii) of Theorem 2.1 as representatives of the two sets of results. It is known that when $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ then in case (i) above f is a constant multiple of the Gaussian. In *great contrast* (see [15]) there are uncountably many functions which satisfy the estimates in case (iii) when $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$.

§2.1. Modified version of the Beurling's theorem

We will state and prove a modified version of Theorem 1.2. We need the following preparations. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. For a suitable function g on $\mathbb{R}^n$, the Radon transform Rg is a function on $S^{n-1}\times\mathbb{R}$, defined by

$$Rg(\omega,r) = R_\omega g(r) = \int_{x\cdot\omega=r} g(x)\,d\sigma_x, \tag{2.2}$$

where $d\sigma_x$ denotes the $(n-1)$-dimensional Lebesgue measure on the hyperplane $x\cdot\omega = r$ and $x\cdot\omega$ is the canonical inner product of x and ω, i.e., $x.\omega = \sum_{i=1}^n x_i\omega_i$. Note that when $g\in L^1(\mathbb{R}^n)$, then for any fixed $\omega\in S^{n-1}$, $Rg(\omega,r)$ exists for almost every $r\in\mathbb{R}$ and is an L^1-function on $\mathbb{R}$. It is also well known that (See [10], p. 185.)

$$\widehat{R_\omega g}(\lambda) = \widehat{g}(\lambda\omega). \tag{2.3}$$

Here $\widehat{R_\omega g}(\lambda) = \int_{\mathbb{R}} R_\omega g(r)e^{-i\lambda r}dr$ and $\widehat{g}(\lambda\omega) = \int_{\mathbb{R}^n} g(x)e^{-ix\cdot\lambda\omega}dx$.

We also need the following lemma:

Lemma 2.2. *Let $f_1(x) = P_1(x)e^{-\alpha_1 x^2}$ and $f_2(x) = P_2(x)e^{-\alpha_2 x^2}$ be two functions on $\mathbb{R}$ where P_1, P_2 are polynomials and α_1,α_2 are positive constants. Suppose that for some $\delta > 0$*

$$I_1 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_1(x)||\widehat{f_2}(y)|e^{|xy|}|Q(y)|^\delta}{(1+|x|+|y|)^N}dx\ dy < \infty$$

and

$$I_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f_2(x)||\widehat{f_1}(y)|e^{|xy|}|Q(y)|^\delta}{(1+|x|+|y|)^N}dx\ dy < \infty$$

where N is a positive integer and Q is a polynomial. Then $\alpha_1 = \alpha_2$.

Proof. We note that $\widehat{f_1}(y) = Q_1(y)e^{-\frac{1}{4\alpha_1}y^2}$ and $\widehat{f_2}(y) = Q_2(y)e^{-\frac{1}{4\alpha_2}y^2}$ where Q_1 and Q_2 are polynomials with $\deg Q_1 = \deg P_1$ and $\deg Q_2 = \deg P_2$. Then

$$I_1 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-\alpha_1 x^2+|xy|-\frac{1}{4\alpha_2}y^2}|Q(y)|^\delta |P_1(x)||Q_2(y)|}{(1+|x|+|y|)^N} dx\ dy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{\alpha_1}|x|-\frac{1}{2\sqrt{\alpha_2}}|y|)^2} e^{(1-\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}})|x||y|}|Q(y)|^\delta |P_1(x)||Q_2(y)|}{(1+|x|+|y|)^N} dx\ dy.$$

Similarly we get

$$I_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{\alpha_2}|x|-\frac{1}{2\sqrt{\alpha_1}}|y|)^2} e^{(1-\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}})|x||y|}|Q(y)|^\delta |P_2(x)||Q_1(y)|}{(1+|x|+|y|)^N} dx\ dy.$$

We fix an $\epsilon > 0$ and consider the set $A_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid |\sqrt{\alpha}_1|x| - \frac{1}{2\sqrt{\alpha}_2}|y|| \le \epsilon\}$, which is clearly of infinite Lebesgue measure.

Let $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_r$ and $\nu_1 \le \nu_2 \le \cdots < \nu_s$ be the set of positive roots of the polynomials $P_1(x)$ and the polynomial $Q(y)Q_1(y)$ respectively. Let $M = \max\{\gamma_r, \nu_s\}$. Then the set $B_\epsilon = \{(x,y) \in A_\epsilon \mid x > 2M, y > 2M\}$ is also evidently a set of infinite Lebesgue measure in the first quadrant i.e. in $\mathbb{R}^+ \times \mathbb{R}^+$. On B_ϵ the integrand in I_1 does not vanish for any x, y.

If we assume that $\alpha_1 < \alpha_2$, then $\frac{\sqrt{\alpha}_1}{\sqrt{\alpha}_2} < 1$. In this case on B_ϵ, $e^{(1-\frac{\sqrt{\alpha_1}}{\sqrt{\alpha_2}})|x||y|}$ grows exponentially and $e^{-(\sqrt{\alpha_1}|x|-\frac{1}{2\sqrt{\alpha_2}}|y|)^2} \ge e^{-\epsilon}$. Hence there exists an $M_1 > 0$ such that the integrand in I_1 is greater than M_1 outside a compact subset of B_ϵ. Therefore $I_1 = \infty$. This contradicts the hypothesis that $I_1 < \infty$. Through similar steps we can prove that when $\alpha_2 < \alpha_1$, then $I_2 = \infty$. This completes the proof. □

With this preparation we will now prove the following modified Beurling's theorem for $\mathbb{R}^n$.

Theorem 2.3. *Suppose $f \in L^2(\mathbb{R}^n)$. Let for some $\delta > 0$*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||\widehat{f}(y)|e^{|x||y|}|Q(y)|^\delta}{(1+|x|+|y|)^N}\ dx\ dy < \infty, \tag{2.4}$$

where Q is a polynomial of degree m. Then $f(x) = P(x)e^{-a|x|^2}$ for some $a > 0$ and polynomial P with $\deg P < \frac{N-n-m\delta}{2}$.

Proof. **Step 0:** As $\widehat{f}$ is not identically zero and as Q is a polynomial, the product $|\widehat{f}(y)||Q(y)|^\delta$ is different from zero on a set of positive measure. Therefore we can assume that for some $y_0 \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{|f(x)|e^{|x||y_0|}}{(1+|x|+|y_0|)^N} dx < \infty.$$

As $f \in L^2(\mathbb{R}^n)$, it is a locally integrable function on $\mathbb{R}^n$ and hence for any $0 < r < |y_0|$, $\int_{\mathbb{R}^n} |f(x)|e^{r|x|}dx < \infty$. This shows in particular that $f \in L^1(\mathbb{R}^n)$. Indeed for the exponential weight $e^{|y_0||x|}$ it is easy to see that $\widehat{f}$ is holomorphic in a tubular neighbourhood in $\mathbb{C}^n$ around $\mathbb{R}^n$.

In (2.4) we use polar coordinates for y, to see that there exists a subset S of S^{n-1} with full surface measure such that for every $\omega_2 \in S$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{|f(x)||\widehat{f}(s\omega_2)||s|^{n-1}|Q(s\omega_2)|^\delta e^{|x||s|}}{(1+|x|+|s|)^N}\, ds\, dx < \infty. \tag{2.5}$$

In view of (2.3) this is the same as for every $\omega_2 \in S$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{|f(x)||\widehat{R_{\omega_2} f}(s)||s|^{n-1}|Q(s\omega_2)|^\delta e^{|x||s|}}{(1+|x|+|s|)^N}\, ds\, dx < \infty. \tag{2.6}$$

Step 1: In this step we will show that for any $\omega_1 \in S^{n-1}$ and $\omega_2 \in S$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{R_{\omega_1}|f|(r)|\widehat{R_{\omega_2} f}(s)||s|^{n-1}|Q(s\omega_2)|^\delta e^{|r||s|}}{(1+|r|+|s|)^N}\, ds\, dr < \infty. \tag{2.7}$$

We will break the above integral into the following three parts and show that each part is finite. That is we will show:

(i)

$$\int_{\mathbb{R}} \int_{|s|>L} \frac{R(|f|)(\omega_1, r)|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^\delta}{(1+|r|+|s|)^N}\, ds\, dr < \infty$$

for $L > 0$ such that $L + L^2 > N$.

(ii)

$$\int_{|r|>M} \int_{|s|\le L} \frac{R(|f|)(\omega_1, r)|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^\delta}{(1+|r|+|s|)^N}\, ds\, dr < \infty$$

for $M = 2(L+1)$ and L as in (i).

(iii)

$$\int_{|r|\le M}\int_{|s|\le L}\frac{R(|f|)(\omega_1,r)|\widehat{R_{\omega_2}f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^{\delta}}{(1+|r|+|s|)^N}\,ds\,dr<\infty$$

for M, L used in (i) and (ii).

Proof of (i): It is given that $L+L^2>N$. We will show that for any s such that $|s|\ge L$,

$$\frac{e^{|s||x|}}{(1+|x|+|s|)^N}\ge\frac{e^{|s||\langle x,\omega_1\rangle|}}{(1+|\langle x,\omega_1\rangle|+|s|)^N}. \tag{2.8}$$

Let $F(z)=\frac{e^{\alpha z}}{(1+\alpha+z)^N}$ for $\alpha>0$ and $\alpha+\alpha^2>N$. Then $F'(z)>0$ for any $z\ge 0$. Therefore, if $z_1\ge z_2\ge 0$, then

$$\frac{e^{\alpha z_1}}{(1+\alpha+z_1)^N}\ge\frac{e^{\alpha z_2}}{(1+\alpha+z_2)^N}. \tag{2.9}$$

Note that $|x|\ge|\langle x,\omega_1\rangle|$ for all $x\in\mathbb{R}^n$ and $\omega_1\in S^{n-1}$. Now take $z_1=|x|$ and $z_2=|\langle x,\omega_1\rangle|$. Then $z_1\ge z_2\ge 0$. We take $\alpha=|s|\ge L$ to get (2.8).

From (2.6) we get:

$$\int_{\mathbb{R}}\int_{x\cdot\omega_1=r}\int_{\mathbb{R}}\frac{|f(x)||\widehat{R_{\omega_2}f}(s)|e^{|x||s|}|s|^{n-1}|Q(s\omega_2)|^{\delta}}{(1+|x|+|s|)^N}ds\,d\sigma_1\,dr<\infty, \tag{2.10}$$

where $d\sigma_1$ denotes the Lebesgue measure on the hyper plane $\{x:x\cdot\omega_1=r\}$. We use the inequality (2.8) to obtain:

(2.11)
$$\int_{\mathbb{R}}\int_{x\cdot\omega_1=r}\int_{|s|>L}\frac{|f(x)||\widehat{R_{\omega_2}f}(s)|e^{|\langle x,\omega_1\rangle||s|}|s|^{n-1}|Q(s\omega_2)|^{\delta}}{(1+|\langle x,\omega_1\rangle|+|s|)^N}\,ds\,d\sigma_1\,dr<\infty.$$

Now we put $\langle x,\omega_1\rangle=r$ in the above integral and use the definition of Radon transform to obtain,

$$\int_{\mathbb{R}}\int_{|s|>L}\frac{R(|f|)(\omega_1,r)|\widehat{R_{\omega_2}f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^{\delta}}{(1+|r|+|s|)^N}\,ds\,dr<\infty. \tag{2.12}$$

This proves (i).

Proof of (ii): Let

$$I_2=\int_{|r|>M}\int_{|s|\le L}\frac{R(|f|)(\omega_1,r)|\widehat{R_{\omega_2}f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^{\delta}}{(1+|r|+|s|)^N}ds\,dr.$$

It is clear that,

$$\begin{aligned} I_2 &\leq C \int_{|r|>M} \frac{R(|f|)(\omega_1, r)|e^{L|r|}}{(1+|r|)^N} dr \\ &= C \int_{|r|>M} \int_{x\cdot\omega_1=r} \frac{|f(x)|e^{L|r|}}{(1+|r|)^N}\ d\sigma_1\ dr \\ &= CI_3, \text{say.} \end{aligned}$$

We have observed in Step 0 that $\widehat{f}$ is real analytic on $\mathbb{R}^n$ and hence $\widehat{f}(y) \neq 0$ for almost every $y \in \mathbb{R}^n$. Therefore, y_0 in Step 0 can be taken so that $|y_0| > 2L$ and we have

$$\int_{\mathbb{R}} \int_{x\cdot\omega_1=r} \frac{|f(x)|e^{|x||y_0|}}{(1+|x|+|y_0|)^N}\ d\sigma_1\ dr < \infty.$$

Since $|y_0| + |y_0|^2 > N$ by our choice of y_0, we can apply the argument of case (i) (2.9) with $\alpha = |y_0|$, $z_1 = |x|$ and $z_2 = |\langle x, \omega\rangle|$ for $\omega \in S^{n-1}$ and get from above:

$$\int_{|r|>M} \int_{x\cdot\omega_1=r} \frac{|f(x)|e^{|r||y_0|}}{(1+|r|+|y_0|)^N}\ d\sigma_1\ dr < \infty.$$

Again noting that $M + M^2 > N$ and applying the argument of case (i) (2.9) with $\alpha = |r| > M$ and $z_1 = |y_0|, z_2 = 2L$ we conclude:

$$\int_{|r|>M} \int_{x\cdot\omega_1=r} \frac{|f(x)|e^{2L|r|}}{(1+|r|+2L)^N}\ d\sigma_1\ dr < \infty.$$

From this it is easy to see that $I_3 < \infty$. This completes the proof of (ii).

Proof of (iii): As the domain $[-M, M] \times [-L, L]$ is compact and as

$$\frac{|\widehat{R_{\omega_2} f}(s)|e^{|r||s|}|s|^{n-1}|Q(s\omega_2)|^\delta}{(1+|r|+|s|)^N}$$

is continuous in this domain, the integral is bounded by $C\int_{-M}^{M} R|f|(\omega_1, r)\ dr$. Now recall that $f \in L^1(\mathbb{R}^n)$. Therefore,

$$\begin{aligned} \int_{-M}^{M} R|f|(\omega_1, r)\ dr &\leq \int_{\mathbb{R}} R|f|(\omega_1, r)\ dr \\ &= \int_{\mathbb{R}} \int_{x\cdot\omega_1=r} |f(x)|\ d\sigma_1\ dr \\ &= \int_{\mathbb{R}^n} |f(x)|\ dx < \infty. \end{aligned} \tag{2.13}$$

Thus from (i), (ii) and (iii) we obtain (2.7). This completes Step 1.

Step 2: Using $|R_\omega f(r)| \le R_\omega|f|(r)$ we see from (2.7) that for almost every $\omega \in S^{n-1}$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|R_\omega f(r)||\widehat{R_\omega f}(s)||s|^{n-1}|Q(s\omega)|^\delta e^{|r||s|}}{(1+|r|+|s|)^N}\, dr\, ds < \infty. \tag{2.14}$$

Now as for fixed ω, $|s|^{n-1}|Q(s\omega)|^\delta$ is a proper map in s and as $R_\omega f$ as well as $\widehat{R_\omega f}$ are locally integrable functions we can apply the 1-dimensional case of Theorem 1.2 to conclude that $R_\omega f(r) = A_\omega(r)e^{-\alpha r^2}$, for some polynomial A_ω which depends on ω with $\deg A_\omega < \frac{N-m\delta-n}{2}$ and α is a positive constant. A priori, α also should depend on ω. But we will see below that α is actually independent of ω. It is clear that $\widehat{R_\omega f}(s) = P_\omega(s)e^{-\frac{1}{4\alpha}s^2}$, where $\deg P_\omega$ is same as $\deg A_\omega$. Consider $\omega_1, \omega_2 \in S$ with $\omega_1 \ne \omega_2$ for which R_{ω_1}, R_{ω_2} satisfy (2.7). From the argument above it follows that $R_{\omega_1} f(r) = A_{\omega_1}(r)e^{-\alpha_1 r^2}$ and $\widehat{R_{\omega_2} f}(s) = P_{\omega_2}(s)e^{-\frac{1}{4\alpha_2}s^2}$ for some positive constants α_1, α_2. Therefore by Lemma 2.2, $\alpha_1 = \alpha_2 = \alpha$, say and $\widehat{R_\omega f}(s) = P_\omega(s)e^{-\frac{1}{4\alpha}s^2}$.

Step 3: We will show that $P_\omega(s) = P(s\omega)$ is a polynomial in $s\omega$, that is P is a polynomial in $\mathbb{R}^n$. We recall that $\widehat{R_\omega f}(s) = \widehat{f}(s\omega)$ is a holomorphic function in a neighbourhood around 0 (see Step 0). We can write $P_\omega(s) = \widehat{f}(s\omega)e^{\frac{1}{4\alpha}s^2} = \widehat{f}(s\omega)e^{\frac{1}{4\alpha}|s\omega|^2} = F(s\omega)$, say.

We write $F(s\omega) = \sum_{j=0}^k a_j(\omega)s^j$, where $k = \max_{\omega\in S^{n-1}} \deg P_\omega < \frac{N-m\delta-n}{2}$. Then for $j = 0, 1, \ldots, k$

$$\frac{1}{j!}\frac{d^j}{ds^j}F(s\omega)\bigg|_{s=0} = a_j(\omega).$$

The left hand side is the restriction of a homogenous polynomial of degree j to S^{n-1}. Therefore $F(s\omega)$ is a polynomial of degree $\le k$ in $\mathbb{R}^n$. Therefore $\widehat{f}(x) = P(x)e^{-\frac{1}{4\alpha}|x|^2}$, where $\deg P < \frac{N-m\delta-n}{2}$. □

§2.2. Modified version of the $L^p - L^q$ Morgan's theorem

We will state and prove a modified version of $L^p - L^q$ Morgan's theorem on $\mathbb{R}^n$.

Theorem 2.4. *Let f be a measurable function on $\mathbb{R}^n$. Suppose for some $a, b > 0$, $p, q \in [1, \infty]$, $\alpha > 2$ and $\beta > 0$ with $1/\alpha + 1/\beta = 1$, f satisfies the following conditions*:

(i) $\int_{\mathbb{R}^n} e^{pa|x|^\alpha}|f(x)|^p\, dx < \infty$,

(ii) $\int_{\mathbb{R}^n} e^{qb|y|^\beta}|\widehat{f}(y)|^q|Q(y)|^\delta\, dy < \infty$, *where* $Q(y)$ *is a polynomial in* y *of degree* k *and* $\delta > 0$.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ *then* $f = 0$ *almost everywhere. If* $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ *then there are infinitely many linearly independent functions which satisfy* (i) *and* (ii).

Proof. First we will see that the theorem is true for $n = 1$. From the hypothesis (i) it is clear that $f \in L^1(\mathbb{R})$ and hence $\widehat{f}$ is continuous. Also as $|Q(y)|^\delta$ is a proper map, we immediately get

$$\int_{\mathbb{R}} e^{qb|y|^\beta}|\widehat{f}(y)|^q\, dy < \infty.$$

That is, f satisfies all the hypothesis of Theorem 2.1 case (v) and hence the theorem for $n = 1$ follows.

Now we assume that $n \geq 2$. Let us consider the case $p = q = 1$ for the sake of simplicity. For each $\omega \in s^{n-1}$

$$\begin{aligned}\int_{\mathbb{R}} e^{a|r|^\alpha}|R_\omega f(r)|\, dr &\leq \int_{\mathbb{R}} e^{a|r|^\alpha} R_\omega|f|(r)\, dr \\ &= \int_{\mathbb{R}}\int_{x\cdot\omega=r} |f(x)|\, d\sigma\, dr \\ &\leq \int_{\mathbb{R}}\int_{x\cdot\omega=r} e^{a|x|^\alpha}|f(x)|\, d\sigma\, dr \\ &= \int_{\mathbb{R}^n} e^{a|x|^\alpha}|f(x)|\, dx < \infty.\end{aligned} \tag{2.15}$$

Here $d\sigma$ denotes the measure on the hyperplane $\{x : x\cdot\omega = r\}$. Using the polar coordinates we get

$$\begin{aligned}&\int_{\mathbb{R}}\int_{S^{n-1}} e^{b|r|^\beta}|\widehat{R_\omega f}(r)||r|^{n-1}|Q(r\omega)|^\delta\, d\omega\, dr \\ &= \int_{\mathbb{R}}\int_{S^{n-1}} e^{b|r|^\beta}|\widehat{f}(r\omega)||r|^{n-1}|Q(r\omega)|^\delta\, d\omega\, dr \\ &= 2\int_{\mathbb{R}^n} e^{b|y|^\beta}|f(y)||Q(y)|^\delta\, dy < \infty.\end{aligned}$$

Hence almost every $\omega \in S^{n-1}$

(2.16)

$$\int_{\mathbb{R}} e^{b|r|^\beta}|\widehat{R_\omega f}(r)||r|^{n-1}|Q(r\omega)|^\delta\, dr = \int_{\mathbb{R}} e^{b|r|^\beta}|\widehat{f}(r\omega)||r|^{n-1}|Q(r\omega)|^\delta\, dr < \infty.$$

We can now apply the one-dimensional case of the theorem proved above to the function $R_\omega f$ to conclude that for almost every $\omega \in S^{n-1}$, $\widehat{R_\omega f}(r) = \widehat{f}(r\omega) = 0$ whenever $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ and hence $f = 0$ almost everywhere.

Given $a, b > 0$ with $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ we can always choose $a' < a$, $b' < b$ such that $(a'\alpha)^{1/\alpha}(b'\beta)^{1/\beta} > (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$. If $p, q > 1$, using Hölder inequality together with the given hypothesis we get

$$\int_{\mathbb{R}^n} e^{a'|x|^\alpha}|f(x)|\,dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^n} e^{b'|y|^\beta}|\widehat{f}(y)||Q(y)|^\delta\,dy < \infty.$$

Hence the first part of the theorem follows.

For the last part: Let us define the function f by

$$f(x) = -i\int_C z^\nu e^{z^q - qAz|x|^2}\,dz$$

where $q = \frac{\alpha}{\alpha-2}, A^\alpha = \frac{1}{4}((\alpha-2)a)^2, \nu = \frac{2m+4-\alpha}{2(\alpha-2)}$, $m \in \mathbb{R}$ and C is a path which lies in the half plane $\Im z > 0$, and goes to infinity, in the directions $\theta = \arg z = \pm\theta_\circ$, where $\frac{\pi}{2}q < \theta_\circ < \frac{1}{2}\pi$. Then Ayadi et al. [2] shows with the help of Morgan's [15] method that for every pair of real numbers (m, m') which are related by $m' = \frac{2m+n(2-\alpha)}{(2\alpha-2)}$ there exists a function f on $\mathbb{R}^n$ such that

$$f = \mathrm{O}(|x|^m e^{-a|x|^\alpha}) \quad \text{and} \quad \widehat{f} = \mathrm{O}(|y|^{m'} e^{-b|y|^\beta}), \tag{2.17}$$

where α, β, a, b are as in the hypothesis of the theorem. As $1/\alpha + 1/\beta = 1$ the relation above can also be written as $m = \frac{2m'+n(2-\beta)}{(2\beta-2)}$.

We will apply this result to construct functions satisfying the equality cases of the hypothesis. Assume that the degree of the polynomial Q is k.

For $p = \infty, q = \infty$, we choose an $m' < \min\{-k\delta, -\frac{n}{2}(2-\beta)\}$ and take the corresponding m. By this choice (as $\beta > 1$ by hypothesis) $m = \frac{2m'+n(2-\beta)}{(2\beta-2)} < 0$ and $m' + k\delta < 0$. We construct a function f satisfying (2.17) for this pair (m, m'). This f will satisfy both the hypothesis for $p = q = \infty$, i.e. $|f(x)| \leq Ce^{-a|x|^\alpha}$ and $|\widehat{f}(y)||Q(y)|^\delta \leq Ce^{-b|y|^\beta}$. If $p \neq \infty, q = \infty$ we take an m' satisfying $m' < \min\{-k\delta, -\frac{n}{p}(\beta-1) - \frac{n}{2}(2-\beta)\}$ and take the corresponding m. Then $m = \frac{2m'+n(2-\beta)}{(2\beta-2)} < -\frac{n}{p}$ $m' + k\delta < 0$. The function in (2.17) for this choice of m, m' is the required function for this case. For $p \in [1, \infty]$ and $q \neq \infty$ we have to choose $m' < \min\{-\frac{n+k\delta}{q}, -\frac{n}{p}(\beta-1) - \frac{n}{2}(2-\beta)\}$. $\square$

§3. Heisenberg Groups

Main results in this section are analogues of Theorem 1.2 and Theorem 2.1 (v) for the Heisenberg groups. Let us first recall some basic facts of the Heisenberg groups. The n-dimensional Heisenberg group H^n is $\mathbb{C}^n \times \mathbb{R}$ equipped with the following group law

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\Im(z.\bar{w})\right),$$

where $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$. For each $\lambda \in \mathbb{R} \setminus \{0\}$ there exists an irreducible unitary representation π_λ realized on $L^2(\mathbb{R}^n)$ given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y))}\phi(\xi + y),$$

for $\phi \in L^2(\mathbb{R}^n)$ and $z = x+iy$. These are all the infinite dimensional irreducible unitary representations of H^n up to unitary equivalence. For $f \in L^1(H^n)$, its group Fourier transform $\widehat{f}(\lambda)$ is defined by

$$\widehat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)\,dz\,dt. \tag{3.1}$$

We define $\pi_\lambda(z) = \pi_\lambda(z,0)$ so that $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z,0)$. For $f \in L^1(\mathbb{C}^n)$, we define the bounded operator $W_\lambda(f)$ on $L^2(\mathbb{R}^n)$ by

$$W_\lambda(f)\phi = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z)\phi\,dz. \tag{3.2}$$

It is clear that $\|W_\lambda(f)\| \le \|f\|_1$ and for $f \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$, it can be shown that $W_\lambda(f)$ is an Hilbert-Schmidt operator and we have the Plancherel theorem

$$\|W_\lambda(f)\|^2_{\mathrm{HS}} = (2\pi)^n|\lambda|^{-n}\int_{\mathbb{C}^n}|f(z)|^2\,dz. \tag{3.3}$$

Thus W_λ is an isometric isomorphism between $L^2(\mathbb{C}^n)$ and $\mathcal{S}_2$, the Hilbert space of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. This $W_\lambda(f)$ is known as the Weyl transform of f. For $f \in L^1(H^n)$, let

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)\,dt$$

be the inverse Fourier transform of f in the t–variable. Then from the definition of $\widehat{f}(\lambda)$, it follows that $\widehat{f}(\lambda) = W_\lambda(f^\lambda)$. For $\lambda = 1$ we define $W(z) = W_1(z)$. For $x \in \mathbb{R}$ and $k \in \mathbb{N}$, the polynomial $H_k(x)$ of degree k is defined by the formula

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}(e^{-x^2}). \tag{3.4}$$

We define the Hermite function $h_k(x)$ by

$$h_k(x) = (2^k k!\sqrt{\pi})^{-1/2} H_k(x) e^{-\frac{x^2}{2}}.$$

For $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$, the normalized Hermite function $\Phi_\mu(x)$ on $\mathbb{R}^n$ is defined by

$$\Phi_\mu(x) = h_{\mu_1}(x_1) \cdots h_{\mu_n}(x_n). \tag{3.5}$$

Hermite functions are eigenfunctions of the Hermite operator $H = -\triangle + |x|^2$ and they form an orthonormal basis for $L^2(\mathbb{R}^n)$. Here $\triangle$ is the Laplacian on $\mathbb{R}^n$. For $\mu, \nu \in \mathbb{N}^n$, the special Hermite function $\Phi_{\mu\nu}$ is defined by

$$\Phi_{\mu,\nu}(z) = (2\pi)^{-\frac{n}{2}} \left(W(z)\Phi_\mu, \Phi_\nu\right). \tag{3.6}$$

These functions form an orthonormal basis for $L^2(\mathbb{C}^n)$ and they are expressible in terms of Laguerre functions. For a detailed account of Hermite and special Hermite functions we refer to [19].

With this preparation we will now prove a version of Theorem 1.2 for H^n.

Theorem 3.1. *Suppose $f \in L^2(H^n)$ and for some $M, N \geq 0$, it satisfies*

$$\int_{H^n} \int_{\mathbb{R}} \frac{|f(z,t)| \|\widehat{f}(\lambda)\|_{HS} e^{|t||\lambda|}}{(1+|z|)^M (1+|t|+|\lambda|)^N} \; |\lambda|^n \, d\lambda \, dz \, dt < \infty.$$

Then $f(z,t) = e^{-at^2}(1+|z|)^M \left(\sum_{j=0}^{m} \psi_j(z) t^j\right)$, where $\psi_j \in L^2(\mathbb{C}^n) \cap L^1(\mathbb{C}^n)$ and $m < \frac{N-n/2-1}{2}$.

Proof. As in the case of $\mathbb{R}^n$, it can be verified that f is integrable in t-variable for almost every z. For each pair (ϕ, ψ), where $\phi, \psi \in L^2(\mathbb{R}^n)$ we consider the function

$$F_{(\phi,\psi)}(t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{C}^n} f(z,t)(1+|z|)^{-M} \overline{(W(z)\phi, \psi)} \, dz.$$

Then it follows that

$$\begin{aligned} |\widehat{F_{(\phi,\psi)}}(\lambda)| &= (2\pi)^{-\frac{n}{2}} \left| \int_{\mathbb{C}^n} f^{-\lambda}(z)(1+|z|)^{-M} \overline{(W(z)\phi, \psi)} \, dz \right| \\ &\leq C \left(\int_{\mathbb{C}^n} |f^{-\lambda}(z)|^2 \, dz \right)^{1/2} \\ &= C|\lambda|^{n/2} \|\widehat{f}(-\lambda)\|_{\mathrm{HS}}. \end{aligned} \tag{3.7}$$

Therefore, from the hypothesis we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|F_{(\phi,\psi)}(t)||\widehat{F_{(\phi,\psi)}}(\lambda)|e^{|t||\lambda|}|\lambda|^{n/2}}{(1+|t|+|\lambda|)^N}\, dt\, d\lambda$$
$$\leq C\int_{H^n}\int_{\mathbb{R}} \frac{|f(z,t)|\|\widehat{f}(\lambda)\|_{\mathrm{HS}}e^{|t||\lambda|}}{(1+|z|)^M\,(1+|t|+|\lambda|)^N}|\lambda|^n\, d\lambda\, dz\, dt < \infty.$$

Now applying Theorem 2.3 to the function $F_{(\phi,\psi)}$ with $\delta = n/2$ we have $F_{(\phi,\psi)}(t) = P_{(\phi,\psi)}(t)e^{-a(\phi,\psi)t^2}$, where $P_{(\phi,\psi)}$ is a polynomial with deg $< \frac{N-n/2-1}{2}$. Let us fix $\psi \in L^2(\mathbb{R}^n)$. Then $\phi \mapsto F_{(\phi,\psi)}(t)$ is conjugate linear for every $t \in \mathbb{R}$. This gives the following identity for all $t \in \mathbb{R}$:

$$P_{(\phi+\phi',\psi)}(t)e^{-a(\phi+\phi',\psi)t^2} = P_{(\phi,\psi)}(t)e^{-a(\phi,\psi)t^2} + P_{(\phi',\psi)}(t)e^{-a(\phi',\psi)t^2},$$

for all $\phi,\phi' \in L^2(\mathbb{R}^n)$. Without loss of generality we assume that $a(\phi,\psi) < a(\phi',\psi)$. Let us first consider the case when $a(\phi,\psi) < a(\phi',\psi) \leq a(\phi+\phi',\psi)$. From the identity above it follows that

$$P_{(\phi+\phi',\psi)}(t) = P_{(\phi,\psi)}(t)e^{(a(\phi+\phi',\psi)-a(\phi,\psi))t^2} + P_{(\phi',\psi)}(t)e^{(a(\phi+\phi',\psi)-a(\phi',\psi))t^2}.$$

Right hand side of the above identity is growing faster than that of left hand side unless $a(\phi,\psi) = a(\phi',\psi) = a(\phi+\phi',\psi)$. Similarly we can deal with the other cases, $a(\phi+\phi',\psi) \leq a(\phi,\psi) < a(\phi',\psi)$ and $a(\phi,\psi) \leq a(\phi+\phi',\psi) < a(\phi',\psi)$ to draw the same conclusion. Thus $a(\phi,\psi)$ is independent of ϕ. Through similar steps we can show that $a(\phi,\psi) = a(\psi)$ is also independent of ψ. Thus $a(\phi,\psi) = a(\psi) = a$, say. We recall that $\{\Phi_{\alpha,\beta} : \alpha,\beta \in \mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. Now we take $\phi = \Phi_\alpha$ and $\psi = \Phi_\beta$. Let $F_{\alpha,\beta} = F_{(\Phi_\alpha,\Phi_\beta)}$ and $P_{\alpha,\beta} = P_{(\Phi_\alpha,\Phi_\beta)}$. Since for almost every $t \in \mathbb{R}$, $(1+|\cdot|)^{-M}f(\cdot,t) \in L^2(C^n)$, the sequence $\{P_{\alpha,\beta}(t)\} \in l^2$ for all t. We write $P_{\alpha,\beta}(t) = \sum_{j=0}^{m} a_j(\alpha,\beta)t^j$, $m < \frac{N-n/2-1}{2}$. Choose $t_i \in \mathbb{R}$ such that $t_i \neq t_j$, for all $0 \leq i,j \leq m$. We consider a system of linear equations given by:

$$\begin{pmatrix} 1 & t_0 & \cdots & t_0^m \\ 1 & t_1 & \cdots & t_1^m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_m & \cdots & t_m^m \end{pmatrix} \begin{pmatrix} \{a_0(\alpha,\beta)\} \\ \{a_1(\alpha,\beta)\} \\ \vdots \\ \{a_m(\alpha,\beta)\} \end{pmatrix} = \begin{pmatrix} \{P_{\alpha,\beta}(t_0)\} \\ \{P_{\alpha,\beta}(t_1)\} \\ \vdots \\ \{P_{\alpha,\beta}(t_m)\} \end{pmatrix}.$$

Since $t_i \neq t_j$ for all $i \neq j$, the determinant of the $(m+1)\times(m+1)$ Vandermonde matrix is nonzero. Therefore, $\{a_j(\alpha,\beta)\}$ will be a linear combination of members from $\{\{P_{\alpha,\beta}(t_j)\} : 0 \leq j \leq m\}$ and hence $\{a_j(\alpha,\beta)\} \in l^2$ for each $0 \leq j \leq m$. With this observation we can write

$$\begin{aligned}(1+|z|)^{-M} f(z,t) &= \left(\sum_{\alpha,\beta} P_{\alpha,\beta}(t)\Phi_{\alpha,\beta}(z)\right) e^{-at^2} \\ &= \left(\sum_{\alpha,\beta}\left(\sum_{j=0}^{m} a_j(\alpha,\beta)t^j\right)\Phi_{\alpha,\beta}(z)\right) e^{-at^2} \\ &= \left(\sum_{j=0}^{m}\left(\sum_{\alpha,\beta} a_j(\alpha,\beta)\Phi_{\alpha,\beta}(z)\right) t^j\right) e^{-at^2} \\ &= \left(\sum_{j=0}^{m}\psi_j(z)t^j\right) e^{-at^2},\end{aligned}$$

where $\psi_j(\cdot) = \sum_{\alpha,\beta} a_j(\alpha,\beta)\Phi_{\alpha,\beta}(\cdot) \in L^2(\mathbb{C}^n)$. It follows from the hypothesis that $\sum_{j=0}^{m}\psi_j(z)t^j \in L^1(\mathbb{C}^n)$ for almost every $t \in \mathbb{R}$ and hence $\psi_j \in L^1(\mathbb{C}^n)$ for $j = 0, \cdots, m$. $\square$

Conversely we suppose that $f(z,t) = e^{-at^2}(1+|z|)^M \left(\sum_{j=0}^{m}\psi_j(z)t^j\right)$, for $\psi_j \in L^2(\mathbb{C}^n) \cap L^1(\mathbb{C}^n), j = 0, \cdots, m$. From (3.3) and the subsequent discussion it follows that

$$(2\pi)^{-\frac{n}{2}}|\lambda|^{\frac{n}{2}}\|\widehat{f}(\lambda)\|_{\text{HS}} \leq Ce^{-\frac{1}{4a}\lambda^2}|P_1(\lambda)|,$$

where P_1 is a polynomial of degree m. A straightforward calculation now shows that f satisfies the hypothesis.

We will conclude this section by proving the following analogue of $L^p - L^q$ Morgan's theorem for H^n.

Theorem 3.2. *Suppose a function $f \in L^2(H^n)$ satisfies*

(i) $\int_{H^n} e^{pa|(z,t)|^\alpha}|f(z,t)|^p\, dz\, dt < \infty$ *and*

(ii) $\int_{\mathbb{R}} e^{q|\lambda|^\beta}\|\widehat{f}(\lambda)\|_{HS}^q|\lambda|^n\, d\lambda < \infty$

where $p, q \in [1,\infty]$, $a, b > 0$, $\alpha > 2$, $\beta > 0$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ then $f = 0$ almost everywhere. But if $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ then there are infinitely many functions on H^n satisfying (i) *and* (ii).

Proof. First we note that $f \in L^1(H^n)$. We can choose $a' < a$, $b' < b$ such that $(a'\alpha)^{1/\alpha}(b'\beta)^{1/\beta} > (\sin \frac{\pi}{2}(\beta - 1))^{1/\beta}$ and use Hölder's inequality to show

(i)′ $\int_{H^n} e^{a'|(z,t)|^\alpha} |f(z,t)| \, dz \, dt < \infty$

(ii)′ $\int_{\mathbb{R}} e^{b'|\lambda|^\beta} \|\widehat{f}(\lambda)\|_{\mathrm{HS}} |\lambda|^{n/2} \, d\lambda < \infty$.

For each $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{N}^n$, we define the auxiliary function

$$F_{\mu,\nu}(t) = \int_{\mathbb{C}^n} f(z,t) \overline{\Phi_{\mu,\nu}(z)} \, dz.$$

Using (i)′ we have

$$\begin{aligned} \int_{\mathbb{R}} e^{a'|t|^\alpha} |F_{\mu,\nu}(t)| \, dt &= \int_{\mathbb{R}} \int_{\mathbb{C}^n} e^{a'|t|^\alpha} |f(z,t)| |\Phi_{\mu,\nu}(z)| \, dz \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{C}^n} e^{a'|(z,t)|^\alpha} |f(z,t)| \, dz \, dt \\ &< \infty. \end{aligned}$$

On the other hand using (ii)′ and the Plancherel formula for the Weyl transform we have

$$\begin{aligned} \int_{\mathbb{R}} e^{b'|\lambda|^\beta} |\widehat{F_{\mu,\nu}}(\lambda)| \, d\lambda &= \int_{\mathbb{R}} e^{b'|\lambda|^\beta} \left| \int_{\mathbb{C}^n} f^{-\lambda}(z) \overline{\Phi_{\mu,\nu}(z)} \, dz \right| d\lambda \\ &\leq C \int_{\mathbb{R}} e^{b'|\lambda|^\beta} \|f^{-\lambda}(\cdot)\|_2 \, d\lambda \\ &= \int_{\mathbb{R}} e^{b'|\lambda|^\beta} \|\widehat{f}(\lambda)\|_{\mathrm{HS}} |\lambda|^{\frac{n}{2}} \, d\lambda < \infty. \end{aligned}$$

Applying Theorem 2.1 (case (iv)) to the function $F_{\mu,\nu}$ we conclude that $F_{\mu,\nu} = 0$ for every $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{N}^n$. Since $\{\Phi_{\mu,\nu} : (\mu, \nu) \in \mathbb{N}^n \times \mathbb{N}^n\}$ form an orthonormal basis for $L^2(\mathbb{C}^n)$ we conclude that $f = 0$ almost everywhere.

Now for the second part of the theorem (i.e. the equality case) we recall that (see 2.17) there are enumerable examples of functions h on $\mathbb{R}$ such that

$$h = O(|t|^m e^{-a|t|^\alpha}) \quad \text{and } \widehat{h} = O(|\lambda|^{m'} e^{-b|\lambda|^\beta})$$

for the pair $m, m' \in \mathbb{R}$ related by $m = \frac{2m' - \beta + 2}{2(\beta - 1)}$. Here a, b, α, β are as in the hypothesis.

We shall use these functions to construct required functions on H^n. For the case $p = q = \infty$, we choose $m' < \min\{-\frac{n}{2}, \frac{\beta - 2}{2}\}$ and take the corresponding m. Using the function h as above for this choice of (m, m') we define a function f on H^n as follows: $f(z,t) = g(z)h(t)$, where g is a smooth function on $\mathbb{C}^n$ with compact support. From the Plancherel formula for the Weyl transform (see (3.3) and the subsequent discussion) it follows that $(2\pi)^{-n/2}|\lambda|^{n/2}\|\widehat{f}(\lambda)\|_{\mathrm{H}S} = \|g\|_2|\widehat{h}(\lambda)|$. Hence f satisfies

(i) $f = O(|(z,t)|^m e^{-a|(z,t)|^\alpha})$

(ii) $|\lambda|^n\|\widehat{f}(\lambda)\|_{\mathrm{HS}} = O(|\lambda|^{n/2+m'}e^{-b|\lambda|^\beta})$.

As both m and $m' + \frac{n}{2}$ are negative by our choice, f will satisfy the purpose. For the case $p \neq \infty$ and $q = \infty$, we choose $m' < \min\{-\frac{n}{2} - (2n+1)\frac{\beta-1}{p} + \frac{\beta-2}{2}\}$. This makes $m < -\frac{(2n+1)}{p}$. Lastly if we choose $m' < \min\{-\frac{(n+1)}{q} + \frac{n}{2}, -(2n+1)\frac{\beta-1}{p} + \frac{\beta-2}{2}\}$, then $m < -\frac{(2n+1)}{p}$. With this choice of m' it is easy to see that f satisfies the required estimates with $q \neq \infty$ and $p \in [1,\infty]$. □

§4. Step Two Nilpotent Lie Groups

Let G be a step two connected simply connected nilpotent Lie group. Then its Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{v}$ is any subspace of $\mathfrak{g}$ complementary to $\mathfrak{z}$. We choose an inner product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. We fix an orthonormal basis $\mathcal{B} = \{e_1, e_2 \cdots, e_m, T_1, \cdots, T_k\}$ so that $\mathfrak{v} = \mathrm{span}_{\mathbb{R}}\{e_1, e_2 \cdots, e_m\}$ and $\mathfrak{z} = \mathrm{span}_{\mathbb{R}}\{T_1, \cdots, T_k\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is an analytic diffeomorphism. We can identify G with $\mathfrak{v}\oplus\mathfrak{z}$ and write $(X+T)$ for $\exp(X+T)$ and denote it by (X,T) where $X \in \mathfrak{v}$ and $T \in \mathfrak{z}$. The product law on G is given by the Baker-Campbell-Hausdorff formula:

$$(X,T)(X',T') = \left(X + X', T + T' + \frac{1}{2}[X,X']\right)$$

for all $X, X' \in \mathfrak{v}$ and $T, T' \in \mathfrak{z}$.

§4.1. Representations of step two nilpotent Lie groups

A complete account of representation theory for general connected simply connected nilpotent Lie groups can be found in [8]. Representations of step two connected simply connected nilpotent groups the Plancherel theorem is described in [17]. We briefly recall the basic facts to make this paper self contained. Let $\mathfrak{g}^*$, $\mathfrak{z}^*$ be the real dual of $\mathfrak{g}$ and $\mathfrak{z}$ respectively. For each $\nu \in \mathfrak{z}^*$ consider the bilinear form B_ν on $\mathfrak{v}$ defined by

$$B_\nu(X,Y) = \nu([X,Y]) \text{ for all } X, Y \in \mathfrak{v}.$$

Let

$$\mathfrak{r}_\nu = \{X \in \mathfrak{v} : \nu([X,Y]) = 0 \text{ for all } Y \in \mathfrak{v}\}.$$

Let $X_i = e_i$ for $1 \leq i \leq m$ and $X_{m+i} = T_i$ for $1 \leq i \leq k$. Then $\mathcal{B} = \{X_1, \cdots, X_m, X_{m+1}, \cdots, X_{m+k}\}$. Let $\mathcal{B}^* = \{X_1^*, \cdots, X_m^*, X_{m+1}^*, \cdots, X_{m+k}^*\}$ be the dual basis of $\mathcal{B}$. Let $\mathfrak{m}_\nu$ be the orthogonal complement of $\mathfrak{r}_\nu$ in $\mathfrak{v}$. Then the set $\mathcal{U} = \{\nu \in \mathfrak{z}^* : \dim\ (\mathfrak{m}_\nu) \text{ is maximum}\}$ is a Zariski open subset of $\mathfrak{z}^*$. Since B_ν is an alternating bilinear form, $\nu \in \mathcal{U}$ has an even number of jump indices independent of ν. The set of jump indices is denoted by $S = \{j_1, j_2 \cdots, j_{2n}\}$. Let $T = \{n_1, n_2, \cdots, n_r, m+1, \cdots, m+k\}$ be the complement of S in $\{1, 2, \cdots, m, m+1, \cdots, m+k\}$. Let

$$V_S = \mathrm{span}_{\mathbb{R}}\{X_{j_1}, \cdots, X_{j_{2n}}\},$$

$$V_T = \mathrm{span}_{\mathbb{R}}\{X_{m+1}, \cdots, X_{m+k}, X_{n_i} : n_i \in T\} \text{ and } \widetilde{V}_T = \mathrm{span}_{\mathbb{R}}\{X_{n_i} : n_i \in T\},$$

$$V_T^* = \mathrm{span}_{\mathbb{R}}\{X_{m+1}^*, \cdots, X_{m+k}^*, X_{n_i}^* : n_i \in T\} \text{ and } \widetilde{V}_T^* = \mathrm{span}_{\mathbb{R}}\{X_{n_i}^* : n_i \in T\}.$$

The irreducible unitary representations relevant to Plancherel measure are parametrized by the set $\Lambda = \widetilde{V}_T^* \times \mathcal{U}$.

If there exist $\nu \in \mathfrak{z}^*$ such that B_ν is nondegenerate then we call the group, a step two nilpotent group with MW– condition or step two MW group. In this case $T = \{m+1, \cdots, m+k\}$ and $\mathcal{U} = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegeneate}\}$. The irreducible unitary representations relevant to Plancherel measure will be parametrized by $\Lambda = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegenerate}\}$.

For

$$(X,T) = \exp\left(\sum_{j=1}^m x_j X_j + \sum_{j=1}^k t_j X_{j+m}\right), \quad x_j, t_j \in \mathbb{R},$$

we define its norm by

$$|(X,T)| = (x_1^2 + \cdots + x_m^2 + t_1^2 + \cdots + t_k^2)^{1/2}.$$

The map

$$(x_1, \cdots, x_m, t_1 \cdots, t_k) \longrightarrow \sum_{j=1}^m x_j X_j + \sum_{j=1}^k t_j X_{j+m}$$
$$\longrightarrow \exp\left(\sum_{j=1}^m x_j X_j + \sum_{j=1}^k t_j X_{j+m}\right)$$

takes Lebesgue measure $dx_1 \cdots dx_m dt_1 \cdots dt_k$ of $\mathbb{R}^{m+k}$ to Haar measure on G. Any measurable function f on G will be identified with a function on $\mathbb{R}^{m+k}$. We identify $\mathfrak{g}^*$ with $\mathbb{R}^{m+k}$ with respect to the basis $\mathcal{B}^*$ and introduce the Euclidean norm relative to this basis.

4.1.1. Step two groups without MW-condition. In this case $\mathfrak{r}_\nu \neq \{0\}$ for each $\nu \in \mathcal{U}$. Then $B_\nu|_{\mathfrak{m}_\nu}$ is nondegenerate and hence $\dim \mathfrak{m}_\nu$ is $2n$. From the properties of an alternating bilinear form there exists an orthonormal basis

$$\{X_1(\nu), Y_1(\nu), \cdots, X_n(\nu), Y_n(\nu), Z_1(\nu), \cdots, Z_r(\nu)\}$$

of $\mathfrak{v}$ and positive numbers $d_i(\nu) > 0$ such that
(i) $\mathfrak{r}_\nu = \mathrm{span}_{\mathbb{R}}\ \{Z_1(\nu), \cdots, Z_r(\nu)\}$,

(ii) $\nu([X_i(\nu), Y_j(\nu)]) = \delta_{i,j} d_j(\nu), 1 \leq i, j \leq n$.

We call the basis

$$\{X_1(\nu), \cdots, X_n(\nu), Y_1(\nu), \cdots, Y_n(\nu), Z_1(\nu), \cdots, Z_r(\nu), T_1, \cdots, T_k\}$$

almost symplectic basis. Let $\xi_\nu = \mathrm{span}_{\mathbb{R}}\{X_1(\nu) \cdots, X_n(\nu)\}$ and $\eta_\nu = \mathrm{span}_{\mathbb{R}}$ $\{Y_1(\nu), \cdots, Y_n(\nu)\}$. Then we have the decomposition $\mathfrak{g} = \xi_\nu \oplus \eta_\nu \oplus \mathfrak{r}_\nu \oplus \mathfrak{z}$. We denote the element $\exp(X + Y + Z + T)$ of G by (X, Y, Z, T) for $X \in \xi_\nu, Y \in \eta_\nu, Z \in \mathfrak{r}_\nu, T \in \mathfrak{z}$. Further we can write

$$(X, Y, Z, T) = \sum_{j=1}^{n} x_j(\nu) X_j(\nu) + \sum_{j=1}^{n} y_j(\nu) Y_j(\nu) + \sum_{j=1}^{r} z_j(\nu) Z_j(\nu) + \sum_{j=1}^{k} t_j T_j$$

and denote it by (x, y, z, t) suppressing the dependence of ν which will be understood from the context. If we take $\lambda \in \Lambda$ then it can be written as $\lambda = (\mu, \nu)$, where $\mu \in \widetilde{V}_T^* = \mathrm{span}_{\mathbb{R}}\ \{X_{n_i}^* : 1 \leq i \leq r\}$ and $\nu \in \mathcal{U}$. Therefore, $\lambda = (\mu, \nu) \equiv \sum_{i=1}^{r} \mu_i X_{n_i}^* + \sum_{i=1}^{m} \nu_i T_i^*$. Let $\lambda' \in \mathfrak{g}^*$ such that $\lambda'(X_{j_i}) = 0$ for $1 \leq i \leq 2n$ and the restriction of λ' to V_T^* is $\lambda = (\mu, \nu)$. Let $\widetilde{\mu}_i = \lambda'(Z_i(\nu))$ and consider the map

$$A_\nu : \widetilde{V}_T^* \to \mathrm{span}_{\mathbb{R}}\ \{Z_1(\nu)^*, \cdots, Z_r(\nu)^*\} \tag{4.1}$$

given by $A_\nu(\mu_1, \cdots, \mu_r) = (\widetilde{\mu}_1, \cdots, \widetilde{\mu}_r)$. Then it has been shown in [17] that $|\mathrm{det} J_{A_\nu}| = \frac{\mathrm{P}f(\nu)}{d_1(\nu) \cdots d_n(\nu)}$, where J_{A_ν} is the Jacobian matrix of A_ν and $\mathrm{P}f(\nu)$ is the Pfaffian of ν. Consider the map

$$D_\nu : \{X_{j1}, \cdots, X_{j2n}\} \to \{X_1(\nu), \cdots, X_n(\nu), Y_1(\nu), \cdots, Y_n(\nu)\} \tag{4.2}$$

then it has been shown $|\det(J_{D_\nu})| = |\det(J_{A_\nu})|^{-1}$ in [17].

We take $\lambda = (\mu, \nu) \in \Lambda$. Using the almost symplectic basis we describe an irreducible unitary representation $\pi_{\mu,\nu}$ of G realized on $L^2(\eta_\nu)$ by the following action:

$$\begin{aligned}&(\pi_{\mu,\nu}(x,y,z,t)\phi)(\xi)\\ &\quad= \exp\left(i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{r}\widetilde{\mu}_j z_j + i\sum_{j=1}^{n} d_j(\nu)\left(x_j\xi_j + \frac{1}{2}x_j y_j\right)\right)\phi(\xi + y)\end{aligned}$$

for all $\phi \in L^2(\eta_\nu)$.

We define the Fourier transform of $f \in L^1(G)$ by

$$\widehat{f}(\mu,\nu) = \int_{\mathfrak{z}}\int_{\mathfrak{r}_\nu}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,z,t)\pi_{\mu,\nu}(x,y,z,t)\, dx\, dy\, dz\, dt$$

for $\lambda = (\mu, \nu) \in \Lambda$. For $\widetilde{\mu} \in \mathfrak{r}_\nu^*, \nu \in \mathfrak{z}^*$ we let

$$f^\nu(x,y,z) = \int_{\mathfrak{z}} \exp\left(i\sum_{j=1}^{k}\nu_j t_j\right) f(x,y,z,t)\, dt \text{ and}$$

$$f^{\widetilde{\mu},\nu}(x,y) = \int_{\mathfrak{r}_\nu}\int_{\mathfrak{z}} \exp\left(i\sum_{j=1}^{k}\nu_j t_j + i\sum_{j=1}^{r}\widetilde{\mu}_j z_j\right) f(x,y,z,t)\, dt\, dz.$$

If $f \in L^1 \cap L^2(G)$ then $\widehat{f}(\mu,\nu)$ is an Hilbert–Schmidt operator and we have (see [17])

$$(2\pi)^{-n}\prod_{j=1}^{n} d_j(\nu)\|\widehat{f}(\mu,\nu)\|_{\mathrm{HS}}^2 = \int_{\eta_\nu}\int_{\xi_\nu} |f^{\widetilde{\mu},\nu}(x,y)|^2\, dx\, dy. \tag{4.3}$$

Now integrating both sides on $\widetilde{V}_T^*$ with respect to the usual Lebesgue measure on it and applying the transformation given by the function A_ν in (4.1) we get

$$\begin{aligned}(2\pi)^{-(n+r)}\mathrm{P}f(\nu)\int_{\widetilde{V}_T^*}\|\widehat{f}(\mu,\nu)\|_{\mathrm{HS}}^2 d\mu &= (2\pi)^{-r}\int_{\mathfrak{r}_\nu^*}\int_{\eta_\nu}\int_{\xi_\nu} |f^{\widetilde{\mu},\nu}(x,y)|^2\, dx\, dy\, d\widetilde{\mu}\\ &= \int_{\mathfrak{r}_\nu}\int_{\eta_\nu}\int_{\xi_\nu} |f^\nu(x,y,z)|^2\, dx\, dy\, dz\\ &= \int_{\mathfrak{v}} |f^\nu(x,y,z)|^2\, dx\, dy\, dz.\end{aligned}$$

The Plancherel formula takes the following form:

$$(2\pi)^{-(n+r+k)} \int_{\Lambda} \|\widehat{f}(\mu,\nu)\|^2_{\mathrm{HS}} \, \mathrm{P}f(\nu) \, d\mu \, d\nu = \int_G |f(x,y,z,t)|^2 \, dx \, dy \, dz \, dt$$

which holds for all L^2-functions by density argument.

4.1.2. Step two MW groups. In this case the representations are parametrized by the Zariski open set $\Lambda = \{\nu \in \mathfrak{z}^* : B_\nu \text{ is nondegenerate}\}$ and is given by:

$$(\pi_\nu(x,y,t)\phi)(\xi) = \exp\left(i\sum_{j=1}^{k} \nu_j t_j + i\sum_{j=1}^{n} d_j(\nu)\left(x_j\xi_j + \frac{1}{2}x_j y_j\right)\right)\phi(\xi+y)$$

for all $\phi \in L^2(\eta_\nu)$. In this case $\mathrm{P}f(\nu) = \prod_{j=1}^{n} d_j(\nu)$. We define the Fourier transform of $f \in L^1(G)$ by

$$\widehat{f}(\nu) = \int_{\mathfrak{z}}\int_{\eta_\nu}\int_{\xi_\nu} f(x,y,t)\pi_\nu(x,y,t) \, dx \, dy \, dt$$

for all $\nu \in \Lambda$. We also define

$$f^\nu(x,y) = \int_{\mathfrak{z}} \exp\left(i\sum_{j=1}^{k} \nu_j t_j\right) f(x,y,t) \, dx \, dy \, dt$$

for all $\nu \in \Lambda$. If $f \in L^1 \cap L^2(G)$ then $\widehat{f}(\nu)$ is an Hilbert-Schmidt operator and

$$\mathrm{P}f(\nu)\|\widehat{f}(\nu)\|^2_{\mathrm{HS}} = (2\pi)^n \int_{\eta_\nu}\int_{\xi_\nu} |f^\nu(x,y)|^2 \, dx \, dy = (2\pi)^n \int_{\mathfrak{v}} |f^\nu(x,y)|^2 \, dx \, dy.$$

The Plancherel formula takes the following form:

$$(2\pi)^{-(n+k)} \int_{\Lambda} \|\widehat{f}(\nu)\|^2_{\mathrm{HS}} \, \mathrm{P}f(\nu) \, d\nu = \int_G |f(x,y,t)|^2 \, dx \, dy \, dt \tag{4.4}$$

which holds for all L^2-functions by density argument.

§5. Beurling's and $L^p - L^q$-Morgan's Theorem for Step Two Nilpotent Lie Groups

In what follows we will use the coordinates given by the following basis of $\mathfrak{g}$.

$$\{X_{j_1}, \cdots, X_{j_n}, X_{j_{n+1}}, \cdots, X_{j_{2n}}, X_{n_1}, \cdots, X_{n_r}, X_{m+1}, \cdots, X_{m+k}\}.$$

Precisely

$$(x, y, z, t) \equiv \sum_{i=1}^{n} x_i X_{j_i} + \sum_{i=1}^{n} y_i X_{j_{n+i}} + \sum_{i=1}^{r} z_i X_{n_i} + \sum_{i=1}^{k} t_i X_{m+i}.$$

We shall first take up the following analogue of Beurling's theorem for step two nilpotent groups.

Theorem 5.1. *Suppose $f \in L^2(G)$ and for some $M, N \geq 0$, it satisfies*

$$\int_\Lambda \int_{\mathfrak{g}} \frac{|f(x,y,z,t)| \|\widehat{f}(\mu,\nu)\|_{HS} e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^M \, (1+|(z,t)|+|(\mu,\nu)|)^N} |\mathrm{P}f(\nu)| \; dx\; dy\; dz\; dt\; d\mu\; d\nu < \infty.$$

Then

$$f(x,y,z,t) = (1+|(x,y)|)^M \left(\sum_{|\gamma|+|\delta| \leq l} \Psi_{\gamma,\delta}(x,y) z^\gamma t^\delta \right) e^{-a(|z|^2+|t|^2)}$$

where $\Psi_{\gamma,\delta} \in L^2(V_S) \cap L1(V_S)$ and l is an nonnegative integer.

Proof. As in the case of $\mathbb{R}^n$ we can verify that f is integrable in (z,t) for almost every x, y. For each Schwartz function Φ on V_S let us consider the function F_Φ defined by

$$F_\Phi(z,t) = \int_{V_S} f(x,y,z,t)(1+|(x,y)|)^{-M} \overline{\Phi(x,y)} \; dx \; dy.$$

It follows that

$$|F_\Phi(z,t)| \leq C \int_{V_S} |f(x,y,z,t)| \; dx \; dy.$$

For all $(\mu, , \nu) \in \widetilde{V}_T^* \times \mathcal{U}$

$$\widehat{F_\Phi}(\mu,\nu) = \int_{V_S} f^{\mu,\nu}(x,y)(1+|(x,y)|)^{-M} \overline{\Phi(x,y)} \; dx \; dy$$

where

$$f^{\mu,\nu}(x,y) = \int_{\widetilde{V}_T} e^{i\mu(z)+i\nu(t)} f(x,y,z,t) \; dz \; dt.$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\widehat{F_\Phi}(\mu,\nu)| &\leq C \left(\int_{V_S} |f^{\mu,\nu}(x,y)|^2 \; dx \; dy \right)^{1/2} \\ &= \left(\int_{V_S} \left| \int_{\widetilde{V}_T} e^{i\mu(z)} f^\nu(x,y,z) \; dz \right|^2 \; dx \; dy \right)^{1/2}. \end{aligned}$$

Writing down the above integral with respect to almost symplectic basis we have

$$|\widehat{F_\Phi}(\mu,\nu)| \leq \left(\int_{\xi_\nu \oplus \eta_\nu} |f^{\widetilde{\mu},\nu}(x(\nu),y(\nu))|^2 \, dx(\nu)\, dy(\nu)\right)^{1/2}$$
$$= (2\pi)^{-n/2} \left(\prod_{j=1}^{n} d_j(\nu)\right)^{1/2} \|\widehat{f}(\mu,\nu)\|_{\mathrm{HS}}.$$

Therefore,

$$\int_\Lambda \int_{\widetilde{V}_T \oplus \mathfrak{z}} \frac{|F_\Phi(z,t)||\widehat{F}_\Phi(\mu,\nu)| e^{|\mu||z|+|t||\nu|}}{(1+|(z,t)|+|(\mu,\nu)|)^N \, (1+\prod_{j=1}^{n} d_j(\nu))} \, |\mathrm{P}f(\nu)| \, dz\, dt\, d\mu\, d\nu$$
$$\leq C \int_\Lambda \int_{V_S \oplus \widetilde{V}_T \oplus \mathfrak{z}} \frac{|f(x,y,z,t)| \|\widehat{f}(\mu,\nu)\|_{\mathrm{HS}} e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^M \, (1+|(z,t)|+|(\mu,\nu)|)^N} |\mathrm{P}f(\nu)| dx\, dy\, dz\, dt\, d\mu\, d\nu$$
$$= \int_\Lambda \int_{\mathfrak{g}} \frac{|f(x,y,z,t)| \|\widehat{f}(\mu,\nu)\|_{HS} e^{|z||\mu|+|t||\nu|}}{(1+|(x,y)|)^M \, (1+|(z,t)+|(\mu,\nu)|)^N} \, |\mathrm{P}f(\nu)| \, dx\, dy\, dz\, dt\, d\mu\, d\nu$$
$$< \infty.$$

Since $\mathcal{U}$ is a set of full measure on $\mathfrak{z}^*$, and $\mathrm{P}f(\nu)$, $\prod_{j=1}^{n} d_j(\nu)$ are polynomial in ν using Theorem 2.3 we have for each Schwartz function Φ

$$F_\Phi(z,t) = P_\Phi(z,t) e^{-a(\Phi)|(z,t)|^2}$$

where $a(\Phi) > 0$ and

$$P_\Phi(z,t) = \sum_{|\gamma|+|\delta| \leq l} a_{(\gamma,\delta)}(\Phi) z^\gamma t^\delta$$

and l is independent of Φ. It is easy to see that $a(\Phi) = a$ is independent of Φ. Finally choosing Φ_α from the orthonormal basis $\{\Phi_\alpha(x,y) : \alpha \in \mathbb{N}^{2n}\}$ for $L^2(V_S)$ we can show as in the proof of Theorem 3.1 that

$$f(x,y,z,t) = (1+|(x,y)|)^M \left(\sum_{|\gamma|+|\delta|\leq l} \Psi_{\gamma,\delta}(x,y) z^\gamma t^\delta\right) e^{-a|(z,t)|^2},$$

where $\Psi_{\gamma,\delta} \in L^2(V_S)$. As in the case of Heisenberg group it can be verified that $\Psi_{\gamma,\delta} \in L^1(V_S)$. □

From the fact that $\mathrm{P}f(\nu)$ and $\Pi_{j=1}^{n} d_j(\nu)$ both are homogenous polynomial in ν of degree n one can verify as in the case of Heisenberg group that the resulting function of the theorem satisfies the hypothesis with the restriction $l < 1/2(N - n/2 - (r+k))$. Note that for H^n, $r = 0$ and $k = 1$.

Sharpness of the estimate in Beurling's theorem

We will show that the condition used in Beurling's theorem is optimal. For the sake of simplicity we consider the Heisenberg group H^n. We suppose a function $f \in L^1 \bigcap L^2(H^n)$ satisfies

$$\int_{H^n}\int_{\mathbb{R}} \frac{|f(z,t)| \|\widehat{f}(\lambda)\|_{\mathrm{HS}} e^{c|t||\lambda|}}{(1+|z|)^M (1+|t|+|\lambda|)^N} |\lambda|^n \, d\lambda \, dz \, dt < \infty \tag{5.1}$$

for some $c > 0$.

(i) If $c > 1$, then f satisfies the hypothesis of Theorem 3.1 and hence $f(z,t) = g(z)e^{-at^2}$ for some $g \in L^1 \bigcap L^2(\mathbb{C}^n)$ and $a > 0$. Since by the Plancherel theorem (3.3) $\|\widehat{f}(\lambda)\|_{\mathrm{HS}} = (2\pi)^{n/2}|\lambda|^{-n/2}\|g\|_2 e^{-\frac{1}{4a}\lambda^2}$, it is easy to see that f cannot satisfy (5.1) unless $f = 0$ almost everywhere.

(ii) Now we suppose $c < 1$. We choose $a, b > 0$ such that $ab = c^2$ and we construct the function $f(z,t) = g(z)P(t)e^{-at^2}$, where $g \in L^1 \bigcap L^2(\mathbb{C}^n)$ and P is a polynomial of any degree. Then f will satisfy (5.1). Clearly for fixed $z \in \mathbb{C}^n$ these functions are linearly independent in the variable t.

Consequences of Beurling's theorem

Let us note the following consequences of Beurling's theorem.

Theorem 5.2 (Morgan's theorem, weak version)**.** *Let f be a measurable function G. suppose for some $a, b > 0$, $\alpha \geq 2$, $\beta > 0$*

(i) $|f(x,y,z,t)| \leq Ce^{-a|(x,y,z,t)|^\alpha}$

(ii) $|\mathrm{P}f(\nu)|^{1/2}\|\hat{f}(\mu,\nu)\|_{HS} \leq Ce^{-b|(\mu,\nu)|^\beta}$

where $1/\alpha + 1/\beta = 1$ and $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \geq 1$. Then $f = 0$ almost everywhere unless $\alpha = \beta = 2$ and $ab = 1/4$ in which case $f(x,y,z,t) = \psi(x,y)e^{-a|(z,t)|^2}$ for some $\psi \in L^2(V_S)$ and $|\psi(x,y)| \leq Ce^{-a|(x,y)|^2}$.

Proof. It is clear from the hypothesis that $f \in L^2(G)$. Since $\alpha \geq 2$ we have $|(x,y,z,t)|^\alpha \geq |(x,y)|^\alpha + |(z,t)|^\alpha$. Therefore from hypothesis (i) we get $|f(x,y,z,t)| \leq Ce^{-a|(x,y)|^\alpha}e^{-a|(z,t)|^\alpha}$. Now the theorem can be obtained from Theorem 5.1 by applying the inequality $|\xi|^\alpha/\alpha + |\eta|^\beta/\beta \geq |\xi\eta|$ and using the fact $e^{-a|(x,y)|^\alpha} \in L^1(V_S)$. □

In the proof above we have used the fact that $\alpha \geq 2$ to split the function $e^{-a|(x,y,z,t)|^\alpha}$ as a product of a function in $L^1(V_S)$ and $e^{-a|(z,t)|^\alpha}$. This motivates us to formulate the following version of Morgan's theorem.

Theorem 5.3. *Let $f \in L^2(G)$. Suppose for some $a, b > 0$, $\alpha, \beta > 0$*

(i) $|f(x,y,z,t)| \leq g(x,y)e^{-a|(z,t)|^\alpha}$, $g \in L^1(V_S)$

(ii) $|\mathrm{P}f(\nu)|^{1/2}\|\hat{f}(\mu,\nu)\|_{HS} \leq Ce^{-b|(\mu,\nu)|^\beta}$

where $1/\alpha + 1/\beta = 1$ and $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \geq 1$. Then $f = 0$ almost everywhere unless $\alpha = \beta = 2$ and $ab = 1/4$ in which case $f(x,y,z,t) = \psi(x,y)e^{-a|(z,t)|^2}$ for some $\psi \in L^2(V_S)$ and $|\psi(x,y)| \leq g(x,y)$.

We omit the proof as it is evident from the proof of Theorem 5.2 and the comment after that.

Theorem 5.4 (Cowling-Price)**.** *Suppose $f \in L^1 \cap L^2(G)$ and it satisfies the following conditions.*

(i) $\int_G e^{pa|(x,y,z,t)|^2}|f(x,y,z,t)|^p \, dx\, dy\, dz\, dt < \infty$ *and*

(ii) $\int_\Lambda e^{bq|(\mu,\nu)|^2}\|\widehat{f}(\mu,\nu)\|^q_{HS}|\mathrm{P}f(\nu)| \, d\mu\, d\nu < \infty$.

Then for $ab \geq 1/4$ and $\min\{p,q\} < \infty$, $f = 0$ almost everywhere.

Proof. Using Hölder's inequality we can find $M, N > 0$ such that

$(i)'$ $\int_{V_S\oplus\widetilde{V}_T\oplus\mathfrak{z}} \frac{e^{a|(z,t)|^2}|f(x,y,z,t)|}{(1+|(x,y)|)^M(1+|(z,t)|)^N} \, dx\, dy\, dz\, dt < \infty$ and

$(ii)'$ $\int_\Lambda \frac{e^{b|(\mu,\nu)|^2}\|\widehat{f}(\mu,\nu)\|_{\mathrm{HS}}}{(1+|(\mu,\nu)|)^N}|\mathrm{P}f(\nu)| \, d\mu\, d\nu < \infty$.

Therefore using Theorem 5.1 we can conclude that $f = 0$ almost everywhere when $ab \geq 1/4$ and $\min\{p,q\} < \infty$. □

Theorem 5.5 (Hardy's theorem)**.** *Suppose f is a measurable function on G which satisfies the following conditions*:

(i) $|f(x,y,z,t)| \leq g(x,y)(1 + |(z,t)|)^m e^{-a|(z,t)|^2}$, *where $g \in L^1 \cap L^2(V_S)$ and*

(ii) $|\mathrm{P}f(\nu)|^{1/2}\|\widehat{f}(\mu,\nu)\|_{HS} \leq (1 + |(\mu,\nu)|)^m e^{-b|(\mu,\nu)|^2}$.

Then $f = 0$ *almost everywhere if* $ab > 1/4$ *and if* $ab = 1/4$ *then* $f(x, y, z, t) = P(x, y, z, t)e^{-a|(z,t)|^2}$, *where* $P(x, y, z, t) = \left(\sum_{|\alpha|+|\delta| \le m} \psi_{\alpha,\delta}(x, y) z^\delta t^\alpha \right)$ *and* $\psi_{\alpha,\delta} \in L^2(V_S)$

We omit the proof which is a straight forward application of the theorem above.

We shall now prove an exact analogue of $L^p - L^q$-Morgan's theorem for step two nilpotent Lie groups.

Theorem 5.6. *Let* $f \in L^2(G)$. *Suppose for some* $a, b > 0$, $\alpha > 2, \beta > 0$

(i) $\int_G e^{pa|(x,y,z,t)|^\alpha} |f(x, y, z, t)|^p \, dv \, dt < \infty$ *and*

(ii) $\int_\Lambda e^{qb|(\mu,\nu)|^\beta} \|\widehat{f}(\mu, \nu)\|^q_{HS} |\mathrm{P}f(\nu)| \, d\mu \, d\nu < \infty$,

where $1/\alpha + 1/\beta = 1$ *and* $p, q \in [1, \infty]$. *Then* $f = 0$ *almost everywhere whenever* $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin \frac{\pi}{2}(\beta - 1))^{1/\beta}$.

Proof. As in the case of $\mathbb{R}^n$ we see that $f \in L^1(G)$. We note that it is sufficient to consider the case $p = q = 1$ as in the case of Heisenberg groups. Since $\prod_{j=1}^n d_j(\nu)$, $\mathrm{P}f(\nu)$ are polynomials in ν, for any $b' < b$, applying Minkowski's integral inequality with respect to the measures $dx \, dy$ and $d\mu \, e^{b'|\nu|^\beta} |\mathrm{P}f(\nu)| d\nu$ we get

$$\begin{aligned}
&\left(\int_{V_S} \left(\int_\Lambda |f^{\mu,\nu}(x, y)| e^{b'|\nu|^\beta} \, d\mu \, |\mathrm{P}f(\nu)| d\nu \right)^2 dx \, dy \right)^{1/2} \\
&\le \int_\Lambda e^{b'|\nu|^\beta} \left(\int_{V_S} |f^{\mu,\nu}(x, y)|^2 \, dx \, dy \right)^{1/2} d\mu \, |\mathrm{P}f(\nu)| d\nu \\
&\le \int_\Lambda e^{b'|(\mu,\nu)|^\beta} \left(\int_{V_S} |f^{\mu,\nu}(x, y)|^2 \, dx \, dy \right)^{1/2} d\mu \, |\mathrm{P}f(\nu)| d\nu \\
&= C \int_\Lambda e^{b'|(\mu,\nu)|^\beta} \|\widehat{f}(\mu, \nu)\|_{\mathrm{HS}} \left(\prod_{j=1}^n d_j(\nu) \right)^{1/2} d\mu \, |\mathrm{P}f(\nu)| d\nu \\
&\le \int_\Lambda e^{b|(\mu,\nu)|^\beta} \|\widehat{f}(\mu, \nu)\|_{\mathrm{HS}} |\mathrm{P}f(\nu)| d\mu \, d\nu \\
&< \infty.
\end{aligned}$$

This implies that for almost every $(x,y)\in V_S$

$$\int_{\Lambda} e^{b'|\nu|^\beta}|f^{\mu,\nu}(x,y)||\mathrm{P}f(\nu)|\;d\mu\;d\nu<\infty. \tag{5.2}$$

Since $\Lambda=\mathcal{U}\times\widetilde{V}_T^*$, it follows that

$$\int_{\mathcal{U}} e^{b'|\nu|^\beta}|f^{\mu,\nu}(x,y)||\mathrm{P}f(\nu)|\;d\nu<\infty \tag{5.3}$$

for almost every $(x,y)\in V_S$ and $\mu\in\widetilde{V}_T^*$. From the hypothesis (i) with $p=1$, it is easy to see that for almost every $(x,y)\in V_S$

$$\int_{\mathfrak{z}}\int_{\widetilde{V}_T} e^{a|(z,t)|^\alpha}|f(x,y,z,t)|\;dz\;dt<\infty.$$

Therefore for almost every $(x,y)\in V_S$

$$\int_{\mathfrak{z}} e^{a|t|^\alpha}|f^{\mu}(x,y,t)|\;dt\le\int_{\mathfrak{z}}\int_{\widetilde{V_T}} e^{a|(z,t)|^\alpha}f(x,y,z,t)\;dz\;dt<\infty$$

where $f^\mu(x,y,t)=\int_{\widetilde{V_T}}e^{\mu(z)}f(x,y,z,t)\;dz$.

As $\mathcal{U}$ is a set of full measure, we can now apply Theorem 2.4 to the function $f^\mu(x,y,t)$ to conclude that for almost every $(x,y)\in V_S$, $f(x,y,z,t)=0$ whenever $(a\alpha)^{1/\alpha}(b'\beta)^{1/\beta}>(\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$. Since given $a,b>0$ with $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta}>(\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$, it is always possible to choose $b'<b$ satisfying $(a\alpha)^{1/\alpha}(b'\beta)^{1/\beta}>(\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$, the theorem follows. □

Remark 5.7.

1. The Gelfand-Shilov theorem and the Morgan's theorem (in their sharpest forms) are particular cases of Theorem 5.6 ($p=q=1$ and $p=q=\infty$ respectively) and thus are accommodated in that theorem.

2. In Theorem 3.2 we have seen example of functions which satisfy the hypothesis with $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta}=(\sin\frac{\pi}{2}(\beta-1))^{1/\beta}$ in the case of H^n. Similar construction can be carried out in this case also.

3. All the theorems proved above for step two groups without MW condition can be formulated and proved for step two MW groups with obvious and routine modifications.

§6. Concluding Remarks

For general nilpotent Lie groups, there are a few attempts (see [14, 4, 5]) in recent times to prove theorems of this genre. The basic step in these works is to build a new function on the central variable which satisfies the hypothesis. But in the process the sharpness of the result is lost and hence it is not possible to get the case of optimality. The explicit formula for $\|\hat{f}(\lambda)\|_{\text{HS}}$ which is crucial in the proof of Beurling's theorem, is also unavailable in this generality. Therefore it is unlikely that the method pursued in those papers will generalize to the case of all nilpotent Lie groups. We refer to the remark in [14, p. 493] in this context.

Our aim in this paper is to obtain the most natural analogue of Beurling's and $L^p - L^q$ Morgan's theorem which can accommodate the case of optimality and from which we get back the strongest version of the other theorems in this genre as consequences. This is the reason we restrict ourselves to the step two nilpotent Lie groups.

We conclude the paper with a brief discussion on comparison of the results obtained in this paper with the existing theorems of this genre. Beurling's theorem, i.e. analogues of Theorem 1.2 is not considered so far for any nilpotent Lie groups. However, for some restricted class of nilpotent Lie groups there are some analogues of Theorem 1.1. In [5] an analogue of Theorem 1.1 is proved for the nilpotent Lie groups of the form $\mathbb{R}^n \rtimes \mathbb{R}$. In [16] a version of Theorem 1.1 is formulated for stratified step two nilpotent Lie groups where the estimate involves the matrix coefficients of the Fourier transform, instead of the operator valued Fourier transform, which seems to be more restrictive. In one hand these theorems does not accommodate (being analogue of Theorem 1.1) the characterization of the optimal case and in the other, one cannot get back other QUP-results in full generality from these theorems. A version of the $L^p - L^q$ Morgan's theorem is proved in [6] only for Heisenberg groups.

Other theorems of this genre which follow from either Beurling's or $L^p - L^q$-Morgan's theorem were proved independently by many authors in nilpotent Lie groups. Nevertheless none of these works dealt with the characterization of the optimal case. There are also some unnatural restrictions on the hypothesis (e.g. on the parameters p, q and α, β). In [17] Ray proved the Cowling–Price theorem for step two nilpotent Lie groups without MW-conditions with the assumption $1 \le p \le \infty, q \ge 2$ and $ab > 1/4$. This was generalized in [4] for any nilpotent Lie group with the restriction $2 \le p, q \le \infty$ and $ab > 1/4$. In contrast we have in this paper the Cowling-Price theorem with the original condition $1 \le p, q \le \infty$ and $ab \ge 1/4$ as a consequence of Beurling's theorem.

In [17] Ray also proved a version of Morgan's theorem which is similar but slightly weaker than Theorem 5.3. Again this can be obtained as a consequence of our Beurling's theorem. We may emphasize here that only a weak version of Morgan's theorem follows from Beurling's theorem, while the actual Morgan's theorem follows from $L^p - L^q$-Morgan's theorem (see Remark 5.7). In [1] Astengo et al. proved a version of Hardy's theorem where they put condition on the operator norm of the Fourier transform, instead of the usual pointwise estimate. We note that only by a slight modification of our proof, a Beurling's theorem can be obtained where Hilbert-Schmidt norm of the Fourier transform is replaced by its operator norm. (We formulated the theorem using Hilbert-Schmidt because it appears to be more natural.) As a consequence we can get the theorem in [1].

Recently an analogue of Beurling's theorem is proved for Riemannian symmetric spaces in [18]. Due to the structural difference, the statement as well as the method of proving the theorem is different and it involves decomposing the hypothesis in K-types and treating each component separately.

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