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Vector Valued Hyperfunctions and Boundary Values of Vector Valued Harmonic and Holomorphic Functions

By

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Abstract

We develop the theory of hyperfunctions with values in a locally convex non-necessarily metrizable space E and find necessary conditions and sufficient conditions such that a reasonable theory of E-valued hyperfunctions exists. In particular, we show that it exists for various spaces of distributions but there is no such theory for the spaces of real analytic functions and distributions with compact support. We also show that vector valued hyperfunctions can be interpreted as boundary values of vector valued harmonic or holomorphic functions and, in many cases, as suitable cohomology groups.

§1. Introduction

Hyperfunctions were defined and developed by Sato [53] (comp. [54] or [45]) in the late fifties and early sixties of the twentieth century. They have become important and useful tools in the theory of differential equations (see [33]). Soon it turned out that also vector valued hyperfunctions would be interesting, for instance, since some partial differential equations can be interpreted

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as ordinary vector valued equations (e.g., [49], [50], [20]). The analogous theory of vector valued distributions was developed at early stage of the theory by Schwartz himself via tensor products [55]. In the case of hyperfunctions an essential difficulty appears: hyperfunctions have no natural linear topology! Nevertheless Ion and Kawai [28] developed such a theory for hyperfunctions with values in Fréchet spaces (= metrizable complete locally convex spaces) using the vector valued Dolbeault complex. Despite of some efforts to extend the theory beyond the class of metrizable spaces (see [29], [30]) as far as we know this is the only fully correct theory of vector valued hyperfunctions. Nevertheless it is of some interest to consider E-valued hyperfuntions for non-metrizable E (for instance, for various spaces of distributions or spaces of real analytic functions).

The aim of this paper is not only to develop a theory of vector valued hyperfunctions far beyond the class of metrizable spaces but also to find the natural limits of such a theory. Inside a large natural class of locally convex spaces we characterize those spaces E for which a reasonable theory of E-valued hyperfuntions exists at all (see Theorem 8.9). To make this statement more precise: we believe that a reasonable theory of E-valued hyperfunctions should produce a flabby sheaf such that the set of sections supported by a compact subset $K \subseteq \mathbb{R}^d$ should be equal to $L(\mathscr{A}(K), E)$, the space of linear continuous operators on the space of germs of analytic functions on K (or "the space of E-valued analytic functionals on K"). As we will prove E-valued hyperfunctions satisfying these minimal requirements can be constructed for instance if E is the space of distributions or tempered distributions as well as for distributional kernels of linear partial differential operators with constant coefficients over convex sets. On the other hand such a theory is *impossible* for E being the space of distributions with compact support or the space of real analytic functions over a compact set with non-empty interior or over an open subset of $\mathbb{R}^d$. More generally, for a wide class of locally convex spaces — the so-called ultrabornological PLS-spaces described later on (which covers most of the natural non-Banach sheaves of analysis) the theory of E-valued hyperfunctions is possible if and only if E has the so-called property (PA) (see Theorem 8.9). Let us add that by now we have a quite extensive knowledge which spaces have (PA) and which have not (see Section 4).

The existence of E-valued hyperfunctions is intimately connected to the solvability of the E-valued Laplace equation. A locally convex space E is called (*weakly*) *d-admissible*, $d \in \mathbb{N}$, $d \geq 1$, if for any (bounded) open set $\Omega \subset \mathbb{R}^d$ the d-dimensional Laplace operator is surjective on the space of E-valued smooth functions on Ω, i.e.

$$\Delta_d : C^\infty(\Omega, E) \to C^\infty(\Omega, E) \text{ is surjective.}$$

Clearly every locally convex space is 1-admissible. Surprisingly, we will show that if E is $(d+1)$-admissible then a reasonable theory of d-dimensional E-valued hyperfunctions is possible. On the other hand, existence of such a theory implies that E is weakly d-admissible. Therefore we devote the whole Section 4 to study which spaces are d-admissible. In fact, we consider first in Section 3 a more general question, namely if

$$P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E) \text{ is surjective}$$

for a general hypoelliptic or elliptic linear partial differential operator $P(D)$ with constant coefficients. We also get analogous statements for hypoelliptic matrices $P(D)$. Our main tools here are new results on surjectivity of tensor products obtained in [7] which allow to clarify via the method of Vogt (see [61], [64]) for which spaces E the operator $P(D)$ is surjective on the space of smooth E-valued functions. This section contains many results on surjectivity of various differential operators on spaces of vector valued smooth functions and therefore it is interesting in itself. In the above mentioned class of PLS-spaces, d-admissible and weakly d-admissible spaces coincide for all $d \geq 2$ and they are exactly described as the spaces having the so-called property (PA), see Corollary 4.1.

In the scalar case, hyperfunctions may be defined either as the sheaf generated by the analytic functionals (which are always compactly supported in the scalar case) or as the sheaf of the d-th relative cohomology groups supported in $\mathbb{R}^d$ with values in the Oka sheaf of holomorphic functions of d variables. We present the vector valued case of both approaches in Section 6 and Section 7 correspondingly. Then both approaches are translated to the boundary value approach for harmonic (Section 6) and holomorphic functions (Section 7). We will profit a lot from Bengel's point of view, i.e., considering harmonic functionals instead of analytic functionals, which lead to a special case of P-functionals of Bengel (see [2], [54] and also [35]). We explain this identification in Section 5.

In Section 8 we find necessary conditions on a locally convex space E such that the theory of E-valued hyperfunctions exists. Summarizing, if E is one of the spaces listed in Corollary 4.8 then such a theory exists and can be built both using the duality method or boundary values of harmonic functions (see Section 6) as well as cohomology groups with values in the E-valued Oka sheaf or boundary values of holomorphic functions (see Section 7). On the other hand, no construction of a reasonable sheaf of E-valued hyperfunctions exists for the spaces E listed in Cor. 4.9 (a) and (b). More precisely, if E is an ultrabornological PLS-space then a reasonable theory of E-valued hyperfunctions exists if and only if E has the property (PA) mentioned above (Thm. 8.9)

§2. Notation and Preliminaries

By E we will always denote a complete locally convex space.

By $L(E,F)$ we denote the space of continuous linear operators from E to F always equipped with the topology of uniform convergence on bounded sets, where E and F are locally convex spaces (to emphasize this we write $L_b(E,F)$). By E'_b, E'_{co} we denote the dual spaces with the strong and the compact open topologies, respectively.

By $\mathscr{A}(\Omega)$ we denote the space of real analytic functions on an open set $\Omega \subset \mathbb{R}^d$. This space is equipped with the natural topology (see [44] or [13]) of the projective limit of inductive limits of Banach spaces:

$$\mathscr{A}(\Omega) = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} H^\infty(U_{N,n}),$$

where $(K_N)_{N\in\mathbb{N}}$ is a compact exhaustion of Ω, $(U_{N,n})_{n\in\mathbb{N}}$ is a basis of complex open neighborhoods of K_N in $\mathbb{C}^d$ (without loss of generality we may assume that they are domains of holomorphy and $U_{N,n+1} \Subset U_{N,n}$) and $H^\infty(U_{N,n})$ is the Banach space of bounded holomorphic functions on $U_{N,n}$. Let us observe that $\mathscr{A}(\Omega) = \operatorname{proj}_{N\in\mathbb{N}} H(K_N)$, $H(K_N)$ the space of germs of holomorphic functions on $K_N \subset \mathbb{C}^d$. For any compact set $K \subset \mathbb{C}^d$ the space $H(K)$ is a DFN-space, i.e., the dual of a nuclear Fréchet space. By $\mathscr{A}(K)$ we denote the space of germs of real analytic functions on $K \subseteq \mathbb{R}^d$, clearly $\mathscr{A}(K) \simeq H(K)$ topologically. By $H(U)$ and $H(U,E)$ we define the spaces of scalar and E-valued holomorphic functions on $U \subseteq \mathbb{C}^d$ or on an open subset U of a Stein manifold. Let us denote by $\mathscr{E}(E)$ and $\mathscr{O}(E)$ the sheaves of E-valued smooth and holomorphic functions, respectively. Analogously, $\mathscr{E}^{(p,q)}(E)$ denotes the sheaf of (p,q)-differential forms with smooth coefficients with values in E.

We will write points $\xi \in \mathbb{R}^{d+1}$ as $\xi = (x,y) \in \mathbb{R}^d \times \mathbb{R}$. For $U \subset \mathbb{R}^{d+1}$ open let

$$C_\Delta(U,E) := \{f \in C^\infty(U,E) \mid \Delta_{d+1} f = 0\}$$

denote the space of E-valued harmonic functions on U and let

$$\widetilde{\mathcal{C}}_\Delta(U,E) := \{f \in C^\infty(U,E) \mid \Delta_{d+1} f = 0, f(x,y) = f(x,-y) \text{ if } (x,y) \in U\}$$

be the space of harmonic functions on U which are even with respect to y if $U \subset \mathbb{R}^{d+1}$ is symmetric (with respect to y), that is, if $(x,-y) \in U$ for any $(x,y) \in U$. The space $\widetilde{\mathcal{C}}_\Delta(U,E)$ is equipped with the topology induced from $C^\infty(U,E) = C^\infty(U)\hat{\otimes}_\varepsilon E$ (see [31] for topological tensor products). Analogously, we define $\tilde{C}^\infty(U,E)$ to be the space of smooth functions even with respect to the last variable.

Let G always denote the canonical even elementary solution of Δ_{d+1}, i.e.

$$\begin{aligned} G(x,y) &:= \ln(|(x,y)|)/(2\pi) \text{ if } d=1 \text{ and} \\ G(x,y) &:= -|(x,y)|^{1-d}/((d-1)c_{d+1}) \text{ if } d \geq 2 \end{aligned} \tag{2.1}$$

where c_{d+1} is the area of the unit sphere in $\mathbb{R}^{d+1}$.

Let us recall that, by a result of Grothendieck [24, part I, p. 39, part II, p. 82], if E is complete then a function $f : \Omega \to E$ is infinitely many times differentiable if and only if $u \circ f \in C^\infty(\Omega)$ for any functional $u \in E'$. This implies immediately the following lemma:

Lemma 2.1. *Let E be a complete locally convex space, $\Omega \subset \mathbb{R}^d$ open and let $f : \Omega \to E$. Then $f \in C^\infty(\Omega, E)$ and $Pf = 0$ for a given linear differential operator P if and only if*

$$u \circ f \in \ker P \subset C^\infty(\Omega)$$

for each $u \in E'$.

Remark 2.2. In fact, it suffices that E is locally complete. Moreover, if $P = \bar{\partial}$ this is nothing else but a classical result of Dunford and Grothendieck that a vector valued function is holomorphic if and only if it is weakly holomorphic (see [24, part I, Th. 1]).

Let us recall that a locally convex space X is a PLS-space (PLN-space) if $X = \text{proj}_{N\in\mathbb{N}} X_N$, where X_N are DFS-spaces, i.e., the strong duals of Fréchet Schwartz spaces, (DFN-spaces, i.e., the strong duals of nuclear Fréchet spaces). Clearly the space of distributions $\mathscr{D}'(\Omega)$, the spaces of Beurling type ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$ (see [10]) and the space of real analytic functions $\mathscr{A}(\Omega)$ are PLN-spaces. Every Fréchet-Schwartz space is a PLS-space. In fact, all non-Banach spaces appearing naturally in analysis are either PLS-spaces or LFS-spaces (=inductive limits of sequences of Fréchet Schwartz spaces). For more details on PLS-spaces see [13].

We will also use some homological tools for locally convex spaces like the functor Proj^1. Let $X = \text{proj}_{N\in\mathbb{N}} X_N$, where (X_N) is a sequence of locally convex spaces with a sequence of linking maps $i_N^{N+1} : X_{N+1} \to X_N$. We define

$$\operatorname{Proj}^1{}_{N\in\mathbb{N}}(X_N) := \prod_{N\in\mathbb{N}} X_N / \operatorname{im}\sigma, \quad \sigma : \prod_{N\in\mathbb{N}} X_N \to \prod_{N\in\mathbb{N}} X_N,$$
$$\sigma((x_N)) := (i_N^{N+1} x_{N+1} - x_N)_{N\in\mathbb{N}}.$$

For reduced spectra of DFS-spaces or Banach spaces (i.e., $i_N : X \to X_N$ has a dense range for any $N \in \mathbb{N}$), Proj^1 depends only on X and not on the spectrum itself. It is worth noting that for any PLS-space X the functor $\operatorname{Proj}^1 X = 0$ if and only if X is ultrabornological. For more details on Proj^1 functor and other derived functors see [67].

We will use later the so-called ε-product of locally convex spaces which is a type of a tensor product. If E and F are complete locally convex spaces then $E\varepsilon F := L(E'_{co}, F)$ equipped with the topology of uniform convergence on equicontinuous subsets of E'_{co}. For instance, if E or F is nuclear then $E\varepsilon F$ is the completion of $E \otimes F$ with its unique natural topology. For more details see [31].

Let E and F be locally convex spaces. If for any locally convex space G every short exact sequence with continuous linear and open onto its image maps

$$0 \to E \to G \xrightarrow{q} F \to 0$$

splits (i.e., q has a continuous linear right inverse) then we write $\operatorname{Ext}^1(F, E) = 0$. If E and F are PLS-spaces and the same holds for all PLS-spaces G, then we denote it by $\operatorname{Ext}^1_{PLS}(F, E) = 0$. In order to distinguish cohomology groups from spaces of holomorphic functions we denote the former by the letter $\mathscr{H}$ while the latter by H.

For the classical theory of hyperfunctions see [54] or [33]. For the sheaf theory see [11]. For the relative cohomology see also [32] (comp. [54]). For the theory of locally convex spaces see [48]. For the theory of topological tensor products see [31].

§3. Surjectivity of Differential Operators on Spaces of Vector Valued Smooth Functions

In this section we study the general problem of surjectivity of hypoelliptic partial differential operators with constant coefficients acting on the space of vector valued smooth functions. For any Fréchet space E if an operator $T : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective then $T \otimes \mathrm{id} : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ is also surjective. This follows from the classical theory of tensor products. The case of dual Fréchet spaces E was solved in [61]. The latter paper contains also some more general examples (for instance $\mathscr{D}'(U)$ or $\mathscr{D}(U)$). We consider systematically the case when E is either a PLS-space or an LFS-space. We are mostly interested in classical spaces of analysis. Moreover, we consider not only individual operators but also systems (matrices) of such differential operators. The suitable new tools are provided by the papers [6] and [7] as well as [17].

Apart from the applications of the presented results to the problem of vector valued hyperfunctions presented later on, the results have clear applications to the question of parameter dependence of solutions of systems of partial differential equations with constant coefficients (for this problem see [12], [57], [58], [41], [42], [43], [5], [6], [7]).

The proofs of this section follow the ideas of Vogt's paper [61] supplemented by some new tools for PLS-spaces.

We start with some preliminary result.

Proposition 3.1. *Let F be an ultrabornological locally convex space and let $E = F'_b$. Then for any complete Montel webbed space G we have algebraically $L(E'_{co}, G) = L(F, G)$.*

Proof. Clearly, by taking adjoint maps (see [31, 9.3.7, 16.7.6])

$$L(E'_{co}, G) \cong L(G'_{co}, E) = L(G'_b, F'_b).$$

For every operator $T \in L(G'_b, F'_b)$ its dual $T' : F''_b \to G$ is weak*-weak continuous, so it restricts to a weak-weak continuous map $T'|_F : F \to G$. The correspondence $T \to T'|_F$ is injective. By the webbed closed graph theorem, $T'|_F \in L(F, G)$. Of course, $(T'|_F)' = T$. □

A matrix $P_0(D)$ of linear partial differential operators with constant coefficients is called *hypoelliptic* iff

$$\{T \in \mathcal{D}'(\Omega)^{s_0} \mid P_0(D)T = 0\} \subset C^\infty(\Omega)^{s_0} \text{ for any open } \Omega \subset \mathbb{R}^d.$$

The main result is the following theorem which for the splitting at $P_0(D)$ is essentially due to Vogt, the rest follows from [17]:

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^d$ and let $P_0(D)$ be a hypoelliptic matrix of linear partial differential operators with constant coefficients*

$$P_0(D) : C^\infty(\Omega)^{s_0} \to C^\infty(\Omega)^{s_1}.$$

Assume that the following complex is exact, where $P_i(D)$ are matrices of linear partial differential operators with constant coefficients (and $n \geq 1$):

$$(3.2)\quad 0 \to \ker P_0(D) \to C^\infty(\Omega)^{s_0} \xrightarrow{P_0(D)} C^\infty(\Omega)^{s_1} \xrightarrow{P_1(D)} C^\infty(\Omega)^{s_2} \to \cdots \xrightarrow{P_n(D)} C^\infty(\Omega)^{s_{n+1}}.$$

Then for a complete locally convex space E the corresponding vector valued complex

$$(3.3)\quad 0 \to \ker P_0(D)\varepsilon E \to C^\infty(\Omega,E)^{s_0} \xrightarrow{P_0(D)} C^\infty(\Omega,E)^{s_1} \xrightarrow{P_1(D)} C^\infty(\Omega,E)^{s_2} \to \cdots \xrightarrow{P_n(D)} C^\infty(\Omega,E)^{s_{n+1}}$$

is exact if and only if

$$\mathrm{Proj}^1{}_{N\in\mathbb{N}} L(E'_{co}, K_N) = 0,$$

where $\ker P_0(D) = \mathrm{proj}_{N\in\mathbb{N}}\ K_N =: K$*, the projective spectrum is a reduced spectrum of Banach spaces.*

If $E = F'_b$, where F is ultrabornological, then the condition is equivalent to

$$\mathrm{Proj}^1{}_{N\in\mathbb{N}} L(F, K_N) = 0.$$

The idea of the proof is inspired by some extension of the proof of [63, Lemma 3.1] and the result is in fact a reformulation of the basic idea of [61].

It is worth noting that $\Omega \subseteq \mathbb{R}^n$ is $P_0(D)$-convex means that the complex (3.2) is exact for $s_{n+1} = 0$ (i.e., the complex ends with the trivial space). If $s_0 = s_1 = 1$ then $P_0(D)$-convexity means that $P_0(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective. Exactness of the complex (3.3) means in that case that $P_0(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ is surjective.

Proof. Let

$$\Omega_K \subset\subset \Omega_{K+1} \subset\subset \cdots \subset\subset \Omega, \qquad \Omega = \bigcup_{N\in\mathbb{N}} \Omega_N$$

be a compact exhaustion of Ω. Moreover,

$$C^\infty(\Omega)^{s_0} = \mathrm{proj}_{N\in\mathbb{N}}\ C^\infty(\Omega_N)^{s_0}$$

and the spectrum $(C^\infty(\Omega_N)^{s_0}, i_N^{N+1})$, $i_N^{N+1} : C^\infty(\Omega_{N+1})^{s_0} \to C^\infty(\Omega_N)^{s_0}$ the restriction map, is reduced.

By [67, Th. 3.2.8], we have the exact fundamental resolution

$$0 \longrightarrow C^\infty(\Omega)^{s_0} \longrightarrow \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0}$$
$$\xrightarrow{\sigma} \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0} \longrightarrow 0$$

where $\sigma((f_N)) := (i_N^{N+1} f_{N+1} - f_N)$.

Let us take $\varphi_N \in C^\infty(\mathbb{R}^d)$, $\operatorname{supp}\varphi_N \Subset \Omega_{N+1}$, $\varphi_N|_{\Omega_N} \equiv 1$, $0 \le \varphi_N \le 1$. For $M \ge N+1$ we define $s_M^{N+1} : C^\infty(\Omega_{N+1})^{s_0} \to C^\infty(\Omega_M)^{s_0}$,

$$s_M^{N+1}(f)(x) := \begin{cases} \varphi_N f(x) & \text{for } x \in \Omega_{N+1}, \\ 0 & \text{for } x \notin \Omega_{N+1}. \end{cases}$$

Clearly, $s_{N+1}^{N+1}(f)|_{\Omega_N} = f|_{\Omega_N}$. The map $R: \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0} \to \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0}$,

$$R((f_N)) := (\sum_{j\le K} s_K^j(f_j) - f_K)_{K\in\mathbb{N}},$$

is a linear continuous right inverse for σ. Therefore, for every locally convex space F and every $T \in L(F, \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0}) = \prod_{N\in\mathbb{N}} L(F, C^\infty(\Omega_N)^{s_0})$ there is $S \in L(F, \prod_{N\in\mathbb{N}} C^\infty(\Omega_N)^{s_0})$ such that $\sigma \circ S = T$. That means

$$\operatorname{Proj}^1{}_{N\in\mathbb{N}} L(E'_{co}, C^\infty(\Omega_N)^{s_0}) = 0. \tag{3.4}$$

A reduced projective spectrum $(K_N)_{N\in\mathbb{N}}$ of Banach spaces such that $\operatorname{proj}_{N\in\mathbb{N}} K_N = K =: \ker P_0(D)$ may be defined as follows:

Let K_N be the closure of $\{f|_{\overline{\Omega}_N} \mid f \in K\}$ in $C(\overline{\Omega}_N)$. Since $C(\overline{\Omega}_N)$ is continuously embedded in $\mathscr{D}'(\Omega_N)^{s_0}$, K_N is contained in the kernel of

$$P_0(D) : \mathscr{D}'(\Omega_N)^{s_0} \to \mathscr{D}'(\Omega_N)^{s_1}.$$

By hypoellipticity, $v_N : K_N \to C^\infty(\Omega_N)^{s_0}$ and, by the closed graph theorem, v_N is continuous. Thus $\operatorname{proj}_{N\in\mathbb{N}} K_N = K$ and the projective spectrum (K_N) is equivalent to the projective spectrum induced by $(C^\infty(\Omega_N))_{N\in\mathbb{N}}$ on K. Since all reduced spectra of Banach spaces (K_N) with the projective limit equal K are equivalent it suffices to show the result for the spectrum (K_N).

Since the projective spectrum induced by $(C^\infty(\Omega_N)^{s_0})_{N\in\mathbb{N}}$ on K is equivalent to $(K_N)_{N\in\mathbb{N}}$ we have the following commutative diagram by [17, Th. 3.6]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & V_1 & \longrightarrow & U_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_N & \longrightarrow & V_N & \xrightarrow{q_N} & U_N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle j_N^{N+1}} & & \\
0 & \longrightarrow & K_{N+1} & \longrightarrow & V_{N+1} & \xrightarrow{q_{N+1}} & U_{N+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & C^\infty(\Omega)^{s_0} & \xrightarrow{P_0(D)} & \mathrm{im}\ P_0(D) & \longrightarrow & 0,
\end{array}$$

where all rows are exact and the space at the bottom of each column is the projective limit of the column above. Moreover, the spectra of Fréchet spaces $(V_N)_{N\in\mathbb{N}}$, $(U_N)_{N\in\mathbb{N}}$ are defined via the definition of graded exactness (comp. [17, Prop. 3.1]) and thus they are equivalent to the spectra $(C^\infty(\Omega_N))_{N\in\mathbb{N}}$ and the spectrum induced on $\mathrm{im}\ P_0(D) = \ker P_1(D)$ by the same spectrum (call the latter spectrum by $(N(\Omega_N))_{N\in\mathbb{N}}$). Without loss of generality, we may assume that

$$j_N^{N+1} = v_N \circ t_N, \quad t_N : U_{N+1} \to N(\Omega_N), \quad v_N : N(\Omega_N) \to U_N.$$

Clearly, $N(\Omega_N) \subseteq C^\infty(\Omega_N)$ is nuclear. By [63, Cor. 1.2, Th. 1.8], the map v_N lifts with respect to q_N since K_N is a Banach space. Thus

$$q_N^* : L(E'_{co}, V_N) \to L(E'_{co}, U_N), \qquad q_N^*(T) := q_N \circ T,$$

satisfies $\mathrm{im}\ q_N^* \supseteq j_N^{N+1}\big(L(E'_{co}, U_{N+1})\big)$. Therefore

$$0 \to (L(E'_{co}, K_N))_{N\in\mathbb{N}} \to (L(E'_{co}, V_N))_{N\in\mathbb{N}} \to (L(E'_{co}, U_N))_{N\in\mathbb{N}} \to 0$$

is an exact sequence of *spectra* (see [67]), thus we have the following exact sequence (apply (3.4) and see [67, Cor. 3.1.5]):

$$\begin{aligned}0 \to L(E'_{co}, K) \to L(E'_{co}, C^\infty(\Omega)^{s_0}) &\xrightarrow{P_0(D)} L(E'_{co}, \mathrm{im}\ P_0(D)) \\ &\to \mathrm{Proj}^1\ L(E'_{co}, K_N) \to 0\end{aligned}$$

since $C^\infty(\Omega, E)^{s_0}$ is naturally identified with $L(E'_{co}, C^\infty(\Omega)^{s_0})$. We have thus proved that

$$\mathrm{Proj}^1\ L(E'_{co}, K_N) = 0$$

is equivalent to the exactness of the sequence

$$0 \to \ker\, P_0(D)\varepsilon E \to C^\infty(\Omega, E)^{s_0} \xrightarrow{P_0(D)} \ker\, P_1(D)\varepsilon E \to 0.$$

By [17, Th. 5.4 and Th. 3.6], for $1 \le k \le n-1$ the sequence

$$0 \to \ker\, P_k(D) \to C^\infty(\Omega)^{s_k} \xrightarrow{P_k(D)} \ker\, P_{k+1}(D) \to 0 \tag{3.5}$$

splits. In fact [17, Th. 5.4] assumes that all spaces in the spectrum are of the form $C^\infty(\Omega)^s$ but in the proof the last space is irrelevant. For the sake of completeness we give below the full proof of the splitting of (3.5).

By [17, Th. 3.6], for $1 \le k \le n-1$ the sequence (3.5) is graded exact whenever $\ker\, P_k(D)$ and $\ker\, P_{k+1}(D)$ are equipped with the grading induced from $C^\infty(\Omega)^{s_{k+1}}$ ("graded" notions are explained in [17]).

Theorem 4.5(2) in [17] says that such a sequence splits whenever the following four conditions are satisfied:

(1) $\ker\, P_k(D)$ is a strict graded space;

(2) $\ker\, P_{k+1}(D)$ is graded isomorphic to a graded subspace of $s^{\mathbb{N}}$;

(3) $C^\infty(\Omega)^{s_k}$ is graded isomorphic to $s^{\mathbb{N}}$;

(4) $\ker\, P_k(D)$ satisfies the conditions (a) and (b) of [17, Th. 4.1 (6)].

Now, (3) is exactly [17, Th. 2.4] and, of course, (2) follows. Since, by the same arguments as above,

$$0 \to \ker\, P_{k-1}(D) \to C^\infty(\Omega)^{s_{k-1}} \xrightarrow{P_{k-1}(D)} \ker\, P_k(D) \to 0 \tag{3.6}$$

is graded for $k \ge 1$, thus $\ker\, P_k(D)$ is a graded quotient of $C^\infty(\Omega)^{s_{k-1}}$ thus strict as a graded quotient of a graded space isomorphic to a strict graded space $s^{\mathbb{N}}$. So it suffices to show (4). We prove it by [17, Th. 4.7 (iii)]. We apply it to the graded exact sequence (3.6) and observe that $\ker\, P_{k-1}(D)$ has a grading consisiting of Fréchet spaces with the property (Ω) by [17, Th. 5.5 (b)] — see the definition on page 226 of [17]. This completes the proof that (3.5) splits.

Thus also

$$0 \to \ker\, P_k(D)\varepsilon E \to C^\infty(\Omega, E)^{s_k} \xrightarrow{P_k(D)} \ker\, P_{k+1}(D)\varepsilon E \to 0$$

is exact (and splits). This completes the proof.

In case $E = F'_b$ with ultrabornological F we can apply Proposition 3.1 and replace in all places $L(E'_{co}, X)$ by $L(F, X)$. □

Remark 3.3. The same result holds for $P_i(D)$ replaced by hypoelliptic matrices of convolution operators T_i and $C^\infty(\Omega)$ replaced by the spaces $\mathscr{E}_{(\omega)}(\Omega_k)$ of ultradifferentiable functions of Beurling type (or $\mathscr{E}_{(M_p)}(\Omega_k)$) for the non-quasianalytic case whenever T_0 is (ω)-hypoelliptic (see [8]). Indeed, it suffices to choose (X_N) in the proof of Theorem 3.2 in a suitable way.

The following Corollary generalizes [61, Prop. 2.2].

Corollary 3.4. *Let $P_0(D) : C^\infty(\Omega)^{s_0} \to C^\infty(\Omega)^{s_1}$ be a hypoelliptic matrix of linear partial differential operators with constant coefficients such that $\Omega \subseteq \mathbb{R}^d$ and* (3.2) *is exact. Let E be a complete locally convex space.*

(*a*) *The complex* (3.3) *is exact if and only if* $\operatorname{Ext}^1(E'_{co}, \ker P_0(D)) = 0$.

(*b*) *If $E = F'_b$, where F is an ultrabornological locally convex space then the complex* (3.3) *is exact if and only if* $\operatorname{Ext}^1(F, \ker P_0(D)) = 0$.

(*c*) *If E is an ultrabornological PLS-space, then the complex* (3.3) *is exact if and only if* $\operatorname{Ext}^1_{PLS}((\ker P_0(D))'_b, E) = 0$.

Proof. (a) and (b): Follows from Theorem 3.2 and [14, Lemma 1.1] since $\ker P_0(D) \subseteq C^\infty(\Omega)^{s_0}$ is a nuclear space.

(c): Follows from Theorem 3.2 and [7, Theorem 3.4], note that $(\ker P_0(D))'_b$ is an LN-space and $\ker P_0(D)$ is Montel. □

Corollary 3.5. *Let $P_0(D)$ and Ω be as in Corollary* 3.4. *If E_1 is a complete quotient of E such that all compact sets in E_1 are images of compact sets in E* (*for instance, if E_1 is a complete quotient of a PLS-space E, see* [18, *Lemma* 1.5]) *then exactness of the complex* (3.3) *implies exactness of the complex*:

$$0 \to \ker P_0(D)\varepsilon E_1 \to C^\infty(\Omega, E_1)^{s_0} \xrightarrow{P_0(D)} C^\infty(\Omega, E_1)^{s_1} \to \cdots \xrightarrow{P_n(D)} C^\infty(\Omega, E_1)^{s_{n+1}}.$$

The same holds if F_1 is an ultrabornological subspace of an ultrabornological space F and $E = F'_b$, $E_1 = (F_1)'_b$.

Proof. In the first case $(E_1)'_{co} \subseteq E'_{co}$ topologically. In the second case $F_1 \subseteq F$ topologically. The result follows by [14, Cor. 1.2] and Corollary 3.4. □

The definitions of the properties (PA) and $(P\Omega)$ were introduced in [7] and [6], respectively. For the reader's convenience we recall them. A PLS-space $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}}(X_{N,n}, \|\cdot\|_{N,n})$ satisfies (PA) if and only if

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \forall\ \eta > 0\ \exists\ k, C,\ \ r_0 > 0\ \forall\ r > r_0\ \forall\ x' \in X_N' :$$

$$\|x' \circ i_N^M\|_{M,m}^* \le C \left(r^\eta \|x' \circ i_N^K\|_{K,k}^* + \frac{1}{r}\|x'\|_{N,n}^* \right), \tag{3.7}$$

where $\|\cdot\|^*$ denotes the dual norm for $\|\cdot\|$. Analogously, a PLS-space X has $(P\Omega)$ if

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ \eta > 0\ \exists\ k, C,\ \ r_0 > 0\ \forall\ r < r_0\ \forall\ x' \in X_N' :$$

the condition (3.7) holds.

Let us observe that a PLS-space X with (PA) or $(P\Omega)$ is ultrabornological since $\operatorname{Proj}^1 X = 0$ (see [7, Prop. 4.2] and [6, Cor. 5.2]).

The property (Ω) for kernels of hypoelliptic linear partial differential operators of constant coefficients is due to Petzsche [52].

Corollary 3.6. *Let $P_0(D)$ be as in Corollary* 3.4 *and let Ω be convex. If E is a PLS-space with property (PA) then the complex* (3.3) *is exact.*

Proof. If $P_0(D)$ is an individual operator and Ω is convex, then the kernel of $P_0(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ has property $(P\Omega)$ by [6, Cor. 8.4]. Exactly the same proof works for matrices $P_0(D)$. Since $P_0(D)$ is hypoelliptic the kernel $\ker P_0(D)$ in $\mathscr{D}'(\Omega)^{s_0}$ and in $C^\infty(\Omega)^{s_0}$ is exactly the same. By the proof of [19, Cor. 2] and the webbed open mapping theorem the topologies coincide as well. So the kernel in $C^\infty(\Omega)$ has $(P\Omega)$ and because of metrizability also (Ω) . By [7, Theorem 4.1], $\operatorname{Ext}^1_{PLS}((\ker P_0(D))_b', E) = 0$. This completes the proof by Corollary 3.4 (c). $\square$

Lemma 3.7. *Let E be an ultrabornological PLS-space and let F be a Fréchet space having property $(\underline{DN})$. Assume that there is an unbounded increasing sequence of positive real numbers $\alpha := (\alpha_j)$, $\sup \frac{\alpha_{2j}}{\alpha_j} < \infty$, such that $\Lambda_\infty(\alpha) \hookrightarrow F^{\mathbb{N}}$ and F is $\Lambda_1(\alpha)$-nuclear. Then $\operatorname{Ext}^1_{PLS}(F_b', E) = 0$ implies that E has (PA).*

Proof. By Remark 5.3 (d) in [64], it follows that F satisfies the assumptions of [64, Th. 5.2]. In the proof of the latter theorem it is shown that there is $\nu_0 \in \mathbb{N}$ such that for every $\mu \in \mathbb{N}$ there is $\kappa \in \mathbb{N}$, $\theta \in (0,1)$ and a constant C such that there is an increasing sequence of real numbers (α_j), $\lim_j \alpha_j = +\infty$, $\limsup_j \frac{\alpha_{j+1}}{\alpha_j} = D < +\infty$ and there is a sequence $(x_j) \subseteq F'$ such that

$$\|x_j\|_\kappa^* = 1, \qquad \|x_j\|_\mu^* = e^{\alpha_j \theta}, \qquad \|x_j\|_{\nu_0}^* = e^{\alpha_j}.$$

On the other hand, [7, Th. 3.1] implies the condition (G):

$$\begin{aligned} &\forall\ N,\nu\ \exists\ M \geq N, \mu \geq \nu\ \forall\ K \geq M, \kappa \geq \mu\ \exists\ n\ \forall\ m \geq n\ \exists\ k \geq m, S \\ &\quad \forall\ y \in E'_N, x \in F'_\nu : \quad \|y \circ i_N^M\|_{M,m}^* \|x \circ j_\nu^\mu\|_\mu^* \\ &\qquad\qquad\qquad\qquad \leq S\left(\|y\|_{N,n}^* \|x\|_\nu^* + \|y \circ i_N^K\|_{K,k}^* \|x \circ j_\nu^\kappa\|_\kappa^*\right); \end{aligned}$$

We take arbitrary N, choose $\nu = \nu_0$, choose M, μ by (G), then take K, $\kappa = \kappa_0$, θ. Finally we take n, m, k, S according (G). Putting x_j as x in (G) we get

$$\|y \circ i_N^M\|_{M,m}^* e^{\alpha_j \theta} \leq S(\|y\|_{N,n}^* e^{\alpha_j} + \|y \circ i_N^K\|_{K,k}^*).$$

Dividing by $e^{\alpha_j \theta}$ we get

$$\|y \circ i_N^M\|_{M,m}^* \leq S(\|y\|_{N,n}^* e^{\alpha_j(1-\theta)} + \|y \circ i_N^K\|_{K,k}^* e^{-\alpha_j \theta}).$$

Let us choose

$$(e^{\theta-1})^{\alpha_{j+1}} \leq r \leq (e^{\theta-1})^{\alpha_j},$$

then for big j:

$$e^{-\alpha_j \theta} \leq e^{-\alpha_{j+1}\theta \frac{1}{2D}} \leq r^{\frac{\theta}{1-\theta} \cdot \frac{1}{2D}}.$$

Thus for $\eta := \frac{\theta}{1-\theta} \cdot \frac{1}{2D}$ and r small enough we get:

$$\|y \circ i_N^M\|_{M,m}^* \leq CS\left(\|y\|_{N,n}^* \frac{1}{r} + r^\eta \|y \circ i_N^K\|_{K,k}^*\right).$$

We have proved that

$$\begin{aligned} &\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n, \eta_0\ \forall\ m\ \exists\ k(m), S\ \forall\ \eta < \eta_0, r \in]0,1[\ \forall\ y \in E'_N \\ &\quad \|y \circ i_N^M\|_{M,m}^* \leq S\left(\|y\|_{N,n}^* \frac{1}{r} + r^\eta \|y \circ i_N^K\|_{K,k}^*\right). \end{aligned}$$

The last part of the proof of [7, Th. 4.4] shows that this implies (PA) for E. □

The exactness in (a) below for $\mathscr{E}^{(0,p)}(X,E)$, $p > 0$, is due to Palamodov [51].

Corollary 3.8. *Let E be an ultrabornological PLS-space.*

(a) If X is a d-dimensional Stein manifold then the E-valued Dolbeault complex

$$0 \to H(X,E) \to \mathscr{E}^{(0,0)}(X,E) \xrightarrow{\bar\partial} \mathscr{E}^{(0,1)}(X,E) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{E}^{(0,d)}(X,E) \to 0$$

is exact if and only if E has (PA).

(b) If $V \subseteq \mathbb{C}^d$ is an open pseudoconvex set then the E-valued complex

$$\begin{aligned}0 \to H(V\setminus\mathbb{R}^d,E) \to \mathscr{E}^{(0,0)}(V\setminus\mathbb{R}^d,E) \xrightarrow{\bar\partial} \mathscr{E}^{(0,1)}(V\setminus\mathbb{R}^d,E) \xrightarrow{\bar\partial} \cdots\\ \xrightarrow{\bar\partial} \mathscr{E}^{(0,d-1)}(V\setminus\mathbb{R}^d,E)\end{aligned}$$

is exact if and only if E has (PA).

Proof. (a): Since the corresponding scalar-valued sequence is exact then the splitting for $\mathscr{E}^{(0,p)}$, $p>0$, holds always and follows from [17, Cor. 5.6]. It is known that $H(X)$ has (Ω) [66, p. 78]. By [7, Th. 4.1] $\mathrm{Ext}^1_{PLS}(H(X)'_b,E)=0$. This completes the proof of the sufficiency part by Corollary 3.4 (c).

Necessity. It is proved in [64, Sect. 7B] that $F=H(X)$ satisfies assumptions of Lemma 3.7. Then the result follows from it and Corollary 3.4 (c).

(b): Since $\mathscr{H}^p_{V\cap\mathbb{R}^d}(V,\mathscr{O})=0$ for $p\neq d$ by Sato's theorem [32, Th. 2.7], $\mathscr{H}^p(V\setminus\mathbb{R}^d,\mathscr{O})=0$ for $p<d-1$. Thus the corresponding scalar valued sequence is exact. We apply Corollary 3.4 (c).

Sufficiency. For $d=1$ the space $H(V\setminus\mathbb{R}^d)$ has (Ω) since $V\setminus\mathbb{R}^d$ is a Stein manifold, use [66, p. 78]. For $d>1$ the cohomology group $\mathscr{H}^1_{\mathbb{R}^d\cap V}(V,\mathscr{O})=0$ by [32, Th. 2.7]. By [32, Th. 1.1], we have an exact sequence

$$H(V) \to H(V\setminus\mathbb{R}^d) \to \mathscr{H}^1_{\mathbb{R}^d\cap V}(V,\mathscr{O}),$$

thus $H(V\setminus\mathbb{R}^d)$ has (Ω) as a quotient of $H(V)\in(\Omega)$ (see [66, p. 78]). By [7, Th. 4.1], $\mathrm{Ext}^1_{PLS}(H(V\setminus\mathbb{R}^d)'_b,E)=0$.

Necessity. By [60, Satz 1.5, Cor. 5.3, Satz 5.4], we observe that $H(V\setminus\mathbb{R}^d)$ is $\Lambda_1(\alpha)$-nuclear for $\alpha_j := j^{1/d}$ and it has $(\underline{DN})$. Since $\mathbb{C}^d \subset \bigcup_j x_j + (V\setminus\mathbb{R}^d)$ for suitable chosen x_j then

$$\Lambda_\infty(\alpha) \simeq H(\mathbb{C}^d) \hookrightarrow \prod_j H(x_j+(V\setminus\mathbb{R}^d)) \simeq H(V\setminus\mathbb{R}^d)^{\mathbb{N}}.$$

Thus we apply Lemma 3.7 and Corollary 3.4 (c). □

The following result is a generalization of [61, Prop. 4.2] (comp. [64, Th. 7.1]).

Corollary 3.9. *Let $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ be an elliptic linear partial differential operator with constant coefficients and let $\Omega \subseteq \mathbb{R}^d$, $d > 1$, be an arbitrary open set. If E is an ultrabornological PLS-space then*

$$P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$$

is surjective if and only if E has the property (PA).

Proof. Sufficiency. By [61, Prop. 3.4], ker $P(D)$ has property (Ω) . By [7, Th. 4.1], $\mathrm{Ext}^1_{PLS}((\ker P(D))'_b, E) = 0$ and the result follows from Corollary 3.4 (c).

Necessity. Let us denote $K := \ker P(D)$, $K = \mathrm{proj}\ _{\nu\in\mathbb{N}} K_\nu$, where (K_ν) is a reduced projective spectrum of Banach spaces. It is proved in [64, proof of Th. 7.1] that the kernel ker $P(D) \subseteq C^\infty(\Omega)$ satisfies assumptions of Lemma 3.7. Apply Corollary 3.4 (c). □

Now, we consider the dual case, i.e. LFS-spaces. The next result generalizes [61, Prop. 1.1].

Corollary 3.10. *Let $P_0(D)$ be a hypoelliptic matrix of linear partial differential operators with constant coefficients like in Corollary* 3.4*. Similarly, let $\Omega \subseteq \mathbb{R}^d$, $d > 1$, satisfy the assumptions of Cor.* 3.4 *and E be a complete locally convex space containing a complemented copy of φ (the countable direct sum of $\mathbb{C}$) then the complex* (3.3) *is never exact.*

Proof. Clearly, E'_{co} contains a complemented copy of ω (the space of all sequences). Thus, by Cor. 3.4 (a), Cor. 3.5 and the exactness of (3.3), imply that

$$\mathrm{Ext}^1(\omega, \ker P_0(D)) = 0.$$

By [63, Lemma 3.2] and the remarks before [63, Th. 3.3] it follows that for any compact set $K \subseteq \Omega$ the space of restrictions of elements in ker $P_0(D)$ to K is finite dimensional. This cannot be true for $d > 1$. □

§4. Admissible Spaces

Recall that a locally convex space E is called (*weakly*) *d-admissible*, $d \in \mathbb{N}$, $d \geq 1$, if for any (bounded) open set $\Omega \subset \mathbb{R}^d$ the d-dimensional Laplace operator is surjective on the space of E-valued smooth functions on Ω, i.e.

$$\Delta_d : C^\infty(\Omega, E) \to C^\infty(\Omega, E) \text{ is surjective.}$$

We will provide many concrete examples of locally convex spaces which are d-admissible (or which are not). The results of this section are direct consequences of the previous section.

For ultrabornological PLS-spaces E d-admissibility is independent of d by Corollary 3.9:

Corollary 4.1. *The following are equivalent for an ultrabornological PLS-space E:*

a) $P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ is surjective for some elliptic operator $P(D)$ and some open set $\Omega \subseteq \mathbb{R}^n$ and some $n \in \mathbb{N}$, $n > 1$.

b) E is weakly d-admissible for some $d \geq 2$

c) E is d-admissible for any $d \in \mathbb{N}$

d) E has (PA)

As we will see later on from Theorem 6.9 and Corollary 8.5 it follows that $(d+1)$-admissibility implies weak d-admissibility. But the general relation between (weak) d-admissibility for various d is unclear.

Problem 4.2. *Does there exist a locally convex space E such that for $d \geq 2$ E is d-admissible but not $(d+1)$-admissible and vice versa? The same problem for weak admissibility.*

We prepare now some auxiliary results on (PA) for spaces of operators.

Proposition 4.3. *Let X be a regular LFS-space and Y be a PLS-space then $L_b(X, Y)$ is a PLS-space. In particular, this is so if X and Y are Fréchet Schwartz spaces.*

Proof. If $X = \text{ind}_{N} \text{proj}_{n} X_{N,n}$, $Y = \text{proj}_{N} \text{ind}_{n} Y_{N,n}$ where $X_{N,n}$, $Y_{N,n}$ are Banach spaces and $X_N := \text{proj}_{n} X_{N,n}$, $Y_N := \text{ind}_{n} Y_{N,n}$ are Fréchet Schwartz spaces and DFS-spaces, respectively, then algebraically

$$L(X, Y) = \text{proj}_{N} \text{ind}_{n} L(X_{N,n}, Y_{N,n}).$$

Clearly, X is reflexive, Montel and, by [67, remarks on p. 110, Cor. 6.7, Cor. 3.3.10], its dual is an ultrabornological barrelled reflexive PLS-space. Therefore

$$L_b(X, Y) = X' \varepsilon Y,$$

where $X' = \text{proj}_{N} \text{ind}_{n} X'_{N,n}$. By [31, Sec. 16], topologically

$$X' \varepsilon Y = \text{proj}_{N} X'_N \varepsilon Y_N$$

and by [3, 4.3] $X'_N \varepsilon Y_N$ are DFS-spaces.

For the Fréchet Schwartz case it suffices to observe that any Fréchet Schwartz space is a PLS-space. Indeed, it follows from the result of Heinrich [25] that if $T : E \to F$ is compact, E, F Banach spaces, then T factorizes through two compact operators. Using that inductively one can prove that T factorizes through an LS-space and we apply that for compact linking maps $i_n^{n+1} : Y_{n+1} \to Y_n$, where $Y = \operatorname{proj}\ _n Y_n$ is an arbitrary Fréchet Schwartz space. □

Remark 4.4. It is worth noting that if Y is an ultrabornological PLS-space then $L_b(X, Y) \simeq L_b(Y', X')$ via taking adjoints. The above proof shows that

$$L_b(X, Y) = \operatorname{proj}\ _N \operatorname{ind}\ _n L_b(X_{N,n}, Y_{N,n})$$

with the notation from the proof.

Proposition 4.5. *The PLS-space $L(\Lambda^1_\infty(\alpha), \Lambda^\infty_\infty(\beta))$ has (PA) (here the superscript means the type of norms used).*

Proof. It is easy to show that $L(\Lambda^1_\infty(\alpha), \Lambda^\infty_\infty(\beta))$ is a Köthe type PLS-space of matrices (i.e. sequences indexed by $(u, v) \in \mathbb{N} \times \mathbb{N}$) with the Köthe type matrix:

$$a_{N,n;v,u} := \exp(N\beta_u - n\alpha_v).$$

By [7, Th. 4.3], it suffices to check (PA) on unit vectors only, i.e., after taking logarithms to show that

(4.8)

$$\begin{aligned}
&\forall\ N\ \exists\ M\ \ \forall\ K\ \ \exists\ n\ \forall\ m, \theta \in]0, 1[\ \ \exists\ k, C\ \ \forall\ u, v : \ \ -M\beta_u + m\alpha_v \\
&\qquad \le \max\left[(-\theta N\beta_u + \theta n\alpha_v - (1-\theta)K\beta_u + (1-\theta)k\alpha_v)\,; -N\beta_u + n\alpha_v\right] + C.
\end{aligned}$$

Assume that

$$-M\beta_u + m\alpha_v > -N\beta_u + n\alpha_v$$

thus

$$\beta_u < \left(\frac{m-n}{M-N}\right)\alpha_v.$$

Then

$$[-M + \theta N + (1-\theta)K]\,\beta_u < [-M + \theta N + (1-\theta)K]\left(\frac{m-n}{M-N}\right)\alpha_v$$

and by choosing k big enough it is smaller than

$$[\theta n + (1-\theta)k - m]\,\alpha_v$$

for every v. This implies (4.8). □

Corollary 4.6. *If X and Y are nuclear Fréchet spaces such that X has (DN) and Y has (Ω) , then the PLS-space $L_b(X,Y)$ has (PA).*

Proof. By the results in [59] (comp. [17, Prop. 1.3]) we have the following short exact sequences

$$0 \to X \xrightarrow{j} s \to s \to 0, \qquad 0 \to s \to s \xrightarrow{q} Y \to 0.$$

Every operator $T : X \to Y$ extends to $T_1 : s \to Y$ via the embedding j using the splitting result [48, 30.1]. Analogously, by the same result, T_1 lifts to $T_2 : s \to s$ via q. We have proved that the map $\Phi : L_b(s,s) \to L_b(X,Y)$, $\Phi(V) := q \circ V \circ j$ is a surjection. By [64], $\mathrm{Proj}^1\ L(X, Y_N) = 0$ thus $L_b(X,Y)$ is ultrabornological with its PLS-topology. By the webbed open mapping theorem Φ is open and $L_b(X,Y)$ is a topological quotient of the PLS-space $L_b(s,s)$. Thus (PA) for $L_b(X,Y)$ follows from Proposition 4.5 and the fact that (PA) is inherited by quotients. □

Corollary 4.7. *Let X and Y be nuclear Fréchet spaces.*

1. *For $X = \Lambda_r(\alpha)$ the PLS-space $L_b(X,Y)$ has (PA) if and only if Y has (Ω) and $r = \infty$.*
2. *For $Y = \Lambda_r(\alpha)$ the PLS-space $L_b(X,Y)$ has (PA) if and only if X has (DN).*

Proof. It follows from the fact that (PA) for $L_b(X,Y)$ implies $\mathrm{Proj}^1\ L(X, Y_N) = 0$ and results of [64] as well as the fact that X' is a complemented subspace of $L_b(X,Y)$ so if the latter space has (PA) the space $X = X''$ must have (DN). □

Using the results of Section 3 we could describe spaces E for which the complex (3.3) is exact in the hypoelliptic or elliptic case for general Ω or convex Ω. Since we are mainly interested in the case of one elliptic operator $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ we make such a survey only for that case. For the definition of various spaces appearing below see [13], [10], [38] and [21]. Sequence space representations are known for many more spaces of analysis. So we can give only a selection of corresponding examples here. Clearly, many of them are already contained in [61].

Corollary 4.8. *Let $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ be a linear elliptic operator with constant coefficients and let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set.*

The following spaces E are ultrabornological PLS-spaces with property (PA) so, in particular, the map $P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ is surjective and E is a d-admissible space for any $d \geq 1$:

- *an arbitrary Fréchet Schwartz space*;
- *the strong dual of a power series space of infinite type $\Lambda'_\infty(\alpha)$*;
- *a PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ whenever $s = \infty$ or $\Lambda_{r,s}(\alpha, \beta)$ is a Fréchet space*;
- *the strong dual of any space of holomorphic functions $H(U)'$, where U is a Stein manifold with the strong Liouville property* (*for instance, for $U = \mathbb{C}^d$*);
- *the space of germs of holomorphic functions $H(K)$ where K is a completely pluripolar compact subset of a Stein manifold* (*for instance K consists of one point*);
- *the space of tempered distributions $\mathscr{S}'$ and the space of Fourier ultra-hyperfunctions $\mathcal{P}'_{**}$*;
- *the spaces of distributions $\mathscr{D}'(U)$ and ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(U)$ for any open set $U \subseteq \mathbb{R}^n$*;
- *the weighted distribution spaces $(K\{pM\})'$ of Gelfand and Shilov if the weight M satisfies*
$$\sup_{|y| \leq 1} M(x+y) \leq C \inf_{|y| \leq 1} M(x+y) \text{ if } x \in \mathbb{R}^d.$$
- *the kernel of any linear partial differential operator with constant coefficients in $\mathscr{D}'(U)$ or in $\mathscr{D}'_{(\omega)}(U)$ when U is convex*;
- *the space $L_b(X, Y)$ where X has (DN), Y has (Ω) and both are nuclear Fréchet spaces. In particular, $L_b(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ if both spaces are nuclear.*

Proof. By Corollary 3.9, surjectivity of $P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ for an ultrabornological PLS-space E is equivalent with the condition (PA) for E. By [7, Th. 4.3] and the remarks preceding that theorem the space $\Lambda'_r(\alpha)$ has (PA) iff $r = \infty$, the space $\Lambda_{r,s}(\alpha, \beta)$ has (PA) iff $s = \infty$ or it is a Fréchet space and the spaces $\mathscr{D}'(U)$, $\mathscr{D}'_{(\omega)}(U)$ have (PA) for any open set $U \subseteq \mathbb{R}^n$. Moreover, it is known that $\mathscr{S}'$, $\mathcal{P}'_{**}$ and $(K\{pM\})'$ are isomorphic to some $\Lambda'_\infty(\alpha)$ (see [38] and [62]) so they have (PA) as well. By [7, Prop. 5.4], also the kernel of any linear partial differential operator $P(D)$ with constant coefficients in $\mathscr{D}'(U)$, U convex, has (PA). An analogous proof works for kernels of $P(D)$ in $\mathscr{D}'_{(\omega)}(U)$. Finally, if U is a Stein manifold, then $H(U)'$ has (A) or, equivalently, (PA) if and only if U has the strong Liouville property, i.e, every bounded plurisubharmonic function on U is constant [68, Th. 2.3.7]. The last statement follows from Corollary 4.6. $\square$

Corollary 4.9. *Let $P(D) : C^\infty(\Omega) \to C^\infty(\Omega)$ be a linear elliptic operator with constant coefficients and let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set.*

(a) The following ultrabornological PLS-spaces E do not have (PA), so, in particular, the map $P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ for $d > 1$ is not surjective and E is not a weakly d-admissible space for any $d > 1$:

- *the strong dual of a power series space of finite type $\Lambda'_0(\alpha)$;*
- *the space of ultradifferentiable functions of Roumieu type $\mathscr{E}_{\{\omega\}}(U)$, where ω is a non-quasianalytic weight and $U \subseteq \mathbb{R}^n$ is an arbitrary open set;*
- *the strong dual of any space of holomorphic functions $H(U)'$ where U is a Stein manifold which does not have the strong Liouville property (for instance, $U = \mathbb{D}^n$ the polydisc, $U = \mathbb{B}_n$ the unit ball etc.);*
- *the space of germs of holomorphic functions $H(K)$ where K is compact and not completely pluripolar (for instance, $K = \overline{\mathbb{D}}^n$ or $K = \overline{\mathbb{B}}_n$);*
- *the space of distributions (or ultradistributions) with compact support $\mathscr{E}'(U)$ (or $\mathscr{E}'_{(\omega)}(U)$) for $U \subseteq \mathbb{R}^n$ open*
- *the space of real analytic functions $\mathscr{A}(U)$ for any open set $U \subseteq \mathbb{R}^n$.*

(b) For the following LFS-spaces E the map $P(D) : C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ is not surjective for any $d > 1$ and E is not weakly d-admissible for any $d > 1$:

- *the spaces of test functions $\mathscr{D}(U)$;*
- *the spaces of test functions for ultradistributions $\mathscr{D}_{(\omega)}(U)$, the space of ultradistributions of Roumieu type with compact support $\mathscr{E}'_{\{\omega\}}(U)$, where ω is a non-quasianalytic weight, $U \subseteq \mathbb{R}^n$ is an arbitrary open set;*
- *the strong dual $\mathscr{A}(U)'_b$ for an arbitrary open set $U \subseteq \mathbb{R}^n$.*

Proof. (a) This follows as in the proof of 4.8. Notice that the space of real analytic functions $\mathscr{A}(U)$ or $\mathscr{E}_{\{\omega\}}(U)$ contains a complemented copy of some $\Lambda_0'(\alpha)$ (see [15, Prop. 5.6] and [62]) thus the result follows by the corresponding result on duals of finite type power series spaces.

(b) The spaces $\mathscr{D}(U)$, $\mathscr{D}_{(\omega)}(U)$, $\mathscr{E}'(U)$ and $\mathscr{E}'_{\{\omega\}}(U)$, ω a non-quasinalytic weight, contain complemented copies of φ (see [62]) so by Corollary 3.10 the result follows.

The space $\mathscr{A}(U)$ is ultrabornological (see [13, Ex. 3.4 (b)]) thus, by Corollary 3.4

$$\operatorname{Ext}^1(\mathscr{A}(U), \ker P(D)) = 0$$

is a necessary condition for $\mathscr{A}(U)_b'$. By [14, Th. 2.3], this implies that ker $P(D)$ has the property $(\overline{\overline{\Omega}})$. This cannot be true by [65, Th. 3]. $\square$

Problem 4.10. *Is any PLS-type non-Fréchet power series space* $\Lambda_{r,s}(\alpha, \beta)$*,* $s = 0$ *a d-admissible space for* $d > 1$*?*

Remark 4.11. Every Fréchet space E is d-admissible and $P(D)\colon C^\infty(\Omega, E) \to C^\infty(\Omega, E)$ as above is surjective.

§5. Vector Valued Analytic Functionals and the Grothendieck-Tillmann Duality

The crucial role in the theory of hyperfunctions is played by the so-called analytic functionals, i.e, elements of $\mathscr{A}(K)'$ for $K \Subset \mathbb{R}^d$. Thus for vector valued hyperfunctions we need vector valued analytic functionals, i.e., elements of $L(\mathscr{A}(K), E)$. We will explain here a method which allows us to replace holomorphic functions with harmonic ones in the definition of analytic functionals.

E is always a complete locally convex space in this section.

For a compact set $K \subset \mathbb{R}^d$ let $\widetilde{\mathcal{C}}_\Delta(K) := \operatorname{ind}_{\mathbb{R}^{d+1} \supset U \supset K} \widetilde{\mathcal{C}}_\Delta(U)$ denote the harmonic germs near K which are even with respect to the last variable. We start with an easy lemma (comp. [16, Prop. 2.3]).

Lemma 5.1. *For any compact set* $K \subset \mathbb{R}^d$*,* $\mathscr{A}(K)$ *is isomorphic to* $\widetilde{\mathcal{C}}_\Delta(K)$ *via the solution of the Cauchy problem*

$$\Delta_{d+1}(f) = 0, \quad f(x,0) = g(x), \quad \partial_y f(x,0) = 0, \ \textit{near } K.$$

Thus $L(\mathscr{A}(K), E)$ *can be identified with* $L(\widetilde{\mathcal{C}}_\Delta(K), E)$ *if* $K \subset \mathbb{R}^d$ *is compact.*

The space $L(\widetilde{\mathcal{C}}_\Delta(K), E)$ may be identified with the quotient space

$$\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)$$

by a vector valued version of the Grothendieck-Tillmann-duality (see [23], [56], [2]) which is the basic general tool for our approach and which we will introduce now. For $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$ we define $H(f) : \widetilde{\mathcal{C}}_\Delta(K) \to E$ as follows. For $g \in \widetilde{\mathcal{C}}_\Delta(K)$ let

$$H(f)(g) := \int f(\xi)\Delta_{d+1}(\phi g)(\xi) d\xi \tag{5.9}$$

if $g \in \widetilde{\mathcal{C}}_\Delta(U)$ for a neighborhood U of K and a test function $\phi \in \mathscr{D}(U)$ which is 1 near K. The definition of $H(f)(g)$ is independent of ϕ since for any $e' \in E'$

$$e'\left(H(f)(g)\right) = \int e' \circ f(\xi)\Delta_{d+1}(\phi g)(\xi) d\xi = H(e' \circ f)(g) \tag{5.10}$$

and since for f scalar valued (i.e., $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K)$) the number $H(f)(g)$ is independent of ϕ by [2, Satz 2b)].

Theorem 5.2. *For any compact set $K \subset \mathbb{R}^d$ the mapping*

$$H : \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E) \to L_b(\widetilde{\mathcal{C}}_\Delta(K), E)$$

is a topological isomorphism.

Proof. First, we show that H as defined above is a continuous map:

$$H : \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E) \to L_b(\widetilde{\mathcal{C}}_\Delta(K), E).$$

Let $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$. Then $H(f) \in L(\widetilde{\mathcal{C}}_\Delta(K), E)$ since for $g \in \widetilde{\mathcal{C}}_\Delta(U)$ and $J := \mathrm{supp}(\mathrm{grad}(\phi)) \subset U$ compact we have for any continuous seminorm p on E

$$p(H(f)(g)) \leq C_1 \sup_{\xi \in \widetilde{J}} |g(\xi)| \sup_{\xi \in J} |p(f(\xi))|$$

if $\widetilde{J}$ is a compact neighborhood of J in U. This also shows the continuity of H.

We will show now, that $H(f) = 0$ if and only if $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)$. If $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)$ then $e' \circ f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1})$ and hence $e'\left(H(f)(g)\right) = H(e' \circ f)(g) = 0$ for any $e' \in E'$ by (5.10) and the scalar result (see [56], [2]).

Conversely, if $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$ and $H(f) = 0$ then $H(e' \circ f) = 0$ and by the scalar result (see [56], [2]) $e' \circ f$ can be (uniquely) extended to $f_{e'} \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1})$ and

$$f_{e'}(\xi) = \int_K G(\xi - \eta) e' \circ \Delta(\phi f)(\eta) d\eta \tag{5.11}$$

for any $e' \in E'$ and any $\xi \in K$ (and G from (2.1)) where $\phi \in \mathscr{D}(\mathbb{R}^{d+1})$ is fixed and 1 near K. Since $\Delta(\phi f)$ is continuous on $J := \operatorname{supp}(\operatorname{grad}(\phi))$, $\Delta(\phi f)(J)$ is compact in E and the right hand side of (5.11) defines a linear form on E' which is continuous for the Mackey topology $\tau(E', E)$. Hence (5.11) defines an E-valued function f on $\mathbb{R}^{d+1}$ and $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)$ by Lemma 2.1.

Finally, we show that H is surjective. For $T \in L(\widetilde{\mathcal{C}}_\Delta(K), E)$ and $\xi \in \mathbb{R}^{d+1} \setminus K$ let

$$g_\xi \in \widetilde{\mathcal{C}}_\Delta(K), \quad g_\xi(x, y) := (G(\xi - (x, y)) + G(\xi - (x, -y)))/2.$$

We define

$$S(T)(\xi) := T(g_\xi) \qquad \text{if } \xi \in \mathbb{R}^{d+1} \setminus K.$$

Clearly, $S(T)(\xi) \in E$ and $S(T) \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$ (use e.g. Lemma 2.1). The mapping

$$S : L_b(\widetilde{\mathcal{C}}_\Delta(K), E) \to \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$$

is continuous since $\{g_\xi \mid \xi \in J\}$ is bounded in $\widetilde{\mathcal{C}}_\Delta(K)$ if $J \subset \mathbb{R}^{d+1} \setminus K$ is compact.

We will show that $H \circ S$ is the identity mapping on $L(\widetilde{\mathcal{C}}_\Delta(K), E)$ (hence, H is surjective). This is equivalent to the condition that

$$e'(H \circ S(T)(g)) = e'(T(g)) \quad \text{for any } g \in \widetilde{\mathcal{C}}_\Delta(K) \text{ and } e' \in E'.$$

Since

$$e'(H \circ S(T)(g)) = \int e'(T(g_\xi))\Delta(\phi g)(\xi) d\xi = H \circ S(e' \circ T)(g)$$

it suffices to show the result for $E = \mathbb{C}$. Since the set of point evaluations of derivatives $\{\delta^{(a)}_{(x_0,0)} \mid x_0 \in K, a \in \mathbb{N}_0^d\}$ is total in $\widetilde{\mathcal{C}}_\Delta(K)'_b$ we need to show that $(H \circ S)(\delta^{(a)}_{(x_0,0)})(g) = \langle \delta^{(a)}_{(x_0,0)}, g\rangle$ if $g \in \widetilde{\mathcal{C}}_\Delta(K)$. Since

$$S(\delta^{(a)}_{(x_0,0)})(\xi) = G^{(a)}(\xi - (x_0, 0)) \text{ if } x_0 \in K, a \in \mathbb{N}_0^d$$

we get

$$\begin{aligned}(H \circ S)(\delta^{(a)}_{(x_0,0)})(g) &= \int G^{(a)}(\xi - (x_0, 0))\Delta(\phi g)(\xi) d\xi \\ &= \langle G^{(a)}(\,\cdot\, - (x_0, 0)), \Delta(\phi g)\rangle_{\mathscr{D}'(\mathbb{R}^{d+1})} \\ = \langle \Delta G^{(a)}(\,\cdot\, - (x_0, 0)), \phi g\rangle_{\mathscr{D}'(\mathbb{R}^{d+1})} &= \langle \delta^{(a)}_{(x_0,0)}, \phi g\rangle = \langle \delta^{(a)}_{(x_0,0)}, g\rangle.\end{aligned}$$

The theorem is proved. $\square$

It is well known that $\mathscr{A}(\mathbb{R}^d)$ is densely embedded in $\mathscr{A}(K)$ if $K \subset \mathbb{R}^d$ is compact. We may thus identify elements of $L(\mathscr{A}(J), E)$ and $L(\mathscr{A}(K), E)$ for different compact sets $K, J \subset \mathbb{R}^d$ by means of their restrictions to $\mathscr{A}(\mathbb{R}^d)$. We then have the following result defining the *support* of a vector-valued analytic functional:

Proposition 5.3. *Let $K, J \subset \mathbb{R}^d$ be compact.*
a) $L(\mathscr{A}(K), E) \cap L(\mathscr{A}(J), E) = L(\mathscr{A}(K \cap J), E)$
b) For any $T \in L(\mathscr{A}(K), E)$ there is a minimal compact $J \subset K$ such that $T \in L(\mathscr{A}(J), E)$. The set J is called the support of T.

Proof. a) Let $T \in L(\mathscr{A}(K), E) \cap L(\mathscr{A}(J), E) = L(\widetilde{\mathcal{C}}_\Delta(K), E) \cap L(\widetilde{\mathcal{C}}_\Delta(J), E)$ (see Lemma 5.1). Then

$$\begin{aligned} H^{-1}(T) \in \big(\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)\big) \cap \big(\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus J, E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E)\big) \\ = \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus (K \cap J), E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}, E) \end{aligned}$$

and $T \in L(\widetilde{\mathcal{C}}_\Delta(K \cap J), E) = L(\mathscr{A}(K \cap J), E)$ by Theorem 5.2.

b) This is evident by Theorem 5.2 since for any $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$ there is a minimal J such that $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus J, E)$. $\square$

Notice that there is an essential difference between the scalar and the vector valued case. Every $f \in \mathscr{A}(\mathbb{R}^d)'$ has a compact support but in general (even for Fréchet spaces) $T \in L(\mathscr{A}(\mathbb{R}^d), E)$ need not be compactly supported, that is, in general there is no compact $K \subset \mathbb{R}^d$ such that $T \in L(\mathscr{A}(K), E)$.

Example 5.4. Let $T(f) := (f(k))_{k\in\mathbb{N}}$ for $f \in \mathscr{A}(\mathbb{R})$. Clearly, $T \in L(\mathscr{A}(\mathbb{R}), \mathbb{C}^{\mathbb{N}})$, but T is not compactly supported since $T(f_n) \to e_j$ (the canonical j^{th} unit vector) for $f_n(x) := \exp(-n(x-j)^2)$ while $f_n \to 0$ in $\mathscr{A}(K)$ for any compact $K \subset (\mathbb{R}^d \setminus \{j\})$. Hence the statement of Ito [30, Theorem 2.7] is false (see also the remark before [30, Theorem 2.5]).

§6. The Duality Method

In this section we will introduce vector valued hyperfunctions as the sheaf generated by equivalence classes of compactly supported vector valued analytic functionals, this method being sometimes called the duality method (see [30]) which was introduced by Martineau [45]. When doing so we will profit a lot of Bengel's point of view of hyperfunctions (i.e. considering harmonic functionals instead of analytic functionals (see [2], [54] and also [35, 36])). Moreover, we will constantly use the Grothendieck-Tillmann-duality of harmonic functionals and harmonic functions explained in the previous section (see Theorem 5.2). At the end of this section we interpret hyperfunctions as boundary values of harmonic functions.

Definition 6.1. For an open and bounded set $\Omega \subset \mathbb{R}^d$ and a locally convex space E we define the space of E-valued hyperfunctions on Ω by

$$\mathscr{B}(\Omega, E) := L(\mathscr{A}(\overline{\Omega}), E)/L(\mathscr{A}(\partial\Omega), E).$$

Since $\mathscr{A}(\overline{\Omega})$ embeds injectively into $\mathscr{A}(\partial\Omega)$ thus $L_b(\mathscr{A}(\partial\Omega), E) = \mathscr{A}(\partial\Omega)'\hat{\otimes}_\varepsilon E$ is dense in $L_b(\mathscr{A}(\overline{\Omega}), E) = \mathscr{A}(\overline{\Omega})'\hat{\otimes}_\varepsilon E$ (in the topology of uniform convergence on bounded sets) that is why on $\mathscr{B}(\Omega, E)$ there is no reasonable locally convex topology. For $T \in L(\mathscr{A}(\bar{\Omega}), E)$ we denote by $[T]$ the corresponding element of $\mathscr{B}(\Omega, E)$.

In the scalar case (i.e. $E = \mathbb{C}$), restrictions and a sheaf structure may be defined on $\mathcal{B}_{\Omega_1} := \{\mathscr{B}(\Omega) := \mathscr{B}(\Omega, \mathbb{C}) \mid \Omega \subset \Omega_1 \text{ open}\}$ for bounded open Ω_1 since it is easily seen that for $\Omega \subset \Omega_1$ the canonical injective mapping

$$I : \mathscr{A}(\overline{\Omega})'/\mathscr{A}(\partial\Omega)' \to \mathscr{A}(\overline{\Omega}_1)'/\mathscr{A}(\overline{\Omega}_1 \setminus \Omega)' \tag{6.1}$$

is surjective, hence an isomorphism. We do not know if the corresponding condition holds always for the vector valued case (comp. Remark 6.3 and also Theorem 8.4). The proof of the corresponding vector valued result is more subtle (compare also the remarks before Remark 6.3). We get it only under the assumption that

$$\Delta_{d+1} : C^\infty(U, E) \to C^\infty(U, E) \text{ is surjective for any open } U \subset \mathbb{R}^{d+1}, \tag{6.2}$$

i.e., that E is $(d+1)$-admissible. First we show that $\mathscr{B}(\Omega, E)$ can be defined also using a set Ω_1 bigger than Ω.

Lemma 6.2. *Let E be $(d+1)$-admissible. Let $\Omega_2 \subset \Omega_1 \subset \mathbb{R}^d$ be open and bounded. Then the canonical mapping*

$$I : L(\mathscr{A}(\overline{\Omega}_2), E)/L(\mathscr{A}(\partial\Omega_2), E) \to L(\mathscr{A}(\overline{\Omega}_1), E)/L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E)$$

is a bijection.

Proof. The map I is well defined since the continuous and dense embedding of $\mathscr{A}(\overline{\Omega}_1)$ into $\mathscr{A}(\overline{\Omega}_2)$ defines the embedding of $L(\mathscr{A}(\overline{\Omega}_2), E)$ into $L(\mathscr{A}(\overline{\Omega}_1), E)$, and $L(\mathscr{A}(\partial\Omega_2), E)$ is mapped into $L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E)$ in this way.

If $T \in L(\mathscr{A}(\overline{\Omega}_2), E) \cap L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E)$ then $T \in L(\mathscr{A}(\partial\Omega_2), E)$ by Proposition 5.3 since $\partial\Omega_2 = \overline{\Omega}_2 \cap (\overline{\Omega}_1 \setminus \Omega_2)$, thus I is injective.

To show that I is surjective it suffices, by Theorem 5.2, to show that the mapping

$$L : \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus (\overline{\Omega}_1 \setminus \Omega_2), E) \times \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}_2, E) \to \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}_1, E),$$
$$L(f_1, f_2) := f_1 + f_2,$$

is surjective. Choose $\varphi \in \tilde{C}^\infty(\mathbb{R}^{d+1} \setminus \partial\Omega_2)$ such that $\varphi \equiv 1$ near Ω_2 and $\varphi \equiv 0$ near $\mathbb{R}^d \setminus \overline{\Omega}_2$. Let $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}_1, E)$. Then $\Delta_{d+1}(\varphi f)$ may be considered as a function in $\tilde{C}^\infty(\mathbb{R}^{d+1} \setminus \partial\Omega_2, E)$ and by (6.2) there is $g \in \tilde{C}^\infty(\mathbb{R}^{d+1} \setminus \partial\Omega_2, E)$ such that $\Delta_{d+1} g = \Delta_{d+1}(\varphi f)$. Then $f_2 := \varphi f - g \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}_2, E)$ and $f_1 := (1-\varphi) f + g \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus (\overline{\Omega}_1 \setminus \Omega_2), E)$ satisfy $f_1 + f_2 = f$. □

Ito states (see [30, p.34, l.2]) that Lemma 6.2 always holds if E is complete, however he does not give a proof that I is surjective. On the other hand, he states as an open problem (see [30, Problem A]) if for two compact sets $K_1, K_2 \subset \mathbb{R}^d$ the mapping

$$L : L(\mathscr{A}(K_1), E) \times L(\mathscr{A}(K_2), E) \to L(\mathscr{A}(K_1 \cup K_2), E)$$

defined by $L(T_1, T_2) := T_1 - T_2$ is surjective. Notice however the following

Remark 6.3. Let $\Omega_2 \subset \Omega_1 \subset \mathbb{R}^d$ be open and bounded. The following are equivalent:

a) The canonical mapping

$$I : L(\mathscr{A}(\overline{\Omega}_2), E)/L(\mathscr{A}(\partial\Omega_2), E) \to L(\mathscr{A}(\overline{\Omega}_1), E)/L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E)$$

is a bijection.

b) The mapping

$$L : L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E) \times L(\mathscr{A}(\overline{\Omega}_2), E) \to L(\mathscr{A}(\overline{\Omega}_1), E)$$

is surjective.

Proof. This is evident since I is always injective. □

If E is $(d+1)$-admissible we can define restrictions on $\mathscr{B}(\Omega, E)$ using Lemma 6.2 as follows:

Definition 6.4. Let E be $(d+1)$-admissible and let

$$q : L(\mathscr{A}(\overline{\Omega}_1), E)/L(\mathscr{A}(\partial\Omega_1), E) \to L(\mathscr{A}(\overline{\Omega}_1), E)/L(\mathscr{A}(\overline{\Omega}_1 \setminus \Omega_2), E)$$

be the canonical quotient map. For $[T] \in \mathscr{B}(\Omega_1, E) = L(\mathscr{A}(\overline{\Omega}_1), E)/L(\mathscr{A}(\partial\Omega_1), E)$ and $\Omega_2 \subset \Omega_1 \subset \mathbb{R}^d$ open and bounded we define the restriction

$$R_{\Omega_1,\Omega_2}([T]) := [T]\big|_{\Omega_2} := I^{-1}(q([T])) \in L(\mathscr{A}(\overline{\Omega}_2), E)/L(\mathscr{A}(\partial\Omega_2), E) = \mathscr{B}(\Omega_2, E) \tag{6.3}$$

with I from Lemma 6.2.

Lemma 6.5. *Let E be $(d+1)$-admissible and let $\Omega \subset \mathbb{R}^d$ be open and bounded. The spaces $\{\mathscr{B}(\omega, E) \mid \omega \subset \Omega$ open$\}$ form a presheaf on Ω (with the restrictions R_{Ω_1,Ω_2} defined in 6.4) satisfying the condition $(S1)$:*
if $\cup_j \omega_j = \omega \subset \Omega$, ω_j open, such that $[T] \in \mathscr{B}(\omega, E)$ satisfies $[T]\big|_{\omega_j} = 0$ for any j then $[T] = 0$.

For the condition (S1) see [11, p. 5].

Proof. Clearly we have $R_{\omega_1,\omega_2} \circ R_{\omega_2,\omega_3} = R_{\omega_1,\omega_3}$ if $\omega \supset \omega_1 \supset \omega_2 \supset \omega_3$ are open. If $[T]$ is as above then the support of T (in the sense of Proposition 5.3) is contained in $\overline{\omega}$ and does not contain any point in ω by assumption, hence $T \in L(\mathscr{A}(\partial\omega), E)$ by Proposition 5.3 b), that is, $[T] = 0$. □

The sheaf $\mathscr{B}(E)$ of E-valued hyperfunctions on $\mathbb{R}^d$ is now defined as follows:

Definition 6.6. Let E be $(d+1)$-admissible and set $\mathscr{B}_1(\Omega, E) := \mathscr{B}(\Omega, E)$ if $\Omega \subset \mathbb{R}^d$ is open and bounded, and $\mathscr{B}_1(\Omega, E) := 0$ if $\Omega \subset \mathbb{R}^d$ is open and unbounded. $\{\mathscr{B}_1(\Omega, E) \mid \Omega \subset \mathbb{R}^d$ open$\}$ is a presheaf when considered with the restrictions R_{Ω_1,Ω_2} from Definition 6.4 if Ω_1 is bounded and $R_{\Omega_1,\Omega_2} := 0$ if Ω_1 is unbounded. The sheaf $\mathscr{B}(E)$ of E-valued hyperfunctions is the associated sheaf.

It is convenient to discuss $\mathscr{B}(E)$ and especially the second sheaf property $(S2)$ (see e.g. [11, p. 6]) using a boundary value representation of $\mathscr{B}(E)$ defined as follows: For open $\Omega \subset \mathbb{R}^d$ let $\mathcal{U}(\Omega)$ denote the open sets $U \subset \mathbb{R}^{d+1}$ which are symmetric with respect to the last variable and satisfy $U \cap \mathbb{R}^d = \Omega$. For $U \in \mathcal{U}(\Omega)$ we define the space of boundary values of harmonic functions by

$$bv(\Omega, E) := \widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U, E). \tag{6.4}$$

Lemma 6.7. *The definition of bv is independent of the choice of $U \in \mathcal{U}(\Omega)$ if E is $(d+1)$-admissible.*

Proof. When showing that $\widetilde{\mathcal{C}}_\Delta(U_1 \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U_1, E)$ is naturally isomorphic to $\widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U, E)$ for $U_1, U \in \mathcal{U}(\Omega)$ we may assume that $U \subset U_1$ and that $U_1 = (\mathbb{R}^{d+1} \setminus \mathbb{R}^d) \cup \Omega$. The canonical mapping

$$J : \widetilde{\mathcal{C}}_\Delta(U_1 \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U_1, E) \to \widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U, E)$$

defined by $[f] \mapsto [f\big|_{U \setminus \mathbb{R}^d}]$ is clearly well defined and injective.

Let $f \in \widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)$ and choose $\varphi \in \tilde{C}^\infty(U)$ such that $\varphi \equiv 1$ near Ω and $\varphi \equiv 0$ near $\partial U \setminus \mathbb{R}^d$. Then $\Delta_{d+1}(\varphi f) \in \tilde{C}^\infty(\mathbb{R}^{d+1} \setminus \partial\Omega, E)$. Since E is $(d+1)$-admissible there is $g \in \tilde{C}^\infty(\mathbb{R}^{d+1} \setminus \partial\Omega, E)$ such that

$$\Delta_{d+1} g = \Delta_{d+1}(\varphi f) \text{ on } \mathbb{R}^{d+1} \setminus \partial\Omega.$$

Then $F(x,y) := (\varphi f)(x,y) - g(x,y) \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}, E)$ and $[F\big|_{U \setminus \mathbb{R}^d}] = [f]$. Hence, J is surjective. □

By Lemma 6.7 we may define restrictions in $bv(\Omega, E)$ as follows:

Definition 6.8. Let $\Omega \supset \Omega_1$ be open and let $[f] \in bv(\Omega, E) = \widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U, E)$ where $U \in \mathcal{U}(\Omega)$. Then $U_1 := U \cap (\Omega_1 \times \mathbb{R}) \in \mathcal{U}(\Omega_1)$ and we may define

$$R_{\Omega,\Omega_1}([f]) := [f]\big|_{\Omega_1} := [f\big|_{U_1}] \in \widetilde{\mathcal{C}}_\Delta(U_1 \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U_1, E) = bv(\Omega_1, E).$$

For a sheaf $\mathcal{F}$ on $\mathbb{R}^d$ let $\mathcal{F}_0(\Omega)$ denote the sections of $\mathcal{F}$ with compact support in Ω.

Theorem 6.9. *Let E be $(d+1)$-admissible.*
a) bv is a sheaf on $\mathbb{R}^d$
b) bv is flabby
c) $bv_0(\mathbb{R}^d, E)$ is isomorphic to $\{L(\mathscr{A}(K), E) \mid K \subset \Omega$ compact$\}$.
d) bv is isomorphic to the sheaf $\mathscr{B}(E)$ of E-valued hyperfunctions.

Proof. a) We clearly have $R_{\Omega_1,\Omega_2} \circ R_{\Omega_2,\Omega_3} = R_{\Omega_1,\Omega_3}$ if $\Omega_1 \supset \Omega_2 \supset \Omega_3$ are open. $R_{\Omega,\Omega_j}([f]) = 0$ iff $f \in \widetilde{\mathcal{C}}_\Delta((U \setminus \mathbb{R}^d) \cup \Omega_j, E)$ and hence bv satisfies $(S1)$.

Let $\cup_j \Omega_j = \Omega$ and let $[f_j] \in \widetilde{\mathcal{C}}_\Delta(U_j \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U_j, E) = bv(\Omega_j, E))$ such that $[f_j]\big|_{\Omega_j \cap \Omega_k} = [f_k]\big|_{\Omega_j \cap \Omega_k}$. Then $f_j\big|_{U_j \cap U_k \setminus \mathbb{R}^d} - f_k\big|_{U_j \cap U_k \setminus \mathbb{R}^d} =: g_{jk} \in \widetilde{\mathcal{C}}_\Delta(U_j \cap U_k, E)$ and exactly as in [26, 1.4.5] there are $g_j \in \widetilde{\mathcal{C}}_\Delta(U_j, E)$ such that $g_{jk} = g_k - g_j$ on $U_j \cap U_k$ (use that E is $(d+1)$-admissible). Then $F_j := f_j + g_j$ defines a function $F \in \widetilde{\mathcal{C}}_\Delta((\cup_j U_j) \setminus \mathbb{R}^d, E)$ such that $[F]\big|_{\Omega_j} = [f_j]$ for any j. This proves $(S2)$.

b) For $[f] \in \widetilde{\mathcal{C}}_\Delta(U \setminus \mathbb{R}^d, E)/\widetilde{\mathcal{C}}_\Delta(U, E), U \in \mathcal{U}(\Omega)$, the function $F \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}, E)$ constructed in the proof of Lemma 6.7 defines an extension $[F] \in bv(\mathbb{R}^d, E)$ of $[f]$ to $\mathbb{R}^d$.

c) This follows from Theorem 5.2 since $[f] \in bv_0(\mathbb{R}^d, E)$ iff $f \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus K, E)$ for some compact $K \subset \mathbb{R}^d$.

d) We have the following series of isomorphisms for bounded open $\Omega \subset \mathbb{R}^d$

$$\mathscr{B}(\Omega, E) \simeq L(\widetilde{\mathcal{C}}_\Delta(\overline{\Omega}), E)/L(\widetilde{\mathcal{C}}_\Delta(\partial\Omega), E) \simeq$$

$$\simeq \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \overline{\Omega}, E)/\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \partial\Omega, E) \simeq \widetilde{\mathcal{C}}_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}), E)/\widetilde{\mathcal{C}}_\Delta(\Omega \times \mathbb{R}, E)$$

by Theorem 5.2, where the last mapping is defined by restriction and is surjective by the proof of Lemma 6.7. This proves d) since these isomorphisms are compatible with the respective restrictions and sheafs on $\mathbb{R}^d$ are uniquely determined by their sections on bounded open sets. □

Corollary 6.10. *The sheaf* $\{\mathscr{B}(\omega, E) \mid \omega \subset \Omega$ *open*$\}$ *is a flabby sheaf on* Ω (*with the restrictions from Definition* 6.4) *if* E *is* $(d+1)$-*admissible and if* $\Omega \subset \mathbb{R}^d$ *is bounded and open.*

Theorem 6.9 provides a complete answer to a problem stated by Ito (see [30, Problem B]) (compare the discussion in Section 4).

The following result will be needed in the homological approach to vector valued hyperfunctions discussed in the next section.

Theorem 6.11. *Let* E *be* $(2d+1)$-*admissible. The following hyperfunction* $\bar{\partial}$-*complex is an exact sequence of sheaves*:

$$0 \to \mathscr{O}(E) \to \mathscr{B}^{(0,0)}(E) \xrightarrow{\bar{\partial}} \mathscr{B}^{(0,1)}(E) \to \cdots \to \mathscr{B}^{(0,d)}(E) \to 0.$$

Proof. Notice that here $\mathscr{B}(E)$ is the sheaf of E-valued hyperfunctions on $\mathbb{C}^d = \mathbb{R}^{2d}$ existing by Theorem 6.9.

To prove the exactness at the first place we argue with Weyl's lemma for vector valued hyperfunctions: if $U \subset \mathbb{C}^d = \mathbb{R}^{2d}$ is open and $[f] \in \mathscr{B}^{(0,0)}(U, E) = \widetilde{\mathcal{C}}_\Delta(U \times \mathbb{R}_*, E)/\widetilde{\mathcal{C}}_\Delta(U \times \mathbb{R}, E)$ (by Theorem 6.9) satisfies $\bar{\partial}[f] = 0$ then $\Delta_{2d}[f] = 0$, that is, $\Delta_{2d} f = u \in \widetilde{\mathcal{C}}_\Delta(U \times \mathbb{R}, E)$. For U_1 open and bounded with $\overline{U}_1 \subset U$ define $v(x, y) := -\int_0^y \int_0^\tau u(x, t)dt\, d\tau$ and $g \in C^\infty(U_1 \times \mathbb{R}, E)$ as follows:

$$g(x, y) := v(x, y) + p_0(x) + p_1(x)y,$$

where

$$\Delta_{2d} p_0 = u(\,\cdot\,, 0) =: v_0 \quad \text{and} \quad \Delta_{2d} p_1 = \partial_y u(\,\cdot\,, 0) =: v_1 \text{ on } U_1.$$

Notice that $p_j := G * (\phi v_j), j = 0, 1$, solves these equations on U_1 if $\phi \in C_0^\infty(U)$ is 1 near $\overline{U}_1$. Then $\Delta_{2d+1} g = 0$ and $\Delta_{2d} g = u$ on $U_1 \times \mathbb{R}$. Hence $[f]\big|_{U_1} = [f\big|_{U_1 \times \mathbb{R}_*} - g]$ and we may assume that $\Delta_{2d} f = 0$ on $U_1 \times \mathbb{R}_*$. Since also $\Delta_{2d+1} f = 0$ on $U_1 \times \mathbb{R}_*$, $f(x,y) = p_0(x) + p_1(x) y$ on $U_1 \times \mathbb{R}_+$ (and on $U_1 \times \mathbb{R}_-$, respectively) with $p_j \in C^\infty(U_1, E)$. Hence, $[f] \in C^\infty(U_1, E)$ and $f \in \mathscr{O}(U_1, E)$ since $\bar{\partial}[f] = 0$.

The rest of the theorem may be proved similarly as [54, Theorem 142]. To use this proof also in the vector valued case, one needs the flabbiness of the sheaf $\mathscr{B}(E)$ (guaranteed by Theorem 6.9) and the convolution $T * H$ for $T \in L(\mathscr{A}(K), E), K \subset \mathbb{R}^d$ compact, and $H \in \mathscr{D}'(\mathbb{R}^d)$. For $H \in \mathscr{E}'(\mathbb{R}^d)$ we use the usual formula

$$\langle T * H, g \rangle := \langle T, \check{H} * g \rangle \text{ if } g \in \mathscr{A}(\mathbb{R}^d)$$

and get $T * H \in L(\mathscr{A}(K \cup J), E)$ if $\operatorname{supp}(H) = J$. This definition is extended to general $H \in \mathscr{D}'(\mathbb{R}^d)$ as explained on [54, page 62] and the convolution has the usual properties. □

§7. Hyperfunctions and Cohomology Groups

Sato [53] used the relative cohomology groups $\mathscr{H}^d_\Omega(V, \mathscr{O})$ as the definition of hyperfunctions on open sets $\Omega \subset \mathbb{R}^d$ (where $V \subset \mathbb{C}^d$ is an open neighborhood of Ω). This approach is developed here for the E-valued case, when E is an ultrabornological PLS-spaces. It is worth noting that most of the tools used here do not work in case E does not have (PA) (see Cor. 3.8) or if E is a complete locally convex space containing a complemented copy of φ (see Cor. 3.10).

The basic tool for this section is the vector-valued Dolbeault-Grothendieck resolution which is proved as in the scalar case (see [26, Theorem 2.3.3] and also [29, Th. 2.1.2]):

Theorem 7.1. *For any complete locally convex space E the following sequence of sheaves (the Dolbeault-Grothendieck resolution) is exact in $\mathbb{C}^d$ (i.e., it is a soft resolution of $\mathscr{O}(E)$):*

$$0 \to \mathscr{O}(E) \to \mathscr{E}^{(0,0)}(E) \xrightarrow{\bar{\partial}} \mathscr{E}^{(0,1)}(E) \xrightarrow{\bar{\partial}} \cdots \to \mathscr{E}^{(0,d)}(E) \to 0.$$

Sato's idea and the connection to the harmonic boundary values from Theorem 6.9 is especially transparent for $d = 1$. So we consider this (very special) case first.

Proposition 7.2. *Let E be 2-admissible. For $\Omega \subset \mathbb{R}$ open let $V \subset \mathbb{C}$ be a complex neighborhood of Ω containing Ω as a closed set. Then*

$$\tilde{\mathscr{B}}(\Omega, E) := \mathscr{H}^1_\Omega(V, \mathscr{O}(E)) \simeq H(V \setminus \Omega, E)/H(V, E) \tag{7.1}$$

and this defines a (flabby) sheaf on $\mathbb{R}$ which is isomorphic to $\mathscr{B}(E)$.

Proof. By [32, Th. 1.1] we have an exact sequence

$$\begin{aligned} 0 \to \mathscr{H}^0_\Omega(V, \mathscr{O}(E)) \to \mathscr{H}^0(V, \mathscr{O}(E)) \to \mathscr{H}^0(V \setminus \Omega, \mathscr{O}(E)) \\ \to \mathscr{H}^1_\Omega(V, \mathscr{O}(E)) \to \mathscr{H}^1(V, \mathscr{O}(E)) \to \cdots \end{aligned}$$

We clearly have $\mathscr{H}^0_\Omega(V, \mathscr{O}(E)) = 0$. The groups $\mathscr{H}^p(U, \mathscr{O}(E))$ may be calculated for $p = 0, 1$ and open $U \subset \mathbb{C}$ using the E-valued Dolbeault complex which is a soft resolution of $\mathscr{O}(E)$ of length 1. Specifically, $\mathscr{H}^1(V, \mathscr{O}(E)) = 0$ since it is isomorphic to the first cohomology group of the complex

$$0 \to H(V, E) \to \mathscr{E}(V, E) \xrightarrow{\bar\partial} \mathscr{E}(V, E) \to 0$$

which is exact since E is 2-admissible. We thus have the exact sequence

$$0 \to H(V, \mathscr{O}(E)) \to H(V \setminus \Omega, \mathscr{O}(E)) \to \mathscr{H}^1_\Omega(V, \mathscr{O}(E)) \to 0$$

showing the isomorphism in (7.1). The sheaf properties may be proved as in Theorem 6.9. It can be proved analogously as in Theorem 5.2 that $H(\mathbb{C} \setminus K, E)/H(\mathbb{C}, E) \simeq L(H(K), E)$. By Lemma 5.1, $L(H(K), E) \simeq L(\tilde{C}_\Delta(K), E)$ and the latter space is isomorphic to $\tilde{C}_\Delta(\mathbb{R}^2 \setminus K, E)/\tilde{C}_\Delta(\mathbb{R}^2, E)$, by Theorem 5.2. Thus the required isomorphism follows. □

Now, the proof of the Malgrange vanishing theorem [40, Lemme 3] may be transferred to the vector valued situation as follows:

Theorem 7.3. *Let E be a complete locally convex $2d$-admissible space. Then for any open set $U \subset \mathbb{C}^d$ we have $\mathscr{H}^p(U, \mathscr{O}(E)) = 0$ for $p \geq d$.*

Proof. The E-valued Dolbeault complex is a soft resolution of $\mathscr{O}(E)$ of length d. Thus $\mathscr{H}^p(U, \mathscr{O}(E))$ can be calculated using this complex and $\mathscr{H}^p(U, \mathscr{O}(E)) = 0$ for $p > d$ [11, Th. II.9.8, Th. II.4.1]. The vanishing for $p = d$ means that

$$\bar\partial : \mathscr{E}^{(0,d-1)}(U, E) \to \mathscr{E}^{(0,d)}(U, E)$$

has to be surjective. Clearly $\mathscr{E}^{(0,d-1)}(U,E)$ is just $(C^\infty(U,E))^{d-1}$ and $\mathscr{E}^{(0,d)}(U, E) \simeq C^\infty(U,E)$, where

$$\bar\partial((f_k)_{k=1}^d) = \sum_{k=1}^d (-1)^{k+1}\bar\partial_k f_k,$$

$\bar\partial_k$ is the Cauchy-Riemann operator with respect to k-th variable, i.e., $\bar\partial_k f := \frac{\partial f}{\partial \bar z_k}$. By the assumption

$$\Delta_{2d} : C^\infty(U,E) \to C^\infty(U,E)$$

is surjective. Thus for any $g \in C^\infty(U,E)$ there is $F \in C^\infty(U,E)$ such that $\Delta_{2d}F = 4g$. We define $f_k := (-1)^{k+1}\partial_k F = \frac{\partial F}{\partial z_k}$. Now, clearly

$$\bar\partial((f_k)_{k=1}^d) = \sum_{k=1}^d \bar\partial_k\partial_k F = \frac{1}{4}\Delta_{2d}F = g.$$

□

The next result means that $\mathbb{R}^d$ is "vector-valued" purely d-codimensional.

Theorem 7.4. *Let E is a PLS-space with the property $(PA)$and let $d \geq 2$. Let Ω be an open set in $\mathbb{R}^d$ and let $V \subset \mathbb{C}^d$ be a complex neighbourhood of Ω containing Ω as a closed set. Then $\mathscr{H}^p_\Omega(V,\mathscr{O}(E)) = 0$ for $p \neq d$ and $\mathscr{H}^d_\Omega(V,\mathscr{O}(E)) = \mathscr{H}^{d-1}(V\setminus\Omega,\mathscr{O}(E))$.*

Proof. The proof is similar to the proof of [28, Theorem 2.4]. Clearly $\mathscr{H}^0_\Omega(V,\mathscr{O}(E)) = 0$. Let $p \geq 1$. By [32, Th. 1.1] we have an exact sequence

$$\cdots \to \mathscr{H}^{p-1}(V\setminus\Omega,\mathscr{O}(E)) \to \mathscr{H}^p_\Omega(V,\mathscr{O}(E)) \to \mathscr{H}^p(V,\mathscr{O}(E)) \to \dots.$$

By the excision theorem [32, Th. 1.1] it suffices to take any open neighborhood V. We thus can assume that V is a pseudoconvex neighborhood by [22]. By Theorem 7.1 and Corollary 3.8 (a), $\mathscr{H}^p(V,\mathscr{O}(E)) = 0$ for $p \geq 1$. By Theorem 7.1 and Corollary 3.8 (b), $\mathscr{H}^{p-1}(V\setminus\Omega,\mathscr{O}(E)) = 0$ for $d-1 \geq p \geq 2$. Therefore $\mathscr{H}^p_\Omega(V,\mathscr{O}(E)) = 0$ for $d-1 \geq p \geq 2$. Since $\mathscr{H}^{p-1}(V\setminus\Omega,\mathscr{O}(E)) = 0$ for $p \geq d+1$ by Theorem 7.3 (which can be applied since E is $(2d)$-admissible by Corollary 4.1) and Corollary 3.8 (a), $\mathscr{H}^p_\Omega(V,\mathscr{O}(E)) = 0$ for $p \geq d+1$.

For $p = d$ we complete the above exact sequence by one term on the left side and get

$$\begin{aligned}\cdots \to \mathscr{H}^{d-1}(V,\mathscr{O}(E)) \to \mathscr{H}^{d-1}(V\setminus\Omega,\mathscr{O}(E)) \to \mathscr{H}^d_\Omega(V,\mathscr{O}(E))\\ \to \mathscr{H}^d(V,\mathscr{O}(E)) \to \dots\end{aligned}$$

which shows that $\mathscr{H}^d_\Omega(V,\mathscr{O}(E)) = \mathscr{H}^{d-1}(V\setminus\Omega,\mathscr{O}(E))$ since also $\mathscr{H}^{d-1}(V,\mathscr{O}(E)) = 0$ by Corollary 3.8 (a) since $d\geq 2$.

We are thus left with the calculation of $\mathscr{H}^1_\Omega(V,\mathscr{O}(E))$. For this we consider the beginning of the above exact sequence

$$\begin{aligned} 0 = \mathscr{H}^0_\Omega(V,\mathscr{O}(E)) &\to H(V,E)\to H(V\setminus\Omega,E) \\ &\to \mathscr{H}^1_\Omega(V,\mathscr{O}(E))\to \mathscr{H}^1(V,\mathscr{O}(E)) = 0. \end{aligned}$$

Since $\mathscr{H}^1_\Omega(V,\mathscr{O}) = 0$ for $d\geq 2$, the restriction $H(V)\to H(V\setminus\Omega)$ is onto, hence a topological isomorphism. Therefore, the restriction $H(V,E)\to H(V\setminus\Omega,E)$ is also onto and thus $\mathscr{H}^1_\Omega(V,\mathscr{O}(E)) = 0$. This completes the proof of the theorem. □

As in [32, Th. 2.9] it can be proved that $\tilde{\mathscr{B}}(\Omega,E)$ defined as the relative cohomology groups

$$\mathscr{H}^d_\Omega(V,\mathscr{O}(E)) = \mathscr{H}^{d-1}(V\setminus\mathbb{R}^d,\mathscr{O}(E)) \text{ for } d\geq 2$$

for $V\cap\mathbb{R}^d = \Omega$ forms a flabby sheaf. The space $\tilde{\mathscr{B}}(\Omega,E)$ does not depend on the open complex neighborhood V of Ω and V can be taken to be a Stein manifold. By Corollary 3.8 (a) we can calculate $\mathscr{H}^{d-1}(V\setminus\mathbb{R}^d,\mathscr{O}(E))$ using a covering of $V\setminus\mathbb{R}^d$ consisting of the following pseudoconvex sets:

$$V_j := \{z\in V : \operatorname{Im} z_j \neq 0\}.$$

Thus we will get (as in the proof of [32, Th. 2.12] or [28, Sec. 3]):

Corollary 7.5. *If E is a PLS-space with the property (PA), then*

$$\tilde{\mathscr{B}}(\Omega,E) = H\left(\bigcap_{j=1}^d V_j, E\right) \Big/ \sum_{k=1}^d H\left(\bigcap_{j=1,j\neq k}^d V_j, E\right).$$

For $d=1$ this was proved already in Proposition 7.2.

Theorem 7.6. *If E is an ultrabornological PLS-space with property (PA) then*

$$\mathscr{H}^d_\Omega(\mathbb{C}^d,\mathscr{O}(E)) \simeq L(\mathscr{A}(\bar{\Omega}),E)/L(\mathscr{A}(\partial\Omega),E)$$

for every bounded open set $\Omega\subseteq\mathbb{R}^d$. Thus the sheaves $\tilde{\mathscr{B}}(E)$ and $\mathscr{B}(E)$ are isomorphic.

Proof. The sheaves $\mathscr{B}^{(0,p)}(E)$ of differential forms of type $(0,p)$ with coefficients in the sheaf of E-valued hyperfunctions in $(2d)$ real variables (existing by Section 6 and Corollary 4.1) are flabby, thus by Theorem 6.11 and [54, Cor. to Thm. B 32] for any compact set $K \subseteq \mathbb{R}^d$ the groups $\mathscr{H}^p_K(\mathbb{C}^d, \mathscr{O}(E))$ are the cohomology groups of the complex:

$$\begin{aligned} 0 \to 0 = \Gamma_K(\mathbb{C}^d, \mathscr{O}(E)) \to \mathscr{B}^{(0,0)}_K(\mathbb{C}^d, E) \xrightarrow{\bar\partial} \mathscr{B}^{(0,1)}_K(\mathbb{C}^d, E) \\ \to \cdots \to \mathscr{B}^{(0,d)}_K(\mathbb{C}^d, E) \to 0. \end{aligned}$$

Now, observe, that

$$\mathscr{B}^{(0,p)}_K(\mathbb{C}^d, E) = L(\mathscr{A}^{(0,d-p)}(K), E),$$

here $\mathscr{A}^{(0,d-p)}(K)$ denotes the $(0, d-p)$-type differential forms with coefficients being germs in $\mathbb{R}^{2d} = \mathbb{C}^d$ of analytic functions in $(2d)$ real variables over a compact set $K \subseteq \mathbb{R}^d \subseteq \mathbb{C}^d$.

As it is proved in [54, proof of Th. 411], the following is an exact sequence of DFN-spaces:

$$0 \to H(K) \to \mathscr{A}^{(0,0)}(K) \xrightarrow{\bar\partial_0} \mathscr{A}^{(0,1)}(K) \xrightarrow{\bar\partial_1} \cdots \xrightarrow{\bar\partial_{d-1}} \mathscr{A}^{(0,d)}(K) \to 0 \tag{7.2}$$

where $H(K)$ is the space of germs of holomorphic functions (in d complex variables) over $K \subseteq \mathbb{R}^d \subseteq \mathbb{C}^d$ and $\mathscr{A}^{(0,p)}(K)$ is a product of spaces of germs of holomorphic functions (in $(2d)$ variables) over $K \subseteq \mathbb{R}^{2d} \subseteq \mathbb{C}^{2d}$. Thus im $\bar\partial_p$ is a closed subspace of $\mathscr{A}^{(0,p+1)}(K)$ so its dual is a quotient of $\mathscr{A}^{(0,p+1)}(K)'$. By Lemma 5.1 and Theorem 5.2, $\mathscr{A}(K)'$ is a quotient of $\widetilde{C}_\Delta(\mathbb{R}^{2d+1} \setminus K)$ and hence a quotient of $C_\Delta(\mathbb{R}^{2d+1} \setminus K)$, the latter space has (Ω) by [61, Prop. 3.4]. Thus $\mathscr{A}(K)'$ has (Ω) and $(\mathrm{im}\ \bar\partial_p)', p \geq 0$, has (Ω) as a quotient of a space with (Ω). By [7, Th. 4.1], every operator $T : \ker \bar\partial_{p+1} \to E$ extends to $\tilde T : \mathscr{A}^{(0,p)}(K) \to E$, which means that the following sequence is exact:

$$\begin{aligned} 0 \to L(\mathscr{A}^{(0,d)}(K), E) \xrightarrow{{}^t\bar\partial_{d-1}} L(\mathscr{A}^{(0,d-1)}(K), E) \to \cdots \\ \xrightarrow{{}^t\bar\partial_0} L(\mathscr{A}^{(0,0)}(K), E) \to L(H(K), E). \end{aligned} \tag{7.3}$$

Now, notice that $H(K)$ is a complemented subspace of $\mathscr{A}(K) = \mathscr{A}^{(0,0)}(K)$. A continuous linear projection Π onto $H(K)$ may be defined by $\Pi(f) := F$ where F is the solution of the Cauchy-problem

$$(\partial/\partial x_j + i\partial/\partial y_j)F = 0, j \leq d, \text{ on a neighborhood } V \subset \mathbb{R}^{2d} \text{ of } K$$

$$\text{and } F(x,0) = f(x,0) \text{ near } K \subset \mathbb{R}^d$$

via power series expansion. Thus the sequence (7.3) can be prolonged to

$$0 \to L(\mathscr{A}^{(0,d)}(K), E) \overset{{}^t\bar{\partial}_{d-1}}{\to} L(\mathscr{A}^{(0,d-1)}(K), E) \to \cdots \\ \overset{{}^t\bar{\partial}_0}{\to} L(\mathscr{A}^{(0,0)}(K), E) \to L(H(K), E) \to 0$$

which is exact. Hence

$$\begin{aligned} \mathscr{H}^d_K(\mathbb{C}^d, \mathscr{O}(E)) &\simeq \mathscr{B}^{(0,d)}_K(\mathbb{C}^d, E)/\bar{\partial}\mathscr{B}^{(0,d-1)}_K(\mathbb{C}^d, E) \\ &\simeq L(\mathscr{A}^{(0,0)}(K), E)/{}^t\bar{\partial}_0 L(\mathscr{A}^{(0,1)}(K), E) \simeq L(H(K), E) \end{aligned} \tag{7.4}$$

since $\bar{\partial}$ from (7.2) equals ${}^t\bar{\partial}_0$.

By the long exact sequence for relative cohomology (see [32, Theorem 1.1 (iii)]) we have the exact sequence

$$\cdots \to \mathscr{H}^d_{\bar{\Omega}}(\mathbb{C}^n, \mathscr{O}(E)) \to \mathscr{H}^d_{\Omega}(\mathbb{C}^n \setminus \partial\Omega, \mathscr{O}(E)) \to H^{d+1}_{\partial\Omega}(\mathbb{C}^n, \mathscr{O}(E)) \to \cdots$$

if $\Omega \subset \mathbb{R}^d$ is open and bounded. By the flabby resolution from Theorem 6.11 we see that $\mathscr{H}^{d+1}_{\partial\Omega}(\mathbb{C}^n, \mathscr{O}(E)) = 0$. Therefore, the restriction $\tilde{\mathscr{B}}_0(\mathbb{C}^d, E) \to \tilde{\mathscr{B}}(\Omega, E)$ is surjective and $\tilde{\mathscr{B}}(E)$ forms a flabby sheaf.

Since the sheaf of cohomology groups is flabby, there is an isomorphism

$$\mathscr{H}^d_{\Omega}(\mathbb{C}^d, \mathscr{O}(E)) \simeq \mathscr{H}^d_{\bar{\Omega}}(\mathbb{C}^d, \mathscr{O}(E))/\mathscr{H}^d_{\partial\Omega}(\mathbb{C}^d, \mathscr{O}(E)).$$

This completes the proof by (7.4). $\square$

§8. Necessity

We will discuss the necessity of the conditions which were used in this paper to construct vector valued hyperfunctions.

The following lemma is a basic tool in our considerations. Its proof uses the main idea from [36, 3.7].

Lemma 8.1. *Let $\Omega, \Omega_1 \subset \mathbb{R}^d$ be open and bounded and let $\overline{\Omega} \subset \Omega_1$. Let $u \in C^\infty((\Omega_1 \times]0,\infty[) \cup \Omega, E)$ be harmonic on $\Omega_1 \times]0,\infty[$. Then there is $g \in C^\infty(\Omega, E)$ such that $\Delta_d g = u(\,\cdot\,, 0)$ on Ω.*

Proof. For $(x, y) \in \Omega \times [0, \infty[$ let

$$v(x, y) := -\int_1^y \int_1^\tau u(x, t)\,dt\,d\tau.$$

Then $v \in C^\infty(\Omega \times [0,\infty[, E)$. Let $g(x,y) := v(x,y) - p_0(x) - p_1(x)y$ where

$$\text{(8.1)}\quad \Delta_d p_0 = -u(\,\cdot\,,1) + \partial_y u(\,\cdot\,,1) =: v_0 \text{ and } \Delta_d p_1 = -\partial_y u(\,\cdot\,,1) =: v_1 \text{ on } \Omega.$$

Since $v_0, v_1 \in C^\infty(\Omega_1, E)$ by assumption, (8.1) may be solved on Ω by means of the convolution $p_j := G * (\varphi v_j)\big|_\Omega \in C^\infty(\Omega, E)$, where $\varphi \in C_0^\infty(\Omega_1)$ is 1 near $\overline{\Omega}$. An easy calculation shows that

(8.2)
$$\Delta_d v(x,y) = -\int_1^y \int_1^\tau \Delta_d u(x,t)dt\; d\tau$$

$$\text{(8.3)}\qquad = \int_1^y \int_1^\tau \partial_t^2 u(x,t)dt\; d\tau = u(x,y) + v_0(x) + v_1(x)y \;\text{ if } (x,y) \in \Omega \times]0,\infty[$$

since u is harmonic on $\Omega \times]0,\infty[$. This implies that $\Delta_d g(x,y) = u(x,y)$ if $(x,y) \in \Omega \times]0,\infty[$, hence $\Delta_d g(x,0) = u(x,0)$ since $g, u \in C^\infty(\Omega \times [0,\infty[)$. □

For any complete locally convex space E we always have the following canonical representation of $f \in C_0^\infty(\mathbb{R}^d, E)$ as boundary value of a harmonic function:

Lemma 8.2. *For $f \in C_0^\infty(\mathbb{R}^d, E)$ let $S(f) := G*(f \otimes \delta_y)$ where δ_y is the Dirac measure at zero with respect to the y-variable. Then $S(f) \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \operatorname{supp}(f), E)$ and $S(f)\big|_{\mathbb{R}^d \times \pm]0,\infty[}$ can be (uniquely) extended to $S(f)_\pm \in C^\infty(\mathbb{R}^d \times \pm [0,\infty[, E)$ and $\partial_y S(f)_+(x,0) - \partial_y S(f)_-(x,0) = f(x)$ on $\mathbb{R}^d$.*

Proof. We use similar arguments as in [37, 1.2] where the corresponding result was proved in the scalar case.

Clearly, $S(f) \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1} \setminus \operatorname{supp}(f), E)$ and

$$\partial_y S(f)(x,y) =: v_f(x,y) = \frac{\operatorname{sign}(y)}{c_{d+1}} \int_{\mathbb{R}^d} f(x - y\xi)(1 + |\xi|^2)^{-(d+1)/2} d\xi \text{ if } y \neq 0$$

(see (2.1) and use [27, 3.3.2]). The function v_f is odd and defined also for $y = \pm 0$ since $(1 + |\xi|^2)^{-(d+1)/2} \in L_1(\mathbb{R}^d)$. Hence we can restrict our considerations to $y \geq 0$.

Let p be a continuous seminorm on E. Then

$$\sup_x p(v_f(x,y) - f(x)/2) \leq \frac{1}{c_{d+1}} \int \sup_x p\big(f(x - y\xi) - f(x)\big)\big)(1 + |\xi|^2)^{-(d+1)/2} d\xi \to 0$$

if $y \downarrow 0$ by Lebesgue's dominated convergence theorem. Hence v_f extends to a continuous function on $\mathbb{R}^d \times \pm[0,\infty[$ and $v_f(x,+0) = f(x)/2$ for any $f \in C_0^\infty(\mathbb{R}^d, E)$.

Since

$$\partial_y^2 v_f(x,y) = -\Delta_d v_f(x,y) = -v_{\Delta_d f}(x,y)$$

we get

$$\partial_y v_f(x,\eta) - \partial_y v_f(x,y) = -\int_\eta^y \partial_y^2 v_f(x,t)dt = \int_\eta^y v_{\Delta_d f}(x,t)dt. \tag{8.4}$$

Hence $\partial_y v_f(x,y)$ extends continuously to $\mathbb{R}^d \times [0,\infty[$ for any $f \in C_0^\infty(\mathbb{R}^d, E)$. Observe that

$$\partial_y^{2b+j}\partial_x^a v_f(x,y) = (-1)^b \partial_y^j v_{\Delta_d^b \partial_x^a f}(x,y), j = 0,1, \text{ if } y > 0$$

(see [37, (1.9)]). This shows that $v_f \in C^\infty(\mathbb{R}^d \times [0,\infty[, E)$. Since $S(f)(x,y) - S(f)(x,\eta) = \int_\eta^y v_f(x,t)dt$ it follows as above that $S(f) \in C^\infty(\mathbb{R}^d \times [0,\infty[)$. $\square$

The following theorem is the main result of this section.

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open. Assume that there is a flabby sheaf $\mathcal{F}$ on Ω such that*

(8.5)
$$\mathcal{F}_0(K) := \{T \in \mathcal{F}(\Omega) \,|\, \mathrm{supp}_{\mathcal{F}}(T) \subset K\} = L(\mathscr{A}(K), E) \textit{ for any compact } K \subset \Omega.$$

Then

$$\Delta_d : C^\infty(\omega, E) \to C^\infty(\omega, E) \textit{ is surjective} \tag{8.6}$$

if ω is open and $\overline{\omega} \subset \Omega$.

Proof. Let ω be open with $\overline{\omega} \subset \Omega$ and let $f \in C^\infty(\omega, E)$.

a) First we represent f as a restriction of some $\overline{u} \in \mathcal{F}_0(\Omega)$. Let $\omega = \cup_j \omega_j$ where $\overline{\omega}_j \subset \omega$. Choose $\varphi_j \in C_0^\infty(\omega)$ such that $\varphi_j = 1$ near $\overline{\omega}_j$. Then $\varphi_j f \in C_0^\infty(\omega, E)$ and hence $T_j := H(S(\varphi_j f)) \in L(\mathscr{A}(\mathrm{supp}(\varphi_j)), E) \subset \mathcal{F}_0(\Omega)$ by Lemma 8.2, Theorem 5.2 and (8.5). Thus $u_j := R_{\Omega,\omega_j}(T_j) \in \mathcal{F}(\omega_j)$ is defined. By the same references we have

$$R_{\omega_k,\omega_j\cap\omega_k} u_k - R_{\omega_j,\omega_j\cap\omega_k} u_j = R_{\Omega,\omega_j\cap\omega_k}(H(S((\varphi_k - \varphi_j)f))) = 0$$

since $(\varphi_k f - \varphi_j f)\big|_{\omega_j\cap\omega_k} = 0$. Since $\mathcal{F}$ is a sheaf on Ω there is $u \in \mathcal{F}(\omega)$ by $(S2)$ such that $R_{\omega,\omega_j} u = u_j$ for any j. Since $\mathcal{F}$ is flabby, there is $\overline{u} \in \mathcal{F}_0(\Omega)$ with $\mathrm{supp}_{\mathcal{F}}(u) \subset \overline{\omega}$ such that $R_{\Omega,\omega}\overline{u} = u$.

b) We then show that a representation v of $\overline{u}$ in $\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}\setminus\overline{\omega},E)$ may be extended to $v_\pm \in C^\infty(\omega\times\pm[0,\infty[,E)$: By (8.5) we know that $\overline{u}\in L(\mathscr{A}(\overline{\omega}),E)$ and hence $H^{-1}(\overline{u}) =: [v] \in \widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}\setminus\overline{\omega},E)/\,\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1},E)$ by Theorem 5.2. Let $S(\varphi_k f) =: [v_k]$. Since $R_{\Omega,\omega_k}(\overline{u}-T_k)=0$ we conclude that $(v-v_k)\in\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}\setminus(\overline{\omega}\setminus\omega_k),E)$. However, v_k can be extended to $v_{k,\pm}\in C^\infty(\mathbb{R}^d\times\pm[0,\infty[,E)$ by Lemma 8.2 such that $(\partial_y v_k(\,\cdot\,,+0)-\partial_y v_k(\,\cdot\,,-0)) = \varphi_k f$. Hence v can be extended to $v_\pm\in C^\infty(\omega\times\pm[0,\infty[,E)$ such that $\partial_y v(\,\cdot\,,+0)-\partial_y v(\,\cdot\,,-0)) = f$ on ω.

Then

$$h(x,y) := \partial_y v(x,y) - \partial_y v(x,-y) \in C^\infty(\omega\times[0,\infty[,E)$$

and Lemma 8.1 implies that there is $g\in C^\infty(\omega,E)$ such that $\Delta_d g = h(x,0) = f$. The theorem is proved. ☐

If we restrict our consideration directly to the models for vector valued hyperfunctions from Section 6, we do not need the flabbiness of $\mathcal{F}$ to obtain the conclusion of Theorem 8.3:

Theorem 8.4. *Let $\emptyset\neq\Omega\subseteq\mathbb{R}^d$ be a bounded open set. If either condition* (a) *or* (b) *below holds then*

$$\Delta_d : C^\infty(\omega,E)\to C^\infty(\omega,E)$$

is surjective for any open ω with $\bar{\omega}\subseteq\Omega$.

(a) *The canonical mapping*

$$I : L(A(\overline{\omega}),E)/L(A(\partial\omega),E)\to L(A(\overline{\Omega}),E)/L(A(\overline{\Omega}\setminus\omega),E)$$

is a bijection for any open $\omega\subset\Omega$ and the spaces $\{\mathcal{B}(\omega,E)\mid\omega\subset\Omega$ open$\}$ define a sheaf on Ω (with the restrictions from Definition 6.4)

(b) *For any open $\omega\subset\Omega$ the quotients $\widetilde{\mathcal{C}}_\Delta(U\setminus\mathbb{R}^d,E)/\widetilde{\mathcal{C}}_\Delta(U,E)$ are independent of $U\in\mathcal{U}(\omega)$ and the spaces $\{bv(\omega,E)\mid\omega\subset\Omega$ open$\}$ define a sheaf on Ω (with the restrictions from Definition* 6.8)

Proof. The assumption (8.5) of Theorem 8.3 is clearly satisfied in both cases (use also Theorem 5.2 in case (b)). By the first part of the proof of Theorem 8.3 we get: for any open $\omega\subset\Omega$ with $\bar{\omega}\subseteq\Omega$ and any $f\in C^\infty(\omega,E)$ there are $[w_f]\in bv(\omega,E)=\widetilde{\mathcal{C}}_\Delta(\Omega\times\mathbb{R}_*,E)/\widetilde{\mathcal{C}}_\Delta((\Omega\times\mathbb{R}_*)\cup\omega,E)$ (in case (b) this equation holds by assumption since $(\Omega\times\mathbb{R}_*)\cup\omega\in\mathcal{U}(\omega)$) and $[v_f]\in\mathcal{B}(\omega,E)$ (in case (a)) representing f. Notice that $v_f\in\widetilde{\mathcal{C}}_\Delta(\mathbb{R}^{d+1}\setminus\overline{\omega},E)$ by Theorem 5.2.

By the second part of the proof of Theorem 8.3, v_f and w_f can be extended from $\mathbb{R}^d \times \pm]0,\infty[$ as C^∞- functions to $\omega \times [0,\infty[$ and the claim follows by that proof. $\square$

Using Corollary 4.1 we get:

Corollary 8.5. *If the assumptions of Theorem* 8.3 *or Theorem* 8.4 (a) *or* (b) *are satisfied for any bounded set* $\Omega \subseteq \mathbb{R}^d$ *for some* $d \geq 2$ *then* E *is a weakly d-admissible space. In particular, if* E *is an ultrabornological PLS-space, then* E *has* (PA).

In the case of one variable (i.e. $d = 1$) Theorem 8.4 only gives the fact that the operator ∂_x^2 is surjective on $C^\infty(\omega, E)$ if ω is ∂_x^2-convex, which is clearly true for any E. Hence we have to improve the argument for this case and we will consider differential operators of infinite order defined as follows:

$$P(z) := \prod_{j\in\mathbb{N}} (1 - iz/j^2) \text{ for } z \in \mathbb{C}.$$

Then P has the expansion

$$P(z) = \sum_{k\in\mathbb{N}_0} c_k z^k \text{ where } |c_k| \leq C^{k+1}(k!)^2 \tag{8.7}$$

and

$$|P(z)| \leq C_1 e^{C_1|z|^{1/2}} \tag{8.8}$$

for some C, C_1 by [34, Prop. 4.6].

Let $\omega \subseteq \mathbb{R}$ be an open set. Let us define the Gevrey class connected with the weight β, $\beta(t) = t^{1/2}$:

$$\gamma^2(\omega, E) := \{f \in C^\infty(\omega, E) \mid \sup_K p(f^{(a)}(x))/(Aa^2)^a < \infty$$

$$\text{for any compact } K \subset \omega, \text{ any continuous seminorm } p \text{ on } E \text{ and any } A > 0\}.$$

In fact, $\gamma^2(\omega, E) = \mathscr{E}_{(\beta)}(\omega, E)$ using the definition from [10]. Let

$$\gamma_0^2(\omega) := \{f \in \gamma^2(\omega, \mathbb{C}) \mid \operatorname{supp}(f) \subset\subset \Omega\}$$

endowed with the natural inductive limit topology.

Theorem 8.6. *The operator* $P(D) : \gamma_0^2(\omega)' \to \gamma_0^2(\omega)'$ *is a well-defined hypoelliptic operator which is surjective for convex* ω. *Moreover* $\ker P(D) \simeq \Lambda_\infty(\alpha)$ *for* $\alpha_j = j^2$.

Proof. $P(D)$ is hypoelliptic on $\gamma_0^2(\omega)'$ by [8, Theorem 2.1] since the slowly decreasing condition [8, (2.1)] follows from (8.8) by application of a standard minimum modulus theorem (see e.g. [39, Lemma 1.11]). This means that any $T \in \gamma_0^2(\omega)'$ with $P(D)T = 0$ satisfies $T \in \gamma^2(\omega)$. The slowly decreasing condition implies surjectivity of $P(D)$ on $\gamma_0^2(\omega)'$ for convex ω by [9, 2.9, 3.4]. By [46, Th. 3.2], we get the representation of $\ker P(D)$. $\square$

From the above result and the fact that

$$\gamma^2(\omega, E) = \{f \in C^\infty(\omega, E) : \ \forall\, u \in E', u \circ f \in \gamma^2(\omega)\}$$

it follows that

$$f \in \gamma^2(\omega, E) \qquad \text{if } f \in C^\infty(\omega, E) \text{ and } P(D)f = 0. \tag{8.9}$$

Notice that the operator $J(D) := P(-iD)$ comes from the entire function $J(z) = P(-iz)$ and that $J(z) = 0$ iff $z = j^2$ for some $j \in \mathbb{N}$. Hence $J(D)$ is hyperbolic in $\gamma^2(\mathbb{R})'$ by [1] (with respect to $x > 0$ and $x < 0$), especially there is an elementary solution $F \in \gamma^2(\mathbb{R})'$ with $\operatorname{supp}(F) \subset]-\infty, 0]$.

Lemma 8.7. *Let $\Omega, \Omega_1 \subset \mathbb{R}$ be open and bounded and let $\overline{\Omega} \subset \Omega_1$. Let $u \in \gamma^2((\Omega_1 \times]0,\infty[) \cup \Omega, E)$ be holomorphic on $\Omega_1 \times]0,\infty[$. Then there is $g \in \gamma^2(\Omega, E)$ such that $P(D_x)g = u(\,\cdot\,, 0)$ on Ω where $P(D_x)$ is the operator defined above.*

Proof. Let $F \in \gamma_0^2(\mathbb{R})'$ be the elementary solution for $J(D)$ above, i.e., $\operatorname{supp} F \subset]-\infty, 0]$. Choose $\varphi \in \gamma^2(\mathbb{R})$ such that $\varphi(x) = 1$ if $x \le 1$ and $\varphi(x) = 0$ if $x \ge 2$. For $(\xi, \eta) \in \Omega_1 \times]0, \infty[$ let

$$v(\xi,\eta) := (\delta_x \otimes F_y) * (\varphi(y)u(x,y))(\xi,\eta) = \langle F_y, \varphi(\eta - y)u(\xi, \eta - y)\rangle_y. \tag{8.10}$$

The convolution is defined and has the usual properties since $g_\eta(y) := \varphi(\eta - y)u(\xi, \eta - y) = 0$ if $\eta - 2 \ge y$ and since $g_\eta \in \gamma^2$ near $]-\infty, 0]$ if $\eta > 0$. Since u is holomorphic on $\Omega_1 \times]0,\infty[$ and hence $P(D_x)u = P(-iD_y)u = J(D_y)u$ on $\Omega_1 \times]0,\infty[$, we get

$$\begin{aligned} P(D_\xi)v(\xi,\eta) &= (\delta_x \otimes F_y) * (\varphi(y)P(D_x)u(x,y))(\xi,\eta) \\ &= (\delta_x \otimes F_y) * (\varphi(y)J(D_y)u(x,y))(\xi,\eta) \end{aligned}$$

Hence

$$P(D_\xi)v(\xi,\eta) = (\delta_x \otimes F_y) * J(D_y)(\varphi(y)u(x,y))(\xi,\eta) + (\delta_x \otimes F_y) * w(\xi,\eta) \tag{8.11}$$

where $w(x,y) := \varphi(y)J(D_y)u(x,y) - J(D_y)(\varphi(y)u(x,y))$.

The first term of (8.11) gives if $0 < \eta < 1$

(8.12)
$$\begin{aligned}(\delta_x \otimes F_y) * J(D_y)(\varphi(y)u(x,y))(\xi,\eta) &= (\delta_x \otimes J(D_y)F_y) * (\varphi(y)u(x,y))(\xi,\eta)\\ &= \varphi(\eta)u(\xi,\eta) = u(\xi,\eta).\end{aligned}$$

Now, we modify the second term in (8.11). We notice that $w(x,y) = 0$ if $y > 2$ or $y < 1$. Let $\psi = \psi(x) \in \gamma_0^2(\Omega_1)$ such that $\psi = 1$ near $\overline{\Omega}$. Then $w\psi \in \gamma_0^2(\Omega_1 \times [1,2], E)$ and

$$h := (\delta_x \otimes F_y) * (w\psi) \in \gamma^2(\Omega_1 \times \mathbb{R}, E)$$

with $\operatorname{supp}(h) \subset \operatorname{supp}(\psi) \times \mathbb{R}$. Let $K \in \gamma_0^2(\mathbb{R})'$ be an elementary solution for $P(D)$ which exists by Theorem 8.6. We define

$$H := (K_x \otimes \delta_y) * h \in \gamma^2(\mathbb{R}^2, E).$$

By (8.11) and (8.12) we get

$$\begin{aligned}P(D_x)(v(x,y) - H(x,y)) &= u(x,y) + (\delta_x \otimes F_y) * w(x,y)\\ &\quad - (P(D_x)K_x \otimes \delta_y) * h(x,y)\\ &= u(x,y) + (\delta_x \otimes F_y) * w(x,y)\\ &\quad - (\delta_x \otimes F_y) * (w\psi)(x,y)\\ &= u(x,y) \qquad \text{if } (x,y) \in \Omega\times]0,1[.\end{aligned} \tag{8.13}$$

Since $u \in \gamma^2(\Omega \times [0,\infty[, E)$, this function may be extended to $U \in \gamma^2(\Omega\times]-1,\infty[, E)$. Indeed, by [47, Theorem 3.1] there exists a continuous linear extension operator

$$T : \gamma^2\left(\Omega \times [0,\infty[\right) \to \gamma^2\left(\Omega\times]-1,\infty[\right)$$

(notice that we only need to show that this extension operator exists locally with respect to the first variable x) which implies the vector valued extension result. Using U instead of u in (8.10) we see that v may be extended to $V \in \gamma^2(\Omega\times]-1,\infty[, E)$, hence both sides of (8.13) are continuous on $\Omega \times [0,1[$ and therefore $P(D_x)(V(x,0) - H(x,0)) = u(x,0)$ on Ω. □

Theorem 8.8. *Let $\emptyset \neq \Omega \subset \mathbb{R}$ be a bounded open set. Assume that there is a flabby sheaf $\mathcal{F}$ on Ω satisfying* (8.5) *from Theorem* 8.3. *Then*

$$P(D) : \gamma^2(\omega, E) \to \gamma^2(\omega, E)$$

is surjective if ω is open and $\bar{\omega} \subset \Omega$. In particular, if E is an ultrabornological PLS-space then E has (PA).

Proof. We repeat the proof of Theorem 8.3 using instead of Theorem 5.2 its analogue for holomorphic functions which gives the topological isomorphism

$$H : H(\mathbb{C} \setminus \bar{\omega}, E)/H(\mathbb{C}, E) \to L_b(\mathscr{A}(\bar{\omega}), E).$$

Also there is an analogue of Lemma 8.2 stating that for any $f \in \gamma_0^2(\mathbb{R}, E)$

$$S(f) := G * (f \otimes \delta_y) \in H(\mathbb{C} \setminus \mathrm{supp}(f), E)$$

and that $S(f)\big|_{\mathbb{R}\times\pm]0,\infty[}$ can be uniquely extended to $g_\pm \in \gamma^2(\mathbb{R} \times \pm[0, \infty[, E)$ and

$$g_+(x, 0) - g_-(x, 0) = f(x) \qquad \text{on } \mathbb{R}.$$

Finally, we replace Lemma 8.1 by Lemma 8.7 and we get the surjectivity of $P(D)$ on $\gamma^2(\omega, E)$. By Remark 3.3 we get $\mathrm{Ext}^1_{PLS}(\ker P(D)'_b, E) = 0$. By Theorem 8.6, $\ker P(D) \simeq \Lambda_\infty(\alpha)$ for stable α. By [7, Th. 4.4], E has (PA). □

We can formulate now the final result of our investigations, combining Theorem 8.8 and Corollary 8.5, and Lemma 4.1 and Theorem 6.9, respectively:

Theorem 8.9. *Let E be an ultrabornological PLS-space. Then the following assertions are equivalent*:

(a) *For any $1 \le d < \infty$ there is a flabby sheaf $\mathcal{F}$ on $\mathbb{R}^d$ such that*

$$\begin{aligned} \mathcal{F}_0(K) &:= \{T \in \mathcal{F}(\mathbb{R}^d) \mid \mathrm{supp}_{\mathcal{F}}(T) \subset K\} \\ &= L(\mathscr{A}(K), E) \textit{ for any compact } K \subset \mathbb{R}^d. \end{aligned}$$

(b) *For some $1 \le d < \infty$ there is a flabby sheaf $\mathcal{F}$ on some open set $\emptyset \ne \Omega \subseteq \mathbb{R}^d$ such that*

$$\begin{aligned} \mathcal{F}_0(K) &:= \{T \in \mathcal{F}(\Omega) \mid \mathrm{supp}_{\mathcal{F}}(T) \subset K\} \\ &= L(\mathscr{A}(K), E) \textit{ for any compact } K \subset \Omega. \end{aligned}$$

(c) *E has (PA).*

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