*Publ. RIMS, Kyoto Univ.*
**45** (2009), 1–24

# $\mathcal{L}$-Invariant of the Symmetric Powers of Tate Curves

By

Haruzo Hida*

## Contents



In my earlier paper [H07] and in my talk at the workshop on "Arithmetic Algebraic Geometry" at RIMS in September 2006, we made explicit a conjectural formula of the $\mathcal{L}$-invariant of symmetric powers of a Tate curve over a totally real field (generalizing the conjecture of Mazur-Tate-Teitelbaum, which is now a theorem of Greenberg-Stevens). In this paper, we prove the formula for Greenberg's $\mathcal{L}$-invariant when the symmetric power is of adjoint type, assuming a standard conjecture (see Conjecture 0.1) on the ring structure of a Galois deformation ring of the symmetric powers.

Let $p$ be an odd prime and $F$ be a totally real field of degree $d < \infty$ with integer ring $O$. Order all the prime factors of $p$ in $O$ as $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$. Throughout this paper, we study an elliptic curve $E_{/F}$ over $O$ with split multiplicative reduction at $\mathfrak{p}_j|p$ for $j = 1, 2, \ldots, b$ and ordinary good reduction at $\mathfrak{p}_j|p$ for $j > b$. Write $F_j = F_{\mathfrak{p}_j}$ for the $\mathfrak{p}_j$-adic completion of $F$ and $q_j \in F_j^\times$ with $j \leq b$ for the Tate period of $E_{/F_j}$. Put $Q_j = N_{F_{\mathfrak{p}_j}/\mathbb{Q}_p}(q_j)$. When $b = 0$, as a convention, we assume that $E_{/F}$ has good ordinary reduction at every $p$-adic place of $F$. We assume throughout the paper that $E$ *does not have complex*

Communicated by A. Tamagawa. Received February 21, 2007. Revised July 3, 2007.
2000 Mathematics Subject Classification(s): 11F11, 11F41, 11F80, 11G05, 11R23, 11R42.
*Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, U.S.A.
The author is partially supported by the NSF grant: DMS 0244401 and DMS 0456252.

*multiplication*, and for simplicity, we also assume that $E$ is *semi-stable over* $O$. Some cases of complex multiplication are treated in [HMI] Section 5.3.3. Take an algebraic closure $\overline{F}$ of $F$. Writing $\rho_E : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbb{Q}_p)$ for the Galois representation on $T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the Tate module $T_pE = \varprojlim_n E[p^n]$, at each prime factor $\mathfrak{p}|p$, we have $\rho_E|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \sim \left(\begin{smallmatrix} \beta_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p} \end{smallmatrix}\right)$ for an unramified character $\alpha_\mathfrak{p}$. Since $\beta_\mathfrak{p}$ restricted to the inertia subgroup $I_\mathfrak{p} \subset \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ is equal to the $p$-adic cyclotomic character $\mathcal{N}$, we have $\alpha_\mathfrak{p}^i \neq \beta_\mathfrak{p}^j$ for any pair of integers $(i, j)$ except for $i = j = 0$. Write $\rho_{n,0}$ for the symmetric $n$-th tensor power of $\rho_E$, which is an $(n+1)$-dimensional Galois representation semi-stable over $O$. More generally, we write $\rho_{n,m}$ for $\rho_{n,0} \otimes \mathcal{N}^{-m} : \mathrm{Gal}(\overline{F}/F) \to G_n(\mathbb{Q}_p)$, where $\mathcal{N}$ is the $p$-adic cyclotomic character. By semi-stability, the sets of ramification primes for $\rho_E$ and $\rho_{n,m}$ are equal.

Consider $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We then define $J_n = Sym^{\otimes n}(J_1)$. Since ${}^t\alpha J_1 \alpha = \det(\alpha) J_1$ for $\alpha \in GL(2)$, we have ${}^t\rho_{n,0}(\sigma) J_n \rho_{n,0}(\sigma) = \mathcal{N}^n(\sigma) J_n$. Define an algebraic group $G_n$ over $\mathbb{Z}_p$ by

$$G_n(A) = \left\{\alpha \in GL_{n+1}(A) \middle| {}^t\alpha J_n \alpha = \nu(\alpha) J_n\right\}$$

with the similitude homomorphism $\nu : G_n \to \mathbb{G}_m$. Then $G_n$ is a quasi-split orthogonal or symplectic group according as $n$ is even or odd. The representation $\rho_{n,0}$ of $\mathrm{Gal}(\overline{F}/F)$ actually factors through $G_n(\mathbb{Q}_p) \subset GL_{n+1}(\mathbb{Q}_p)$. Two representations $\rho$ and $\rho' : G \to G_n(A)$ for a group $G$ are isomorphic if $\rho(g) = x\rho'(g)x^{-1}$ for $x \in G_n(A)$ independent of $g \in G$. If $\rho$ is isomorphic to $\rho'$, we write $\rho \cong \rho'$.

Let $S$ be the set of prime ideals of $O$ prime to $p$ where $E$ has bad reduction (and by semi-stability, $S \sqcup \{\mathfrak{p}|p\} \sqcup \{\infty\}$ gives the set of ramified primes for $\rho_{n,0}$). Let $K/\mathbb{Q}_p$ be a finite extension with $p$-adic integer ring $W$. We may take $K = \mathbb{Q}_p$, but it is useful to formulate the result allowing other choices of $K$. Start with $\rho_{n,0}$ and consider the deformation ring $(R_n, \boldsymbol{\rho}_n)$ which is universal among the following deformations: Galois representations $\rho_A : \mathrm{Gal}(\overline{F}/F) \to G_n(A)$ for Artinian local $K$-algebras $A$ with residue field $K = A/\mathfrak{m}_A$ such that

(K$_n$1) unramified outside $S$, $\infty$ and $p$;

(K$_n$2) $\rho_A|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \cong \begin{pmatrix} \alpha_{0,A,\mathfrak{p}} & * & \cdots & * \\ 0 & \alpha_{1,A,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,A,\mathfrak{p}} \end{pmatrix}$ with $\alpha_{j,A,\mathfrak{p}} \equiv \beta_\mathfrak{p}^{n-j}\alpha_\mathfrak{p}^j \mod \mathfrak{m}_A$ with $\alpha_{j,A,\mathfrak{p}}|_{I_\mathfrak{p}}$ $(j = 0, 1, \ldots, n)$ factoring through $\mathrm{Gal}(F_\mathfrak{p}^{ur}[\mu_{p^\infty}]/F_\mathfrak{p}^{ur})$ for the maximal unramified extension $F_\mathfrak{p}^{ur}/F_\mathfrak{p}$ for all prime factors $\mathfrak{p}$ of $p$;

($K_n$3) $\nu \circ \rho_A = \mathcal{N}^n$ for the $p$-adic cyclotomic character $\mathcal{N}$;

($K_n$4) $\rho_A \equiv \rho_{n,0} \mod \mathfrak{m}_A$.

Since $\rho_{n,0}$ is absolutely irreducible as long as $E$ does not have complex multiplication (because $\mathrm{Im}(\rho_E)$ is open in $GL_2(\mathbb{Z}_p)$ by a result of Serre) and all $\alpha_{\mathfrak{p}}^i \beta_{\mathfrak{p}}^{n-i}$ for $i = 0, 1, \dots, n$ are distinct, the deformation problem specified by ($K_n$1–4) is representable by a universal couple $(R_n, \boldsymbol{\rho}_n)$ (see [Ti]). In other words, for any $\rho_A$ as above, there exists a unique $K$-algebra homomorphism $\varphi : R_n \to A$ such that $\varphi \circ \boldsymbol{\rho}_n \cong \rho_A$.

Write now

$$\boldsymbol{\rho}_n|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \cong \begin{pmatrix} \boldsymbol{\delta}_{0,\mathfrak{p}} & * & \cdots & * \\ 0 & \boldsymbol{\delta}_{1,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\delta}_{n,\mathfrak{p}} \end{pmatrix}$$

with $\boldsymbol{\delta}_{j,\mathfrak{p}} \equiv \beta_{\mathfrak{p}}^{n-j}\alpha_{\mathfrak{p}}^j \mod \mathfrak{m}_n$ (for $\mathfrak{m}_n = \mathfrak{m}_{R_n}$).

Let $\Gamma_{\mathfrak{p}}$ be the maximal torsion-free quotient of $\mathrm{Gal}(F_{\mathfrak{p}}^{ur}[\mu_{p^\infty}]/F_{\mathfrak{p}}^{ur})$. Then the character $\widehat{\boldsymbol{\delta}}_{j,\mathfrak{p}} = \boldsymbol{\delta}_{j,\mathfrak{p}}(\beta_{\mathfrak{p}}^{n-j}\alpha_{\mathfrak{p}}^j)^{-1}$ restricted to $I_{\mathfrak{p}}$ factors through $\Gamma_{\mathfrak{p}}$, giving rise to an algebra structure of $R_n$ over $W[[\Gamma_{\mathfrak{p}}]]$. Take the product $\boldsymbol{\Gamma} = \prod_{\mathfrak{p}|p} \Gamma_{\mathfrak{p}}^{n+1}$ of $n+1$ copies of $\Gamma_{\mathfrak{p}}$ over all prime factors $\mathfrak{p}$ of $p$ in $F$. We write general elements of $\boldsymbol{\Gamma}$ as $x = (x_{j,\mathfrak{p}})_{j,\mathfrak{p}}$ with $x_{j,\mathfrak{p}}$ in the $j$-th component $\Gamma_{\mathfrak{p}}$ in $\boldsymbol{\Gamma}$ ($j = 0, 1, \dots, n$). Consider the character $\widehat{\boldsymbol{\delta}} : \boldsymbol{\Gamma} \to R_n^\times$ given by $\widehat{\boldsymbol{\delta}}(x) = \prod_{j=0}^n \prod_{\mathfrak{p}|p} \widehat{\boldsymbol{\delta}}_{j,\mathfrak{p}}(x_{j,\mathfrak{p}})$. Choosing a generator $\gamma_i = \gamma_{\mathfrak{p}}$ (for $\mathfrak{p} = \mathfrak{p}_i$) of the topologically cyclic group $\Gamma_{\mathfrak{p}}$, we identify $W[[\boldsymbol{\Gamma}]]$ with a power series ring $W[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}}$ by associating the generator $\gamma_{\mathfrak{p}}$ of the $j$-th component: $\Gamma_{\mathfrak{p}}$ of $\boldsymbol{\Gamma}$ with $1 + X_{j,\mathfrak{p}}$. The character $\widehat{\boldsymbol{\delta}} : W[[\Gamma]] \to R_n$ extends uniquely to an algebra homomorphism $\widehat{\boldsymbol{\delta}} : W[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}} \to R_n$ by the universality of the (continuous) group ring $W[[\boldsymbol{\Gamma}]]$. Thus $R_n$ is naturally an algebra over $K[[X_{j,\mathfrak{p}}]]_{j,\mathfrak{p}}$. This algebra structure of $R_n$ over the local Iwasawa algebra $W[[\boldsymbol{\Gamma}]]$ is a standard one which has been studied for long (about 20 years) in many places (for example, [Ti] Chapter 8 and [MFG] 5.2.2). The $(n+1)e$ variables $X_{j,\mathfrak{p}}$ may not be independent in $R_n$, and we expect that only a half of them survives. More precisely, we have the following conjectural statement:

**Conjecture 0.1.** Suppose that $n$ is odd. Then $R_n$ is isomorphic to the power series ring $K[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p, j:\mathrm{odd}}$ of $e\frac{n+1}{2}$ variables.

When $n = 1$, we write $\boldsymbol{\beta}_i = \boldsymbol{\delta}_{0,\mathfrak{p}_i}$, $\boldsymbol{\alpha}_i = \boldsymbol{\delta}_{1,\mathfrak{p}_i}$ and $T_i = X_{1,\mathfrak{p}_i}$. If $n = 1$ and $F = \mathbb{Q}$, via the solution of the Shimura-Taniyama conjecture, this conjecture follows from Kisin's work (generalizing earlier works of Wiles, Taylor-Wiles and Skinner-Wiles). Assuming potential modularity of $\rho_E$ (see [Ta]) with additional assumptions that $\mathrm{Im}(\overline{\rho})$ is nonsoluble and that the semi-simplification of $\overline{\rho}|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})}$ is non-scalar for all prime $\mathfrak{p}|p$ in $F$, we will prove this conjecture for $n = 1$ in this paper (see Proposition 2.1). Assuming Hilbert modularity over $F$ of $E$ and the following two conditions:

(ai) The $\mathbb{F}_p$-linear Galois representation $\overline{\rho} = (T_pE \mod p)$ is absolutely irreducible over $\mathrm{Gal}(\overline{F}/F[\mu_p])$.

(ds) $\overline{\rho}^{ss}$ has a non-scalar value over $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ for all prime factors $\mathfrak{p}|p$,

the conjecture for $n = 1$ follows from a result of Fujiwara (see [F] and [F1]) and Skinner-Wiles [SW1] as described in [HMI] Theorem 3.65 and Proposition 3.78.

In the special case of rational elliptic curve $E_{/\mathbb{Q}}$ with multiplicative reduction at $p$, the following conjecture (generalizing the one by Mazur-Tate-Teitelbaum in [MTT]) was proven by R. Greenberg for his $\mathcal{L}$-invariant of symmetric powers of $E$. His proof is described in his remark in page 170 of [Gr]. Although his proof might also be generalized to our setting, our point of view is different from [Gr], relating the following conjecture to Conjecture 0.1, and indeed, if one can generalize Greenberg's proof to cover the following conjecture, it might supply us with a proof of Conjecture 0.1 (we hope to discuss this point in our future work).

**Conjecture 0.2.** Let the notation and the assumption be as in Theorem 0.3. Suppose that the $n$-th symmetric power motive $Sym^{\otimes n}(H_1(E))(-m)$ with Tate twist by an integer $m$ is critical at 1. Then if $\mathrm{Ind}_F^{\mathbb{Q}}(Sym^{\otimes n}(\rho_E)(-m))$ has an exceptional zero at $s = 1$, we have

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_{n,m})$$
$$= \begin{cases} \left(\prod_{i=1}^{b} \frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)}\right)\mathcal{L}(m) & \text{for a constant } \mathcal{L}(m) \in \mathbb{Q}_p^\times \text{ if } n = 2m \text{ with odd } m, \\ \prod_{i=1}^{b} \frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)} & \text{if } n \neq 2m. \end{cases}$$

We have $\mathcal{L}(m) = 1$ if $b = e$, and the value $\mathcal{L}(1)$ when $b < e$ is given by

$$\mathcal{L}(1) = \det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}\right)_{i>b, j>b}\Big|_{X_1=X_2=\cdots=X_e=0} \prod_{i>b} \frac{\log_p(\gamma_i)}{[F_i : \mathbb{Q}_p]\alpha_i([p, F_i])}$$

for the local Artin symbol $[p, F_i]$, where $\gamma_{\mathfrak{p}}$ is the generator of $\mathcal{N}(\mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}}))$ by which we identify the group algebra $W[[\Gamma_{\mathfrak{p}}]]$ with $W[[X_{\mathfrak{p}}]]$.

The analytic $\mathcal{L}$-invariant of $p$-adic analytic $L$-functions (when $n = 1$) is studied by C.-P. Mok [M] following the method of [GS], and his result confirms the conjecture in some special cases (see a remark in [H07] after Conjecture 1.3).

The motive $Sym^{\otimes n}(H_1(E))(-m)$ is critical at 1 if and only if the following two conditions are satisfied:

- $0 \leq m < n$;
- either $n$ is odd or $n = 2m$ with odd $m$.

We will specify $\mathcal{L}(m)$ in Definition 1.11 assuming Conjecture 0.1. There is a wild guess that $\mathcal{L}(m)$ might be independent of $m$ only depending on $E$. We hope to discuss this matter in our future work.

We will prove in this paper (for Greenberg's $\mathcal{L}$-invariant of $\rho_{2n,n}$) that Conjecture 0.1 implies the above conjecture for $\rho_{2m,m}$. Here are some additional remarks about the conjecture:

(1) When $n = 2m$ with even $m$, the motive associated to $Sym^{\otimes n}(\rho_E)(-m)$ is not critical at $s = 1$; so, the situation is drastically different (and in such a case, we do not make any conjecture; see [H00] Examples 2.7 and 2.8).

(2) The above conjecture applies to arithmetic and analytic $p$-adic $L$-functions.

We let $\sigma \in \mathrm{Gal}(\overline{F}/F)$ act on the Lie algebra of $G_{n/K}$

$$\mathfrak{s}_n(K) = \{x \in M_{n+1}(K) | \mathrm{Tr}(x) = 0 \text{ and } {}^t x J_n + J_n x = 0\}$$

by conjugation: $x \mapsto \sigma x = \rho_{n,0}(\sigma) x \rho_{n,0}(\sigma)^{-1}$. This representation $Ad(\rho_{n,0})$ is isomorphic to $\bigoplus_{0<j\leq n, j:\mathrm{odd}} \rho_{2j,j}$ and is called the adjoint square representation of $\rho_{n,0}$. By using a canonical isomorphism between the tangent space of $\mathrm{Spf}(R_n)$ and a certain Selmer group of $Ad(\rho_{n,0})$, we get

**Theorem 0.3.** *Let $m$ be an odd positive integer. Assume Conjecture* 0.1 *for all odd integers $n$ with $0 < n \leq m$. Then Conjecture* 0.2 *holds for Greenberg's $\mathcal{L}$-invariant of $\rho_{2m,m}$.*

All the assumptions in [Gr] (particularly, $\mathrm{Sel}_F(\rho_{2m,m}) = 0$: Lemma 1.2) made to define the invariant can be verified under Conjecture 0.1 for $\rho_{2m,m}$. The assumption in the theorem that $E$ has split (multiplicative) reduction at $\mathfrak{p}_j$ with $j \leq b$ is inessential, because $Ad(\rho_{n,0}) \cong Ad(Sym^{\otimes n}(\rho_E \otimes \chi))$ (for a $K^\times$-values Galois character $\chi$) and we can bring any elliptic curve with multiplicative reduction at $\mathfrak{p}_j$ to an elliptic curve with split multiplicative reduction at $\mathfrak{p}_j$ by a quadratic twist. We will prove this theorem as Theorem 1.14 later.

Conjecture 0.1 and Conjecture 0.2 are logically close. Since $\rho_{2m,m}$ is self dual, the complex $L$-function $L(s,\rho_{2m,m})$ has functional equation of the form $s \leftrightarrow 1-s$, and the complex $L$-value $L(1,\rho_{2m,m})$ should not vanish at $s=1$ (the abscissa of convergence). Conjecturally, this should imply $\mathrm{Sel}_F(\rho_{2m,m})=0$, since $\rho_{2m,m}$ with odd $m$ is critical at 1. This vanishing is essential for Greenberg's definition of his $\mathcal{L}$-invariant to work (especially in his definition of the subspace $\widetilde{\mathbf{T}} \subset H^1(\mathrm{Gal}(\overline{F}/F),\rho_{2m,m})$ ($\widetilde{\mathbf{T}}$ is written later as $\mathbf{H}_F$ in this paper; see [Gr] page 163–4). Conjecture 0.1 for an integer $n \geq m$ implies $\mathrm{Sel}_F(\rho_{2m,m})=0$ for odd $m>0$ (see Lemma 1.2). Indeed, at least in appearance, a much weaker infinitesimal version than Conjecture 0.1 asserting that $R_n$ shares the tangent space with $K[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p,0<j\leq n,j:\mathrm{odd}}$ (that is, $K[[X_{j,\mathfrak{p}}]]/(X_{j,\mathfrak{p}})^2 \cong R_n/\mathfrak{m}_n^2$) is sufficient for this vanishing $\mathrm{Sel}_F(\rho_{2m,m})=0$ and to prove Conjecture 0.2. However, for example, if $m=1$ and $n=1$, any characteristic 0 $p$-adic (motivic) Galois deformation $\rho$ over $\mathbb{Z}_p$ (not over $\mathbb{Q}_p$ in Conjecture 0.1) of $\overline{\rho} := (\rho_E \mod p)$ has its $p$-adic $L$-function $L_p(s,\rho_{2,1})$ with an exceptional zero at $s=1$. Thus the weaker infinitesimal statement at each $\rho$ should actually imply the stronger statement as in Conjecture 0.1 (if we admit the "$R=T$" theorem as in [MFG] Theorem 5.29 for $F=\mathbb{Q}$ or [HMI] Theorem 3.50 for general $F$ for nearly ordinary deformations). In this sense, the two conjectures are almost equivalent if we include motivic deformations $\rho$ of $\overline{\rho}$ in the scope of Conjecture 0.2 not limiting ourselves to elliptic curves. This point will be discussed in more details in our future work.

## §1. Symmetric Tensor $\mathcal{L}$-Invariant

We recall briefly an $F$-version (given in [HMI] Definition 3.85) of Greenberg's formula of the $\mathcal{L}$–invariant for a general $p$-adic totally $p$-ordinary Galois representation $V$ (of $\mathrm{Gal}(\overline{F}/F)$) with an exceptional zero. This definition is equivalent to the one in [Gr] if we apply it to $\mathrm{Ind}_F^{\mathbb{Q}} V$ as proved in [HMI] (in Definition 3.85). When $V=\rho_{2m,m}$ with odd $m$, the definition can be outlined as follows. Under some hypothesis, he found a unique subspace $\mathbf{H} \subset H^1(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\rho_{2m,m})$ of dimension $e$. By Shapiro's lemma, $H^1(\mathbb{Q},\mathrm{Ind}_F^{\mathbb{Q}}\rho_{2m,m}) \cong H^1(F,\rho_{2m,m})$, and one can give a definition of the image $\mathbf{H}_F$ of $\mathbf{H}$ in $H^1(F,\rho_{2m,m})$ without reference to the induction $\mathrm{Ind}_F^{\mathbb{Q}}\rho_{2m,m}$ ([HMI] Definition 3.85) as we recall the precise definition later (see Lemma 1.7). The space $\mathbf{H}_F$ is represented by cocycles $c : \mathrm{Gal}(\overline{F}/F) \to \rho_{2m,m}$ such that

(1) $c$ is unramified outside $p$;

(2) $c$ restricted to the decomposition subgroup $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \cong D_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}/F)$

at each $\mathfrak{p}|p$ has values in $\mathcal{F}^-_{\mathfrak{p}}\rho_{2m,m}$ and $c|_{D_{\mathfrak{p}}}$ modulo $\mathcal{F}^+_{\mathfrak{p}}\rho_{2m,m}$ becomes unramified over $F_{\mathfrak{p}}[\mu_{p^\infty}]$ for all $\mathfrak{p}|p$.

Here $\mathcal{F}^-_{\mathfrak{p}}\rho_{2m,m} = \mathcal{F}^0_{\mathfrak{p}}\rho_{2m,m}$, $\mathcal{F}^+_{\mathfrak{p}}\rho_{2m,m} = \mathcal{F}^1_{\mathfrak{p}}\rho_{2m,m}$, and $\mathcal{F}^j\rho_{2m,m}$ is the decreasing filtration on $\rho_{2m,m}$ such that $I_{\mathfrak{p}}$ acts by $\mathcal{N}^j$ on $\mathcal{F}^j_{\mathfrak{p}}\rho_{2m,m}/\mathcal{F}^{j+1}_{\mathfrak{p}}\rho_{2m,m}$.

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, and put $F_\infty/F$ for the composite of $F$ and $\mathbb{Q}_\infty$. By the condition (2), $(c|_{D_{\mathfrak{p}'}} \mod \mathcal{F}^+_{\mathfrak{p}'}\rho_{2m,m})$ with a prime $\mathfrak{p}'|p$ may be regarded as a homomorphism $a : D_{\mathfrak{p}'} \to K$ because $\mathcal{F}^-_{\mathfrak{p}'}\rho_{2m,m}/\mathcal{F}^+_{\mathfrak{p}'}\rho_{2m,m}$ is isomorphic to the trivial $D_{\mathfrak{p}'}$-module $K$. Hence $a$ becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension $F_\infty/F$. In other words, the cohomology class $[c]$ is in $\mathrm{Sel}_{F_\infty}(\rho_{2m,m})$ but not in $\mathrm{Sel}_F(\rho_{2m,m})$. In other words, we have

$$\mathbf{H}_F \cong \mathrm{Sel}^{cyc}_F(\rho_{2m,m}) := \mathrm{Res}^{-1}(\mathrm{Sel}_{F_\infty}(\rho_{2m,m}))$$

for the restriction map $\mathrm{Res} : H^1(F, \rho_{2m,m}) \to H^1(F_\infty, \rho_{2m,m})$ (see the definition of various Selmer groups given in the following section).

Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of $\mathbf{H}_F$ over $K$. Write $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \to K$ for $c_{\mathfrak{p}} \mod \mathcal{F}^+_{\mathfrak{p}'}\rho_{2m,m}$ regarded as a homomorphism (identifying $\mathcal{F}^-_{\mathfrak{p}}\rho_{2m,m}/\mathcal{F}^+_{\mathfrak{p}}\rho_{2m,m}$ with $K$). We now have two $e \times e$ matrices with coefficients in $K$: $A = (a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}]))_{\mathfrak{p},\mathfrak{p}'|p}$ and $B = \left(\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}$. Under Conjecture 0.1 for $\rho_{n,0}$ for all odd $n \le m$, we can show that $B$ is invertible. Then Greenberg's $\mathcal{L}$-invariant is defined by

$$\mathcal{L}(\mathrm{Ind}^{\mathbb{Q}}_F \rho_{2m,m}) = \det(AB^{-1}). \tag{1.1}$$

The determinant $\det(AB^{-1})$ is independent of the choice of the basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$. Though $L(s, \mathrm{Ind}^{\mathbb{Q}}_F \rho) = L(s, \rho)$ for a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(K)$ in a compatible system, the (nonvanishing) modification Euler $p$-factors $\mathcal{E}^+(\rho)$ and $\mathcal{E}^+(\mathrm{Ind}^{\mathbb{Q}}_F \rho)$ (cf. [Gr] (6)) to define the corresponding $p$-adic $L$-functions could be different (see [H07] (1.1)). Thus the $\mathcal{L}(\rho)$ and $\mathcal{L}(\mathrm{Ind}^{\mathbb{Q}}_F \rho)$ could be slightly different. As in [H07] (1.1), we have the following relation

$$\mathcal{L}(\rho_{2m,m}) = \left(\prod_{\mathfrak{p}|p} f_{\mathfrak{p}}\right) \mathcal{L}(\mathrm{Ind}^{\mathbb{Q}}_F \rho_{2m,m}), \tag{1.2}$$

where $f_{\mathfrak{p}} = [O/\mathfrak{p} : \mathbb{F}_p]$.

Choose a generator $\gamma$ of $\mathcal{N}(\mathrm{Gal}(F_\infty/F)) \subset \mathbb{Z}_p^\times$ for the $p$-adic cyclotomic character $\mathcal{N}$, and identify $\Lambda = W[[\mathrm{Gal}(F_\infty/F)]]$ with $W[[T]]$ by $\gamma \mapsto 1 + T$. The Selmer group $\mathrm{Sel}_{F_\infty}(\rho^*_{2m,m}) := \mathrm{Sel}_{F_\infty}(Sym^{\otimes 2m}(T_pE)(-m) \otimes (\mathbb{Q}_p/\mathbb{Z}_p))$ has its Pontryagin dual which is a $\Lambda$-module of finite type. Choose a characteristic power series $\Phi^{arith}(T) \in \Lambda$ of the Pontryagin dual. Put $L^{arith}_p(s, \rho_{2m,m}) = \Phi^{arith}(\gamma^{1-s} - 1)$. We consider the following condition stronger than (ds):

(ds$_m$) $\overline{\rho}^{ss}_{m,0}$ *(for* $\overline{\rho}_{m,0} = Sym^{\otimes m}(\overline{\rho})$*) is a direct sum of* $m+1$ *distinct characters of* $D_{\mathfrak{p}}$ *for all prime factors* $\mathfrak{p}|p$.

For the known cases of the following conjecture, see [Gr] Proposition 4 and [H07] Theorem 5.3.

**Conjecture 1.1** (Greenberg)**.** Suppose (ds$_m$) and that $\overline{\rho}_{m,0}$ is absolutely irreducible. Then $L_p^{arith}(s,\rho_{2m,m})$ has zero of order equal to $e = \left|\{\mathfrak{p}|p\}\right|$ and for the constant $\mathcal{L}(\rho_{2m,m}) \in K^\times$ given in (1.1) and (1.2), we have

$$\lim_{s\to 1} \frac{L_p^{arith}(s,\rho_{2m,m})}{(s-1)^d} = \mathcal{L}(\rho_{2m,m})\left||\mathrm{Sel}_F(\rho^*_{2m,m})|\right|_p^{-1/[K:\mathbb{Q}_p]}$$

up to units.

This conjecture has been proven by Greenberg (see [Gr] Proposition 4) for more general ordinary Galois representation than $\rho_{2m,m}$ under some (mild, believable but possibly restrictive) assumptions. Especially the assumption (5) in [Gr] proposition 4 is difficult to verify just by assuming (ds$_m$) and absolute irreducibility of $\overline{\rho}_{n,0}$ and could be far deeper (even for those of adjoint type like $\rho_{2m,m}$) than the modularity statement like Conjecture 0.1; so, unfortunately, the above statement remains to be a conjecture.

In the above conjecture, the modifying Euler factor at the $p$-adic places $\mathfrak{p}_j$ of good reduction ($j > b$):

$$\mathcal{E}^+(\rho_{2m,m}) = \prod_{j>b}\left(\prod_{i=1}^{m}(1-\alpha_j^{-2i}N(\mathfrak{p}_i)^{i-1})(1-\alpha_j^{-2i}N(\mathfrak{p}_i)^{i})\right)$$

does not appear, where $\alpha_j = \alpha_j(Frob_{\mathfrak{p}_j})$. However, if we replace Greenberg's Selmer group $\mathrm{Sel}_F(\rho^*_{2m,m})$ by the Bloch-Kato Selmer group $S_F(\rho^*_{2m,m})$ over $F$ (crystalline at $\mathfrak{p}_j$ for $j > b$), we expect to have the relation

$$\left||\mathrm{Sel}_F(\rho^*_{2m,m})|\right|_p^{-1/[K:\mathbb{Q}_p]} = \mathcal{E}^+(\rho_{2m,m})\left||S_F(\rho^*_{2m,m})|\right|_p^{-1/[K:\mathbb{Q}_p]}$$

up to $p$-adic units (as described in [MFG] page 284 for $\rho_{2,1}$). Thus if one uses the formulation of Bloch-Kato, we should have the modifying Euler factor in the formula, and the size of the Bloch-Kato Selmer group is expected to be equal to the primitive archimedean $L$-values (divided by a suitable period; see Greenberg's Conjecture 0.1 in [H06]).

### §1.1. Selmer groups

First we recall Greenberg's definition of Selmer groups. Write $F^{(S)}/F$ for the maximal extension unramified outside $S$, $p$ and $\infty$. Put $\mathfrak{G} = \mathrm{Gal}(F^{(S)}/F)$ and $\mathfrak{G}_M = \mathrm{Gal}(F^{(S)}/M)$. Let $V$ be a potentially ordinary representation of $\mathfrak{G}$ on a $K$-vector space $V$. Thus $V$ has decreasing filtration $\mathcal{F}^i_{\mathfrak{p}}V$ such that an open subgroup of $I_{\mathfrak{p}}$ (for each prime factor $\mathfrak{p}|p$) acts on $\mathcal{F}^i_{\mathfrak{p}}V/\mathcal{F}^{i+1}_{\mathfrak{p}}V$ by the $i$-th power $\mathcal{N}^i$ of the $p$-adic cyclotomic character $\mathcal{N}$. We fix a $W$-lattice $T$ in $V$ stable under $\mathfrak{G}$.

Put $\mathcal{F}^+_{\mathfrak{p}}V = \mathcal{F}^1_{\mathfrak{p}}V$ and $\mathcal{F}^-_{\mathfrak{p}}V = \mathcal{F}^0_{\mathfrak{p}}V$. Writing $\mathcal{F}^\bullet_{\mathfrak{p}}T = T \cap \mathcal{F}^\bullet_{\mathfrak{p}}V$ and $\mathcal{F}^\bullet_{\mathfrak{p}}V/T = \mathcal{F}^\bullet_{\mathfrak{p}}V/\mathcal{F}^\bullet_{\mathfrak{p}}T$, we have a 3-step filtration for $A = V$, $T$ or $V/T$:

$$A \supset \mathcal{F}^-_{\mathfrak{p}}A \supset \mathcal{F}^+_{\mathfrak{p}}A \supset \{0\}. \tag{ord}$$

Its dual $V^*(1) = \mathrm{Hom}_K(V,K) \otimes \mathcal{N}$ again satisfies (ord).

Let $M/F$ be a subfield of $F^{(S)}$, and put $\mathfrak{G}_M = \mathrm{Gal}(F^{(S)}/M)$. We write $\mathfrak{p}$ for a prime of $M$ over $p$ and $\mathfrak{q}$ for general primes outside $p$ of $M$. We write $I_{\mathfrak{p}}$ and $I_{\mathfrak{q}}$ for the inertia subgroup in $\mathfrak{G}_M$ at $\mathfrak{p}$ and $\mathfrak{q}$, respectively. We put

$$L_{\mathfrak{p}}(A) = \mathrm{Ker}\left(\mathrm{Res} : H^1(M_{\mathfrak{p}}, A) \to H^1\left(I_{\mathfrak{p}}, \frac{A}{\mathcal{F}^+_{\mathfrak{p}}(A)}\right)\right),$$

and

$$L_{\mathfrak{q}}(A) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{q}}, A) \to H^1(I_{\mathfrak{q}}, A)).$$

Then we define the Selmer submodule in $H^1(M, A)$ by

$$\mathrm{Sel}_M(A) = \mathrm{Ker}\left(H^1(\mathfrak{G}_M, A) \to \prod_{\mathfrak{q}} \frac{H^1(M_{\mathfrak{q}}, A)}{L_{\mathfrak{q}}(A)} \times \prod_{\mathfrak{p}} \frac{H^1(M_{\mathfrak{p}}, A)}{L_{\mathfrak{p}}(A)}\right) \tag{1.3}$$

for $A = V, V/T$. The classical Selmer group of $V$ is given by $\mathrm{Sel}_M(V/T)$, equipped with discrete topology. We define the "minus", the "locally cyclotomic" and the "strict" Selmer groups $\mathrm{Sel}^-_M(A)$, $\mathrm{Sel}^{cyc}_M(A)$ and $\mathrm{Sel}^{st}_M(A)$, respectively, replacing $L_{\mathfrak{p}}(A)$ by

$$\begin{aligned}
L^-_{\mathfrak{p}}(A) &= \mathrm{Ker}\left(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1\left(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}^-_{\mathfrak{p}}(A)}\right)\right) \supset L_{\mathfrak{p}}(A)\\
L^{cyc}_{\mathfrak{p}}(A) &= \mathrm{Ker}\left(\mathrm{Res} : L^-_{\mathfrak{p}}(A) \to H^1\left(I_{\mathfrak{p},\infty}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}}(A)}\right)\right) \subset L^-_{\mathfrak{p}}(A)\\
L^{st}_{\mathfrak{p}}(A) &= \mathrm{Ker}\left(\mathrm{Res} : L^-_{\mathfrak{p}}(A) \to H^1\left(M_{\mathfrak{p}}, \frac{V}{\mathcal{F}^+_{\mathfrak{p}}(A)}\right)\right) \subset L_{\mathfrak{p}}(A),
\end{aligned}$$

where $I_{\mathfrak{p},\infty}$ is the inertia group of $\mathrm{Gal}(\overline{M}_{\mathfrak{p}}/M_{\mathfrak{p}}[\mu_{p^\infty}])$. Then we have

$$\mathrm{Sel}_F^{cyc}(A) = \mathrm{Res}^{-1}_{F_\infty/F}(\mathrm{Sel}_{F_\infty}(A)).$$

**Lemma 1.2.** *We have*

$$\mathrm{Sel}_F^{cyc}(Ad(\rho_{n,0})) \cong \bigoplus_{0<m\le n, m:odd} \mathrm{Sel}_F^{cyc}(\rho_{2m,m}) \cong \mathrm{Hom}_K(\mathfrak{m}_n/\mathfrak{m}_n^2, K),$$

*where $\mathfrak{m}_n$ is the maximal ideal of $R_n$. If we suppose Conjecture 0.1 for odd $n > 0$, we have $\mathrm{Sel}_F(\rho_{2m,m}) = 0$ for all odd $m$ with $0 < m \le n$.*

*Proof.* Let $V = Ad(\rho_{n,0})$. Then we have the filtration:

$$V \supset \mathcal{F}_{\mathfrak{p}}^- V \supset \mathcal{F}_{\mathfrak{p}}^+ V \supset \{0\},$$

where taking a basis so that the semi-simplification of $\rho_{n,0}|_{D_{\mathfrak{p}}}$ is diagonal with diagonal character $\beta_{\mathfrak{p}}^n, \beta_{\mathfrak{p}}^{n-1}\alpha_{\mathfrak{p}}, \ldots, \alpha_{\mathfrak{p}}^n$ in this order from top to bottom, $\mathcal{F}_{\mathfrak{p}}^- V$ is made up of upper triangular matrices and $\mathcal{F}_{\mathfrak{p}}^+ V$ is made up of upper nilpotent matrices, and on $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$, $D_{\mathfrak{p}}$ acts trivially (getting eigenvalue 1 for $Frob_{\mathfrak{p}}$). We consider the space $Der_K(R_n, K)$ of continuous $K$-derivations of $R_n$. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \mod t^2)$. Then writing each $K$-algebra homomorphism $\phi : R_n \to K[\varepsilon]$ as $\phi(r) = \phi_0(r) + \partial_\phi(r)\varepsilon$ and sending $\phi$ to $\partial_\phi \in Der_K(R_n, K)$, we have $\mathrm{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong Der_K(R_n, K) = \mathrm{Hom}_K(\mathfrak{m}_n/\mathfrak{m}_n^2, K)$. By the universality of $(R_n, \boldsymbol{\rho}_n)$, we have

$$\mathrm{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong \frac{\{\rho : \mathrm{Gal}(\overline{F}/F) \to G_n(K[\varepsilon]) | \rho \text{ satisfies } (\mathrm{K}_n 1\text{–}4)\}}{\cong}$$

by $\mathrm{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \ni \phi \mapsto \rho_\phi = \phi \circ \boldsymbol{\rho}_n = \rho_{n,0} + \varepsilon\partial_\phi \boldsymbol{\rho}_n$. Pick $\rho = \rho_\phi$ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$ with $\rho_1(\sigma) = \frac{\partial\rho}{\partial t} = \partial_\phi\boldsymbol{\rho}_n(\sigma)$. Then $c_\rho = (\partial_\phi\boldsymbol{\rho}_n)\rho_{n,0}^{-1}$ can be easily checked to be an inhomogeneous 1-cocycle having values in $M_{n+1}(K) \supset V$. Here $\sigma \in \mathrm{Gal}(\overline{F}/F)$ acts on $x \in M_{n+1}(K)$ by $x \mapsto \rho_{n,0}(\sigma)x\rho_{n,0}(\sigma)^{-1}$.

Since $\nu \circ \rho = \nu \circ \rho_{n,0}$ by $(\mathrm{K}_n 3)$, we have $\det(\rho) = \det(\rho_{n,0})$, which implies $\mathrm{Tr}(c_\rho) = 0$; so, $c_\rho$ has values in $\mathfrak{sl}_{n+1}(K)$. For $\partial \in Der_K(R_n, K)$ and $X \in GL_{n+1}(R_n)$ with ${}^tXJ_nX = J_n$, writing $\overline{X} = (X \mod \mathfrak{m}_n) \in GL_{n+1}(K)$

$$0 = \partial(X^{-1}X) = \overline{X}^{-1}\partial X + (\partial X^{-1})\overline{X}.$$

Since ${}^t\boldsymbol{\rho}_n J_n \boldsymbol{\rho}_n = \mathcal{N}^n J_n = {}^t\rho_{n,0}J_n\rho_{n,0}$, we have ${}^t\rho_{n,0}^{-1}\,{}^t\boldsymbol{\rho}_n J_n \boldsymbol{\rho}_n \rho_{n.0}^{-1} = J_n$. Let $X = \boldsymbol{\rho}_n\rho_{n.0}^{-1}$. Differentiating the identity: ${}^tXJ_nX = J_n$ by $\partial$, we have

$({}^t\partial X J_n)\overline{X} + {}^t\overline{X}(J_n \partial X) = 0$, which is equivalent to $c_\rho(\sigma) \in \mathfrak{s}_n(K) = V$. By the reducibility condition (K$_n$2), $[c_\rho]$ vanishes in $\frac{H^1(M_{\mathfrak{p}},V)}{L^-_{\mathfrak{p}}(V)}$. By the local cyclotomy condition in (K$_n$2), $[c_\rho]$ vanishes in $\frac{H^1(M_{\mathfrak{p}},V)}{L^{cyc}_{\mathfrak{p}}(V)}$. If $E$ has multiplicative reduction at $\mathfrak{q}$ (so, $\mathfrak{q} \in S$), the unramifiedness of $c_\rho$ follows from the following lemma. Thus the cohomology class $[c_\rho]$ of $c_\rho$ is in $\mathrm{Sel}^{cyc}_F(V)$. We see easily that $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$.

We can reverse the above argument starting with a cocycle $c$ giving an element of $\mathrm{Sel}^{cyc}_F(V)$ to construct a deformation $\rho_c = \rho_{n,0} + \varepsilon(c\rho_{n,0})$ with values in $G_n(K[\varepsilon])$. Thus we have

$$\frac{\{\rho : \mathrm{Gal}(\overline{F}/F) \to G_n(K[\varepsilon]) | \rho \text{ satisfies the conditions } (\mathrm{K}_n 1\text{–}4)\}}{\cong} \cong \mathrm{Sel}^{cyc}_F(V).$$

Recall that the isomorphism $Der_K(R_n, K) \cong \mathrm{Sel}^{cyc}_F(V)$ is given by

$$Der_K(R_n, K) \ni \partial \mapsto [c_\partial] \in \mathrm{Sel}^{cyc}_F(V)$$

for the cocycle $c_\partial = c_\rho = (\partial \boldsymbol{\rho}_n)\rho^{-1}_{n,0}$, where $\rho = \rho_{n,0} + \varepsilon(\partial \boldsymbol{\rho}_n)$.

Suppose Conjecture 0.1. Since the algebra structure of $R_n$ over $W[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p}$ is given by $\boldsymbol{\delta}_{j,\mathfrak{p}}(\beta^{n-j}_{\mathfrak{p}}\alpha^j_{\mathfrak{p}})^{-1}$ and $\boldsymbol{\delta}_{n-j,\mathfrak{p}}\boldsymbol{\delta}_{j,\mathfrak{p}} = \mathcal{N}^n$, the $K$-derivation $\partial = \partial_\phi : R_n \to K$ corresponding to a $K[\varepsilon]$-deformation $\rho$ is a $W[[X_{j,\mathfrak{p}}]]$-derivation for odd $j$ if and only if $\partial\boldsymbol{\rho}_n|_{I_{\mathfrak{p}}}$ is upper nilpotent, which is equivalent to $[c_\partial] \in \mathrm{Sel}_F(V)$. Thus we have $\mathrm{Sel}_F(V) \cong Der_{W[[X_{\mathfrak{p}}]]}(R_n, K) = 0$. Since $V \cong \bigoplus_{0<m\le n, m:\text{odd}} \rho_{2m,m}$ as global Galois modules, we have $\mathrm{Sel}_F(V) \cong \bigoplus_{0<m\le n, m:\text{odd}} \mathrm{Sel}_F(\rho_{2m,m})$, and we conclude $\mathrm{Sel}_F(\rho_{2m,m}) = 0$. $\square$

**Lemma 1.3.** *Let $\mathfrak{q}$ be a prime outside $p$ at which $E$ has potentially multiplicative reduction. Then for a deformation $\rho$ of $\rho_{n,0}$ satisfying* (K$_n$1–4), *the cocycle $c_\rho$ (defined in the above proof) is unramified at $\mathfrak{q}$.*

*Proof.* Since $Ad((\rho_E \otimes \eta)_{n,0}) \cong Ad(\rho_{n,0})$ twisting by a character $\eta$, we may assume that the restriction of $\rho_E$ to the inertia group $I_{\mathfrak{q}}$ has values in the upper unipotent subgroup having the form $\left(\begin{smallmatrix}1 & \xi_q(\sigma)\\ 0 & 1\end{smallmatrix}\right)$ for $\sigma \in I_{\mathfrak{q}}$ up to conjugation. Thus we may assume

$$\rho_{n,0}|_{I_{\mathfrak{q}}} = \begin{pmatrix} 1 & n\xi_q & \binom{n}{2}\xi_q^2 & \cdots & \xi_q^n \\ 0 & 1 & (n-1)\xi_q & \cdots & \xi_q^{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \xi_q \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Since $I_{\mathfrak{q}} \ni \sigma \mapsto \log(\rho_{n,0}(\sigma))$ is a homomorphism of $I_{\mathfrak{q}}$ into the Lie algebra $\mathfrak{u}_n$ of the unipotent radical of the Borel subgroup of $G_n$ containing the image of $I_\mathfrak{q}$, it factors through the tame inertia group $\cong \widehat{\mathbb{Z}}^{(q)}(1)$. By the theory of Tate curves, $\rho_{n,0}$ ramifies at $\mathfrak{q}$ and hence $\xi_q$ is nontrivial. The $p$-factor of $\widehat{\mathbb{Z}}^{(q)}$ is of rank 1 isomorphic to $\mathbb{Z}_p(1)$. Then $\rho(I_\mathfrak{q})$ is cyclic, and therefore $\dim_K \rho(I_\mathfrak{q}) = 1 = \dim_K \rho_{n,0}(I_\mathfrak{q})$. Thus the deformation $\rho$ is constant over the inertia subgroup, and hence $c_\rho$ restricted to $I_\mathfrak{q}$ is trivial. $\square$

**Corollary 1.4.** *Let $n$ be an odd positive integer. Suppose Conjecture* 0.1 *for all odd integers $m$ with $0 < m \le n$. Then we have $\dim_K \mathrm{Sel}_F^{cyc}(\rho_{2n,n}) = e$.*

*Proof.* Let $V = \rho_{2n,n}$. By Lemma 1.2, we have $\dim_K \mathrm{Sel}_F^{cyc}(Ad(\rho_{m,0})) = e \cdot \frac{m+1}{2}$. Since

$$\mathrm{Sel}_F^{cyc}(Ad(\rho_{n,0})) = \mathrm{Sel}_F^{cyc}(Ad(\rho_{n-2,0})) \oplus \mathrm{Sel}_F^{cyc}(V),$$

we find that $\dim_K \mathrm{Sel}_F^{cyc}(V) = e$. $\square$

Let $\rho_{n,m} = Sym^{\otimes n}(\rho_E)(-m)$, and write $V$ for either the representation space of $\rho_{n,m}$ or that of $Ad(\rho_{n,0})$. For each prime $\mathfrak{q} \in S \cup \{\mathfrak{p}|p\}$, we put

(1.4)

$$\overline{L}_\mathfrak{q}(V) = \begin{cases} \mathrm{Ker}(H^1(F_j, V) \to H^1(F_j, \frac{V}{\mathcal{F}^+_{\mathfrak{p}_j}(V)})) \subset L_{\mathfrak{p}_j}(V) & \text{if } \mathfrak{q} = \mathfrak{p}_j \text{ with } j \le b, \\ L_\mathfrak{q}(V) & \text{otherwise} \end{cases}$$

Once $\overline{L}_\mathfrak{q}(V)$ is defined, we define $\overline{L}_\mathfrak{q}(V^*(1)) = \overline{L}_\mathfrak{q}(V)^\perp$ under the local Tate duality between $H^1(F_\mathfrak{q}, V)$ and $H^1(F_\mathfrak{q}, V^*(1))$, where $V^*(1) = \mathrm{Hom}_K(V, \mathbb{Q}_p(1))$ as Galois modules. Then we define the balanced Selmer group $\overline{\mathrm{Sel}}_F(V)$ (resp. $\overline{\mathrm{Sel}}_F(V^*(1))$) by the same formula as in (1.3) replacing $L_\mathfrak{p}(V)$ (resp. $L_\mathfrak{p}(V^*(1))$) by $\overline{L}_\mathfrak{p}(V)$ (resp. $\overline{L}_\mathfrak{p}(V^*(1))$). By definition, $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$. We will show in Lemma 1.6, $\overline{L}_\mathfrak{p}(V) = L_\mathfrak{p}(V)$ for $V = Ad(\rho_{n,0})$ and $\rho_{2n,n}$ for odd $n$, and we actually have $\mathrm{Sel}_F(V) = \overline{\mathrm{Sel}}_F(V)$.

**Lemma 1.5.** *Let $V$ be $Ad(\rho_{n,0})$ or $\rho_{n,m}$. If $V$ is critical at $s = 1$,*

$$\text{(V)} \qquad \mathrm{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_{\mathfrak{q} \in S} \frac{H^1(F_\mathfrak{q}, V)}{L_\mathfrak{q}(V)} \times \prod_{\mathfrak{p}|p} \frac{H^1(F_\mathfrak{p}, V)}{\overline{L}_\mathfrak{p}(V)}.$$

*Proof.* Since $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$, the assumption implies $\overline{\mathrm{Sel}}_F(V) = 0$. Then the Poitou-Tate exact sequence tells us the exactness of the following sequence:

$$\overline{\mathrm{Sel}}_F(V) \to H^1(\mathfrak{G}, V) \to \prod_{\mathfrak{l} \in S \sqcup \{\mathfrak{p}|p\}} \frac{H^1(F_\mathfrak{l}, V)}{\overline{L}_\mathfrak{l}(V)} \to \overline{\mathrm{Sel}}_F(V^*(1))^*.$$

It is an old theorem of Greenberg (which assumes criticality at $s = 1$) that

$$\dim \overline{\mathrm{Sel}}_F(V) = \dim \overline{\mathrm{Sel}}_F(V^*(1))^*$$

(see [Gr] Proposition 2 or [HMI] Proposition 3.82); so, we have the assertion (V). In [HMI], Proposition 3.82 is formulated in terms of $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} V)$ and $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} V^*(1))$ defined in [HMI] (3.4.11), but this does not matter because we can easily verify $\overline{\mathrm{Sel}}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}}?) \cong \overline{\mathrm{Sel}}_F(?)$ (similarly to [HMI] Corollary 3.81). $\square$

## §1.2. Greenberg's $\mathcal{L}$-invariant

In this subsection, we let $V = \rho_{2n,n}$ or $Ad(\rho_{n,0})$ for odd $n$ (so, $V$ is critical at $s = 1$). Write $t(\mathfrak{p})$ for $\dim \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$ (thus, $t(\mathfrak{p}) = 1$ or $\frac{n+1}{2}$ according as $V = \rho_{2n,n}$ or $Ad(\rho_{n,0})$). We recall a little more detail of the $F$-version of Greenberg's definition of $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} V)$ (which is equivalent to the one given in [Gr] if we apply Greenberg's definition to $\mathrm{Ind}_F^{\mathbb{Q}} V$ as explained in [HMI] 3.4.4 without assuming the simplifying condition). Let $F_{\mathfrak{p}}^{gal}$ be the Galois closure of $F_{\mathfrak{p}}/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. Write $D_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $D_{\mathfrak{p}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ and $D_{\mathfrak{p}}^{gal} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}}^{gal})$. Write $D_L = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ for an intermediate field $L$ of $F_{\mathfrak{p}}^{gal}/\mathbb{Q}_p$. For a $D_L$-module $M$ (which is a $K$-vector space), the group $D_L$ acts on $H^\bullet(F_{\mathfrak{p}}^{gal}, M)$ naturally through the finite quotient $\mathrm{Gal}(F_{\mathfrak{p}}^{gal}/L)$. Since, for $q > 0$,

$$H^q(\mathrm{Gal}(F_{\mathfrak{p}}^{gal}/L), H^0(D_{\mathfrak{p}}^{gal}, M)) = 0,$$

by the inflation-restriction sequence, taking $L = \mathbb{Q}_p$ and $L = F_{\mathfrak{p}}$, we verify that $H^1(F_{\mathfrak{p}}^{gal}, M))^{D_p}$ is canonically isomorphic to a subspace of $H^1(F_{\mathfrak{p}}, M)$ even if $F_{\mathfrak{p}}/\mathbb{Q}_p$ is not a normal extension. We regard $H^1(F_{\mathfrak{p}}^{gal}, M)^{D_p}$ as a subspace of $H^1(F_{\mathfrak{p}}, M)$.

The long exact sequence associated to the short one $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \hookrightarrow V/\mathcal{F}_{\mathfrak{p}}^+ V \twoheadrightarrow V/\mathcal{F}_{\mathfrak{p}}^- V$ gives a homomorphism

$$H^1\left(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}\right)^{D_p} = \mathrm{Hom}\left((D_{\mathfrak{p}}^{gal})^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}\right)^{D_p} \xrightarrow{\iota_{\mathfrak{p}}} H^1(F_{\mathfrak{p}}^{gal}, V)/\overline{L}_{\mathfrak{p}}(V),$$

where $D_p$ acts on $H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$ regarding $\frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$ as the trivial $D_p$-module; so, its action on $\phi \in \mathrm{Hom}((D_{\mathfrak{p}}^{gal})^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$ is given by $\phi \mapsto \tau \cdot \phi(\sigma) = \phi(\tau\sigma\tau^{-1})$. Note that canonically

$$H^1\left(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}\right)^{D_p} \underset{\mathrm{Res}}{\xleftarrow{\sim}} \mathrm{Hom}\left(D_p^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}\right)$$
$$\cong \mathrm{Hom}\left(\mathbb{Q}_p^\times, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}\right) \cong (\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)^2 \cong K^{2t(\mathfrak{p})}$$

by $\phi \mapsto (\frac{\phi([\gamma,F_{\mathfrak{p}}])}{\log_p(\gamma)}, \phi([p,F_{\mathfrak{p}}]))$. Here, as before, $[x,F_{\mathfrak{p}}]$ is the local Artin symbol. Identifying $H^1(F_{\mathfrak{p}}^{gal}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})^{D_p}$ with $\mathrm{Hom}(D_p^{ab}, \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V})$, a homomorphism $\phi : D_p^{ab} \to \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$ in $\mathrm{Ker}(\iota_{\mathfrak{p}})$ is unramified if $\mathfrak{p} = \mathfrak{p}_i$ with $i > b$; so, the image of $\iota_{\mathfrak{p}}$ is one-dimensional (those ramified classes modulo unramified ones). In other words, the image of $\iota_{\mathfrak{p}}$ is isomorphic to $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \cong K^{t(\mathfrak{p})}$. Even if $\mathfrak{p} = \mathfrak{p}_j$ with $j \le b$, if $\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V)$, by the same argument, the image of $\iota_{\mathfrak{p}}$ is isomorphic to $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \cong K^{t(\mathfrak{p})}$. The fact $\overline{L}_{\mathfrak{p}_j}(V) = L_{\mathfrak{p}_j}(V)$ follows from the following $F$-version of the argument in [Gr] page 160:

**Lemma 1.6.** *Let $V = \rho_{2n,n}$ or $Ad(\rho_{n,0})$ for odd $n$. Then we have $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$.*

Thus for $K$-vector space $V$ with Galois action, we have $\overline{\mathrm{Sel}}_F(V) = \mathrm{Sel}_F(V)$.

*Proof.* Since we have $\overline{L}_{\mathfrak{p}}(V) = L_{\mathfrak{p}}(V)$ by definition if $\mathfrak{p} = \mathfrak{p}_j$ with $j > b$; so, we may assume that $j \le b$. Write $H^\bullet(M)$ for $H^\bullet(F_{\mathfrak{p}}, M)$ for $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$-modules $M$. We need to show the image $\overline{L}_{\mathfrak{p}}(V)$ of $H^1(\mathcal{F}_{\mathfrak{p}}^+ V)$ in $H^1(V)$ is equal to $L_{\mathfrak{p}}(V) := \mathrm{Ker}(r : H^1(V) \to H^1(I_{\mathfrak{p}}, \overline{V}))$ for $\overline{V} = V/\mathcal{F}^+ V$. We can factor the map $r$ as $r = \mathrm{Res} \circ \gamma$ for $\gamma : H^1(V) \to H^1(\overline{V})$ and $\mathrm{Res} : H^1(\overline{V}) \to H^1(I_{\mathfrak{p}}, \overline{V})$. Since $\mathrm{Ker}(\gamma) = \overline{L}_{\mathfrak{p}}(V)$, we need to show that $\mathrm{Im}(\gamma) \cap \mathrm{Ker}(\mathrm{Res}) = 0$.

Writing $Y = \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^2 V$ and $\overline{Y} = \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$, we have exact sequences of $D_{\mathfrak{p}}$-modules: $Y \hookrightarrow V/\mathcal{F}_{\mathfrak{p}}^2 V \twoheadrightarrow V/\mathcal{F}_{\mathfrak{p}}^- V$ and $\overline{Y} \hookrightarrow \overline{V} \twoheadrightarrow V/\mathcal{F}_{\mathfrak{p}}^- V$. Since $H^0(V/\mathcal{F}_{\mathfrak{p}}^- V) = 0$, by the long exact sequences of the above two short exact sequences, we find that the natural maps $H^1(Y) \to H^1(V/\mathcal{F}_{\mathfrak{p}}^2 V)$ and $H^1(\overline{Y}) \to H^1(\overline{V})$ are injective. Identify $H^1(\overline{Y})$ with its image in $H^1(\overline{V})$. We have

$$\mathrm{Im}(\gamma) = \mathrm{Im}(\overline{\gamma} : H^1(Y) \to H^1(\overline{Y})) \subset H^1(\overline{V}).$$

By the inflation-restriction sequence,

$$\mathrm{Ker}(\mathrm{Res}) = H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{V}^{I_{\mathfrak{p}}}) = \overline{V}^{I_{\mathfrak{p}}}/(Frob_{\mathfrak{p}} - 1)\overline{V}^{I_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V.$$

Similarly

$$\begin{aligned}\mathrm{Ker}(\mathrm{Res}_Y : H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) &= H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \\ &= \overline{Y}^{I_{\mathfrak{p}}}/(Frob_{\mathfrak{p}} - 1)\overline{Y}^{I_{\mathfrak{p}}} = \mathcal{F}_{\mathfrak{p}}^- Y/\mathcal{F}_{\mathfrak{p}}^+ Y = \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V.\end{aligned}$$

Thus inside $H^1(\overline{V})$, $\mathrm{Ker}(\mathrm{Res}) = \mathrm{Ker}(\mathrm{Res}_Y)$, and we may replace $V$ by $Y$ in our argument. We therefore need to show that

$$\mathrm{Im}(\overline{\gamma} : H^1(Y) \to H^1(\overline{Y})) \cap \mathrm{Ker}(\mathrm{Res} : H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) = 0.$$

We have the long exact sequence attached to the short one $\mathcal{F}^+_{\mathfrak{p}}Y \hookrightarrow Y \twoheadrightarrow \overline{Y}$:

$$0 \to \overline{Y} = H^0(\overline{Y}) \to H^1(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^1(Y) \xrightarrow{\overline{\gamma}} H^1(\overline{Y}) \to H^2(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^2(Y) = 0.$$

By the non-splitting of the short sequence, $H^0(\overline{Y})$ injects into $H^1(\mathcal{F}^+_{\mathfrak{p}}Y)$. By the local Tate duality,

$$\dim_K H^2(Y) = \dim_K H^0(\mathrm{Hom}_K(Y, K(1))) = 0 \text{ and } \dim_K H^2(\mathcal{F}^+_{\mathfrak{p}}Y) = t(\mathfrak{p}).$$

This shows that $\dim_K H^1(Y) = 2t(\mathfrak{p})d$ and $\dim_K \mathrm{Im}(\overline{\gamma}) = t(\mathfrak{p})d$, because by Kummer's theory

$$H^1(K(1)) = K \otimes_{\mathbb{Z}_p} \varprojlim_n F_{\mathfrak{p}}^\times/(F_{\mathfrak{p}}^\times)^{p^n} \cong K^{d+1}$$

and $H^1(K) \cong \mathrm{Hom}((F_{\mathfrak{p}})^\times, K) \cong K^{d+1}$ for $d = [F_{\mathfrak{p}}, \mathbb{Q}_p]$. By the inflation-restriction sequence, we have

$$L_{\mathfrak{p}}(\overline{Y}) := \mathrm{Ker}(H^1(\overline{Y}) \to H^1(I_{\mathfrak{p}}, \overline{Y})) \cong H^1(D_{\mathfrak{p}}/I_{\mathfrak{p}}, \overline{Y}^{I_{\mathfrak{p}}}) \cong \overline{Y}.$$

Thus $\dim L_{\mathfrak{p}}(\overline{Y}) + \dim \mathrm{Im}(\overline{\gamma}) = \dim H^1(I_{\mathfrak{p}}, \overline{Y})$. Thus we need to show $L_{\mathfrak{p}}(\overline{Y}) + \mathrm{Im}(\overline{\gamma}) = H^1(I_{\mathfrak{p}}, \overline{Y})$. By the local Tate duality, noting $Y^*(1) \cong Y$, this statement is equivalent to

$$\mathrm{Ker}(\delta : H^1(\mathcal{F}^+_{\mathfrak{p}}Y) \to H^1(Y)) \cap L_{\mathfrak{p}}(\overline{Y})^\perp = 0.$$

Here $L_{\mathfrak{p}}(\overline{Y})^\perp = H^1_{fl}(\mathcal{F}^+_{\mathfrak{p}}Y) = \overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O_{\mathfrak{p}}^\times/(O_{\mathfrak{p}}^\times)^{p^n} \subset H^1(\overline{Y}(1))$, because $\overline{Y}^*(1) = \overline{Y}(1) = K(1)^{t(\mathfrak{p})}$. Since $\mathrm{Ker}(\delta)$ gives rise to the subspace spanned by extension class of $K(1)^{t(\mathfrak{p})} = \mathcal{F}^+_{\mathfrak{p}}Y \hookrightarrow Y \twoheadrightarrow \overline{Y} \cong K^{t(\mathfrak{p})}$, it is given by the cocycles in $\xi_q \otimes \overline{Y}$ for the Tate period $q$ of $E$ at $\mathfrak{p} = \mathfrak{p}_j$ (where $\xi_q$ is as in the proof of Lemma 1.3). Defining $\xi_n : D_{\mathfrak{p}} \to \mu_{p^n}$ by $\xi_n(\sigma) = (q^{1/p^n})^{\sigma-1}$, the map $\xi_q = \varprojlim_n \xi_n$ having values in $\mathbb{Z}_p(1) \subset K(1)$ is an explicit form of the cocycle $\xi_q$ (see [H07] Section 4). In particular, $(\overline{Y} \otimes \xi_q) \cap H^1_{fl}(\mathcal{F}^+_{\mathfrak{p}}Y)$ is given by

$$(q \otimes \overline{Y}) \cap (\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n O_{\mathfrak{p}}^\times/(O_{\mathfrak{p}}^\times)^{p^n})$$

inside $\overline{Y} \otimes_{\mathbb{Z}_p} \varprojlim_n F_{\mathfrak{p}}^\times/(F_{\mathfrak{p}}^\times)^{p^n}$, which is trivial (because $q$ is a nonunit). □

Suppose $R_n \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then by (V) in Lemma 1.5 (and Lemma 1.2), we have a unique subspace $\mathbf{H}_F$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_{\mathfrak{p}} \mathrm{Im}(\iota_{\mathfrak{p}}) \hookrightarrow \prod_{\mathfrak{p}} \frac{H^1(F_{\mathfrak{p}}, V)}{\overline{L}_{\mathfrak{p}}(V)}.$$

Then by the restriction, $\mathbf{H}_F$ gives rise to a subspace $L = L_V$ of

$$\prod_{\mathfrak{p}} \mathrm{Hom}((D_{\mathfrak{p}}^{gal})^{ab}, \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)^{D_p} \cong \prod_{\mathfrak{p}} \mathrm{Hom}(D_p^{ab}, \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V) \cong \prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)^2$$

isomorphic to $\prod_{\mathfrak{p}} (\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V)$. If a cocycle $c$ representing an element in $\mathbf{H}_F$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$ (Lemma 1.2), this implies $c = 0$; so, the projection of $L$ to the first factor $\prod_{\mathfrak{p}} \frac{\mathcal{F}_{\mathfrak{p}}^- V}{\mathcal{F}_{\mathfrak{p}}^+ V}$ (via $\phi \mapsto (\phi([\gamma, F_{\mathfrak{p}}^{gal}])/\log_p(\gamma))_{\mathfrak{p}}$) is surjective. Thus this subspace $L$ is a graph of a $K$–linear map

$$\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V. \tag{1.5}$$

We then define $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} V) = \det(\mathcal{L}) \in K$. This is a description of the direct construction of $\mathbf{H}_F$. In the following lemma, we verify the equivalence between the earlier definition and this direct one:

**Lemma 1.7.** *Let $V = Ad(\rho_{n,0})$ or $\rho_{2m,m}$ for an odd $m > 0$, and assume that $\mathrm{Sel}_F(V) = 0$. The space $\mathbf{H}_F$ defined above consists of cohomology classes of 1-cocycles $c : \mathrm{Gal}(\overline{F}/F) \to V$ such that*

(1) *$c$ is unramified outside $p$;*

(2) *$c$ restricted to the decomposition subgroup $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}) \cong D_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}/F)$ at each $\mathfrak{p}|p$ has values in $\mathcal{F}_{\mathfrak{p}}^- V$ and $c|_{D_{\mathfrak{p}}}$ modulo $\mathcal{F}_{\mathfrak{p}}^+ V$ becomes unramified over $F_{\mathfrak{p}}[\mu_{p^\infty}]$ for all $\mathfrak{p}|p$.*

We here give a sketch of the proof, assuming $F_{\mathfrak{p}} = F_{\mathfrak{p}}^{gal}$ (leaving the general case to the attentive reader).

*Proof.* Since $Ad(\rho_{n,0}) \cong \bigoplus_{0<j\leq n, j:\mathrm{odd}} \rho_{2j,j}$, we may assume that $V = Ad(\rho_{n,0})$. Recall the decomposition groups $D_p \supset D_{\mathfrak{p}}$ in $\mathrm{Gal}(\overline{F}/\mathbb{Q})$ at $p$, and write $I_p \supset I_{\mathfrak{p}}$ for the corresponding intertia groups. Let $\mathbf{H}'_F \subset H^1(\mathrm{Gal}(\overline{F}/F), V)$ be the subspace spanned by the cohomology classes satisfying (1) and (2). Take a cocycle $c$ satisfying (1) and (2). Note that for any $\sigma \in D_p$, $\sigma(F_{\mathfrak{p}}) = F_{\mathfrak{p}}$ by our simplifying assumption. Since $\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p$ is abelian, we have $\sigma\gamma\sigma^{-1} = \gamma$ for any $\gamma \in \mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$. Since $(c|_{I_{\mathfrak{p}}} \mod \mathcal{F}_{\mathfrak{p}}^+ V) : I_{\mathfrak{p}} \to \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$ factors through $\mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$, for any $\sigma \in D_p$, $c(\sigma\gamma\sigma^{-1}) = c(\gamma)$ for any $\gamma \in I_{\mathfrak{p}}$. Since $D_p = \phi^{\mathbb{Z}} \ltimes I_p$ for a Frobenius element $\phi = Frob_p$, the cocycle $(c|_{D_p} \mod \mathcal{F}_{\mathfrak{p}}^+ V)$ is actually $D_p$-invariant. Thus $c|_{D_{\mathfrak{p}}} \mod \mathcal{F}_{\mathfrak{p}}^+ V$ is in

$H^1(F_{\mathfrak{p}}, \mathcal{F}^+_{\mathfrak{p}}V/\mathcal{F}^-_{\mathfrak{p}}V)^{D_p}$. For $\mathfrak{q} \in S$, $c|_{D_{\mathfrak{q}}}$ is unramified and vanishes on $I_{\mathfrak{q}}$; so, the restriction map in Lemma 1.5

$$\mathrm{Res} : H^1(\mathrm{Gal}(\overline{F}/F), V) \to \prod_{\mathfrak{q}\in S} \frac{H^1(F_{\mathfrak{q}}, V)}{L_{\mathfrak{q}}(V)} \times \prod_{\mathfrak{p}|p} \frac{H^1(F_{\mathfrak{p}}, V)}{L_{\mathfrak{p}}(V)}$$

brings $c$ into $\prod_{\mathfrak{p}|p} \mathrm{Im}(\iota_{\mathfrak{p}})$. Note here $L_{\mathfrak{p}}(V) = \overline{L}_{\mathfrak{p}}(V)$ by the above lemma, and hence the above map Res is the map in Lemma 1.5. Thus we conclude $\mathbf{H}'_F \subset \mathrm{Res}^{-1}(\prod_{\mathfrak{p}|p} \mathrm{Im}(\iota_{\mathfrak{p}}))$.

Conversely, we suppose that the class $[(c|_{D_{\mathfrak{p}}} \mod \mathcal{F}^+_{\mathfrak{p}}V)]$ falls in $\mathrm{Im}(\iota_{\mathfrak{p}})$. Thus the homomorphism $(c|_{D_{\mathfrak{p}}} \mod \mathcal{F}^+_{\mathfrak{p}}V) : D_{\mathfrak{p}} \to \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$ is $D_p$-invariant. Then it extends to a homomorphism $\widetilde{c}_{\mathfrak{p}} : D_p \to \mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V$. Indeed, for any two groups $G \triangleright H$ with finite index and a torsion-free divisible abelian group $X$, every $G$-invariant homomorphism $\phi : H \to X$ extends to a homomorphism $\widetilde{\phi} : G \to X$ by Schur's theory of multipliers (e.g. [MFG] 4.3.5), because the obstruction lies in $H^2(G/H, X)$ which vanishes by the finiteness of $G/H$ and divisibility of $X$. Then $\widetilde{c}_{\mathfrak{p}}$ has to factor through $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ for the maximal abelian extension $\mathbb{Q}_p^{ab}/\mathbb{Q}_p$, which is equal to $\mathbb{Q}_p^{ur}[\mu_{p^\infty}]$ for the maximal unramified extension $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$ (by local class field theory); so, $(c|_{I_p} \mod \mathcal{F}^+_{\mathfrak{p}}V)$ factors through $\mathrm{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ and $c$ satisfies (2). The condition (1) for $c|_{D_{\mathfrak{q}}}$ $(\mathfrak{q} \nmid p)$ is equivalent to the vanishing of $i_{\mathfrak{q}}(c|_{D_{\mathfrak{q}}})$ in $\frac{H^1(F_{\mathfrak{q}},V)}{L_{\mathfrak{q}}(V)}$. Then we get the reverse inclusion. Since Res is an isomorphism if $\mathrm{Sel}_F(V) = 0$ by Lemma 1.5, $\mathbf{H}'_F \xrightarrow{\mathrm{Res}} \prod_{\mathfrak{p}|p} \mathrm{Im}(\iota_{\mathfrak{p}})$ is a surjective isomorphism, and hence $\mathbf{H}_F = \mathbf{H}'_F$. □

If one restricts $c \in \mathbf{H}_F$ to $\mathfrak{G}_\infty = \mathrm{Gal}(F^{(S)}/F_\infty)$, its ramification is exhausted by $\Gamma = \mathrm{Gal}(F_\infty/F)$ (because of the definition of $\mathrm{Sel}_F^{cyc}(\rho_{2n,n})$ and $\mathbf{H}_F$) giving rise to a class $[c] \in \mathrm{Sel}_{F_\infty}(V)$. The kernel of the restriction map: $H^1(\mathfrak{G}, V) \to H^1(\mathfrak{G}_\infty, V)$ is given by $H^1(\Gamma, H^0(\mathfrak{G}_\infty, V)) = 0$ because $H^0(\mathfrak{G}_\infty, V) = 0$. Thus the image of $\mathbf{H}_F$ in $\mathrm{Sel}_{F_\infty}(V/T)$ gives rise to the order $e$ exceptional zero of $L^{arith}(s, \rho_{2n,n})$ at $s = 1$. We have reproved the first half of the following result in [Gr] Proposition 1.

**Proposition 1.8.** *Let $n$ be an odd positive integer. Suppose Conjecture* 0.1 *for all odd $m \leq n$. Then for the number $e$ of prime factors of $p$ in $F$, we have*

$$\mathrm{ord}_{s=1} L_p^{arith}(s, \rho_{2n,n}) \geq e.$$

*Further we have $\mathcal{L}(\rho_{2n,n}) = 0 \iff \mathrm{ord}_{s=1} L_p^{arith}(s, \rho_{2n,n}) > e$.*

The last assertion follows from [Gr] Proposition 3. In [Gr] Proposition 3, Conjecture 0.1 is not assumed. However, in the very definition of Greenberg's $\mathcal{L}$-invariant, the condition (V) in Lemma 1.5 is necessary as explicitly pointed out in pages 163–4 of [Gr]. As is clear from Lemma 1.2, Conjecture 0.1 supplies us the vanishing $\mathrm{Sel}_F(V) = 0$ (which is equivalent to the finiteness of Greenberg's Selmer group $S_A(\mathbb{Q})$ in [Gr]).

## §1.3. Factorization of $\mathcal{L}$-invariants

In this section, we factorize $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} \rho_{2n,n})$ and $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_{n,0}))$ for odd $n$ into the product over multiplicative places and the contribution of the good reduction part. This good reduction part gives $\mathcal{L}(n)$ for $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} \rho_{2n,n})$ in Conjecture 0.2. We keep notation introduced in the previous section; so, $V$ is either $\rho_{2n,n}$ or $Ad(\rho_{n,0})$.

**Proposition 1.9.** *Let $V$ be either $\rho_{2n,n}$ or $Ad(\rho_{n,0})$. Suppose $b > 0$, and fix an index $k$ with $1 \leq k \leq b$. Let $a \in \prod_{i=1}^{e} \mathrm{Hom}(D_{\mathfrak{p}_i}^{gal}, \mathcal{F}_{\mathfrak{p}_i}^{-} V/\mathcal{F}_{\mathfrak{p}_i}^{+} V)^{D_p}$ be induced by $c \in \mathbf{H}_F$ such that $c \in \mathbf{H}_F$ restricts down trivially to $\frac{H^1(F_i,V)}{\overline{L}_{\mathfrak{p}_i}(V)}$ for all $i \neq k$. Then we have $a([\gamma_i, F_i]) = 0$ for all $i \neq k$ and $a([p, F_{k'}]) = 0$ for all $k' \neq k$ with $k' \leq b$.*

*Proof.* For the index $k \leq b$, $\overline{L}_{\mathfrak{p}}(V)$ is exactly $\mathcal{F}_{\mathfrak{p}_k}^{+} H^1(F_k, V)$. Take a cocycle $c \in \mathbf{H}_F$ restricting down to $\frac{H^1(F_k,V)}{\overline{L}_{\mathfrak{p}_k}(V)}$ trivially to $\frac{H^1(F_i,V)}{\overline{L}_{\mathfrak{p}_i}(V)}$ for all $i \neq k$. Since $\mathbf{H}_F \cong \prod_{i=1}^{e} \mathrm{Im}(\iota_{\mathfrak{p}_i})$ by the restriction map (Lemmas 1.2 and 1.5), such cocycles $c$ form a direct summand of $\mathbf{H}_F$ isomorphic to $\mathrm{Im}(\iota_{\mathfrak{p}_k})$.

If $i > b$, $L_{\mathfrak{p}_i}(V)$ is made of classes of cocycles becoming unramified modulo those with values in $\mathcal{F}_{\mathfrak{p}_i}^{+} V$; so, even if $c|_{D_{\mathfrak{p}_i}}$ vanishes in $\frac{H^1(F_i,V)}{L_{\mathfrak{p}_i}(V)}$ (that is, $c|_{D_{\mathfrak{p}_i}} \in L_{\mathfrak{p}_i}(V)$), we cannot pull out much information on the value $a([p, F_i])$ because of the ambiguity modulo unramified cocycles with values in $\mathcal{F}_{\mathfrak{p}_i}^{-} V/\mathcal{F}_{\mathfrak{p}_i}^{+} V$. Anyway, $a([\gamma_i, F_i]) = 0$ because $[\gamma_i, F_i] \in I_{\mathfrak{p}_i}$.

For $i \leq b$ with $i \neq k$, $\overline{L}_{\mathfrak{p}_i}(V)$ is made of cocycles of $D_{\mathfrak{p}_i}$ with values in $\mathcal{F}_{\mathfrak{p}_i}^{+} V$, and the condition that $c|_{D_{\mathfrak{p}_i}} \in \overline{L}_{\mathfrak{p}_i}(V)$ implies the vanishing of $a(\sigma) = c(\sigma)$ mod $\mathcal{F}_{\mathfrak{p}_i}^{+} V$ for all $\sigma \in D_{\mathfrak{p}_i}$. This shows the last assertion: $a([p, F_{k'}]) = 0$. □

By the above lemma, we get immediately the following fact.

**Corollary 1.10.** *Let the notation be as in Proposition 1.9. Then the linear operator $\mathcal{L}$ acting on $\prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{-} V/\mathcal{F}_{\mathfrak{p}}^{+} V$ preserves the following exact sequence:*

$$0 \to \prod_{i>b} \mathcal{F}_{\mathfrak{p}_i}^{-} V/\mathcal{F}_{\mathfrak{p}_i}^{+} V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{-} V/\mathcal{F}_{\mathfrak{p}}^{+} V \to \prod_{k \leq b} \mathcal{F}_{\mathfrak{p}_k}^{-} V/\mathcal{F}_{\mathfrak{p}_k}^{+} V \to 0,$$

*and $\mathcal{L}$ acting on the quotient $\prod_{k \leq b} \mathcal{F}_{\mathfrak{p}_k}^{-} V/\mathcal{F}_{\mathfrak{p}_k}^{+} V$ sends $\mathcal{F}_{\mathfrak{p}_k}^{-} V/\mathcal{F}_{\mathfrak{p}_k}^{+} V$ into itself for each $k \leq b$.*

**Definition 1.11.** Define $\mathcal{L}(n)$ (resp. $\mathcal{L}_k(V)$) by

$$\det\left(\mathcal{L}|_{\prod_{i>b}\mathcal{F}^-_{\mathfrak{p}_i}V/\mathcal{F}^+_{\mathfrak{p}_i}V}\right) \in \mathbb{Q}_p$$

for $V = \rho_{2n,n}$ (resp. the determinant of the linear operator induced by $\mathcal{L}$ on $\prod_{\mathfrak{p}}\mathcal{F}^-_{\mathfrak{p}}V/\mathcal{F}^+_{\mathfrak{p}}V/\prod_{i\neq k}\mathcal{F}^-_{\mathfrak{p}_i}V/\mathcal{F}^+_{\mathfrak{p}_i}V$ for $V = \rho_{2n,n}$ and $V = Ad(\rho_{n,0})$).

**Corollary 1.12.** *Let the notation be as above. Then we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\rho_{2n,n}) = \mathcal{L}(n)\prod_{k=1}^{b}\mathcal{L}_k(\rho_{2n,n})$$

*for odd $n \geq 1$.*

**Proposition 1.13.** *Suppose $n = 1$. Then for $k \leq b$, we have $\mathcal{L}_k(\rho_{2,1}) = \frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}$, where $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$ for the Tate period $q_k$ of $E_{/F_k}$.*

This follows from [H07] Theorem 5.3. In [H07], the above corollary is proved by automorphic means in Section 3 of [H07], but replacing the result of [H07] Section 3 by the above factorization result, the same argument proving Theorem 5.3 there proves the above proposition.

We now generalize Proposition 1.13 to arbitrary odd $n > 1$.

**Theorem 1.14.** *Let $n$ be an odd positive integer, and assume $V = \rho_{2n,n}$. Suppose Conjecture 0.1 for all odd positive $m \leq n$. Then $\mathcal{L}_k(V) = \frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}$ for $k \leq b$, where $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$ for the Tate period $q_k$ of $E$.*

*Proof.* Fix $k \leq b$, and write $\mathfrak{p} = \mathfrak{p}_k$. Write $X_i = X_{i,\mathfrak{p}_j}$ if $i$ is odd. Define $\mathfrak{M}_\ell$ be the ideal generated by $X_i$ for odd $i \neq \ell$ and $X_\ell^2$. We fix an odd $\ell$ with $0 < \ell \leq n$, and write $\mathfrak{M}$ for $\mathfrak{M}_\ell$ and $\widetilde{K} = R_n/\mathfrak{M} \cong K[\varepsilon]$ with $\varepsilon^2 = 0$ by $X_\ell \mapsto \varepsilon$. Let $\overline{\boldsymbol{\rho}} = (\boldsymbol{\rho}_n \mod \mathfrak{M})$, and write $\overline{\boldsymbol{\delta}}_i$ for $\boldsymbol{\delta}_{i,\mathfrak{p}} \mod \mathfrak{M}$. We consider the exact sequence of $\widetilde{K}[D_\mathfrak{p}]$-modules:

$$0 \to \frac{\mathcal{F}^{i+1}_\mathfrak{p}\overline{\boldsymbol{\rho}}}{\mathcal{F}^{i+2}_\mathfrak{p}\overline{\boldsymbol{\rho}}} \to \frac{\mathcal{F}^{i}_\mathfrak{p}\overline{\boldsymbol{\rho}}}{\mathcal{F}^{i+2}_\mathfrak{p}\overline{\boldsymbol{\rho}}} \to \frac{\mathcal{F}^{i}_\mathfrak{p}\overline{\boldsymbol{\rho}}}{\mathcal{F}^{i+1}_\mathfrak{p}\overline{\boldsymbol{\rho}}} \to 0.$$

Writing $\widetilde{K}(\psi)$ for the rank one free $\widetilde{K}$-module on which $D_\mathfrak{p}$ acts by a character $\psi : D_\mathfrak{p} \to \widetilde{K}^\times$, this exact sequence gives the following exact sequence

$$0 \to \widetilde{K}(\overline{\boldsymbol{\delta}}_{i+1}) \to \frac{\mathcal{F}^{i}_\mathfrak{p}\overline{\boldsymbol{\rho}}}{\mathcal{F}^{i+2}_\mathfrak{p}\overline{\boldsymbol{\rho}}} \to \widetilde{K}(\overline{\boldsymbol{\delta}}_i) \to 0.$$

Twisting by $\overline{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}$, we get another exact sequence of $\widetilde{K}[D_{\mathfrak{p}}]$-modules:

$$0 \to \widetilde{K}(\mathcal{N}) \to M \to \widetilde{K}(\overline{\boldsymbol{\delta}}_i\overline{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}) \to 0.$$

By [H07] Lemma 5.1, this sequence gives the top row of the following commutative diagram of $D_{\mathfrak{p}}$-modules with exact rows:

$$\begin{tikzcd}
\widetilde{K}(\mathcal{N}) \arrow[r, hook] \arrow[d, "\bmod X_\ell"'] & M \arrow[r, two heads] \arrow[d, "\bmod X_\ell"'] & \widetilde{K}(\overline{\boldsymbol{\delta}}_i\overline{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}) \arrow[d, "\bmod X_\ell"] \\
K(\mathcal{N}) \arrow[r, hook] & T_pE \otimes_{\mathbb{Z}_p} K \arrow[r, two heads] & K.
\end{tikzcd}$$

Then by taking the induction from $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ to $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$, we get the following new commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} \widetilde{K}(\mathcal{N}) \arrow[r, hook] \arrow[d, "\bmod X_\ell"'] & \mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} M \arrow[r, two heads] \arrow[d, "\bmod X_\ell"'] & \mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} \widetilde{K}(\overline{\boldsymbol{\delta}}_i\overline{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}) \arrow[d, "\bmod X_\ell"] \\
\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} K(\mathcal{N}) \arrow[r, hook] & \mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} T_pE \otimes_{\mathbb{Z}_p} K \arrow[r, two heads] & \mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} K.
\end{tikzcd}$$

By [H07] Lemma 4.8, we have a unique extension $\widetilde{\boldsymbol{\delta}}_j$ of $\overline{\boldsymbol{\delta}}_j$ to $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$ with $\widetilde{\boldsymbol{\delta}}_j \equiv \mathcal{N}^{n-j} \mod \mathfrak{m}_n$. We write this extension as $\widetilde{\boldsymbol{\delta}}_j$. For any potentially ordinary $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$-module $X$, write the maximal quotient of $\mathcal{F}^+X$ on which $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$ acts by $\mathcal{N}$ as $\mathcal{F}^+X/\mathcal{F}^{11}X$. Similarly, we define $\mathcal{F}^+X \subset \mathcal{F}^{00}X \subset X$ by $\mathcal{F}^{00}X/\mathcal{F}^+X = H^0(\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p), X/\mathcal{F}^+X)$. Then the above commutative diagram yields another commutative diagram with exact rows:

$$\begin{tikzcd}
\widetilde{K}(\mathcal{N}) \arrow[r, hook] \arrow[d, "\bmod X_\ell"'] & \mathcal{F}^{00}\,\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} M/\mathcal{F}^{11}\,\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} M \arrow[r, two heads] \arrow[d, "\bmod X_\ell"'] & \widetilde{K}(\widetilde{\boldsymbol{\delta}}_i\widetilde{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}) \arrow[d, "\bmod X_\ell"] \\
K(\mathcal{N}) \arrow[r, hook] & \frac{\mathcal{F}^{00}\,\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} T_pE\otimes_{\mathbb{Z}_p}K}{\mathcal{F}^{11}\,\mathrm{Ind}_{F_{\mathfrak{p}}}^{\mathbb{Q}_p} T_pE\otimes_{\mathbb{Z}_p}K} \arrow[r, two heads] & K.
\end{tikzcd}$$

By Theorem 4.7 of [H07], this implies

$$\frac{\partial\widetilde{\boldsymbol{\delta}}_i\widetilde{\boldsymbol{\delta}}_{i+1}^{-1}\mathcal{N}}{\partial X_\ell}([Q_k,\mathbb{Q}_p]) = 0.$$

Since $\mathcal{N}([Q_k,\mathbb{Q}_p])$ is constant in $\mathbb{Q}_p^\times$, we get

$$\frac{\partial\widetilde{\boldsymbol{\delta}}_i\widetilde{\boldsymbol{\delta}}_{i+1}^{-1}}{\partial X_\ell}([Q_k,\mathbb{Q}_p]) = 0$$

which yields by the Leibnitz formula

$$\left(\delta_i^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_i}{\partial X_\ell}-\delta_{i+1}^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_{i+1}}{\partial X_\ell}\right)([Q_k,\mathbb{Q}_p])=0,$$

where $\delta_i=(\widetilde{\boldsymbol{\delta}}_i \mod \mathfrak{m}_n)=\mathcal{N}^{n-i}$. Since this holds for $i=0,1,\dots,n$, we get

$$\left(\delta_0^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_0}{\partial X_\ell}-\delta_{n}^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_{n}}{\partial X_\ell}\right)([Q_k,\mathbb{Q}_p])=0.$$

Since $\widetilde{\boldsymbol{\delta}}_0\widetilde{\boldsymbol{\delta}}_n=\mathcal{N}$ which is the unique extension of $\overline{\boldsymbol{\delta}}_0\overline{\boldsymbol{\delta}}_n=\mathcal{N}$ to $\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/\mathbb{Q}_p)$ congruent to $\mathcal{N}$ modulo $\mathfrak{m}_n$ (see Lemma 4.8 of [H07]), we have

$$\delta_0^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_0}{\partial X_\ell}=-\delta_n^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_n}{\partial X_\ell},$$

and hence

$$\delta_n^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_n}{\partial X_\ell}([Q_k,\mathbb{Q}_p])=0.$$

This in turn yields

$$\delta_i^{-1}\frac{\partial\widetilde{\boldsymbol{\delta}}_i}{\partial X_\ell}([Q_k,\mathbb{Q}_p])=0$$

for all $i=0,1,\dots,n$.

Write $Q_k=p^a u$ for $a=\mathrm{ord}_p(Q_k)$ and $u\in\mathbb{Z}_p^\times$. Then $\log_p(u)=\log_p(Q_k)$. Write $d_k=[F_k:\mathbb{Q}_p]$ and $N_k=N_{F_k/\mathbb{Q}_p}:F_k^\times\to\mathbb{Q}_p^\times$ for the norm map. Since $[p,\mathbb{Q}_p]^{d_k}=[N_k(p),\mathbb{Q}_p]=[p,F_k]|_{\mathbb{Q}_p^{ab}}$ and $[u,\mathbb{Q}_p]^{d_k}=[N_k(u),\mathbb{Q}_p]=[u,F_k]|_{\mathbb{Q}_p^{ab}}$, for odd $i$, we have

$$\begin{aligned}\widetilde{\boldsymbol{\delta}}_i([N(q_k),\mathbb{Q}_p]^{d_k})&\equiv\boldsymbol{\delta}_i([p,F_k])^a\boldsymbol{\delta}_i([u,F_k])\\&\equiv\boldsymbol{\delta}_i([p,F_k])^a(1+X_i)^{-d_j\log_p(u)/\log_p(\gamma_j)}\mod\mathfrak{M}\end{aligned}$$

(because $\mathcal{N}([u,F_{\mathfrak{p}}])=u^{-d_{\mathfrak{p}}}$ for $d_{\mathfrak{p}}=[F_{\mathfrak{p}}:\mathbb{Q}_p]$). Differentiating this identity with respect to $X_\ell$, we get from $\delta_i([p,F_k])=\mathcal{N}^{n-i}([p,F_k])=1$

$$a\frac{\partial\boldsymbol{\delta}_\ell}{\partial X_\ell}\Big|_{X=0}([p,F_k])-\frac{d_k\log_p(u)}{\log_p(\gamma_k)}=0$$

and

$$a\frac{\partial\boldsymbol{\delta}_i}{\partial X_\ell}\Big|_{X=0}([p,F_k])=0\ \text{ if odd } i\neq\ell.$$

Since $a\neq0$, we have

$$\frac{\partial\boldsymbol{\delta}_i}{\partial X_\ell}\Big|_{X=0}([p,F_k])=0\ \text{ if odd } i\neq\ell,$$

and

$$\frac{\partial \boldsymbol{\delta}_\ell}{\partial X_\ell}\Big|_{X=0}([p,F_k])d_k^{-1}\log_p(\gamma_k) = \frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}.$$

Since $\frac{\partial \boldsymbol{\rho}_n}{\partial X_\ell}\Big|_{X=0}\rho_{n,0}^{-1}$ for odd $\ell$ with $0<\ell\le n$ gives a basis of the $\mathfrak{p}$-part of $\mathbf{H}_F$ isomorphic to $\mathrm{Im}(\iota_\mathfrak{p})$, we find that $\mathcal{L}_k(Ad(\rho_{n,0})) = \left(\frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}\right)^{(n+1)/2}$. Since $\mathrm{Sel}_F^{cyc}(Ad(\rho_{n,0})) \cong \bigoplus_{0<m\le n,\ m:\mathrm{odd}}\mathrm{Sel}_F^{cyc}(\rho_{2m,m})$, we find

$$\mathcal{L}_k(Ad(\rho_{n,0})) = \prod_{0<m\le n,\ m:\mathrm{odd}}\mathcal{L}_k(\rho_{2m,m}) = \left(\frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}\right)^{(n+1)/2}.$$

By induction on $m$ starting with the case $m=1$ treated in Proposition 1.13, we find $\mathcal{L}_k(\rho_{2n,n}) = \frac{\log_p(Q_k)}{\mathrm{ord}_p(Q_k)}$ as desired. □

## §2. Proof of Conjecture 0.1 under Potential Modularity When $n$=1

We suppose that

(NS) $\overline{\rho} = E[p]$ has non-soluble image in $GL_2(\mathbb{F}_p)$;

(DS) the semi-simplification of $\overline{\rho}$ restricted to $D_\mathfrak{p}$ is non-scalar.

We now give a sketch of a proof of Conjecture 0.1 under these two conditions:

**Proposition 2.1.** *Suppose* (NS) *and* (DS)*. If there exists a totally real Galois extension $L/F$ totally split at $p$ such that $\overline{\rho}_L = \overline{\rho}|_{\mathrm{Gal}(\overline{F}/L)}$ is associated to a Hilbert modular form, then we have $R_1 \cong K[[X_{1,\mathfrak{p}}]]_{\mathfrak{p}|p}$.*

By the result of [Ta] and [Ta1], the Galois representation $\overline{\rho}$ is potentially modular in the sense that there exists a totally real Galois extension $L/F$ in which $p$ totally split and $\overline{\rho}_L$ is associated to a Hilbert cusp form of weight 2. Actually, in the above paper of Taylor, details of the proof is given for $F=\mathbb{Q}$, but we should be able to adjust his argument to prove the result for general $F$ (see [V] Theorem 1.1).

*Proof.* To indicate the dependence of $R_n$ on the base-field $L$, we write $(R_{n/L}, \boldsymbol{\rho}_{n/L})$ if we consider the universal couple of $\rho_E|_{\mathrm{Gal}(\overline{L}/F)}$ (under ($\mathrm{K}_n$1–4)). By the potential modularity assumption, $\overline{\rho}_L$ is modular. By further making a soluble base-change, by the potential level-lowering done by [SW], we may assume that $\overline{\rho}_L$ is associated to a Hilbert modular cusp form of weight 2 of level

$\Gamma_0(\mathfrak{N}p)$ satisfying the conditions (h1–4) of [HMI] page 185 for the prime-to-$p$ Artin conductor $\mathfrak{N}$ of $\overline{\rho}_L$. Then by [HMI] Corollary 3.77 and Proposition 3.78, we have $R_{1/L} \cong K[[X_{1,\mathfrak{P}}]]_{\mathfrak{P}|p}$, where $\mathfrak{P}$ runs over all prime factors of $p$ in $L$. For $\sigma \in \mathrm{Gal}(L/F)$, we take a lift $\widetilde{\sigma} \in \mathrm{Gal}(\overline{F}/L)$ inducing $\sigma$ on $L$, for any deformation $\rho$ of $\rho_E$ over $L$, we can define $\rho^\sigma(g) = \rho(\widetilde{\sigma} g \widetilde{\sigma}^{-1})$. The isomorphism class of $\rho^\sigma$ is determined independently of the choice of the lift $\widetilde{\sigma}$ and depends only on $\sigma$. Since $E$ is defined over $F$, $\rho_E^\sigma \cong \rho_E$, $\boldsymbol{\rho}_{n/L}^\sigma$ is another deformation of $\rho_E$ over $L$ satisfying ($\mathrm{K}_n$1–4). Thus we have a unique ring automorphism $[\sigma] \in \mathrm{Aut}(R_{n/L})$ such that $\boldsymbol{\rho}_{n/L}^\sigma \cong [\sigma] \circ \boldsymbol{\rho}_{n/L}$. In this way, $\Delta := \mathrm{Gal}(L/F)$ acts on $R_{n/L}$. Since $\boldsymbol{\delta}_{1,\mathfrak{P}}^\sigma(g) = \boldsymbol{\delta}_{1,\mathfrak{P}}(\widetilde{\sigma} g \widetilde{\sigma}^{-1})$ coincides with $\boldsymbol{\delta}_{1,\mathfrak{P}^\sigma}$, we have $[\sigma](X_{1,\mathfrak{P}}) = X_{1,\mathfrak{P}^\sigma}$. By the $K$-deformation version of Theorem 5.42 in [MFG], we have $R_{1/F} \cong R_{1/L}/\sum_{\sigma\in\Delta} R_{1/L}([\sigma]-1)R_{1/L}$, where $\sum_{\sigma\in\Delta} R_{1/L}([\sigma]-1)R_{1/L}$ is the ideal of $R_{1/L}$ generated by $[\sigma](r) - r$ for all $r \in R_{1/L}$. Then it is clear that $R_{1/F} \cong K[[X_{1,\mathfrak{p}}]]_{\mathfrak{p}|p}$. $\square$

*Remark* 2.1. Since the potential modularity for $\rho_{n,0}$ is proven in [Ta2] under mild assumptions, we expect that the above argument (or a modified version) would prove Conjecture 0.1 for general $n$ in near future.

## References


[CHT] L. Clozel, M. Harris and R. Taylor, Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations, preprint, 2005.

[F] K. Fujiwara, Deformation rings and Hecke algebras in totally real case, preprint, 1999, arXiv.math.NT/0602606.

[F1] ———, Galois deformations and arithmetic geometry of Shimura varieties, in *International Congress of Mathematicians. Vol. II*, 347–371, Eur. Math. Soc., Zürich, 2006.

[GeT] A. Genestier and J. Tilouine, Systèmes de Taylor-Wiles pour $GSp_4$, Astérisque No. 302 (2005), 177–290.

[Gr] R. Greenberg, Trivial zeros of $p$-adic $L$-functions, in *$p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, 149–174, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[GS] R. Greenberg and G. Stevens, $p$-adic $L$-functions and $p$-adic periods of modular forms, Invent. Math. **111** (1993), no. 2, 407–447.

[GS1] ———, On the conjecture of Mazur, Tate, and Teitelbaum, in *$p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, 183–211, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[H89] H. Hida, On nearly ordinary Hecke algebras for GL(2) over totally real fields, in *Algebraic number theory*, 139–169, Adv. Stud. Pure Math., 17, Academic Press, Boston, MA, 1989.

[H94] ———, On the critical values of $L$-functions of GL(2) and GL(2) $\times$ GL(2), Duke Math. J. **74** (1994), no. 2, 431–529.

[H00] ———, Adjoint Selmer groups as Iwasawa modules, Israel J. Math. **120** (2000), part B, 361–427. (a preprint version downloadable at `www.math.ucla.edu/~hida`)

[H02] H. Hida, Control theorems of coherent sheaves on Shimura varieties of PEL type, J. Inst. Math. Jussieu **1** (2002), no. 1, 1–76. (a preprint version downloadable at `www.math.ucla.edu/~hida`)

[H04] ———, Greenberg's $\mathcal{L}$-invariants of adjoint square Galois representations, Int. Math. Res. Not. **2004**, no. 59, 3177–3189. (a preprint version downloadable at `www.math.ucla.edu/~hida`)

[H06] ———, $\mathcal{L}$-invariant of $p$-adic $L$-functions, in *The Conference on L-Functions*, 17–53, World Sci. Publ., Hackensack, NJ, 2007. (a preprint version downloadable at `www.math.ucla.edu/~hida`)

[H07] ———, $\mathcal{L}$-invariants of Tate curves, to appear in Pure and Applied Math Quarterly **5** No.4 (2009) (a preprint version downloadable at `www.math.ucla.edu/~hida`).

[HMI] ———, *Hilbert modular forms and Iwasawa theory*, Oxford Univ. Press, Oxford, 2006.

[K] M. Kisin, Overconvergent modular forms and the Fontaine-Mazur conjecture, Invent. Math. **153** (2003), no. 2, 373–454.

[K1] ———, Geometric deformations of modular Galois representations, Invent. Math. **157** (2004), no. 2, 275–328.

[K2] ———, Moduli of finite flat group schemes, and modularity, preprint, 2005.

[M] C.-P. Mok, Exceptional zero conjectures for Hilbert modular forms, to appear in Compositio Math.

[MFG] H. Hida, *Modular forms and Galois cohomology*, Cambridge Univ. Press, Cambridge, 2000.

[MTT] B. Mazur, J. Tate and J. Teitelbaum, On $p$-adic analogues of the conjectures of Birch and Swinnerton-Dyer, Invent. Math. **84** (1986), no. 1, 1–48.

[PAF] H. Hida, *p-adic automorphic forms on Shimura varieties*, Springer, New York, 2004.

[ST] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) **88** (1968), 492–517 (Serre's Œuvres II, 472–497, No. 79).

[SW] C. M. Skinner and A. J. Wiles, Base change and a problem of Serre, Duke Math. J. **107** (2001), no. 1, 15–25.

[SW1] ———, Nearly ordinary deformations of irreducible residual representations, Ann. Fac. Sci. Toulouse Math. (6) **10** (2001), no. 1, 185–215.

[T] J. Tate, A review of non-Archimedean elliptic functions, in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, 162–184, Int. Press, Cambridge, MA, 1995.

[Ta] R. Taylor, Remarks on a conjecture of Fontaine and Mazur, J. Inst. Math. Jussieu **1** (2002), no. 1, 125–143.

[Ta1] ———, On the meromorphic continuation of degree two $L$-functions, Doc. Math. **2006**, Extra Vol., 729–779 (electronic).

[Ta2] ———, Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations II, preprint, 2006.

[TaW] R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, Ann. of Math. (2) **141** (1995), no. 3, 553–572.

[Ti] J. Tilouine, *Deformations of Galois representations and Hecke algebras*, Published for The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1996.

[Ti1] ———, Nearly ordinary rank four Galois representations and $p$-adic Siegel modular forms, Compos. Math. **142** (2006), no. 5, 1122–1156.

[V] C. Virdol, Non-solvable base change for Hilbert modular forms and zeta functions of twisted quaternionic Shimura varieties, preprint (downloadable at `http://www.math.columbia.edu/~virdol/research.html`).

[W] A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. of Math. (2) **141** (1995), no. 3, 443–551.