

# Representation of Spaces of Entire Functions on Banach Spaces

By

José M. ANSEMIL\*, Richard M. ARON\*\* and Socorro PONTE\*\*\*

## Abstract

We prove the impossibility of expressing the space of entire functions on any infinite dimensional complex Banach space with a Schauder basis  $E$  as a countable union of spaces of entire functions that are bounded on countable open covers of  $E$ .

## §1. Notation and Preliminaries

In the study of the space of holomorphic functions on infinite dimensional complex normed spaces  $\mathcal{H}(E)$ , several natural locally convex topologies appear. Among them, the three most important are the compact open topology  $\tau_0$ , the Nachbin “ported” topology  $\tau_\omega$ , and the bornological topology  $\tau_\delta$  associated to these two topologies. One useful characterization of the  $\tau_\delta$  topology, introduced independently by Nachbin [6] and Cœuré [2] about 30 years ago, is that it is generated by all seminorms  $p : \mathcal{H}(E) \rightarrow \mathbb{C}$  which satisfy the following property:

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\*Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain.

e-mail: [jm\\_ansemil@mat.ucm.es](mailto:jm_ansemil@mat.ucm.es)

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\*\*Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242, USA.

e-mail: [aron@math.kent.edu](mailto:aron@math.kent.edu)

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\*\*\*Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain.

e-mail: [socorro\\_ponte@mat.ucm.es](mailto:socorro_ponte@mat.ucm.es)

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For every increasing countable open cover  $\Theta = \{V_j \mid j \in \mathbb{N}\}$  of  $E$ , there is a positive constant  $C$  and a positive natural number  $j_0$  such that

$$p(f) \leq C \sup_{x \in V_{j_0}} |f(x)| \text{ for all } f \in \mathcal{H}(E).$$

In fact, it is easy to see that the  $\tau_\delta$  topology on  $\mathcal{H}(E)$  can be defined as an inductive limit of the metrizable spaces  $\mathcal{H}_\Theta(E) = \{f \in \mathcal{H}(E) \mid f \text{ is bounded on each } V_j\}$  (with the topology of uniform convergence on the  $V_j$ 's considered and where  $\Theta$  ranges over the family of all increasing countable open covers of  $E$ ). Note that given  $f \in \mathcal{H}(E)$ , then  $f \in \mathcal{H}_\Theta(E)$  where  $\Theta$  is the increasing open cover of  $E$  given by

$$\Theta = \{V_1, V_2, \dots\}$$

with  $V_j = \{x \in E : |f(x)| < j\}$ .

It is natural to wonder whether, in fact,  $(\mathcal{H}(E), \tau_\delta)$  is a *countable* inductive limit of such spaces. The aim of this paper is to prove that this inductive limit cannot be such a countable union, at least in the case when  $E$  is any infinite dimensional complex Banach space with a Schauder basis.

In our main result, below, we will be discussing sequences  $\{\Theta_n\}_{n=0}^\infty$  of increasing countable open covers of  $E$ . That is, for every  $n$ ,

$$\Theta_n = \{V_{n,1}, V_{n,2}, \dots\},$$

where each  $V_{n,j}$  is open,  $V_{n,j} \subset V_{n,j+1}$  for all  $j$  and  $\cup_{j=1}^\infty V_{n,j} = E$ . As usual, for a function  $h$  and a set  $V$ ,  $\|h\|_V = \sup_{x \in V} |h(x)|$ .

## §2. Countable Unions of Spaces of Entire Functions

Our main aim in this paper is to prove the result mentioned in the abstract:

**Theorem.** For any infinite dimensional complex Banach space  $E$  with a Schauder basis and for any sequence  $\{\Theta_n\}_{n=0}^\infty$  of increasing countable open covers of  $E$ ,

$$\mathcal{H}(E) \neq \cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E).$$

In fact, we will show the density of  $\mathcal{H}(E) \setminus \cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E)$  in the space of all entire functions  $\mathcal{H}(E)$ .

The proof of the theorem will use the following two propositions that we believe are also interesting in themselves. The first proposition is an adaptation of an argument of the second author in [1].

**Proposition 2.1.** *Let  $E$  be an infinite dimensional Banach space with a normalized Schauder basis  $\{e_n\}$  with basis constant  $M$ . Then given  $s > 0$ , there is an entire function  $h$  on  $E$  such that  $\|h\|_{B(0, \frac{1}{3M})} < \infty$  and such that for every  $p \in \mathbb{N}$  there is  $q \in \mathbb{N}$  such that  $\|h\|_{B(e_q, \frac{s}{2^p})} = \infty$ .*

*Proof.* Let us fix a natural number  $\gamma_1$  such that  $\frac{1}{2^{\gamma_1}} < s$  and consider the sequence  $\{\gamma_j\}_{j=2}^\infty$  where  $\gamma_j = \gamma_1 + j$ . For each  $n \in \mathbb{N}$ , let  $\phi_n : E \rightarrow \mathbb{C}$  be given by  $\phi_n(x) = 2x_n$ , where  $x = \sum_{n=1}^\infty x_n e_n \in E$ . Note that as we assume that the projections  $x \mapsto x_n$  have norm less than or equal to  $M$ , then our  $\phi_n$  have norm less or equal than  $2M$ . Write  $\mathbb{N} = \cup_j \mathbb{N}_j$  where each  $\mathbb{N}_j$  is an increasing infinite sequence and  $\mathbb{N}_j \cap \mathbb{N}_k = \emptyset$  for  $j \neq k$ . Let  $\mathbb{N}_j = \{n_1^j, n_2^j, \dots\}$ , where we assume that  $n_1^j = 1$ , and let  $h$  be the function defined on  $E$  by

$$h(x) = \sum_{j=1}^\infty \sum_{k=2}^\infty [(\phi_{n_1^j}(x))^{\gamma_j} \phi_{n_k^j}(x)]^{jk}.$$

The function  $h$  is holomorphic because the series converges uniformly on compact subsets of  $E$ . Moreover,  $h$  is bounded in  $B(0, \frac{1}{3M})$ . Indeed for  $x \in B(0, \frac{1}{3M})$  we have

$$\begin{aligned} |h(x)| &\leq \sum_{j=1}^\infty \sum_{k=2}^\infty |(\phi_{n_1^j}(x))^{\gamma_j} \phi_{n_k^j}(x)|^{jk} \leq \\ &\sum_{j=1}^\infty \sum_{k=2}^\infty [(2M\|x\|)^{\gamma_j} 2M\|x\|]^{jk} \leq \\ &\sum_{j=1}^\infty \sum_{k=2}^\infty \left[ \left(\frac{2}{3}\right)^{\gamma_j} \frac{2}{3} \right]^{jk} \leq \sum_{j=1}^\infty \sum_{k=2}^\infty \left(\frac{4}{9}\right)^{kj} < \infty. \end{aligned}$$

Given  $p \in \mathbb{N}$ , take another natural number  $t \geq 2$  and consider

$$w_{p,t} := e_{n_1^p} + \frac{s}{2^{p+1}} e_{n_t^p}$$

Then we have

$$h(w_{p,t}) = \sum_{j=1, j \neq p}^\infty \sum_{k=2}^\infty [(\phi_{n_1^j}(w_{p,t}))^{\gamma_j} \phi_{n_k^j}(w_{p,t})]^{jk} + \sum_{k=2}^\infty [(\phi_{n_1^p}(w_{p,t}))^{\gamma_p} \phi_{n_k^p}(w_{p,t})]^{pk}$$

The first sum is 0 since  $j \neq p$ , and so

$$h(w_{p,t}) = \sum_{k=2}^\infty [(\phi_{n_1^p}(w_{p,t}))^{\gamma_p} \phi_{n_k^p}(w_{p,t})]^{pk} = \left[ (2 \cdot 1)^{\gamma_p} \frac{s}{2^{p+1}} \cdot 2 \right]^{pt}$$

which goes to  $\infty$  with  $t$  because  $2^{\gamma_p \frac{s}{2^p}} > 1$ . Since  $w_{p,t} \in B(e_{n_1^p}, \frac{s}{2^p})$  for all  $t \in \mathbb{N}$ , we have that  $\|h\|_{B(e_{n_1^p}, \frac{s}{2^p})} = \infty$ , this proves the result if we take  $q = n_1^p$ .  $\square$

**Proposition 2.2.** *Let  $E$  be an infinite dimensional Banach space with a normalized Schauder basis  $\{e_n\}$  with basis constant  $M$ . For every  $R > 0$  and  $r > 0$ , there exists an entire function  $f$  such that*

$$\|f\|_{B(0,R)} < \infty \text{ and } \|f\|_{B(3MR e_1, r)} = \infty.$$

*Proof.* Given  $R$  and  $r$  let us define a function  $f$  on  $E$  by

$$f(x) = h\left(\frac{x}{3MR}\right)$$

being  $h$  the function associated with  $s = \frac{r}{3MR}$  in the above proposition. Then  $\|f\|_{B(0,R)} = \|h\|_{B(0, \frac{1}{3M})} < \infty$ . On the other hand, if we take  $p = 1$ , the proof of that proposition gives  $q = n_1^1 = 1$ , which implies that  $\|h\|_{B(e_1, \frac{r}{2})} = \infty$ . Therefore since for  $x \in B(e_1, s)$  we have  $3MRx \in B(3MR e_1, r)$ , we get

$$\|f\|_{B(3MR e_1, r)} \geq \|h\|_{B(e_1, s)} = \infty.$$

□

*Remark.* It is not known if for any pair of disjoint balls  $B_1$  and  $B_2$  in an infinite dimensional normed space  $E$  there is an entire function  $f$  such that

$$\|f\|_{B_1} < \infty \text{ and } \|f\|_{B_2} = \infty.$$

This is somewhat surprising, in light of results such as [1] that prove the existence of a dense set (for all the topologies we are considering) of entire functions  $f$  such that for every  $\varepsilon > 0$ , there is  $x_\varepsilon \in E$ ,  $\|x_\varepsilon\| \leq 1$ , satisfying  $\|f\|_{B(x_\varepsilon, \varepsilon)} = \infty$ . For the normed spaces for which this occurs, we have the following.

**Lemma 2.1.** *Given  $R > 0$  and  $0 < r < R$ , let  $f \in \mathcal{H}(E)$  be chosen to satisfy the conclusion of the above proposition. Then for every  $n \in \mathbb{N}$ , there is  $g_n \in \mathcal{H}(E)$  such that*

$$\|g_n f\|_{B(0,R)} < \frac{1}{2^n} \text{ and } \|g_n f\|_{B(3MR e_1, r)} = \infty.$$

*Proof.* Let  $x_1 = 3MR e_1$ . By the Hahn Banach theorem there are  $u \in (E)_{\mathbb{R}}^*$  and  $\alpha > 0$  such that

$$u(a) < \alpha \leq u(b) \text{ for all } a \in B(0, R) \text{ and } b \in B(x_1, r).$$

Note that  $u(0) = 0$ . Let us consider the functional  $v \in E^*$  given by  $v(x) = u(x) - iu(ix)$  and

$$g_n = \frac{v}{2^n \|f\|_{B(0,R)} \cdot \|v\|_{B(0,R)}}.$$

Then  $g_n \in E^* \subset \mathcal{H}(E)$ ,

$$\|g_n f\|_{B(0,R)} = \left\| \frac{vf}{2^n \|f\|_{B(0,R)} \cdot \|v\|_{B(0,R)}} \right\|_{B(0,R)} \leq \frac{1}{2^n}$$

and

$$\begin{aligned} \|g_n f\|_{B(x_1,r)} &= \left\| \frac{vf}{2^n \|f\|_{B(0,R)} \cdot \|v\|_{B(0,R)}} \right\|_{B(x_1,r)} \geq \\ &= \frac{\alpha}{2^n \|f\|_{B(0,R)} \cdot \|v\|_{B(0,R)}} \|f\|_{B(x_1,r)} = \infty. \end{aligned}$$

□

*Proof of the Theorem.* There is no loss of generality if we assume that the Schauder basis is normalized. Given a countable collection of increasing open covers  $\{\Theta_n\}_{n=0}^\infty$ , where each  $\Theta_n = \{V_{n,j} \mid j \in \mathbb{N}\}$ , we will prove the existence of an entire function  $f$  on  $E$  such that for every  $n$  there is  $j_n$  such that  $\|f\|_{V_{n,j_n}} = \infty$ . As a consequence,  $f$  will not belong to  $\mathcal{H}_{\Theta_n}(E)$  for any  $n$ .

Since  $\cup_{j=1}^\infty V_{0,j} = E$ , there is  $j_0 \in \mathbb{N}$  such that  $0 \in V_{0,j_0}$  and so there is  $R_0 > 0$  such that  $B(0, R_0) \subset V_{0,j_0}$ . It is well-known (see [3], pp. 157-158) that there is an entire function  $f_0$  in  $E$  such that  $\|f_0\|_{B(0,R_0)} = \infty$ .

Let  $x_1 = 3MR_0e_1 \in E$ . Since  $\cup_{j=1}^\infty V_{1,j} = E$ , there is  $j_1 \in \mathbb{N}$  such that  $x_1 \in V_{1,j_1}$  and then, since  $V_{1,j_1}$  is open and  $f_0$  is continuous, there is  $r_1 > 0$  such that  $r_1 < R_0$ ,  $B(x_1, r_1) \subset V_{1,j_1}$  and  $\|f_0\|_{B(x_1,r_1)} < \infty$ .

By Proposition 2.2 we get a function  $f_1 \in \mathcal{H}(E)$  such that

$$\|f_1\|_{B(0,R_0)} < \infty \text{ and } \|f_1\|_{B(x_1,r_1)} = \infty.$$

In this way we obtain sequences:  $\{R_n\} \subset \mathbb{R}^+$  increasing to  $\infty$ ,  $\{x_n\} \subset E$ ,  $\{j_n\} \subset \mathbb{N}$ ,  $\{r_n\} \subset \mathbb{R}^+$ , and  $\{f_n\} \subset \mathcal{H}(E)$  having the following properties: For every  $n \geq 1$ ,  $R_n = (3M + 1)R_{n-1}$ ,  $x_n = 3MR_{n-1}e_1$ ,  $r_n < R_0$ ,  $B(x_n, r_n) \subset V_{n,j_n}$ ,  $\|f_j\|_{B(x_n,r_n)} < \infty$  for all  $j = 0, \dots, n - 1$ ,  $\|f_n\|_{B(0,R_{n-1})} < \infty$ , and  $\|f_n\|_{B(x_n,r_n)} = \infty$ .

By an application of Lemma 2.1 we get, for every  $n \in \mathbb{N}$ , a function  $g_n \in \mathcal{H}(E)$  such that

$$\|g_n f_n\|_{B(0,R_{n-1})} < \frac{1}{2^n} \text{ and } \|g_n f_n\|_{B(x_n,r_n)} = \infty.$$

To conclude the proof, let us verify that the function

$$f = f_0 + g_1 f_1 + g_2 f_2 + \dots$$

satisfies our requirements.

Given a ball  $B$  in  $E$ , choose  $R_k$  in the above sequence such that  $B \subset B(0, R_k)$ . Then

$$\sum_{j=k+1}^{\infty} \|g_j f_j\|_B \leq \sum_{j=k+1}^{\infty} \|g_j f_j\|_{B(0, R_k)} \leq \sum_{j=k+1}^{\infty} \frac{1}{2^j}$$

which implies that the series that defines  $f$  is uniformly convergent on balls, and so  $f$  is an entire function on  $E$ .

On the other hand, since

$$\|f_j\|_{B(x_n, r_n)} < \infty$$

for every  $n \in \mathbb{N}$  and  $j = 0, \dots, n - 1$ , we have that  $\|f_0\|_{B(x_n, r_n)} < \infty$  and  $\|g_j f_j\|_{B(x_n, r_n)} < \infty$  for all  $j \leq n - 1$ . Moreover for  $j = n + 1, n + 2, \dots$  we have that  $B(x_n, r_n) \subset B(0, R_{j-1})$  and so

$$\|g_j f_j\|_{B(x_n, r_n)} \leq \|g_j f_j\|_{B(0, R_{j-1})} < \frac{1}{2^j}.$$

This implies that

$$\|f\|_{B(x_n, r_n)} = \left\| f_0 + \sum_{j=1}^{n-1} g_j f_j + g_n f_n + \sum_{j=n+1}^{\infty} g_j f_j \right\|_{B(x_n, r_n)} \geq \|g_n f_n\|_{B(x_n, r_n)} - \left( \|f_0\|_{B(x_n, r_n)} + \sum_{j=1}^{n-1} \|g_j f_j\|_{B(x_n, r_n)} + \sum_{j=n+1}^{\infty} \|g_j f_j\|_{B(x_n, r_n)} \right) = \infty$$

because  $\|g_n f_n\|_{B(x_n, r_n)} = \infty$ . Since  $B(x_n, r_n)$  is contained in  $V_{n, j_n}$  we have that

$$\|f\|_{V_{n, j_n}} = \infty$$

and also

$$\|f\|_{V_{0, j_0}} = \infty$$

because  $\|f_0\|_{V_{0, j_0}} = \infty$  and  $\sum_{j=1}^{\infty} \|g_j f_j\|_{B(0, R_0)} < \infty$ . Therefore

$$f \notin \mathcal{H}_{\Theta_n}(E) \text{ for all } n = 0, 1, 2, \dots$$

□

In the next proposition we are going to see that not only is

$$\mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E)$$

non-void, but it is even dense in  $\mathcal{H}(E)$  for the strongest topology we are considering on  $\mathcal{H}(E)$ , namely  $\tau_\delta$ .

**Proposition 2.3.** *Let  $E$  be an infinite dimensional Banach space with a Schauder basis. For any sequence  $\{\Theta_n\}_{n=0}^\infty$  of increasing countable open covers of  $E$ , the set*

$$\mathcal{H}(E) \setminus \bigcup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E)$$

is dense in  $(\mathcal{H}(E), \tau_\delta)$ .

*Proof.* Without loss of generality, we may assume that the sequence  $\{\Theta_n\}_{n=0}^\infty$  is decreasing in  $n$ . That is, for every  $n$  and  $j$  we assume  $V_{n,j} \supset V_{n+1,j}$ . Indeed, if this is not the case it is enough to consider a new sequence  $\{\Phi_n\}_{n=0}^\infty$  where  $\Phi_n = \{W_{n,j}\}_{j=1}^\infty$  and  $W_{n,j} = \bigcap_{k=0}^n V_{k,j}$  for all  $j$ . Note that each  $\Phi_n$  is an increasing open covering of  $E$ . (Given  $n \in \mathbb{N}, x \in E$  and  $k = 0, 1, \dots, n$ , let  $j_k$  be chosen so that  $x$  belongs to  $V_{k,j_k}$ ; then  $x$  belongs to  $W_{n, \max\{j_k: k=0, \dots, n\}}$ .) Note also that  $\mathcal{H}(E) \setminus \bigcup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E) \supset \mathcal{H}(E) \setminus \bigcup_{n=0}^\infty \mathcal{H}_{\Phi_n}(E)$ . This assumption allows us to ignore any finite number of terms in the expression of the function  $f$  in the above theorem and this will be used later.

Let us prove first that for every continuous seminorm  $p$  in  $(\mathcal{H}(E), \tau_\delta)$  and for every  $\varepsilon > 0$  there exists  $g_{p,\varepsilon} \in \mathcal{H}(E) \setminus \bigcup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E)$  such that  $p(g_{p,\varepsilon}) < \varepsilon$ . To do this, we will make use of the functions  $f_n, g_n$  and real numbers  $R_m$  that appear in the proof of the theorem.

With the notation of Lemma 2.1 let  $\Theta = \{\overset{\circ}{A}_j\}_{j=1}^\infty$ , where  $A_j = \{x \in E : |f_k(x)g_k(x)| < \frac{1}{2^k} \text{ for all } k \geq j\}$ . We claim that the collection  $\Theta$  is an increasing open cover of  $E$ . Indeed, every  $x \in E$  is in one of the balls  $B(0, R_{j-1})$ , and then for every  $k \geq j$  we have,

$$(1) \quad \|f_k g_k\|_{B(0, R_{j-1})} \leq \|f_k g_k\|_{B(0, R_{k-1})} < \frac{1}{2^k}.$$

This implies that  $B(0, R_{j-1}) \subset A_j$  and so  $x \in \overset{\circ}{A}_j$ . By the definition of the  $\tau_\delta$  topology, given  $p$  there is a positive constant  $C$  and a natural number  $j_0$  such that

$$(2) \quad p(g) \leq C \|g\|_{\overset{\circ}{A}_{j_0}} \text{ for all } g \in \mathcal{H}(E).$$

Given  $\varepsilon$  assume that the above  $j_0$  satisfies  $\sum_{j=j_0}^\infty \frac{1}{2^j} < \frac{\varepsilon}{C}$ . We observe that the function  $g_{p,\varepsilon}$  in  $\mathcal{H}(E) \setminus \bigcup_{n=0}^\infty \mathcal{H}_{\Theta_j}(E)$ , given by

$$g_{p,\varepsilon} = g_{j_0} f_{j_0} + g_{j_0+1} f_{j_0+1} + \dots,$$

satisfies  $p(g_{p,\varepsilon}) < \varepsilon$ . Indeed, from (1) and (2) we get that

$$p(g_{p,\varepsilon}) \leq C \sum_{j=j_0}^\infty \|g_j f_j\|_{\overset{\circ}{A}_{j_0}} \leq C \sum_{j=j_0}^\infty \frac{1}{2^j} < \varepsilon.$$

Finally, let  $f \in \mathcal{H}(E)$ , a  $\tau_\delta$  seminorm  $p$ , and  $\epsilon > 0$  be given, and take  $j_0 \in \mathbb{N}$  as in the above paragraph. Let us assume that  $f \notin \mathcal{H}(E) \setminus \cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E)$ . There is  $n_0 \in \mathbb{N}$ , that we may assume to be greater than  $j_0$ , such that  $f \in \mathcal{H}_{\Theta_{n_0}}(E)$ . By our assumption on the open covers, it follows that  $f \in \mathcal{H}_{\Theta_n}(E)$ , and so  $\|f\|_{V_{n,j}} < \infty$  for all  $n \geq n_0$  and  $j \in \mathbb{N}$ . Given  $p$  and  $\epsilon$ , we have that

$$f + g_{p,\epsilon} \in \mathcal{H}(E) \setminus \cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E).$$

To see this, if  $n \geq n_0$  there is  $j_n$  as in the proof of the theorem such that

$$\|f + g_{p,\epsilon}\|_{V_{n,j_n}} \geq \|g_{p,\epsilon}\|_{V_{n,j_n}} - \|f\|_{V_{n,j_n}} = \infty.$$

Also,  $V_{1,j_{n_0}} \supset V_{2,j_{n_0}} \supset \cdots \supset V_{n_0,j_{n_0}}$  and therefore  $\|f + g_{p,\epsilon}\|_{V_{n,j_{n_0}}} = \infty$  for  $n = 0, \dots, n_0 - 1$ , which implies that  $f + g_{p,\epsilon} \in \mathcal{H}(E) \setminus \cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E)$ . Thus,  $p(f + g_{p,\epsilon} - f) = p(g_{p,\epsilon}) < \epsilon$ .  $\square$

*Remarks.*

1. The above arguments can also be applied to normed spaces with a Schauder basis such that the projections are uniformly bounded. For instance, this happens for  $c_{00}$ , the space of all eventually null sequences with any of the  $p$  norms, and for every normed space which is dense in a Banach space with a Schauder basis, since by the Perturbation of Basis Theorem these spaces also have a Schauder basis with uniformly bounded projections (see, e.g., [4, Prop. 1.a.9.]). In particular this applies to  $C[0, 1]$  with any  $p$  norm,  $p \in [1, \infty)$ , since they are normed spaces which are dense in  $L^p[0, 1]$  which has a Schauder basis.

2. As we have shown in the proof of Proposition 2.3, from a given sequence  $\{\Theta_n\}$  of increasing open coverings of  $E$ , we can construct another sequence  $\{\Phi_n\}$  of increasing open coverings of  $E$  such that for all  $n$ ,

$$\mathcal{H}_{\Phi_n}(E) \subset \mathcal{H}_{\Phi_{n+1}}(E)$$

with continuous inclusions and  $\cup_{n=0}^\infty \mathcal{H}_{\Theta_n}(E) \subset \cup_{n=0}^\infty \mathcal{H}_{\Phi_n}(E)$ . Then our Theorem proves that the  $(LF)$ -space  $\varinjlim \mathcal{H}_{\Phi_n}(E)$  does not agree with  $(H(E), \tau_\delta)$ .  $(LF)$ -spaces have been studied and applied to holomorphy by several authors, particularly Mujica in [5].

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