

Counting Lines and Conics on a Surface

Dedicated to Professor Friedrich Hirzebruch on his 80th birthday

By

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§1. Main Results

This short note is a supplementary remark to the author's article [7]. Our main results are the following two propositions concerning the number of rational curves and elliptic curves on polarized complex algebraic surfaces:

Proposition A. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{\frac{h}{2}+1}$ be a K3 surface of degree $h \geq 4$ and let $r_d = r_d(X)$ denote the number of rational curves of degree d on X , $d \in \mathbb{Z}_{>0}$. If $h > 4N^2$ for a positive integer N , then $r_1 + 2r_2 + \cdots + Nr_N \leq \frac{24Nh}{h - 4N^2}$. In particular, $r_1 \leq \frac{24h}{h - 4}$ for $h \geq 6$ and $r_2 \leq \frac{24h}{h - 16}$ for $h \geq 18$.*

Proposition B. *Let $X \subset \mathbb{P}_{\mathbb{C}}^N$ be a canonically embedded surface of degree K^2 and put $\sigma = c_2/K^2 \geq 1/3$ (as usual, K and c_2 stand for the canonical divisor and the topological Euler number of X). Let $r_d = r_d(X)$ be the number of rational curves of degree d on X and let $s = s(X) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denote the sum $\sum_C KC$ of the degrees of the elliptic curves $C \subset X$.*

(1) *Assume that $\sigma < 1 + \frac{4}{N} + \frac{6}{N^2}$ for some positive integer N . Then*

$$\sum_{d=1}^N dr_d \leq \frac{(3\sigma - 1) \left(1 + \frac{2}{N}\right)}{1 - \sigma + \frac{4}{N} + \frac{6}{N^2}} K^2. \text{ For instance } r_1 \leq \frac{9\sigma - 3}{11 - \sigma} K^2 \text{ if } \sigma < 11$$

(i.e., if X is not a quintic $\subset \mathbb{P}^3$) and $r_2 \leq \frac{6\sigma - 2}{9 - 2\sigma} K^2$ if $\sigma < \frac{9}{2}$.

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(2) If $\sigma < 1$, then $s \leq \frac{3\sigma - 1}{1 - \sigma} K^2$.

Proof of Propositions A and B. Let X be a smooth complex projective surface of non-negative Kodaira dimension. Pick a rational number $\alpha \in [0, 1]$ and a finite sum $C = \sum C_i$ of distinct irreducible curves $C_i \subset X$ of geometric genera g_i . In [7], we defined an “orbibundle” $\tilde{\mathcal{E}}_\alpha$ attached to the triplet (X, C, α) and showed the Miyaoka-Yau-Sakai inequality $3c_2(\tilde{\mathcal{E}}_\alpha) \geq c_1^2(\tilde{\mathcal{E}}_\alpha)$ or, more explicitly,

$$(\dagger) \quad \frac{\alpha^2}{2} (C^2 + 3CK - 6(g - 1)) - 2\alpha (CK - 3(g - 1)) + 3c_2 - K^2 \geq 0.$$

Here $g - 1$ is construed as the sum $\sum (g_i - 1)$; see *ibid.*, §1, Remark G.

Fix a very ample divisor H of degree $h = H^2$ on X . Assume that C_i is rational or elliptic (*i.e.*, $g_i - 1 = -1$ or 0). The formula (\dagger) involves four parameters α, C^2, CK and $g - 1$ (= the number of rational curves with opposite sign). Denoting by δ the total degree $CH = \sum C_i H$, we bound C^2 by δ^2/h (the Hodge index theorem). This substitution simplifies (\dagger) in two important cases:

CASE A: (X, H) is a polarized K3 surface ($K = 0$) of degree h . Then

$$\alpha^2 \left(\frac{\delta^2}{h} - 6(g - 1) \right) + 12\alpha(g - 1) + 144 \geq 0 \quad \text{for } \alpha \in [0, 1].$$

CASE B: X is a canonical surface ($H = K$). For $\alpha \in [0, 1]$, we have

$$\alpha^2 \left(\frac{\delta^2}{K^2} + 3\delta - 6(g - 1) \right) - 2\alpha(2\delta - 6(g - 1)) + 6c_2 - 2K^2 \geq 0.$$

We view the left hand sides of these two inequalities as quadratic functions $Q_1(\alpha), Q_2(\alpha)$ of α , which attain the minima at $\alpha_1, \alpha_2 \in [0, 1] \cap \mathbb{Q}$. We readily elicit Propositions A and B from the inequalities $Q_i(\alpha_i) \geq 0, i = 1, 2$. □

Remarks. (1) A general projective K3 surface X is known to carry countably many nodal rational curves [3] as well as a one-parameter family of nodal elliptic curves [8]; *i.e.*, $\sum_d^\infty dr_d(X) = \infty, s(X) = \infty$.

(2) The author has no idea how close to the best possible our estimates are. Nor does he know if there are any precedent results, apart from a handful of treatises that study either lines on surfaces in $\mathbb{P}_\mathbb{C}^3$ [11], [12], [2], [1] or configurations of disjoint smooth rational curves [6], [9], [10]. As shown in §§2 and 3, the inequality (\dagger) is optimal for countably many examples.

(3) Unfortunately, (†) does not say anything about the classical problem of counting lines on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. We are luckier when dealing with lines on complete intersections of codimension two or more; elementary, but cumbersome, calculation of explicit bounds is left to the reader.

§2. Two Examples

Take four points P_1, P_2, P_3, P_4 in general position on \mathbb{P}^2 . The line L_{ij} connecting P_i and P_j is defined by a linear form λ_{ij} , and the ratios $\lambda_{ij}/\lambda_{kl}$ are rational functions on \mathbb{P}^2 . Fixing a positive integer $n \geq 2$, the n -th roots of these ratios define a Kummer extension K_n of $\mathbb{C}(\mathbb{P}^2)$ with Galois group $(\mathbb{Z}/n\mathbb{Z})^{\oplus 5}$ (cf. Hirzebruch [5]).

Let $X_1 \rightarrow \mathbb{P}^2$ be the blowing up at the four points P_i . X_1 is a del Pezzo surface of degree five (unique up to isomorphisms). Denote by $E_i \subset X_1$ the exceptional curve over P_i and let $\tilde{L}_{ij} \subset X_1$ be the strict transform of the line $L_{ij} \subset \mathbb{P}^2$. The reduced curve $D = \bigcup_{i=1}^4 E_i \cup \bigcup_{i,j} \tilde{L}_{ij}$ is the union of all the (-1) -curves on X_1 and linearly equivalent to $-2K_{X_1}$. D has 15 double points and its smooth part consists of 10 components, each of which isomorphic to \mathbb{P}^1 minus three points. The minimal model of the function field K_n is realized as a finite covering $\pi_n: X_n \rightarrow X_1$ with branch locus D of constant ramification index n . Standard invariants of X_n are given by:

$$K_{X_n} = \left(\frac{1}{2} - \frac{1}{n}\right) \pi_n^* D, \quad c_2(X_n) = n^5 \left(2 - \frac{10}{n} + \frac{15}{n^2}\right).$$

Specifically, X_5 is a ball quotient with $K^2 = 5^4 \times 9 = 3c_2$ (see *ibid*).

The pullback of E_i or \tilde{L}_{ij} via $\pi_n: X_n \rightarrow X_1$ is divisible by n and supported by n^2 disjoint curves, each of which being isomorphic to the Fermat curve $x^n + y^n + z^n = 0$ of genus $(n-1)(n-2)/2$. The half of the ramification locus $H_n = R_n/2 = \pi_n^* D/(2n)$ turns out to be an integral, very ample divisor on X_n . Thus i) (X_n, H_n) is a polarized surface of degree $5n^3$, ii) the ramification locus R_n of $\pi_n: X_n \rightarrow X_1$ consists of $10n^2$ irreducible components, all isomorphic to the Fermat curve of degree n , and iii) $K_{X_n} = (n-2)H_n$. By choosing 2 and 3 as values of n , we obtain two examples for which our upper bound of r_2 in Proposition A and that of s in Proposition B are respectively attained:

Example A. (X_2, H_2) is a K3 surface of degree 40, with the effective divisor $R_2 \sim 2H_2$ consisting of 40 conics. Thus $r_2(X_2) \geq 40 = \frac{24 \times 40}{40 - 16}$.

Example B. (X_3, H_3) is a canonical surface with $K^2 = 3^3 \times 5$, $c_2 = 3^3 \times 3$ (*i.e.*, $\sigma = 3/5$). The divisor R_3 is a union of 90 copies of the Fermat

cubic curve, so that the total degree of the elliptic curves $s(X_3)$ is at least $90 \times 3 = \frac{3\sigma - 1}{3(1 - \sigma)}K^2$.

§3. A Concluding Remark

The *Hirzebruch proportionality principle* explains why (†) turns into an equality for the examples in the previous section.

Let $\mathbb{B}^2 \subset \mathbb{C}^2$ denote the unit ball $\text{PU}(1, 2)/\text{P}(\text{U}(1, 1) \times \text{U}(1))$ equipped with the Bergmann metric. Let $\Gamma_0 \subset \text{PU}(1, 2)$ be a discrete, torsion-free, cocompact subgroup of the holomorphic isometries of \mathbb{B}^2 . Consider a Γ_0 -stable curve $\Delta \subset \mathbb{B}^2$ with only normal crossing singularities (Δ may have countably many irreducible components). Let $\Gamma \subset \text{PU}(1, 2)$ be a subgroup which satisfies the following four conditions:

- (1) Γ contains Γ_0 as a normal subgroup with $\Gamma/\Gamma_0 \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus r}$.
- (2) $X_1 = \Gamma \backslash \mathbb{B}^2$ is nonsingular.
- (3) The action of Γ preserves Δ .
- (4) The projection $\pi: Y = \Gamma_0 \backslash \mathbb{B}^2 \rightarrow X_1 = \Gamma \backslash \mathbb{B}^2$ is a Kummer cover branching along $D = \Gamma \backslash \Delta \subset X_1$ with constant ramification index m .

Then $D \subset X_1$ is necessarily a divisor with only normal crossings. The “orbibundle” $\tilde{\mathcal{E}}_{1-\frac{1}{m}}$ on X constructed in [7] from the pair (X_1, D) is identified with Ω_Y^1 in this case, so that $c_1^2(\tilde{\mathcal{E}}_{1-\frac{1}{m}}) = 3c_2(\tilde{\mathcal{E}}_{1-\frac{1}{m}})$ by the Hirzebruch proportionality theorem [4]. If there is another Kummer cover $p_n: X_n \rightarrow X_1$ branching along the same divisor D , but with a smaller ramification index $n < m$, then the same orbibundle $\tilde{\mathcal{E}}_{1-\frac{1}{m}}$ on (X_1, D) can be viewed as the orbibundle $\tilde{\mathcal{E}}_{1-\frac{n}{m}}$ associated with (X_n, R_n) , where $R_n = p_n^*D/n$ is the ramification locus of $X_n \rightarrow X_1$. This is precisely the case in Examples A and B, where $(m, n) = (5, 2)$ and $(5, 3)$. Recalling that the inequality (†) is essentially the Miyaoka-Yau-Sakai inequality $3c_2(\tilde{\mathcal{E}}_\alpha) \geq c_1^2(\tilde{\mathcal{E}}_\alpha)$, we see that (†) is indeed an equality when $(X, C, \alpha) = (X_n, R_n, \frac{m-n}{m})$.

The construction above produces *countably many* examples of (X, C, α) to which the attached orbibundle $\tilde{\mathcal{E}}_\alpha$ satisfies $3c_2(\tilde{\mathcal{E}}_\alpha) = c_1^2(\tilde{\mathcal{E}}_\alpha)$. It is another question, however, if we can find infinitely many such triples with C being a union of curves of small genera (a union of rational or elliptic curves, for example).

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