

On a Curvature Property of Effective Divisors and Its Application to Sheaf Cohomology

By

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Abstract

Exploring a method of taming the boundary behavior of n -convex exhaustion functions, a curvature property of line bundles associated to effective Cartier divisors is proved. Cohomology vanishing theorems of the Serre type and the Kodaira-Nakano type are obtained as application.

Introduction

Let X be a complex analytic space of dimension n . It is known that X is n -complete in the sense of Andreotti-Grauert [A-G] (see section one for the definition) if every irreducible component of X is noncompact (cf. [O-3], [Dm]). This shows that the vanishing theorem for the cohomology groups of top degrees, due to Y.-T. Siu [S], is essentially contained in [A-G].

In [F], based on the 1-completeness of noncompact Riemann surfaces, an elementary proof was given to a basic fact that, for any Riemann surface R and for any point $p \in R$, the line bundle $[p]$ associated to the divisor p , is positive.

The purpose of the present note is to extend the paper [F] to establish the following.

Theorem 1. *Let X be a compact complex analytic space of dimension n and let D be an effective Cartier divisor of X such that $|D|$, the support of*

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D , intersects every n -dimensional irreducible component of X . Then the line bundle $[D]$ is n -concave (see section one for the definition).

Theorem 1 supplements [O-3] and [Dm]. By applying it we shall show at first the following Serre type vanishing theorem.

Theorem 2. *Let M be a complex manifold, let Z be a complex analytic space, let $f:M \rightarrow Z$ be a proper holomorphic map, let D be an effective divisor of M , let $z \in f(|D|)$, and let n be any positive integer exceeding the dimension of any compact irreducible component of $(f^{-1}(z) \setminus |D|) \cup (f^{-1}(z) \cap |D|)$. Then, for any holomorphic vector bundle $E \rightarrow M$, there exists a positive number m_0 such that*

$$(R^n f_* \mathcal{O}(E \otimes [D]^m))_z = 0$$

holds if $m \geq m_0$. Here $\mathcal{O}(E \otimes [D]^m)$ denotes the sheaf of the germs of holomorphic sections of $E \otimes [D]^m$, and $R^n f_ \mathcal{O}(E \otimes [D]^m)$ the n -th direct image of $\mathcal{O}(E \otimes [D]^m)$ by f .*

For the proof of Theorem 2, we need results from [B] and [O-3].

Further, by employing a method in [O-2], we obtain a refined version of Theorem 2 when E is the canonical bundle of M .

Theorem 3. *In the above situation, suppose moreover that E is the canonical bundle K_M of M , and that M admits a Kähler metric. Then*

$$(R^n f_* \mathcal{O}(K_M \otimes [D]))_z = 0$$

holds.

Theorem 3 may well be regarded as a supplement to Theorem 3.1 and Theorem 4.5 in [O-2], which are extensions of the Kodaira-Nakano vanishing.

§1. q -Complete Spaces and q -Concave Bundles

Let X be a complex analytic space of dimension n . A real valued C^2 function φ defined on an open set $U \subset X$ is said to be q -convex if, for any point $x \in U$, there exist a neighborhood $V \ni x$, a holomorphic embedding ι of V into a domain Ω of $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and a real valued C^2 function Φ on Ω such that $\iota^* \Phi = \varphi|_V$ and that the complex Hessian $\partial \bar{\partial} \Phi$ has everywhere at least $N - q + 1$ positive eigenvalues on Ω .

Definition 1. X is called a q -complete space (in the sense of [A-G]), if there exists a q -convex exhaustion function on X .

Here, an exhaustion function on a topological space is by definition a real valued function whose sublevel sets are all relatively compact.

Let $L \longrightarrow X$ be a holomorphic line bundle and let $\{e_{\alpha\beta}\}_{\alpha,\beta \in \Lambda}$ be a system of transition functions of L associated to an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ such that $\pi^{-1}(U_\alpha)$ is equivalent to the product $U_\alpha \times \mathbb{C}$. Then a fiber metric of L is naturally identified with a system of positive C^∞ functions $h = \{h_\alpha\}$, h_α being defined on U_α , such that $h_\alpha = |e_{\alpha\beta}|^2 h_\beta$ holds on $U_\alpha \cap U_\beta$. The pair (L, h) is called a Hermitian line bundle over X .

Definition 2. A Hermitian line bundle (L, h) is said to be q -concave if the functions $-\log h_\alpha$ are all q -convex.

§2. Proof of Theorem 1

Let X be a compact complex analytic space and let $\text{Sing} X$ be the set of singular points of X , with respect to the reduced structure. We put

$$X_k = \underbrace{\text{Sing}(\text{Sing}(\cdots \text{Sing} X) \cdots)}_k$$

and $X_0 = X$. Let X_k^j be the union of j -dimensional irreducible components of X_k , and let $X^j = \bigcup_k X_k^j$.

Let D be an effective Cartier divisor of X , let $s = \{s_\alpha\}$ be a system of local defining functions of D , s_α being defined on an open set $U_\alpha \subset X$, and let $\{e_{\alpha\beta}\}$ be the system of transition functions of $[D]$ such that $s_\alpha = e_{\alpha\beta} s_\beta$ holds on $U_\alpha \cap U_\beta$, for every α and β .

A fiber metric of $[D]$ is then a system of positive C^∞ functions $h = \{h_\alpha\}$, h_α defined on U_α , such that $h_\alpha |s_\alpha|^2 = h_\beta |s_\beta|^2$ holds on $U_\alpha \cap U_\beta$. We fix a fiber metric h of $[D]$ and denote by $|s|^2$ the function on X defined by $h_\alpha |s_\alpha|^2$ on each U_α .

In order to find a metric $\tilde{h} = \{\tilde{h}_\alpha\}$ of $[D]$ such that $-\log \tilde{h}_\alpha$ are n -convex, we shall at first find a C^∞ function η on X such that $-\log |s|^2 + \eta$ is n -convex outside some compact subset of $X \setminus |D|$. After that, we shall modify $-\log |s|^2 + \eta$ on a compact set, to obtain an n -convex exhaustion function Φ on $X \setminus |D|$ and put $\tilde{h}_\alpha = e^{-\Phi} |s_\alpha|^{-2}$. For this purpose, the following is crucial.

Lemma 1. *Let Y be a complex analytic space of dimension n equipped with an n -convex exhaustion function φ , and let ψ be a C^∞ exhaustion function on Y such that ψ is n -convex outside a compact subset K of Y . Suppose that $\varphi|_{(Y^j \setminus K)}$ and $\psi|_{(Y^j \setminus K)}$ do not have local maximums for $j > 0$. Then there*

exist a compact set $\hat{K} \subset Y$, a positive number ε , a real number C , and a C^∞ n -convex function Φ on Y such that $\Phi|_K = \varepsilon\varphi$ and $\Phi(Y \setminus \hat{K}) = \psi - C$.

Proof. Take any $c_1 \in \mathbb{R}$ such that $\varphi|_K < c_1$ and that $\psi|\varphi^{-1}((c_1, \infty))$ is n -convex. Then, take c_2 and c_3 in such a way that $\psi|\varphi^{-1}((-\infty, c_1]) < c_2$ and $\varphi|\psi^{-1}((-\infty, c_2]) < c_3$.

Then we put, for any $A > 0$,

$$\varphi_A = \begin{cases} A(\psi - c_2) & \text{on } \varphi^{-1}([c_3, \infty)) \\ \max\{\varphi, A(\psi - c_2)\} & \text{on } \varphi^{-1}((c_1, c_3)) \\ \varphi & \text{on } \varphi^{-1}((-\infty, c_1]) \end{cases}$$

Although φ_A is not n -convex in general, it is clearly continuous for sufficiently large A , and has no local maximum because so do φ and $\psi|\varphi^{-1}((c_1, \infty))$. Note that $\varphi_A|(Y^j \setminus K)$ has no local maximum if $j > 0$ by assumption.

After fixing such a number A , take a C^∞ function $\tilde{\varphi}$ by approximating φ_A , such that $\tilde{\varphi} = \varphi_A$ on $Y \setminus \varphi^{-1}((c_1, c_3))$ and $\tilde{\varphi}|(Y^j \setminus K)$ have no local maximum for $j > 0$. We may assume that the critical points of $\tilde{\varphi}|Y^j \cap \varphi^{-1}([c_1, c_3]) \setminus \text{Sing } Y^j$ are isolated and non-degenerate for all j , and that $\tilde{\varphi}$ is n -convex on a neighborhood of Y^0 .

Let Σ be the union of the sets of critical points of $\tilde{\varphi}|(Y \cap \varphi^{-1}([c_1, c_3]) \setminus \text{Sing } Y^j)$ for all $j > 0$.

Since $\tilde{\varphi}|(Y^j \cap \varphi^{-1}([c_1, c_3]) \setminus \text{Sing } Y^j)$ have no local maximums, one can find an arbitrarily small neighborhood U of Σ and a C^∞ diffeomorphism $F_v : Y \rightarrow Y$ fixing the points of Σ and $Y \setminus U$, such that $\tilde{\varphi} \circ F_v$ is n -convex on a neighborhood of Σ . As such a diffeomorphism F_v , it suffices to take one with sufficiently enlarging dilation along a positive direction, compared to the complementary directions, of the Hessian of $\tilde{\varphi}|(Y^j \setminus \text{Sing } Y^j)$ at Σ .

Then it is obvious that one can find a C^∞ convex increasing function λ on $\mathbb{R}$ such that $\lambda \circ \tilde{\varphi} \circ F_v$ satisfies the requirements for Φ for some ε and C . $\square$

The following is contained in [O-3]. Although the notations are slightly different from that of [O-3], the adjustment is routine and may well be left to the reader.

Lemma 2 (See [O-3, Proposition (in §2) and §3]). *Let X be a complex analytic space whose irreducible components are noncompact, and let $\xi : X \rightarrow \mathbb{R}$ be a C^∞ exhaustion function such that ξ is n -convex on a neighborhood of $\text{Sing } X$. Then there exists a convex increasing function λ on $\mathbb{R}$ and a C^∞ n -convex exhaustion function Ψ on X such that $\Psi = \lambda \circ \xi$ holds on a neighborhood of $\text{Sing } X$.*

Therefore, by a construction inductive on the dimension, one can find a C^∞ n -convex exhaustion function φ on $X \setminus |D|$ such that $\varphi|(X^j \setminus |D|)$ have no local maximums outside a compact subset, for any $j > 0$.

Proof of Theorem 1. Let $\eta : X \rightarrow \mathbb{R}$ be any C^∞ function which is n -convex on a neighborhood of $|D|$. Existence of such a function η is obvious. Replacing η by $B\eta$ for a sufficiently large constant B , if necessary, we may assume that there exists a compact set $K_1 \subset X \setminus |D|$ such that $\psi := -\log |s|^2 + \eta$ is n -convex on $X \setminus |D| \setminus K_1$.

Clearly, we may choose K_1 in such a way that $\psi|(X^j \setminus |D| \setminus K_1)$ have no local maximums if $j > 0$.

Hence, in view of the existence of φ as above and Lemma 1, there exists a compact set $K_2 \subset X \setminus |D|$ such that one can extend $\psi|(X \setminus |D| \setminus K_2)$ to a C^∞ n -convex function on $X \setminus |D|$. This was what we wanted to show, as was mentioned before stating Lemma 1.

§3. Proof of Theorem 2

If $n > \dim f^{-1}(z)$, then for any coherent analytic sheaf $\mathcal{F}$ over M one has $(R^n f_* \mathcal{F})_z = 0$ by [A-G], since $f^{-1}(z)$ admits an n -complete neighborhood system (cf. [B]). If z is as in the assumption and $n = \dim f^{-1}(z)$, then $[D]|f^{-1}(z)$ is n -concave by Theorem 1. Hence $[D]$ is n -concave on a neighborhood of $f^{-1}(z)$.

Therefore, since $f^{-1}(z)$ admits a holomorphically convex neighborhood system, the result follows from a vanishing theorem of Serre type on weakly 1-complete manifolds (cf. [O-1, Corollary 1.4]).

§4. Proof of Theorem 3

Similarly as above, it suffices to show the assertion when $n = \dim f^{-1}(z)$. Let s be a canonical section of $[D]$ and let g be any Kähler metric on M . By the assumption on z , one can find a fiber metric h of $[D]$ and a holomorphically convex neighbourhood W of $f^{-1}(z)$ in such a way that $([D], h)$ is n -concave on W . Then, since the function η in the proof of Theorem 1 can be chosen in such a way that all the eigenvalues of $\partial\bar{\partial}\eta$ with respect to any prescribed metric are greater than -1 and at most $n-1$ of them are less than n , it is easy to see that, in the present situation, one can choose h so that there exists a C^∞ (weakly) convex increasing function λ satisfying $\lambda'(t) = 1$ for sufficiently large t , such that the eigenvalues of the curvature form of

$$\hat{h} := h|s|^{-2} \exp(-\lambda(-\log |s|^2))$$

with respect to g , say $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N$, satisfy $\gamma_1 + \cdots + \gamma_n \geq 0$ everywhere and $\gamma_1 + \cdots + \gamma_n > 1$ on some neighborhood of $f^{-1}(z) \cap |D|$.

Let φ be any C^∞ plurisubharmonic function on W such that $g + \partial\bar{\partial}\varphi$ is a complete Kähler metric on W . Then there exists a positive number L , independent of φ , such that the sums of n eigenvalues of the curvature form of $\hat{h} \exp(-L\varphi)$ with respect to $g + \partial\bar{\partial}\varphi$ are nonnegative on W and greater than 1 on some neighborhood of $f^{-1}(z) \cap |D|$.

In this situation, it is clear from Nakano's inequality that there exist no nonzero square integrable $K_M \otimes [D]$ -valued harmonic $(0, n)$ -forms on W with respect to the metrics $g + \partial\bar{\partial}\varphi$ and $\hat{h} \exp(-L\varphi)$.

Thus, the vanishing of the L^2 harmonic forms holds with respect to $(g + \partial\bar{\partial}\varphi^2, \hat{h} \exp(-L\varphi^2))$, for any C^∞ nonnegative plurisubharmonic exhaustion function φ on W .

Recall that the analytic sheaf cohomology groups of holomorphically convex manifolds are Hausdorff spaces by the direct image theorem of Grauert (cf. [G-2] and [F-K]).

Thus we may conclude that, from the vanishing of the L^2 harmonic forms, that $(R^n f_* \mathcal{O}(K_M \otimes [D]))_z = 0$.

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