

# Logarithmic Geometry, Minimal Free Resolutions and Toric Algebraic Stacks

By

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## Abstract

In this paper we will introduce a certain type of morphisms of log schemes (in the sense of Fontaine, Illusie and Kato) and study their moduli. Then by applying this we define the notion of toric algebraic stacks, which may be regarded as torus embeddings in the framework of algebraic stacks and prove some fundamental properties. Also, we study the stack-theoretic analogue of toroidal embeddings.

## Introduction

In this paper we will introduce a certain type of morphisms of log schemes (in the sense of Fontaine, Illusie, and Kato) and investigate their moduli. Then by applying this we define a notion of *toric algebraic stacks* over arbitrary schemes, which may be regarded as *torus embeddings within the framework of algebraic stacks*, and study some basic properties. Our notion of toric algebraic stacks gives a natural generalization of smooth torus embedding (toric varieties) over arbitrary schemes preserving the smoothness, and it is closely related to simplicial toric varieties.

We first introduce a notion of the *admissible and minimal free resolutions* of a monoid. This notion plays a central role in this paper. This leads to define a certain type of morphisms of fine log schemes called *admissible FR* morphisms. (“FR” stands for *free resolution*.) We then study the moduli stack of admissible FR morphisms into a toroidal embedding endowed with the

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canonical log structure. One may think of these moduli as a sort of natural “stack-theoretic generalization” of the classical notion of toroidal embeddings.

As promised above, the concepts of admissible FR morphisms and their moduli stacks yield the notion of *toric algebraic stacks* over arbitrary schemes. Actually in the presented work on toric algebraic stacks, admissible free resolutions of monoids and admissible FR morphisms play the role which is analogous to that of the monoids and monoid rings arising from cones in classical toric geometry. That is to say, the algebraic aspect of toric algebraic stacks is the algebra of admissible free resolutions of monoids. In a sense, our notion of toric algebraic stacks is a hybrid of the definition of toric varieties given in [4] and the moduli stack of admissible FR morphisms. Fix a base scheme  $S$ . Given a simplicial fan  $\Sigma$  with additional data called “level”  $\mathfrak{n}$ , we define a toric algebraic stack  $\mathcal{X}_{(\Sigma, \Sigma_{\mathfrak{n}}^{\circ})}$ . It turns out that this stack has fairly good properties. It is a *smooth Artin stack* of finite type over  $S$  with finite diagonal, whose coarse moduli space is the simplicial toric variety  $X_{\Sigma}$  over  $S$  (see Theorem 4.6). Moreover it has a torus-embedding, a torus action functor and a natural coarse moduli map, which are defined in canonical fashions. The complement of the torus is a divisor with normal crossings relative to  $S$ . If  $\Sigma$  is non-singular and  $\mathfrak{n}$  is a canonical level, then  $\mathcal{X}_{(\Sigma, \Sigma_{\mathfrak{n}}^{\circ})}$  is the smooth toric variety  $X_{\Sigma}$  over  $S$ . Thus we obtain the following diagram of (2)-categories,

$$\begin{array}{ccc}
 & & \text{(Toric algebraic stacks over } S) \\
 & \xrightarrow{a} & \downarrow c \\
 \text{(Smooth toric varieties over } S) & & \text{(Simplicial toric varieties over } S) \\
 & \xrightarrow{b} & 
 \end{array}$$

where  $a$  and  $b$  are fully faithful functors and  $c$  is an essentially surjective functor (see Remark 4.8). One remarkable point to notice is that working in the framework of algebraic stacks (including Artin stacks) allows one to have a generalization of smooth toric varieties over  $S$  that preserves the important features of smooth toric varieties such as the smoothness. There is another point to note. Unlike toric varieties, some properties of toric algebraic stacks depend very much on the choice of a base scheme. For example, the question of whether or not  $\mathcal{X}_{(\Sigma, \Sigma_{\mathfrak{n}}^{\circ})}$  is Deligne-Mumford depends on the base scheme. Thus it is natural to develop our theory over arbitrary schemes.

Over the complex number field (and algebraically closed fields of characteristic zero), one can construct simplicial toric varieties as geometric quotients by means of homogeneous coordinate rings ([8]). In [5], by generalizing Cox’s construction, toric Deligne-Mumford stacks was introduced, whose theory comes

from Cox's viewpoint of toric varieties. On the other hand, roughly speaking, our construction stemmed from the usual definition of toric varieties given in, for example [18], [4], [10], [6, Chapter IV, 2], in log-algebraic geometry, and it also yields a sort of "stacky toroidal embeddings". We hope that toric algebraic stacks provide an ideal testing ground for problems and conjectures on stacks in many areas of mathematics, such as arithmetic geometry, algebraic geometry, mathematical physics, etc.

This paper is organized as follows. In Section 2, we define the notion of admissible and minimal free resolution of monoids and admissible FR morphisms and investigate their properties. It is an "algebra" part of this paper. In Section 3, we construct algebraic moduli stacks of admissible FR morphisms of toroidal embeddings with canonical log structures. In Section 4, we define the notion of toric algebraic stacks and prove fundamental properties by applying Section 2 and 3.

*Applications and further works.* Let us mention applications and further works, which are not discussed in this paper. The presented paper plays a central role in the subsequent papers ([12], [13]). In [12], by using the results and machinery presented in this paper, we study the 2-category of toric algebraic stacks, and show that 2-category of toric algebraic stacks are equivalent to the category of stacky fans. Furthermore we prove that toric algebraic stacks defined in this paper have a quite nice geometric characterization in characteristic zero. In [13], we calculate the integral Chow ring of a toric algebraic stack and show that it is isomorphic to the Stanley-Reisner ring. As a possible application, we hope that our theory might be applied to smooth toroidal compactifications of spaces (including arithmetic schemes) that can not be smoothly compactified in classical toroidal geometry (cf. [4], [10], [20]).

### Notations And Conventions

1. We will denote by  $\mathbb{N}$  the set of *natural numbers*, by which we mean the set of integers  $n \geq 0$ , by  $\mathbb{Z}$  the *ring of rational integers*, by  $\mathbb{Q}$  the *rational number field*, and by  $\mathbb{R}$  the *real numbers*. We write  $\text{rk}(L)$  for the rank of a free abelian group  $L$ .
2. By an *algebraic stack* we mean an algebraic stack in the sense of [19, 4.1]. All schemes, algebraic spaces, and algebraic stacks are assumed to be *quasi-separated*. We call an algebraic stack  $\mathcal{X}$  which admits an étale surjective cover  $X \rightarrow \mathcal{X}$ , where  $X$  is a scheme a *Deligne-Mumford stack*. For details on algebraic stacks, we refer to [19]. Let us recall the definition of coarse

moduli spaces and the fundamental existence theorem due to Keel and Mori ([17]). Let  $\mathcal{X}$  be an algebraic stack. A *coarse moduli map* (or *space*) for  $\mathcal{X}$  is a morphism  $\pi : \mathcal{X} \rightarrow X$  from  $\mathcal{X}$  to an algebraic space  $X$  such that the following conditions hold.

- (a) If  $K$  is an algebraically closed field, then the map  $\pi$  induces a bijection between the set of isomorphism classes of objects in  $\mathcal{X}(K)$  and  $X(K)$ .
- (b) The map  $\pi$  is universal for maps from  $\mathcal{X}$  to algebraic spaces.

Let  $\mathcal{X}$  be an algebraic stack of finite type over a locally noetherian scheme  $S$  with finite diagonal. Then a result of Keel and Mori says that there exists a coarse moduli space  $\pi : \mathcal{X} \rightarrow X$  with  $X$  of finite type and separated over  $S$  (See also [7] in which the Noetherian assumption is eliminated). Moreover  $\pi$  is proper, quasi-finite and surjective, and the natural map  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$  is an isomorphism. If  $S' \rightarrow S$  is a flat morphism, then  $\mathcal{X} \times_S S' \rightarrow S'$  is also a coarse moduli map.

## §1. Toroidal and Logarithmic Geometry

We first review some definitions and basic facts concerning toroidal geometry and logarithmic geometry in the sense of Fontaine, Illusie and K. Kato, and establish notation for them. We refer to [6, Chapter IV. 2], [10], [18] for details on toric and toroidal geometry, and refer to [15], [16] for details on logarithmic geometry.

### §1.1. Toric varieties over a scheme

Let  $N \cong \mathbb{Z}^d$  be a lattice and  $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  its dual. Let  $\langle \bullet, \bullet \rangle : M \times N \rightarrow \mathbb{Z}$  be the natural pairing. Let  $S$  be a scheme. Let  $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$  be a strictly convex rational polyhedral cone and

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

its dual. (In this paper, all cones are assumed to be a strictly convex rational polyhedral, unless otherwise stated.) The *affine toric variety* (or *affine torus embedding*)  $X_{\sigma}$  associated to  $\sigma$  over  $S$  is defined by

$$X_{\sigma} := \text{Spec } \mathcal{O}_S[\sigma^{\vee} \cap M]$$

where  $\mathcal{O}_S[\sigma^{\vee} \cap M]$  is the monoid algebra of  $\sigma^{\vee} \cap M$  over the scheme  $S$ .

Let  $\sigma \subset N_{\mathbb{R}}$  be a cone. (We sometimes use the  $\mathbb{Q}$ -vector space  $N \otimes_{\mathbb{Z}} \mathbb{Q}$  instead of  $N_{\mathbb{R}}$ .) Let  $v_1, \dots, v_m$  be a minimal set of generators  $\sigma$ . Each  $v_i$  spans a *ray*, i.e., a 1-dimensional face of  $\sigma$ . The affine toric variety  $X_{\sigma}$  is smooth over  $S$  if and only if the first lattice points of  $\mathbb{R}_{\geq 0}v_1, \dots, \mathbb{R}_{\geq 0}v_m$  form a part of basis of  $N$  (cf. [10, page 29]). In this case, we refer to  $\sigma$  as a *nonsingular cone*. The cone  $\sigma$  is *simplicial* if it is generated by  $\dim(\sigma)$  lattice points, i.e.,  $v_1, \dots, v_m$  are linearly independent. Let  $\sigma$  be an  $r$ -dimensional simplicial cone in  $N_{\mathbb{R}}$  and  $v_1, \dots, v_r$  the first lattice points of rays in  $\sigma$ . The *multiplicity* of  $\sigma$ , denoted by  $\text{mult}(\sigma)$ , is defined to be the index  $[N_{\sigma} : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_r]$ . Here  $N_{\sigma}$  is the lattice generated by  $\sigma \cap N$ . If the multiplicity of a simplicial cone  $\sigma$  is invertible on  $S$ , we say that the cone  $\sigma$  is *tamely simplicial*. If  $\sigma$  and  $\tau$  are cones, we write  $\sigma \prec \tau$  (or  $\tau \succ \sigma$ ) to mean that  $\sigma$  is a *face* of  $\tau$ .

A *fan* (resp. *simplicial fan*, *tamely simplicial fan*)  $\Sigma$  in  $N_{\mathbb{R}}$  is a set of cones (resp. *simplicial cones*, *tamely simplicial cones*) in  $N_{\mathbb{R}}$  such that:

- (1) Each face of a cone in  $\Sigma$  is also a cone in  $\Sigma$ ,
- (2) The intersection of two cones in  $\Sigma$  is a face of each.

If  $\Sigma$  is a fan in  $N_{\mathbb{R}}$ , we denote by  $\Sigma(r)$  the set of  $r$ -dimensional cones in  $\Sigma$ , and denote by  $|\Sigma|$  the support of  $\Sigma$  in  $N_{\mathbb{R}}$ , i.e., the union of cones in  $\Sigma$ . (Note that the set  $\Sigma$  is not necessarily finite. Even in classical situations, infinite fans are important and they arises in various contexts such as constructions of degeneration of abelian varieties, the construction of hyperbolic Inoue surfaces, etc.)

Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$ . There is a natural patching of affine toric varieties associated to cones in  $\Sigma$ , and the patching defines a scheme of finite type and separated over  $S$ . We denote by  $X_{\Sigma}$  this scheme, and we refer it as the *toric variety* (or *torus embedding*) associated to  $\Sigma$ . A toric variety  $X$  contains a split algebraic torus  $T = \mathbb{G}_{m,S}^n = \text{Spec } \mathcal{O}_S[M]$  as an open dense subset, and the action of  $T$  on itself extends to an action on  $X_{\Sigma}$ .

For a cone  $\tau \in \Sigma$  we define its associated torus-invariant closed subscheme  $V(\tau)$  to be the union

$$\bigcup_{\sigma \succ \tau} \text{Spec } \mathcal{O}_S[(\sigma^{\vee} \cap M)/(\sigma^{\vee} \cap \tau_0^{\vee} \cap M)]$$

in  $X_{\Sigma}$ , where  $\sigma$  runs through the cones which contains  $\tau$  as a face and

$$\tau_0^{\vee} := \{m \in \tau^{\vee} \mid \langle m, n \rangle > 0 \text{ for some } n \in \tau\}$$

(Affine schemes on the right hand naturally patch together, and the symbol

$(\sigma^\vee \cap \tau_0^\vee \cap M)$  means the ideal generated by  $\sigma^\vee \cap \tau_0^\vee \cap M$ ). We have

$$\text{Spec } \mathcal{O}_S[(\sigma^\vee \cap M)/(\sigma^\vee \cap \tau_0^\vee \cap M)] = \text{Spec } \mathcal{O}_S[\sigma^\vee \cap \tau^\perp \cap M] \subset \text{Spec } \mathcal{O}_S[\sigma^\vee \cap M]$$

and the split torus  $\text{Spec } \mathcal{O}_S[\tau^\perp \cap M]$  is a dense open subset of  $\text{Spec } \mathcal{O}_S[\sigma^\vee \cap \tau^\perp \cap M]$ , where  $\tau^\perp = \{m \in M \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle m, n \rangle = 0 \text{ for any } n \in \tau\}$ . (Notice that  $\tau_0^\vee = \tau^\vee - \tau^\perp$ .) For a ray  $\rho \in \Sigma(1)$  (resp. a cone  $\sigma \in \Sigma$ ), we shall call  $V(\rho)$  (resp.  $V(\sigma)$ ) the *torus-invariant divisor* (resp. *torus-invariant cycle*) associated to  $\rho$  (resp.  $\sigma$ ). The complement  $X_\Sigma - T$  set-theoretically equals the union  $\cup_{\rho \in \Sigma(1)} V(\rho)$ . Set  $Z_\tau = \text{Spec } \mathcal{O}_S[\tau^\perp \cap M]$ . Then we have a natural stratification  $X_\Sigma = \sqcup_{\tau \in \Sigma} Z_\tau$  and for each cone  $\tau \in \Sigma$ , the locally closed subscheme  $Z_\tau$  is a  $T$ -orbit.

### §1.2. Toroidal embeddings

Let  $X$  be a normal variety over a field  $k$ , i.e., a geometrically integral normal scheme of finite type and separated over  $k$ . Let  $U$  be a smooth Zariski open set of  $X$ . We say that a pair  $(X, U)$  is a *toroidal embedding* (resp. *good toroidal embedding*, *tame toroidal embedding*) if for every closed point  $x$  in  $X$  there exist an étale neighborhood  $(W, x')$  of  $x$ , an affine toric variety (resp. an affine simplicial toric variety, an affine tamely simplicial toric variety)  $X_\sigma$  over  $k$ , and an étale morphism

$$f : W \longrightarrow X_\sigma.$$

such that  $f^{-1}(T_\sigma) = W \cap U$ . Here  $T_\sigma$  is the algebraic torus in  $X_\sigma$ .

### §1.3. Logarithmic geometry

First of all, we shall recall some generalities on monoids. In this paper, all monoids will assume to be commutative with unit. Given a monoid  $P$ , we denote by  $P^{\text{gp}}$  the Grothendieck group of  $P$ . If  $Q$  is a submonoid of  $P$ , we write  $P \rightarrow P/Q$  for the cokernel in the category of monoids. Two elements  $p, p' \in P$  have the same image in  $P/Q$  if and only if there exist  $q, q' \in Q$  such that  $p + q = p' + q'$ . The cokernel  $P/Q$  has a monoid structure in the natural manner. A monoid  $P$  is *finitely generated* if there exists a surjective map  $\mathbb{N}^r \rightarrow P$  for some positive integer  $r$ . A monoid  $P$  is said to be *sharp* if whenever  $p + q = 0$  for  $p, q \in P$ , then  $p = q = 0$ . We say that  $P$  is *integral* if the natural map  $P \rightarrow P^{\text{gp}}$  is injective. A finitely generated and integral monoid is said to be *fine*. An integral monoid  $P$  is *saturated* if for every  $p \in P^{\text{gp}}$  such that  $np \in P$  for some  $n > 0$ , it follows that  $p \in P$ . An integral monoid  $P$  is

said to be *torsion free* if  $P^{\text{gp}}$  is a torsion free abelian group. We remark that a fine, saturated and sharp monoid is torsion free.

Given a scheme  $X$ , a *prelog structure* on  $X$  is a sheaf of monoids  $\mathcal{M}$  on the étale site of  $X$  together with a homomorphism of sheaves of monoids  $h : \mathcal{M} \rightarrow \mathcal{O}_X$ , where  $\mathcal{O}_X$  is viewed as a monoid under multiplication. A prelog structure is a *log structure* if the map  $h^{-1}(\mathcal{O}_X^*) \rightarrow \mathcal{O}_X^*$  is an isomorphism. We usually denote simply by  $\mathcal{M}$  the log structure  $(\mathcal{M}, h)$  and by  $\overline{\mathcal{M}}$  the sheaf  $\mathcal{M}/\mathcal{O}_X^*$ . A morphism of prelog structures  $(\mathcal{M}, h) \rightarrow (\mathcal{M}', h')$  is a map  $\phi : \mathcal{M} \rightarrow \mathcal{M}'$  of sheaves of monoids such that  $h' \circ \phi = h$ .

For a prelog structure  $(\mathcal{M}, h)$  on  $X$ , we define its *associated log structure*  $(\mathcal{M}^a, h^a)$  to be the push-out of

$$\begin{array}{ccc} h^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on the étale site  $X_{\text{ét}}$ . This gives the left adjoint functor of the natural inclusion functor

$$(\text{log structures on } X) \rightarrow (\text{prelog structures on } X).$$

We say that a log structure  $\mathcal{M}$  is *fine* if étale locally on  $X$  there exists a fine monoid and a map  $P \rightarrow \mathcal{M}$  from the constant sheaf associated to  $P$  such that  $P^a \rightarrow \mathcal{M}$  is an isomorphism. A fine log scheme  $(X, \mathcal{M})$  is *saturated* if each stalk of  $\mathcal{M}$  is a saturated monoid. We remark that if  $\mathcal{M}$  is fine and saturated, then each stalk of  $\overline{\mathcal{M}}$  is fine and saturated.

A morphism of log schemes  $(X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$  is a pair  $(f, h)$  of a morphism of underlying schemes  $f : X \rightarrow Y$  and a morphism of log structures  $h : f^*\mathcal{N} \rightarrow \mathcal{M}$ , where  $f^*\mathcal{N}$  is the log structure associated to the composite  $f^{-1}\mathcal{N} \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ . A morphism  $(f, h) : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$  is said to be *strict* if  $h$  is an isomorphism.

Let  $P$  be a fine monoid. Let  $S$  be a scheme. Set  $X_P := \text{Spec } \mathcal{O}_S[P]$ . The *canonical log structure*  $\mathcal{M}_P$  on  $X_P$  is the fine log structure induced by the inclusion map  $P \rightarrow \mathcal{O}_S[P]$ . Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$  ( $N \cong \mathbb{Z}^d$ ) and  $X_{\Sigma}$  the associated toric variety over  $S$ . Then we have an induced log structure  $\mathcal{M}_{\Sigma}$  on  $X_{\Sigma}$  by gluing the log structures arising from the homomorphism  $\sigma^{\vee} \cap M \rightarrow \mathcal{O}_S[\sigma^{\vee} \cap M]$  for each cone  $\sigma \in \Sigma$ . Here  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ . We shall refer this log structure as the *canonical log structure* on  $X_{\Sigma}$ . If  $S$  is a locally noetherian regular scheme, we have that  $\mathcal{M}_{\Sigma} = \mathcal{O}_{X_{\Sigma}} \cap i_*\mathcal{O}_{\text{Spec } \mathcal{O}_S[M]}^*$  where  $i : \text{Spec } \mathcal{O}_S[M] \rightarrow X_{\Sigma}$  is the torus embedding (cf. [16, 11.6]).

Let  $(X, U)$  be a toroidal embedding over a field  $k$  and  $i : U \rightarrow X$  be the natural immersion. Define a log structure  $\alpha_X : \mathcal{M}_X := \mathcal{O}_X \cap i_* \mathcal{O}_U^* \rightarrow \mathcal{O}_X$  on  $X$ . This log structure is fine and saturated and said to be the *canonical log structure* on  $(X, U)$ .

## §2. Free Resolution of Monoids

### §2.1. Minimal and admissible free resolution of a monoid

**Definition 2.1.** Let  $P$  be a monoid. The monoid  $P$  is said to be *toric* if  $P$  is a fine, saturated and torsion free monoid.

*Remark 2.2.* If a monoid  $P$  is toric, there exists a strictly convex rational polyhedral cone  $\sigma \in \text{Hom}_{\mathbb{Z}}(P^{\text{gp}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $\sigma^{\vee} \cap P^{\text{gp}} \cong P$ . Here the dual cone  $\sigma^{\vee}$  lies on  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Indeed, we see this as follows. There exists a sequence of canonical injective homomorphisms  $P \rightarrow P^{\text{gp}} \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Define a cone

$$C(P) := \{ \sum_{i=0}^n a_i \cdot p_i \mid a_i \in \mathbb{Q}_{\geq 0}, p_i \in P \} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note that it is a *full-dimensional* rational polyhedral cone (but not necessarily strictly convex), and  $P = C(P) \cap P^{\text{gp}}$  since  $P$  is saturated. Thus the dual cone  $C(P)^{\vee} \subset \text{Hom}_{\mathbb{Z}}(P^{\text{gp}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  is a strictly convex rational polyhedral cone (cf. [10, (13) on page 14]). Hence our assertion follows.

- Let  $P$  be a monoid and  $S$  a submonoid of  $P$ . We say that the submonoid  $S$  is *close to* the monoid  $P$  if for every element  $e$  in  $P$ , there exists a positive integer  $n$  such that  $n \cdot e$  lies in  $S$ .

- Let  $P$  be a toric sharp monoid, and let  $r$  be the rank of  $P^{\text{gp}}$ . A toric sharp monoid  $P$  is said to be *simplicially toric* if there exists a submonoid  $Q$  of  $P$  generated by  $r$  elements such that  $Q$  is close to  $P$ .

**Lemma 2.3.** (1) *A toric sharp monoid  $P$  is simplicially toric if and only if we can choose a (strictly convex rational polyhedral) simplicial full-dimensional cone*

$$\sigma \subset \text{Hom}_{\mathbb{Z}}(P^{\text{gp}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*such that  $\sigma^{\vee} \cap P^{\text{gp}} \cong P$ , where  $\sigma^{\vee}$  denotes the dual cone in  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ .*

(2) *If  $P$  is a simplicially toric sharp monoid, then*

$$C(P) := \{ \sum_{i=0}^n a_i \cdot p_i \mid a_i \in \mathbb{Q}_{\geq 0}, p_i \in P \} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

*is a (strictly convex rational polyhedral) simplicial full-dimensional cone.*

*Proof.* We first prove (1). The “if” direction is clear. Indeed, if there exists such a simplicial full-dimensional cone  $\sigma$ , then the dual cone  $\sigma^\vee$  is also a simplicial full-dimensional cone. Let  $Q$  be the submonoid of  $P$ , which is generated by the first lattice points of rays on  $\sigma^\vee$ . Then  $Q$  is close to  $P$ .

Next we shall show the “only if” part. Assume there exists a submonoid  $Q \subset P$  such that  $Q$  is close to  $P$  and generated by  $\text{rk}(P^{\text{gp}})$  elements. By Remark 2.2 there exists a cone  $C(P) = \{\sum_{i=0}^n a_i \cdot p_i \mid a_i \in \mathbb{Q}_{\geq 0}, p_i \in P\}$  such that  $P = C(P) \cap P^{\text{gp}}$ . Note that since  $P$  is sharp,  $C(P)$  is strictly convex and full-dimensional. Thus  $\sigma := C(P)^\vee \subset \text{Hom}_{\mathbb{Z}}(P^{\text{gp}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  is a full-dimensional cone. It suffices to show that  $C(P)$  is simplicial, i.e., the cardinality of the set of rays of  $C(P)$  is equal to the rank of  $P^{\text{gp}}$ . For any ray  $\rho$  of  $\sigma^\vee = C(P)$ ,  $Q \cap \rho$  is non-empty because  $Q$  is close to  $P$ . Thus  $Q$  can not be generated by any set of elements of  $Q$  whose cardinality is less than the cardinality of rays in  $\sigma^\vee$ . Thus we have  $\text{rk}(P^{\text{gp}}) \geq \#\sigma^\vee(1)$  (here we write  $\#\sigma^\vee(1)$  for the cardinality of the set of rays of  $\sigma^\vee$ ). Hence  $\sigma^\vee = C(P)$  is simplicial and thus  $\sigma$  is also simplicial. It follows (1). The assertion of (2) is clear.  $\square$

**Lemma 2.4.** *Let  $P$  be a toric sharp monoid. Let  $F$  be a monoid such that  $F \cong \mathbb{N}^r$  for some  $r \in \mathbb{N}$ . Let  $\iota : P \rightarrow F$  be an injective homomorphism such that  $\iota(P)$  is close to  $F$ . Then the rank of  $P^{\text{gp}}$  is equal to the rank of  $F$ , i.e.,  $\text{rk}(F^{\text{gp}}) = r$ .*

*Proof.* Note first that  $P^{\text{gp}} \rightarrow F^{\text{gp}}$  is injective. Indeed, the natural homomorphisms  $P \rightarrow F$  and  $F \rightarrow F^{\text{gp}}$  are injective. Thus if  $p_1, p_2 \in P$  have the same image in  $F^{\text{gp}}$ , then  $p_1 = p_2$ . Hence  $P^{\text{gp}} \rightarrow F^{\text{gp}}$  is injective. Since  $\iota(P)$  is close to  $F$ , the cokernel of  $P^{\text{gp}} \rightarrow F^{\text{gp}}$  is finite. Hence our claim follows.  $\square$

**Proposition 2.5.** *Let  $P$  be a simplicially toric sharp monoid. Then there exists an injective homomorphism of monoids*

$$i : P \longrightarrow F$$

*which has the following properties.*

- (1) *The monoid  $F$  is isomorphic to  $\mathbb{N}^d$  for some  $d \in \mathbb{N}$ , and the submonoid  $i(P)$  is close to  $F$ .*
- (2) *If  $j : P \rightarrow G$  is an injective homomorphism, and  $G$  is isomorphic to  $\mathbb{N}^d$  for some  $d \in \mathbb{N}$ , and  $j(P)$  is close to  $G$ , then there exists a unique*

homomorphism  $\phi : F \rightarrow G$  such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & F \\ j \downarrow & \searrow \phi & \\ & & G \end{array}$$

commutes.

Furthermore if  $C(P) := \{\sum_{i=0}^n a_i \cdot p_i \mid a_i \in \mathbb{Q}_{\geq 0}, p_i \in P\} \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  (it is a simplicial cone (cf. Lemma 2.3 (2)), then there exists a canonical injective map

$$F \rightarrow C(P)$$

that has the following properties:

(a) The natural diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & F \\ \downarrow & \searrow & \\ C(P) & & \end{array}$$

commutes,

(b) Each irreducible element of  $F$  lies on a unique ray of  $C(P)$  via  $F \rightarrow C(P)$ .

*Proof.* Let  $d$  be the rank of the torsion-free abelian group  $P^{\text{gp}}$ . By Lemma 2.3,  $C(P)$  is a full-dimensional simplicial cone in  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Let  $\{\rho_1, \dots, \rho_d\}$  be the set of rays in  $C(P)$ . Let us denote by  $v_i$  the first lattice point on  $\rho_i$  in  $C(P)$ . Then for any element  $c \in C(P)$  we have a unique representation of  $c$  such that  $c = \sum_{1 \leq i \leq d} a_i \cdot v_i$  where  $a_i \in \mathbb{Q}_{\geq 0}$  for  $1 \leq i \leq d$ . Consider the map  $q_k : P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}; p = \sum_{1 \leq i \leq d} a_i \cdot v_i \mapsto a_k$  ( $a_i \in \mathbb{Q}$  for all  $i$ ). Set  $P_i := q_i(P^{\text{gp}}) \subset \mathbb{Q}$ . It is a free abelian group generated by one element. Let  $p_i \in P_i$  be the element such that  $p_i > 0$  and the absolute value of  $p_i$  is the smallest in  $P_i$ . Let  $F$  be the monoid generated by  $p_1 \cdot v_1, \dots, p_d \cdot v_d$ . Clearly, we have  $F \cong \mathbb{N}^d$  and  $P \subset F \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Note that there exists a positive integer  $b_i$  such that  $p_i = 1/b_i$  for each  $1 \leq i \leq d$ . Therefore  $b_i \cdot p_i \cdot v_i = v_i$  for all  $i$ , thus it follows that  $P$  is close to  $F$ .

It remains to show that  $P \subset F$  satisfies the property (2). Let  $j : P \rightarrow G$  be an injective homomorphism of monoids such that  $j(P)$  is close to  $G$ . Notice that by Lemma 2.4, we have  $G \cong \mathbb{N}^d$ . The monoid  $P$  has the natural injection  $\iota : P \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . On the other hand, for any element  $e$  in  $G$ , there exists a

positive integer  $n$  such that  $n \cdot e$  is in  $j(P)$ . Therefore we have a unique injective homomorphism  $\lambda : G \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  which extends  $P \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  to  $G$ . Indeed, if  $g \in G$  and  $n \in \mathbb{Z}_{\geq 1}$  such that  $n \cdot g \in j(P) = P$ , then we define  $\lambda(g)$  to be  $\iota(n \cdot g)/n$  (it is easy to see that  $\lambda(g)$  does not depend on the choice of  $n$ ). The map  $\lambda$  defines a homomorphism of monoids. Indeed, if  $g_1, g_2 \in G$ , then there exists a positive integer  $n$  such that both  $n \cdot g_1$  and  $n \cdot g_2$  lie in  $P$ , and it follows that  $\lambda(g_1 + g_2) = \iota(n(g_1 + g_2))/n = \iota(n \cdot g_1)/n + \iota(n \cdot g_2)/n = \lambda(g_1) + \lambda(g_2)$ . In addition,  $\lambda$  sends the unit element of  $G$  to the unit element of  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . Since  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ , thus  $\lambda : G \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  is a unique extension of the homomorphism  $\iota : P \rightarrow P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . The injectiveness of  $\lambda$  follows from its definition. We claim that there exists a sequence of inclusions

$$P \subset F \subset G \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since  $P$  is close to  $G$  and  $C(P)$  is a full-dimensional simplicial cone, thus each irreducible element of  $G$  lies in a unique ray of  $C(P)$ . (For a ray  $\rho$  of  $C(P)$ , the first point of  $G \cap \rho$  is an irreducible element of  $G$ .) On the other hand, we have  $\mathbb{Z}_{\geq 0} \cdot p_i \subset q_i(G^{\text{gp}}) \cap \mathbb{Q}_{\geq 0}$ . This implies  $F \subset G$ . Thus we have (2).

By the above construction, clearly there exists the natural homomorphism  $F \rightarrow C(P)$ . The property (a) is clear. The property (b) follows from the above argument. Hence we complete the proof of our Proposition. □

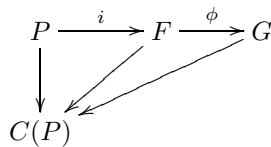
**Definition 2.6.** Let  $P$  be a simplicially toric sharp monoid. If an injective homomorphism of monoids

$$i : P \longrightarrow F$$

that satisfies the properties (1), (2) (resp. the property (1)) in Proposition 2.5, we say that  $i : P \longrightarrow F$  is a *minimal free resolution* (resp. *admissible free resolution*) of  $P$ .

*Remark 2.7.*

- (1) By the observation in the proof of Proposition 2.5, if  $j : P \rightarrow G$  is an admissible free resolution of a simplicially toric sharp monoid  $P$ , then there is a natural commutative diagram



such that  $\phi \circ i = j$ , where  $i : P \rightarrow F$  is the minimal free resolution of  $P$ . Furthermore, all three maps into  $C(P)$  are injective and each irreducible element of  $G$  lies on a unique ray of  $C(P)$ .

- (2) By Lemma 2.4, the rank of  $F$  is equal to the rank of  $P^{\text{gp}}$ .
- (3) We define the *multiplicity* of  $P$ , denoted by  $\text{mult}(P)$ , to be the order of the cokernel of  $i^{\text{gp}} : P^{\text{gp}} \rightarrow F^{\text{gp}}$ . If  $P$  is isomorphic to  $\sigma^\vee \cap M$  where  $\sigma$  is a simplicial cone, it is easy to see that  $\text{mult}(P) = \text{mult}(\sigma)$ .

**Proposition 2.8.** *Let  $P$  be a simplicially toric sharp monoid and  $i : P \rightarrow F \cong \mathbb{N}^d$  its minimal free resolution. Consider the following diagram*

$$\begin{array}{ccc} P & \xrightarrow{i} & F \cong \mathbb{N}^d \\ q \downarrow & & \downarrow \pi \\ Q & \xrightarrow{j} & \mathbb{N}^r, \end{array}$$

where  $Q := \text{Image}(\pi \circ i)$  and  $\pi : \mathbb{N}^d \rightarrow \mathbb{N}^r$  is defined by  $(a_1, \dots, a_d) \mapsto (a_{\alpha(1)}, \dots, a_{\alpha(r)})$ . Here  $\alpha(1), \dots, \alpha(r)$  are positive integers such that  $1 \leq \alpha(1) < \dots < \alpha(r) \leq d$ . Then  $Q$  is a simplicially toric sharp monoid and  $j$  is the minimal free resolution of  $Q$ .

*Proof.* After reordering we assume that  $\alpha(k) = k$  for  $1 \leq k \leq r$ . First, we will show that  $Q$  is a simplicially toric sharp monoid. Since  $Q$  is close to  $\mathbb{N}^r$  via  $j$ ,  $Q$  is sharp. If  $e_i$  denotes the  $i$ -th standard irreducible element, then for each  $i$ , there exists positive integers  $n_1, \dots, n_r$  such that  $n_1 \cdots e_1, \dots, n_r \cdots e_r \in Q$  and  $n_1 \cdots e_1, \dots, n_r \cdots e_r$  generates a submonoid which is close to  $Q$ . Thus it suffices only to prove that  $Q$  is a toric monoid. Clearly,  $Q$  is a fine monoid. To see the saturatedness we first regard  $P^{\text{gp}}$  and  $Q^{\text{gp}}$  as subgroups of  $\mathbb{Q}^d = (\mathbb{N}^d)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\mathbb{Q}^r = (\mathbb{N}^r)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  respectively. It suffices to show  $Q^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^r = Q$ . Since  $P$  is saturated, thus  $P = P^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^d$ . Note that  $q^{\text{gp}} : P^{\text{gp}} \rightarrow Q^{\text{gp}}$  is surjective. It follows that  $P^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^d \rightarrow Q^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^r$  is surjective. Indeed, let  $\xi \in Q^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^r$  and  $\xi' \in P^{\text{gp}}$  such that  $\xi = q^{\text{gp}}(\xi')$ . Put  $\xi' = (b_1, \dots, b_d) \in P^{\text{gp}} \subset (\mathbb{N}^d)^{\text{gp}} = \mathbb{Z}^d$ . Note that  $b_i \geq 0$  for  $1 \leq i \leq r$ . Since  $P^{\text{gp}}$  is a subgroup of  $\mathbb{Z}^d$  of a finite index, there exists an element  $\xi'' = (0, \dots, 0, c_{r+1}, \dots, c_d) \in P^{\text{gp}}$  such that  $\xi' + \xi'' = (b_1, \dots, b_r, b_{r+1} + c_{r+1}, \dots, b_d + c_d) \in \mathbb{Z}_{\geq 0}^d$ . Then  $\xi = q^{\text{gp}}(\xi') = q^{\text{gp}}(\xi' + \xi'')$ . Thus  $P^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^d \rightarrow Q^{\text{gp}} \cap \mathbb{Q}_{\geq 0}^r$  is surjective. Hence  $Q$  is saturated.

It remains to prove that  $Q \subset \mathbb{N}^r$  is the minimal free resolution. In order to prove this, recall that the construction of minimal free resolution of  $P$ . With

the same notation as in the first paragraph of the proof of Proposition 2.5, the monoid  $F$  is defined to be a free submonoid  $\mathbb{N} \cdot p_1 \cdot v_1 \oplus \cdots \oplus \mathbb{N} \cdot p_d \cdot v_d$  of  $C(P)$  where  $p_k \cdot v_k$  is the first point of  $C(P) \cap \tilde{q}_k(P^{\text{sp}})$  for  $1 \leq k \leq d$ . Here the map  $\tilde{q}_k : P^{\text{sp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q} \cdot v_k$  is defined by  $\sum_{1 \leq i \leq d} a_i \cdot v_i \mapsto a_k \cdot v_k$  ( $a_i \in \mathbb{Q}$  for all  $i$ ). We shall refer this construction as the *canonical construction*. After reordering, we have the following diagram

$$\begin{array}{ccccc}
 \mathbb{N} \cdot p_1 \cdot v_1 \oplus \cdots \oplus \mathbb{N} \cdot p_d \cdot v_d & \longrightarrow & C(P) & \longrightarrow & P^{\text{sp}} \otimes_{\mathbb{Z}} \mathbb{Q} \\
 \downarrow \pi & & \downarrow & & \downarrow \\
 \mathbb{N} \cdot p_1 \cdot v_1 \oplus \cdots \oplus \mathbb{N} \cdot p_r \cdot v_r & \longrightarrow & C(Q) & \longrightarrow & Q^{\text{sp}} \otimes_{\mathbb{Z}} \mathbb{Q}
 \end{array}$$

where  $p_k \cdot v_k$  is regarded as a point on a ray of  $C(Q)$  for  $1 \leq i \leq r$ . Then  $p_k \cdot v_k$  is the first point of  $C(Q) \cap \tilde{q}'_k(Q^{\text{sp}})$  for  $1 \leq k \leq r$ , where  $\tilde{q}'_k : Q^{\text{sp}} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q} \cdot v_k$ ;  $\sum_{1 \leq i \leq r} a_i \cdot v_i \mapsto a_k \cdot v_k$  (note that for any  $c \in Q^{\text{sp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , there is a unique representation of  $c$  such that  $c = \sum_{1 \leq i \leq r} a_i \cdot v_i$  where  $a_i \in \mathbb{Q}_{\geq 0}$  for  $1 \leq i \leq r$ ). Then  $Q \rightarrow \mathbb{N}^r \cong \mathbb{N} \cdot p_1 \cdot v_1 \oplus \cdots \oplus \mathbb{N} \cdot p_r \cdot v_r$  is the canonical construction for  $Q$ , and thus it is the minimal free resolution. Hence we obtain our Proposition.  $\square$

**Proposition 2.9.** (1) *Let  $\iota : P \rightarrow F$  be an admissible free resolution. Then  $\iota$  has the form*

$$\iota = n \circ i : P \xrightarrow{i} F \cong \mathbb{N}^d \xrightarrow{n} \mathbb{N}^d \cong F$$

where  $i$  is the minimal free resolution and  $n : \mathbb{N}^d \rightarrow \mathbb{N}^d$  is defined by  $e_i \mapsto n_i \cdot e_i$ . Here  $e_i$  is the  $i$ -th standard irreducible element of  $\mathbb{N}^d$  and  $n_i \in \mathbb{Z}_{\geq 1}$  for  $1 \leq i \leq d$ .

(2) *Let  $\sigma$  be a full-dimensional simplicial cone in  $N_{\mathbb{R}}$  ( $N = \mathbb{Z}^d$ ,  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ ) and  $\sigma^{\vee} \cap M \hookrightarrow F$  the minimal free resolution (note that  $\sigma^{\vee} \cap M$  is a simplicially toric sharp monoid). Then there is a natural inclusion  $\sigma^{\vee} \cap M \subset F \subset \sigma^{\vee}$ . Each irreducible element of  $F$  lies on a unique ray of  $\sigma^{\vee}$ . This gives a bijective map between the set of irreducible elements of  $F$  and the set of rays of  $\sigma^{\vee}$ .*

*Proof.* We first show (1). By Remark 2.2 (1), there exist natural inclusions

$$P \subset F \subset F \subset C(P)$$

where the first inclusion  $P \subset F$  is the minimal free resolution and the composite  $P \subset F \subset F$  is equal to  $\iota : P \rightarrow F$ . Moreover each irreducible element of the

left  $F$  (resp. the right  $F$ ) lies on a unique ray of  $C(P)$ . Let  $\{s_1, \dots, s_d\}$  (resp.  $\{t_1, \dots, t_d\}$ ) denote images of irreducible elements of the left  $F$  (resp. the right  $F$ ) in  $C(P)$ . Since the rank of the free monoid  $F$  is equal to the cardinality of the set of rays of  $C(P)$ , thus there is a positive integer  $n_i$  such that  $n_i \cdot t_i \in \{s_1, \dots, s_d\}$  for  $1 \leq i \leq d$ . After reordering, we have  $n_i \cdot t_i = s_i$  for each  $1 \leq i \leq d$ . Therefore our assertion follows.

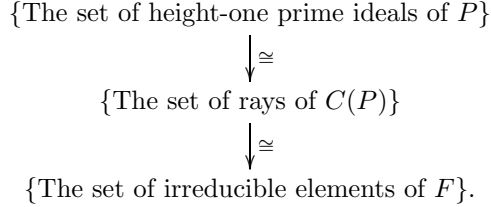
To see (2), consider

$$\sigma^\vee \cap M \subset C(\sigma^\vee \cap M) \subset (\sigma^\vee \cap M)^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q} \subset M \otimes_{\mathbb{Z}} \mathbb{R}$$

where  $C(\sigma^\vee \cap M) = \{\sum_{1 \leq i \leq m} a_i \cdot s_i \mid a_i \in \mathbb{Q}_{\geq 0}, s_i \in \sigma^\vee \cap M\}$  is a simplicial full-dimensional cone by Lemma 2.3. The cone  $\sigma^\vee$  is the completion of  $C(\sigma^\vee \cap M)$  with respect to the usual topology on  $M \otimes_{\mathbb{Z}} \mathbb{R}$ . Then the second assertion follows from Proposition 2.5 (b) and the fact that the rank of the free monoid  $F$  is equal to the cardinality of the set of rays of  $C(\sigma^\vee \cap M)$ .  $\square$

Let  $P$  be a toric monoid. Let  $I \subset P$  be an *ideal*, i.e., a subset such that  $P + I \subset I$ . We say that  $I$  is a *prime ideal* if  $P - I$  is a submonoid of  $P$ . Note that the empty set is a prime ideal. Set  $V = P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ . For a subset  $S \subset V$ , let  $C(S)$  be the (not necessarily strictly convex) cone define by  $C(S) := \{\sum_{1 \leq i \leq n} a_i \cdot s_i \mid a_i \in \mathbb{Q}_{\geq 0}, s_i \in S\}$ . To a prime ideal  $\mathfrak{p} \subset P$  we associate  $C(P - \mathfrak{p})$ . By an elementary observation, we see that  $C(P - \mathfrak{p})$  is a face of  $C(P)$  and it gives rise to a bijective correspondence between the set of prime ideals of  $P$  and the set of faces of  $C(P)$  (cf. [24, Proposition 1.10]).

Let  $P$  be a simplicially toric sharp monoid. Then the cone  $C(P) \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  is a strictly convex rational polyhedral simplicial full-dimensional cone (cf. Lemma 2.3). A prime ideal  $\mathfrak{p} \in P$  is called a *height-one prime ideal* if  $C(P - \mathfrak{p})$  is a  $(\dim P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} - 1)$ -dimensional face of  $C(P)$ , equivalently  $\mathfrak{p}$  is a minimal nonempty prime. In this case, for each height-one prime ideal  $\mathfrak{p}$  of  $P$  there exists a unique ray of  $C(P)$ , which does not lie in  $C(P - \mathfrak{p})$ . Let  $P \rightarrow F$  be the minimal free resolution. Notice that the rank of  $F$  is equal to the cardinality of the set of rays of  $C(P)$ . Therefore taking account of Proposition 2.5 (b), there exists a natural bijective correspondence between the set of rays of  $C(P)$  and the set of irreducible elements of  $F$ . Therefore there exists the natural correspondences



**Definition 2.10.** Let  $P$  be a simplicially toric sharp monoid. Let  $I$  be the set of height-one prime ideals of  $P$ . Let  $j : P \rightarrow F$  be an admissible free resolution of  $P$ . Let us denote by  $e_i$  the irreducible element of  $F$  corresponding to  $i \in I$ . We say that  $j : P \rightarrow F$  is an *admissible free resolution of type*  $\{n_i \in \mathbb{Z}_{\geq 1}\}_{i \in I}$  if  $j$  is isomorphic to the composite  $P \xrightarrow{i} F \xrightarrow{w} F$  where  $i$  is the minimal free resolution and  $w : F \rightarrow F$  is defined by  $e_i \mapsto n_i \cdot e_i$ .

Note that admissible free resolutions of a simplicially toric sharp monoid are classified by their *type*.

We use the following technical Lemma in the subsequent section.

**Lemma 2.11.** *Let  $P$  be a toric monoid and  $Q$  a saturated submonoid that is close to  $P$ . Then the monoid  $P/Q$  (cf. Section 1.3) is an abelian group, and the natural homomorphism  $P/Q \rightarrow P^{\text{gp}}/Q^{\text{gp}}$  is an isomorphism.*

*Proof.* Clearly,  $P/Q$  is finite and thus it is an abelian group. We will prove that  $P/Q \rightarrow P^{\text{gp}}/Q^{\text{gp}}$  is injective. It suffices to show that  $P \cap Q^{\text{gp}} = Q$  in  $P^{\text{gp}}$ . Since  $P \cap Q^{\text{gp}} \supset Q$ , we will show  $P \cap Q^{\text{gp}} \subset Q$ . For any  $p \in P \cap Q^{\text{gp}}$ , there exists a positive integer  $n$  such that  $n \cdot p \in Q$  because  $Q$  is close to  $P$ . Since  $Q$  is saturated, we have  $p \in Q$ . Hence  $P \cap Q^{\text{gp}} = Q$ . Next we will prove that  $P/Q \rightarrow P^{\text{gp}}/Q^{\text{gp}}$  is surjective. Let  $p \in P^{\text{gp}}$ . Take  $p_1, p_2 \in P$  such that  $p = p_1 - p_2$  in  $P^{\text{gp}}$ . It is enough to show that there exists  $p' \in P$  such that  $p' + p_2 \in Q$ . Since  $Q$  is close to  $P$ , thus our assertion is clear. Hence  $P/Q \rightarrow P^{\text{gp}}/Q^{\text{gp}}$  is surjective.  $\square$

**§2.2. MFR morphisms and admissible FR morphisms**

The notions defined below play a pivotal role in our theory.

**Definition 2.12.** Let  $(F, \Phi) : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N})$  be a morphism of fine log-schemes. We say that  $(F, \Phi)$  is an MFR (=Minimal Free Resolution) morphism if for any point  $x$  in  $X$ , the monoid  $F^{-1}\overline{\mathcal{N}}_{\bar{x}}$  is simplicially toric and the homomorphism of monoids  $\overline{\Phi}_{\bar{x}} : F^{-1}\overline{\mathcal{N}}_{\bar{x}} \rightarrow \overline{\mathcal{M}}_{\bar{x}}$  is the minimal free resolution of  $F^{-1}\overline{\mathcal{N}}_{\bar{x}}$ .

**Proposition 2.13.** *Let  $P$  be a simplicially toric sharp monoid. Let  $i : P \rightarrow F$  be its minimal free resolution. Let  $R$  be a ring. Then the map  $i : P \rightarrow F$  defines an MFR morphism of fine log schemes  $(f, h) : (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$ , where  $\mathcal{M}_F$  and  $\mathcal{M}_P$  are log structures induced by charts  $F \rightarrow R[F]$  and  $P \rightarrow R[P]$  respectively.*

*Proof.* Since  $\mathcal{M}_F$  and  $\mathcal{M}_P$  are Zariski log structures arising from  $F \rightarrow R[F]$  and  $P \rightarrow R[P]$  respectively, to prove our claim it suffices to consider only Zariski stalks of log structures i.e., to show that for any point of  $x \in \text{Spec } R[F]$  the homomorphism  $h : f^{-1}\overline{\mathcal{M}}_{P,f(x)} \rightarrow \overline{\mathcal{M}}_{F,x}$  is the minimal free resolution. Suppose that  $F = \mathbb{N}^r \oplus \mathbb{N}^{d-r}$  and  $x \in \text{Spec } R[(\mathbb{N}^{d-r})^{\text{gp}}] \subset \text{Spec } R[\mathbb{N}^{d-r}] = \text{Spec } R[\mathbb{N}^r \oplus \mathbb{N}^{d-r}]/(\mathbb{N}^r - \{0\}) \subset \text{Spec } R[\mathbb{N}^r \oplus \mathbb{N}^{d-r}]$ . Then  $f(x)$  lies in  $\text{Spec } R[P_0^{\text{gp}}] \subset \text{Spec } R[P_0] = \text{Spec } R[P]/(P_1) \subset \text{Spec } R[P]$ , where  $P_0$  is the submonoid of elements whose images of  $u : P \rightarrow F = \mathbb{N}^r \oplus \mathbb{N}^{d-r} \xrightarrow{\text{pr}_1} \mathbb{N}^r$  are zero and  $P_1$  is the ideal generated by elements of  $P$ , whose images of  $u$  are non-zero. Indeed, since  $x \in \text{Spec } R[\mathbb{N}^r \oplus \mathbb{N}^{d-r}]/(\mathbb{N}^r - \{0\})$ , thus  $f(x) \in \text{Spec } R[P]/(P_1)$ . For any  $p \in P_0$ , the image of  $i(p)$  in  $(\mathbb{N}^{d-r})^{\text{gp}}$  is invertible, and thus  $f(x) \in \text{Spec } R[P_0^{\text{gp}}]$ . Note that there exists the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & F = \mathbb{N}^r \oplus \mathbb{N}^{d-r} \\ \downarrow & & \downarrow \text{pr}_1 \\ f^{-1}\overline{\mathcal{M}}_{P,f(x)} & \xrightarrow{\overline{h}} & \overline{\mathcal{M}}_{F,x} = \mathbb{N}^r, \end{array}$$

where the vertical surjective homomorphisms are induced by the standard charts  $P \rightarrow \mathcal{M}_P$  and  $F \rightarrow \mathcal{M}_F$  respectively. Applying Proposition 2.8 to this diagram, it suffices to prove that  $f^{-1}\overline{\mathcal{M}}_{P,f(x)} \rightarrow \overline{\mathcal{M}}_{F,x}$  is injective. Since there are a sequence of surjective maps  $P^{\text{gp}} \rightarrow P^{\text{gp}}/P_0^{\text{gp}} \rightarrow f^{-1}\overline{\mathcal{M}}_{P,f(x)}^{\text{gp}}$  and the inclusion  $f^{-1}\overline{\mathcal{M}}_{P,f(x)} \subset f^{-1}\overline{\mathcal{M}}_{P,f(x)}^{\text{gp}}$ , thus it is enough to prove that for any  $p_1$  and  $p_2$  in  $P$  such that  $u(p_1) = u(p_2)$  the element  $p_1 - p_2 \in P^{\text{gp}}$  lies in  $P_0^{\text{gp}}$ . To this aim, it suffices to show that  $(\{0\} \oplus (\mathbb{N}^{d-r})^{\text{gp}}) \cap P^{\text{gp}} \subset P_0^{\text{gp}}$ . Let  $C(P) \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $C(P_0) \subset P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  be cones spanned by  $P$  and  $P_0$  respectively. Then  $C(P) \cap P^{\text{gp}} = P$  (cf. Remark 2.2), and the cone  $C(P_0)$  is a face of  $C(P)$ . Indeed, identifying  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  with  $F^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , the cone  $C(P)$  and  $C(P_0)$  are generated by irreducible elements of  $F = \mathbb{N}^d$  and  $\{0\} \oplus \mathbb{N}^{d-r}$  respectively. For any  $p \in C(P_0) \cap P^{\text{gp}}$  there exists a positive integer  $n$  such that  $n \cdot p$  lies in  $P_0$ . Taking account of the definition of  $P_0$  and  $C(P_0) \cap P^{\text{gp}} \subset P$ , we have  $p \in P_0$ , and thus  $C(P_0) \cap P^{\text{gp}} = P_0$ . Since  $C(P_0)$  is a cone in  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , we have  $P_0^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cap P^{\text{gp}} = P_0^{\text{gp}}$ . (We regard  $P_0^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  as a subspace of  $P^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ .)

This means that  $(\{0\} \oplus (\mathbb{N}^{d-r})^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}) \cap P^{\text{gp}} = P_0^{\text{gp}}$ . Thus we conclude that  $(\{0\} \oplus (\mathbb{N}^{d-r})^{\text{gp}}) \cap P^{\text{gp}} \subset P_0^{\text{gp}}$ . Hence we complete the proof.  $\square$

**Lemma 2.14.** *Let  $R$  be a ring. Let  $X = \text{Spec } R[\sigma^\vee \cap M]$  be the toric variety over  $R$ , where  $\sigma$  is a full-dimensional simplicial cone in  $N_{\mathbb{R}}$  ( $N = \mathbb{Z}^d$ ). Let  $\mathcal{M}_X$  denote the canonical log structure induced by  $\sigma^\vee \cap M \rightarrow R[\sigma^\vee \cap M]$ . Let  $\overline{\mathcal{M}}_{X,\bar{x}}$  be the stalk at a geometric point  $\bar{x} \rightarrow X$ , and let  $\overline{\mathcal{M}}_{X,\bar{x}} \rightarrow F$  be the minimal free resolution. Then there exists a natural bijective map from the set of irreducible elements of  $F$  to the set of torus-invariant divisors on  $X$  on which  $\bar{x}$  lies.*

*In particular, if  $\sigma^\vee \cap M \rightarrow H$  is the minimal free resolution, then there exists a natural bijective map from the set of irreducible elements of  $H$  to  $\sigma(1)$ .*

*Proof.* Without loss of generality we may suppose that  $\bar{x}$  lies on the subscheme

$$\text{Spec } R[\sigma^\vee \cap M]/(\sigma^\vee \cap M) \subset \text{Spec } R[\sigma^\vee \cap M].$$

Then we have  $\overline{\mathcal{M}}_{X,\bar{x}} = \sigma^\vee \cap M$ . The set of torus-invariant divisors on which  $\bar{x}$  lies is  $\{V(\rho)\}_{\rho \in \sigma(1)}$ , i.e., the set of rays of  $\sigma$ . For each ray  $\rho \in \sigma(1)$  the intersection  $\rho^\perp \cap \sigma^\vee$  is a  $(\dim \sigma - 1)$ -dimensional face. Since  $\sigma^\vee$  is simplicial, there is a unique ray of  $\sigma^\vee$  which does not lie in  $\rho^\perp \cap \sigma^\vee$ . We denote this ray by  $\rho^*$ . Then it gives rise to a bijective map  $\sigma(1) \rightarrow \sigma^\vee(1)$ ;  $\rho \mapsto \rho^*$ . By Proposition 2.5, there is a natural embedding  $\sigma^\vee \cap M \hookrightarrow F \hookrightarrow \sigma^\vee$  and each irreducible element of  $F$  lies on a unique ray of  $\sigma^\vee$ . It gives a bijective map from the set of irreducible elements of  $F$  to  $\sigma^\vee(1)$ . Hence our assertion follows.  $\square$

**Definition 2.15.** Let  $S$  be a scheme. Let  $N = \mathbb{Z}^d$  be a lattice and  $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  its dual. Let  $\Sigma$  be a fan in  $N_{\mathbb{R}}$  and  $X_\Sigma$  the associated toric variety over  $S$ . Let  $\mathfrak{n} := \{n_\rho\}_{\rho \in \Sigma(1)}$  be a set of positive integers indexed by  $\Sigma(1)$ . A morphism of fine log schemes  $(f, \phi) : (Y, \mathcal{M}) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma)$  is called an *admissible FR morphism of type  $\mathfrak{n}$*  if for any geometric point  $\bar{y} \rightarrow Y$  the homomorphism  $f^{-1}\overline{\mathcal{M}}_{P,f(\bar{y})} \rightarrow \overline{\mathcal{M}}_{\bar{y}}$  is isomorphic to the composite  $f^{-1}\overline{\mathcal{M}}_{P,f(\bar{y})} \xrightarrow{i} F \xrightarrow{n} F$  where  $i : f^{-1}\overline{\mathcal{M}}_{P,f(\bar{y})} \rightarrow F$  is the minimal free resolution and  $n : F \rightarrow F$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$ . Here for a ray  $\rho \in \Sigma(1)$  such that  $f(\bar{y}) \in V(\rho)$  we write  $e_\rho$  for the irreducible element of  $F$  corresponding to  $\rho$  (cf. Lemma 2.14).

**Proposition 2.16.** *Let  $P$  be a simplicially toric sharp monoid. Suppose that  $P = \sigma^\vee \cap M$  where  $\sigma \subset N_{\mathbb{R}}$  is a full-dimensional simplicial cone. Let  $\mathfrak{n} := \{n_\rho\}_{\rho \in \sigma(1)}$  be a set of positive integers indexed by  $\sigma(1)$ . Let  $\iota : P \rightarrow$*

$F$  be an admissible free resolution defined to be the composite  $P \rightarrow F \xrightarrow{n} F$  where  $P \rightarrow F$  is the minimal free resolution and  $n : F \rightarrow F$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$  for each ray  $\rho \in \sigma(1)$ . Here  $e_\rho$  is the irreducible element of  $F$  corresponding to  $\rho$  (cf. Definition 2.10, Lemma 2.14). Let  $R$  be a ring. Let  $(f, \phi) : (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$  be the morphism induced by  $\iota$ . Then  $(f, \phi)$  is an admissible FR morphism of type  $\mathbf{n}$ .

*Proof.* Let us denote by  $(g, \psi) : (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[F], \mathcal{M}_F)$  the morphism induced by  $n : F \rightarrow F$ . Notice that for any geometric point  $\bar{x} \rightarrow \text{Spec } R[F]$  the canonical map  $F \rightarrow \overline{\mathcal{M}}_{F, \bar{x}}$  and  $F \rightarrow \overline{\mathcal{M}}_{F, g(\bar{x})}$  are of the form  $F \cong \mathbb{N}^s \oplus \mathbb{N}^t \xrightarrow{\text{pr}_1} \mathbb{N}^s \cong \overline{\mathcal{M}}_{F, \bar{x}}$  and  $F \cong \mathbb{N}^s \oplus \mathbb{N}^t \xrightarrow{\text{pr}_1} \mathbb{N}^s \cong \overline{\mathcal{M}}_{F, g(\bar{x})}$  respectively for some  $s, t \in \mathbb{Z}_{\geq 0}$ . Therefore  $\overline{\psi}_{\bar{x}} : \mathbb{N}^s \cong g^{-1}\overline{\mathcal{M}}_{F, g(\bar{x})} \rightarrow \mathbb{N}^s \cong \overline{\mathcal{M}}_{F, \bar{x}}$  is the homomorphism induced by  $e_\rho \mapsto n_\rho \cdot e_\rho$ . Taking account of Proposition 2.13, our assertion follows from the definition of  $\iota : P \rightarrow F$ .  $\square$

**Proposition 2.17.** *Let  $R$  be a ring. Let  $P$  be a simplicially toric sharp monoid and  $\iota : P \rightarrow F = \mathbb{N}^d$  an admissible free resolution of  $P$ . Let  $(f, h) : (S, \mathcal{M}) \rightarrow (X_P := \text{Spec } R[P], \mathcal{M}_P)$  be a morphism of fine log schemes. Here  $\mathcal{M}_P$  is the canonical log structure on  $\text{Spec } R[P]$  induced by  $P \rightarrow R[P]$ . Let  $c : P \rightarrow \mathcal{M}_P$  be the standard chart. Let  $\bar{s} \rightarrow S$  be a geometric point. Consider the following commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{\iota} & F \\ \bar{c}_{\bar{s}} \downarrow & & \downarrow \alpha \\ f^{-1}\overline{\mathcal{M}}_{P, \bar{s}} & \xrightarrow{\bar{h}_{\bar{s}}} & \overline{\mathcal{M}}_{\bar{s}} \end{array}$$

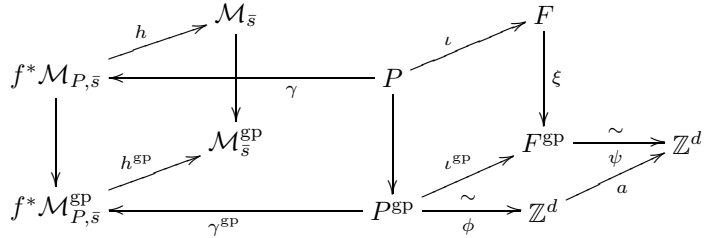
where  $\bar{c}_{\bar{s}}$  is the map induced by the standard chart such that  $\alpha$  étale locally lifts to a chart. Then there exists an fppf neighborhood  $U$  of  $\bar{s}$  in which we have a chart  $\varepsilon : F \rightarrow \mathcal{M}$  such that the following diagram

$$\begin{array}{ccc} P & \xrightarrow{\iota} & F \\ c \downarrow & & \downarrow \varepsilon \\ f^*\mathcal{M}_P & \xrightarrow{h} & \mathcal{M} \end{array}$$

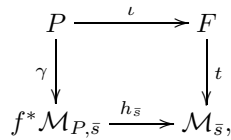
commutes and the composition  $F \xrightarrow{\varepsilon} \mathcal{M} \rightarrow \overline{\mathcal{M}}_{\bar{s}}$  is equal to  $\alpha$ . If the order of the cokernel of  $P^{\text{gp}} \rightarrow F^{\text{gp}}$  is invertible on  $R$ , then we can take an étale neighborhood  $U$  of  $\bar{s}$  with the above property.

*Proof.* Let  $\gamma := c_{\bar{s}} : P \rightarrow f^*\mathcal{M}_{P, \bar{s}}$  be the chart induced by the standard chart  $P \rightarrow \mathcal{M}_P$ . In order to show our assertion, we shall prove that there exists

a homomorphism  $t : F \rightarrow \mathcal{M}_{\bar{s}}$ , which is a lifting of  $\alpha$ , such that  $t \circ \iota = h_{\bar{s}} \circ \gamma$ . To this aim, consider the following diagram



where vertical arrows are natural inclusions and  $\phi$  and  $\psi$  are isomorphisms chosen as follows. By elementary algebra, we can take the isomorphisms  $\phi$  and  $\psi$  so that  $a := \psi \circ \iota^{\text{gp}} \circ \phi^{-1}$  is represented by the  $(d \times d)$ -matrix  $(a_{ij})$  with  $a_{ii} =: \lambda_i \in \mathbb{Z}_{\geq 1}$  for  $0 \leq i \leq d$ , and  $a_{ij} = 0$  for  $i \neq j$ . Let us construct a homomorphism  $F^{\text{gp}} \rightarrow \mathcal{M}_{\bar{s}}^{\text{gp}}$  filling in the diagram. Let  $\{e_i\}_{1 \leq i \leq d}$  be the canonical basis of  $\mathbb{Z}^d$  and put  $m_i := h^{\text{gp}} \circ \gamma^{\text{gp}} \circ \phi^{-1}(e_i)$ . Let  $n_i$  be an element in  $\mathcal{M}_{\bar{s}}^{\text{gp}}$  such that  $q(n_i) = \alpha^{\text{gp}}(\psi^{-1}(e_i))$  for  $0 \leq i \leq d$ , where  $q : \mathcal{M}_{\bar{s}}^{\text{gp}} \rightarrow \overline{\mathcal{M}}_{\bar{s}}^{\text{gp}}$  is the natural projection. Then there exists a unit element  $u_i$  in  $\mathcal{O}_{\bar{S}}^*$  such that  $u_i + \lambda_i \cdot n_i = m_i$  in  $\mathcal{M}_{\bar{s}}^{\text{gp}}$  for  $0 \leq i \leq d$ . The algebra  $\mathcal{O}' := \mathcal{O}_{S, \bar{s}}[T_1, \dots, T_d]/(T_i^{\lambda_i} - u_i)_{i=1}^d$  is a finite flat  $\mathcal{O}_{S, \bar{s}}$ -algebra. If the order of cokernel of  $P^{\text{gp}} \rightarrow F^{\text{gp}}$  is invertible on  $R$ ,  $\mathcal{O}'$  is an étale  $\mathcal{O}_{S, \bar{s}}$ -algebra. After the base change to  $\mathcal{O}'$ , there exists a homomorphism  $\eta : F^{\text{gp}} \rightarrow \mathcal{M}_{\bar{s}}^{\text{gp}}$  which is an extension of  $h^{\text{gp}} \circ \gamma^{\text{gp}}$ . Since  $\mathcal{M}_{\bar{s}} = \mathcal{M}_{\bar{s}}^{\text{gp}} \times_{\overline{\mathcal{M}}_{\bar{s}}^{\text{gp}}} \overline{\mathcal{M}}_{\bar{s}}$ , the map  $t : F \rightarrow \mathcal{M}_{\bar{s}}^{\text{gp}} \times_{\overline{\mathcal{M}}_{\bar{s}}^{\text{gp}}} \overline{\mathcal{M}}_{\bar{s}} = \mathcal{M}_{\bar{s}}$  defined by  $m \mapsto (\eta(\xi(m)), \alpha(m))$  is a homomorphism which makes the diagram



commutative. Since  $P$  and  $F$  are finitely generated, this diagram extends to a chart in some fppf neighborhood of  $\bar{s}$ . The last assertion immediately follows. □

### §3. Moduli Stack of Admissible FR Morphisms into a Toroidal Embedding

#### §3.1. Moduli stack of admissible FR morphisms

Let  $(X, U)$  be a toroidal embedding over a field  $k$ . Let us denote by  $I$  the set of irreducible components of  $X - U$ .

Consider a triple  $(X, U, \mathbf{n})$ , where  $\mathbf{n} = \{n_i \in \mathbb{Z}_{\geq 1}\}_{i \in I}$ . We shall refer to  $(X, U, \mathbf{n})$  as a *toroidal embedding*  $(X, U)$  of level  $\mathbf{n}$ . Let  $\mathcal{M}_X$  be the canonical log structure of  $(X, U)$ . The pair  $(X, \mathcal{M}_X)$  is a fine saturated log scheme (cf. Section 1.3). If we further assume that  $(X, U)$  is a *good* toroidal embedding (cf. Section 1.2), then for any point  $x$  on  $X$ , the stalk  $\overline{\mathcal{M}}_{X, \bar{x}}$  is a *simplicially toric sharp monoid*.

**Proposition 3.1.** *Let  $\overline{\mathcal{M}}_{X, \bar{x}}$  be the stalk at a geometric point  $\bar{x} \rightarrow X$ . Let  $\overline{\mathcal{M}}_{X, \bar{x}} \rightarrow F$  be the minimal free resolution. Then there exists a natural map from the set of irreducible elements of  $F$  to the set of irreducible components of  $X - U$  in which  $\bar{x}$  lies.*

*Proof.* Set  $P = \overline{\mathcal{M}}_{X, \bar{x}}$ . Note first that there exists the natural correspondence between irreducible elements of  $F$  and height-one prime ideals of  $P$  (see Section 2). Therefore it is enough to show that there exists a natural map from the set of height-one prime ideals of  $P$  to the set of irreducible components of  $X - U$  on which  $\bar{x}$  lies. Let  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$  be the set of height-one prime ideals of  $P$ . Let  $D = X - U$  be the reduced closed subscheme (each component is pure 1-codimensional) and  $I_D$  be the ideal sheaf associated to  $D$ . If  $x$  denotes the image of  $\bar{x}$ , and  $\mathcal{O}_{X, x}$  denotes its Zariski stalk, then the natural morphism  $\text{Spec } \mathcal{O}_{X, \bar{x}}/I_D \mathcal{O}_{X, \bar{x}} \rightarrow \text{Spec } \mathcal{O}_{X, x}/I_D \mathcal{O}_{X, x}$  maps generic points to generic points. On the other hand, the log structure  $\mathcal{M}_X$  (cf. Section 1) has a chart  $P \rightarrow \mathcal{O}_{X, \bar{x}}$  and the support of  $\overline{\mathcal{M}}_X$  is equal to  $D$ , the underlying space of  $\text{Spec } \mathcal{O}_{X, \bar{x}}/I_D \mathcal{O}_{X, \bar{x}}$  is naturally equal to  $\text{Spec } \mathcal{O}_{X, \bar{x}}/(\mathfrak{p}_1 \mathcal{O}_{X, \bar{x}} \cap \dots \cap \mathfrak{p}_n \mathcal{O}_{X, \bar{x}})$ . (The closed subscheme  $\text{Spec } k[P]/(\mathfrak{p}_1 k[P] \cap \dots \cap \mathfrak{p}_n k[P])$  has the same underlying space as the complement  $\text{Spec } k[P] - \text{Spec } k[P^{\text{gp}}]$ .) Therefore it suffices to prove that for any height-one prime ideal  $\mathfrak{p}_i$ , the closed subset  $\text{Spec } \mathcal{O}_{X, \bar{x}}/\mathfrak{p}_i \mathcal{O}_{X, \bar{x}}$  is an irreducible component of  $\text{Spec } \mathcal{O}_{X, \bar{x}}/I_D \mathcal{O}_{X, \bar{x}}$ . By [16, Theorem 3.2 (1)] there exists an isomorphism  $\hat{\mathcal{O}}_{X, \bar{x}} \xrightarrow{\sim} k'[[P]][[T_1, \dots, T_r]]$  and the composite  $P \rightarrow \mathcal{O}_{X, \bar{x}} \rightarrow \hat{\mathcal{O}}_{X, \bar{x}}$  can be identified with the natural map  $P \rightarrow k'[[P]][[T_1, \dots, T_r]]$ , where  $k'$  is the residue field of  $\mathcal{O}_{X, \bar{x}}$ . (Strictly speaking, [16] only treats the case of Zariski log structures, but the same proof can apply to the case of étale log structures.) Since  $P - \mathfrak{p}_i$  is a submonoid and moreover it is a toric monoid, thus

$\hat{\mathcal{O}}_{X,\bar{x}}/\mathfrak{p}_i\hat{\mathcal{O}}_{X,\bar{x}}$  is isomorphic to the integral domain  $k[[P - \mathfrak{p}_i]][[t_1, \dots, t_r]]$ . Note that  $\hat{\mathcal{O}}_{X,\bar{x}}$  is a flat  $\mathcal{O}_{X,\bar{x}}$ -algebra and the natural map  $\mathcal{O}_{X,\bar{x}} \rightarrow \hat{\mathcal{O}}_{X,\bar{x}}$  is injective, thus  $\mathcal{O}_{X,\bar{x}}/\mathfrak{p}_i\mathcal{O}_{X,\bar{x}} \rightarrow \hat{\mathcal{O}}_{X,\bar{x}}/\mathfrak{p}_i\hat{\mathcal{O}}_{X,\bar{x}}$  is injective. Therefore  $\mathcal{O}_{X,\bar{x}}/\mathfrak{p}_i\mathcal{O}_{X,\bar{x}}$  is an integral domain. Hence  $\text{Spec } \mathcal{O}_{X,\bar{x}}/\mathfrak{p}_i\mathcal{O}_{X,\bar{x}}$  is irreducible. Thus we obtain the natural map as desired.  $\square$

**Definition 3.2.** Let  $(X, U, \mathbf{n} = \{n_i \in \mathbb{Z}_{\geq 1}\}_{i \in I})$  be a good toroidal embedding of level  $\mathbf{n}$  over  $k$ . (We denote by  $I$  the set of irreducible components of  $X - U$ .) Let  $\mathcal{M}_X$  be the canonical log structure on  $X$ . An *admissible FR morphism* to  $(X, \mathcal{M}_X, \mathbf{n})$  (or  $(X, U, \mathbf{n})$ ) is a morphism  $(f, \phi) : (S, \mathcal{M}_S) \rightarrow (X, \mathcal{M}_X)$  of fine log schemes such that for any geometric point  $\bar{s} \rightarrow S$  the homomorphism  $\bar{\phi} : f^{-1}\overline{\mathcal{M}}_{X,f(\bar{s})} \rightarrow \overline{\mathcal{M}}_{S,\bar{s}}$  is isomorphic to

$$f^{-1}\overline{\mathcal{M}}_{X,f(\bar{s})} \xrightarrow{\iota} F \xrightarrow{n} F,$$

where  $\iota$  is the minimal free resolution and  $n$  is defined by  $e \mapsto n_{i(e)} \cdot e$  where  $e$  is an irreducible element of  $F$  and  $i(e)$  is an irreducible component of  $X - U$  to which  $e$  corresponds via Proposition 3.1. (We shall call such a resolution an *admissible free resolution of type  $\mathbf{n}$*  at  $f(\bar{s})$ .)

We define a category  $\mathcal{X}_{(X,U,\mathbf{n})}$  as follows. The objects are admissible FR morphisms to  $(X, \mathcal{M}_X, \mathbf{n})$ . A morphism  $\{(f, \phi) : (S, \mathcal{M}) \rightarrow (X, \mathcal{M}_X)\} \rightarrow \{(g, \psi) : (T, \mathcal{N}) \rightarrow (X, \mathcal{M}_X)\}$  in  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a morphism of  $(X, \mathcal{M}_X)$ -log schemes  $(h, \alpha) : (S, \mathcal{M}) \rightarrow (T, \mathcal{N})$  such that  $\alpha : h^*\mathcal{N} \rightarrow \mathcal{M}$  is an isomorphism. By simply forgetting log structures, we have a functor

$$\pi_{(X,U,\mathbf{n})} : \mathcal{X}_{(X,U,\mathbf{n})} \rightarrow (X\text{-schemes}),$$

which makes  $\mathcal{X}_{(X,U,\mathbf{n})}$  a fibered category over the category of  $X$ -schemes. This fibered category is a stack with respect to étale topology because it is fibered in groupoid and log structures are pairs of étale sheaves and their homomorphism on étale site. Furthermore according to the fppf descent theory for fine log structures ([21, Theorem A.1]),  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a stack with respect to the fppf topology.

**Theorem 3.3.** Let  $(X, U, \mathbf{n} = \{n_i \in \mathbb{Z}_{\geq 1}\}_{i \in I})$  be a good toroidal embedding of level  $\mathbf{n}$  over a field  $k$ . Then the stack  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a smooth algebraic stack of finite type over  $k$  with finite diagonal. The functor  $\pi_{(X,U,\mathbf{n})} : \mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is a coarse moduli map, which induces an isomorphism  $\pi_{(X,U,\mathbf{n})}^{-1}(U) \xrightarrow{\sim} U$ . If we suppose that  $(X, U)$  is tame (cf. Section 1.2), and  $n_i$  is prime to the characteristic of  $k$  for all  $i \in I$ , then the stack  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a smooth Deligne-Mumford stack of finite type and separated over  $k$ .

In the case of  $n_i = 1$  for all  $i \in I$ , we have the followings.

- (1) The coarse moduli map induces an isomorphism  $\pi_{(X,U,n)}^{-1}(X_{\text{sm}}) \xrightarrow{\sim} X_{\text{sm}}$  when we denote by  $X_{\text{sm}}$  the smooth locus of  $X$ .
- (2) Suppose further that  $(X,U)$  is tame. If there exists another functor  $f : \mathcal{X} \rightarrow X$  such that  $\mathcal{X}$  is a smooth separated Deligne-Mumford stack and  $f$  is a coarse moduli map, then there exists a functor  $\phi : \mathcal{X} \rightarrow \mathcal{X}_{(X,U,n)}$  such that the diagram

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{\phi} & \mathcal{X}_{(X,U,n)} \\
 & \searrow f & \downarrow \pi_{(X,U,n)} \\
 & & X
 \end{array}$$

commutes in the 2-categorical sense and such  $\phi$  is unique up to a unique isomorphism.

*Remark 3.4.* Moreover the stack  $\mathcal{X}_{(X,U,n)}$  has the following nice properties, which we will show later (because we need some preliminaries).

- (1) We will see that  $\mathcal{X}_{(X,U,n)}$  is a *tame* algebraic stack in the sense of [2] (see Corollary 3.10).
- (2) We will prove that the complement  $\mathcal{X}_{(X,U,n)} - U$  with reduced induced stack structure is a *normal crossing divisor* on  $\mathcal{X}_{(X,U,n)}$  (see Section 3.5).

### §3.2.

Before the proof of Theorem 3.3, we shall observe the case when  $(X,U)$  is an *affine simplicial toric variety*  $\text{Spec } R[\sigma^\vee \cap M]$  over a ring  $R$ , where  $\sigma$  is a full-dimensional simplicial cone in  $N_{\mathbb{R}}$  ( $N = \mathbb{Z}^d$ ). (For the application in Section 4, we work over a general ring rather than a field.) Each ray  $\rho \in \sigma$  defines the torus-invariant divisor  $V(\rho)$ . Consider the torus embedding  $X(\sigma, \mathbf{n}) := (\text{Spec } R[\sigma^\vee \cap M], \text{Spec } R[M], \mathbf{n})$  of level  $\mathbf{n} = \{n_\rho\}_{\rho \in \sigma(1)}$ . Set  $P := \sigma^\vee \cap M$  (this is a simplicially toric sharp monoid). Let  $\iota : P \rightarrow F$  be the injective homomorphism defined to be the composite  $P \xrightarrow{i} F \xrightarrow{n} F$ , where  $i$  is the minimal free resolution and  $n : F \rightarrow F$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$ , where  $e_\rho$  denotes the irreducible element of  $F$  corresponding to a ray  $\rho \in \sigma(1)$  (cf. Lemma 2.14). The cokernel  $F^{\text{gp}}/\iota(P)^{\text{gp}}$  is a finite abelian group (and isomorphic to  $F/\iota(P)$  by Lemma 2.11). We view  $F^{\text{gp}}/\iota(P)^{\text{gp}}$  as a finite commutative group scheme over  $R$ . We denote by  $G := (F^{\text{gp}}/\iota(P)^{\text{gp}})^D$  its Cartier

dual over  $R$ . We define an action  $m : \text{Spec } R[F] \times_R G \rightarrow \text{Spec } R[F]$  as follows. Put  $\pi : F \rightarrow F^{\text{gp}}/\iota(P)^{\text{gp}}$  the canonical map. For each  $R$ -algebra  $A$  and  $A$ -valued point  $p : F \rightarrow A$  (a map of monoids) of  $\text{Spec } R[F]$ , a  $A$ -valued point  $g : (F^{\text{gp}}/\iota(P)^{\text{gp}}) \rightarrow A$  (a map of monoids) of  $G$  sends  $p$  to the map  $p^g : F \rightarrow A$  defined by the formula  $p^g(f) = p(f) \cdot g(\pi(f))$  for  $f \in F$ . We denote by  $[\text{Spec } R[F]/G]$  the stack-theoretic quotient associated to the groupoid  $m, \text{pr}_1 : \text{Spec } R[F] \times_R G \rightrightarrows \text{Spec } R[F]$  (cf. [19, (10.13.1)]). By [7, section 3 and Theorem 3.1], the coarse moduli space for this quotient stack is  $\text{Spec } R[F]^G$  where

$$R[F]^G = \{a \in R[F] \mid m^*(a) = \text{pr}_1^*(a)\} \subset R[F].$$

Moreover the natural morphism  $R[P] \rightarrow R[F]^G$  is an isomorphism:

**Claim 3.4.1.** *The natural morphism  $R[P] \rightarrow R[F]^G$  is an isomorphism. In particular, the toric variety  $\text{Spec } R[P]$  is a coarse moduli space for  $[\text{Spec } R[F]/G]$ .*

*Proof.* Note first that  $\Gamma(G) = R[F^{\text{gp}}]/(f - 1)_{f \in P^{\text{gp}} \subset F^{\text{gp}}}$  is a finite free  $R$ -module, and

$$m^* : R[F] \rightarrow R[F] \otimes_R R[F^{\text{gp}}]/(f - 1)_{f \in P^{\text{gp}} \subset F^{\text{gp}}}$$

maps  $f \in F$  to  $f \otimes f$ . Since  $\text{pr}_1^*$  maps  $f$  to  $f \otimes 1$ , thus  $m^*(\sum_{f \in F} r_f \cdot f) = \text{pr}_1^*(\sum_{f \in F} r_f \cdot f)$  ( $r_f \in R$ ) if and only if  $r_f = 0$  for  $f$  with  $f \notin P^{\text{gp}} \subset F^{\text{gp}}$ . Therefore it suffices to prove that  $P^{\text{gp}} \cap F = P$ . Clearly, we have  $P^{\text{gp}} \cap F \supset P$ , thus we will show  $P^{\text{gp}} \cap F \subset P$ . For any  $f \in P^{\text{gp}} \cap F$ , there exists a positive integer  $n$  such that  $n \cdot f \in P$ . Since  $P$  is saturated, we have  $f \in P$ . Hence our claim follows. □

Let  $\mathcal{M}_P$  be the canonical log structure on the toric variety  $\text{Spec } R[P]$ .

**Proposition 3.5.** *There exists a natural isomorphism  $\Phi : [\text{Spec } R[F]/G] \rightarrow \mathcal{X}_{X(\sigma, n)}$  over  $\text{Spec } R[P]$ . The composite  $\text{Spec } R[F] \rightarrow [\text{Spec } R[F]/G] \rightarrow \mathcal{X}_{X(\sigma, n)}$  corresponds to the admissible  $FR$  morphism  $(\chi, \epsilon) : (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$  which is induced by  $\iota : P \rightarrow F$ .*

*Proof.* Let  $[\text{Spec } R[F]/G] \rightarrow \text{Spec } R[P]$  be the natural coarse moduli map. According to [21, Proposition 5.20 and Remark 5.21], the stack  $[\text{Spec } R[F]/G]$  over  $\text{Spec } R[P]$  is isomorphic to the stack  $\mathcal{S}$  whose fiber over  $f : S \rightarrow \text{Spec } R[P]$  is the groupoid of triples  $(\mathcal{N}, \eta, \gamma)$ , where  $\mathcal{N}$  is a fine log structure on  $S$ ,  $\eta :$

$f^*\mathcal{M}_P \rightarrow \mathcal{N}$  is a morphism of log structures, and  $\gamma : F \rightarrow \overline{\mathcal{N}}$  is a morphism, which étale locally lifts to a chart, such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\iota} & F \\ \bar{c} \downarrow & & \downarrow \gamma \\ f^{-1}\overline{\mathcal{M}}_P & \longrightarrow & \overline{\mathcal{N}} \end{array}$$

commutes. Here we denote by  $c$  the standard chart  $P \rightarrow \mathcal{M}_P$ . Notice that the last condition implies that  $\eta$  is an admissible FR morphism to  $(\text{Spec } R[P], \mathcal{M}_P, \mathfrak{n})$ . Indeed, by Proposition 2.17 there exists fppf locally a chart  $\gamma' : F \rightarrow \mathcal{N}$  such that  $\gamma'$  induces  $\gamma$ , and  $\gamma' \circ \iota : P \rightarrow F \rightarrow \mathcal{N}$  is equal to  $P \rightarrow f^*\mathcal{M}_P \rightarrow \mathcal{N}$ . This means that there exists fppf locally on  $S$  a strict morphism  $(S, \mathcal{N}) \rightarrow (\text{Spec } R[F], \mathcal{M}_F)$  such that the composite  $(S, \mathcal{N}) \rightarrow (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$  is equal to  $(f, \eta) : (S, \mathcal{N}) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$ . By Proposition 2.16  $(\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$  is an admissible morphism to  $(\text{Spec } R[P], \mathcal{M}_P, \mathfrak{n})$ , thus so is  $(f, \eta) : (S, \mathcal{N}) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$ . Therefore there exists a natural morphism  $\Phi : [\text{Spec } R[F]/G] \rightarrow \mathcal{X}_{X(\sigma, \mathfrak{n})}$  which forgets the additional data of the map  $\gamma : F \rightarrow \overline{\mathcal{N}}$ . To show that  $\Phi$  is essentially surjective, it suffices to see that every object in  $\mathcal{X}_{X(\sigma, \mathfrak{n})}$  is fppf locally isomorphic to the image of  $\Phi$ . Let  $(f, \phi) : (S, \mathcal{N}) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$  be an admissible FR morphism to  $(\text{Spec } R[P], \mathcal{M}_P, \mathfrak{n})$ . By Proposition 2.16, for any geometric point  $\bar{s} \rightarrow S$  the map  $f^{-1}\overline{\mathcal{M}}_{P, f(\bar{s})} \rightarrow \overline{\mathcal{N}}_{\bar{s}}$  is isomorphic to  $g^{-1}\overline{\mathcal{M}}_{P, g(\bar{x})} \rightarrow \overline{\mathcal{M}}_{F, \bar{x}}$ , where  $g : \text{Spec } R[F] \rightarrow \text{Spec } R[P]$  is induced by  $\iota : P \rightarrow F$ , and  $\bar{x}$  is a geometric point on  $\text{Spec } R[F]$  which is lying over  $f(\bar{s})$ . Since  $F \rightarrow \overline{\mathcal{M}}_{F, \bar{x}}$  has the form  $F \cong \mathbb{N}^r \oplus \mathbb{N}^{d-r} \xrightarrow{\text{pr}_1} \mathbb{N}^r$ , thus  $P \rightarrow f^{-1}\overline{\mathcal{M}}_{P, f(\bar{s})} \rightarrow \overline{\mathcal{N}}_{\bar{s}}$  has the form  $P \xrightarrow{\iota} F \cong \mathbb{N}^r \oplus \mathbb{N}^{d-r} \xrightarrow{\text{pr}_1} \mathbb{N}^r$ , thus the essentially surjectiveness follows from Proposition 2.17. Finally, we will prove that  $\Phi$  is fully faithful. To this end, it suffices to show that given two objects  $(h_1 : f^*\mathcal{M}_P \rightarrow \mathcal{N}_1, \gamma_1 : F \rightarrow \overline{\mathcal{N}}_1)$  and  $(h_2 : f^*\mathcal{M}_P \rightarrow \mathcal{N}_2, \gamma_2 : F \rightarrow \overline{\mathcal{N}}_2)$  in  $[\text{Spec } R[F]/G](S)$ , any morphism of log structures  $\xi : \mathcal{N}_1 \rightarrow \mathcal{N}_2$ , such that  $\xi \circ h_1 = h_2$ , has the property that  $\bar{\xi} \circ \gamma_1 = \gamma_2$ . It follows from the fact that  $P$  is close to  $F$  via  $\iota$  and every stalk of  $\overline{\mathcal{N}}_2$  is of the form  $\mathbb{N}^r$  for some  $r \in \mathbb{N}$ . Indeed for any  $f \in F$ , there exists a positive integer  $n$  such that  $n \cdot f \in P$ . Since  $\xi \circ h_1 = h_2$ , we have  $\bar{\xi} \circ \gamma_1(n \cdot f) = \gamma_2(n \cdot f)$ . Since every stalk of  $\overline{\mathcal{N}}_2$  is of the form  $\mathbb{N}^r$  for some  $r \in \mathbb{N}$ , we conclude that  $\bar{\xi} \circ \gamma_1(f) = \gamma_2(f)$ .

Finally, we will prove the last assertion. By Proposition 2.16,  $(\chi, \epsilon)$  is an admissible FR morphism of type  $\mathfrak{n}$ . By [21, Proposition 5.20], the projection  $\text{Spec } R[F] \rightarrow [\text{Spec } R[F]/G]$  amounts to the triple  $(\mathcal{M}_F, \epsilon : \chi^*\mathcal{M}_P \rightarrow$

$\mathcal{M}_F, F \rightarrow \overline{\mathcal{M}}_F$ ) over  $\chi : \text{Spec } R[F] \rightarrow \text{Spec } R[P]$ . Hence  $\Phi((\mathcal{M}_F, \epsilon : \chi^* \mathcal{M}_P \rightarrow \mathcal{M}_F, F \rightarrow \overline{\mathcal{M}}_F)) = (\mathcal{M}_F, \epsilon : \chi^* \mathcal{M}_P \rightarrow \mathcal{M}_F)$  and thus our claim follows.  $\square$

**Proposition 3.6.** *With the same notation as above, we have followings:*

- (1)  $\mathcal{X}_{X(\sigma, n)}$  has finite diagonal.
- (2) The natural morphism  $\mathcal{X}_{X(\sigma, n)} \rightarrow \text{Spec } R[P]$  is a coarse moduli map.
- (3)  $\mathcal{X}_{X(\sigma, n)}$  is smooth over  $R$ .

*Proof.* We first prove (1). The base change of the diagonal map  $[\text{Spec } R[F]/G] \rightarrow [\text{Spec } R[F]/G] \times_R [\text{Spec } R[F]/G]$  by the natural morphism  $\text{Spec } R[F] \times_R \text{Spec } R[F] \rightarrow [\text{Spec } R[F]/G] \times_R [\text{Spec } R[F]/G]$  is isomorphic to  $\text{Spec } R[F] \times_R G \rightarrow \text{Spec } R[F] \times_R \text{Spec } R[F]$ , which maps  $(x, g)$  to  $(x, x^g)$ . Since  $\text{pr}_1 : \text{Spec } R[F] \times_R G \rightarrow \text{Spec } R[F]$  is proper and  $\text{pr}_1 : \text{Spec } R[F] \times_R \text{Spec } R[F] \rightarrow \text{Spec } R[F]$  is separated, thus  $\text{Spec } R[F] \times_R G \rightarrow \text{Spec } R[F] \times_R \text{Spec } R[F]$  is proper. Clearly, it is also quasi-finite, so it is a finite morphism and we conclude that  $[\text{Spec } R[F]/G]$  has finite diagonal. Hence by Proposition 3.5  $\mathcal{X}_{X(\sigma, n)}$  has finite diagonal. The assertion (2) follows from Claim 3.4.1 and Proposition 3.5. Next we will prove (3). By Proposition 3.5, we have a finite flat cover  $\text{Spec } R[F] \rightarrow \mathcal{X}_{X(\sigma, n)}$  from a smooth  $R$ -scheme  $\text{Spec } R[F]$ , where  $F$  is isomorphic to  $\mathbb{N}^r$  for some  $r \in \mathbb{N}$ . Let  $V \rightarrow \mathcal{X}_{X(\sigma, n)}$  be a smooth surjective morphism from a  $R$ -scheme  $V$ . Notice that the composite  $V \times_{\mathcal{X}_{X(\sigma, n)}} \text{Spec } R[F] \xrightarrow{\text{pr}_1} V \rightarrow \text{Spec } R$  is smooth, and  $V \times_{\mathcal{X}_{X(\sigma, n)}} \text{Spec } R[F] \rightarrow V$  is a finitely presented flat surjective morphism. Indeed the composite  $V \times_{\mathcal{X}_{X(\sigma, n)}} \text{Spec } R[F] \rightarrow \text{Spec } R[F] \rightarrow \text{Spec } R$  is smooth. Thus by [9, IV Proposition 17.7.7], we see that  $V$  is smooth over  $R$ . Hence  $\mathcal{X}_{X(\sigma, n)}$  is smooth over  $R$ .  $\square$

*Remark 3.7.* Let  $\sigma$  be a simplicial (not necessarily full-dimensional) cone in  $N_{\mathbb{R}}$ . Then there exists a splitting  $N \cong N' \oplus N''$  such that  $\sigma \cong \sigma' \oplus \{0\} \subset N'_{\mathbb{R}} \oplus N''_{\mathbb{R}}$  where  $\sigma'$  is a full-dimensional cone in  $N'_{\mathbb{R}}$ . Thus  $X_{\sigma} \cong X_{\sigma'} \times_R \text{Spec } R[M'']$ , where  $M'' = \text{Hom}_{\mathbb{Z}}(N'', \mathbb{Z})$ . The log structure  $\mathcal{M}_{\sigma}$  is isomorphic to the pullback  $\text{pr}_1^* \mathcal{M}_{\sigma'}$  where  $\text{pr}_1 : X_{\sigma'} \times_R \text{Spec } R[M''] \rightarrow X_{\sigma}$ . Therefore  $\mathcal{X}_{X(\sigma, n)}$  is an algebraic stack of finite type over  $R$  for arbitrary simplicial cone  $\sigma$ . Also, Proposition 3.6 holds for arbitrary simplicial cones.

### §3.3. Proof of Theorem 3.3

We will return to the proof of Theorem 3.3. Let  $R = k$  be a field.

*Proof of algebraicity.* We will prove that  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a smooth algebraic stack of finite type over a field  $k$  with finite diagonal. Clearly, one can assume that  $X(\sigma, \mathbf{n}) := (X, U, \mathbf{n}) = (\mathrm{Spec} R[\sigma^\vee \cap M], \mathrm{Spec} R[M], \mathbf{n})$  (with the same notation and hypothesis as in Section 3.2), where  $\sigma$  is a full-dimensional simplicial cone. Then by Proposition 3.5 and Proposition 3.6 (1) and (3),  $\mathcal{X}_{X(\sigma,\mathbf{n})}$  is a smooth algebraic stack of finite type over  $R$  with finite diagonal because  $\mathcal{X}_{X(\sigma,\mathbf{n})}$  is isomorphic to  $[\mathrm{Spec} R[F]/G]$ . Since the restriction  $\mathcal{M}_X|_U$  is a trivial log structure, thus we see that  $\pi_{(X,U,\mathbf{n})}^{-1}(U) \rightarrow U$  is an *isomorphism*.

If we suppose that  $(X, U)$  is tame and  $n_i$  is prime to the characteristic of  $k$  for all  $i \in I$ , then the order of  $G = F^{\mathrm{gp}}/\iota(P)^{\mathrm{gp}}$  is prime to the characteristic of  $k$ . Therefore the Cartier dual  $(F^{\mathrm{gp}}/\iota(P)^{\mathrm{gp}})^D$  is a finite étale group scheme over  $k$ . Thus  $[\mathrm{Spec} k[F]/G]$  is a *Deligne-Mumford stack* over  $k$  (cf. [19, Remarque 10.13.2]). Therefore in this case  $\mathcal{X}_{(X,U,\mathbf{n})}$  is a Deligne-Mumford stack.

*Coarse moduli map for  $\mathcal{X}_{(X,U,\mathbf{n})}$ .* Next we will prove that the natural map  $\mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is a coarse moduli map for  $\mathcal{X}_{(X,U,\mathbf{n})}$ . By the above argument, we see that  $\mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is a proper quasi-finite surjective morphism. Indeed, in the case when  $X$  is a toric variety,  $\mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is a coarse moduli map by Proposition 3.6 (2). Moreover  $\mathcal{X}_{(X,U,\mathbf{n})}$  is integral and  $\mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is generically an isomorphism. By [22, Corollary 2.9 (ii)], we conclude that  $\mathcal{X}_{(X,U,\mathbf{n})} \rightarrow X$  is a coarse moduli map. □

Assume that  $n_i = 1$  for all  $i \in I$ . Before the proof of (1) of Theorem 3.3, we prove the following Lemma.

**Lemma 3.8.** *Let  $(X, U)$  be a toroidal embedding. Let  $\mathcal{M}_X$  be the canonical log structure of  $(X, U)$ . For every geometric point  $\bar{x}$  on  $X_{\mathrm{sm}}$ , the monoid  $\overline{\mathcal{M}}_{X,\bar{x}}$  is free.*

*Proof.* Clearly, our claim is local with respect to étale topology. Hence, we may assume that for every closed point  $x$  of  $X$ , there exist an étale neighborhood  $W \rightarrow X$ , a simplicially toric sharp monoid  $P$ , and a smooth morphism of  $w : W \rightarrow \mathrm{Spec} k[P]$ . Note that  $\mathcal{M}_X|_W$  is induced by the log structure on  $\mathrm{Spec} k[P]$  defined by  $P \rightarrow k[P]$ . Since  $X$  is smooth, thus after shrinking  $\mathrm{Spec} k[P]$  we may assume that  $\mathrm{Spec} k[P]$  is a smooth affine toric variety over  $k$ . Therefore  $\mathrm{Spec} k[P]$  has the form  $\mathrm{Spec} k[\mathbb{N}^r \oplus \mathbb{Z}^s]$  (cf. [10, p28]) and the log structure is induced by  $\mathbb{N}^r \rightarrow k[\mathbb{N}^r \oplus \mathbb{Z}^s]$ . Hence our claim is clear. □

*Proof of (1) in Theorem 3.3.* To prove that  $\pi_{(X,U,\mathbf{n})}^{-1}(X_{\mathrm{sm}}) \xrightarrow{\sim} X_{\mathrm{sm}}$ , it suffices to show that  $\pi_{(X,U,\mathbf{n})}^{-1}(X_{\mathrm{sm}})$  is an algebraic spaces because  $\pi_{(X,U,\mathbf{n})}$  is a coarse moduli map. Thus let us show that every MFR morphism  $(f, h) :$

$(S, \mathcal{N}) \rightarrow (X_{\text{sm}}, \mathcal{M}_X |_{X_{\text{sm}}})$  has no non-trivial automorphism in  $\mathcal{X}_{(X,U,n)}$ . By Lemma 3.8 (1), each stalk of  $f^{-1}\overline{\mathcal{M}}_X$  is a free monoid. Hence  $h : f^*\mathcal{M}_X \rightarrow \mathcal{N}$  is an isomorphism because  $(f, h)$  is an MFR morphism and thus  $\bar{h} : f^{-1}\overline{\mathcal{M}}_X \rightarrow \overline{\mathcal{N}}$  is an isomorphism. Therefore it does not have a non-trivial automorphism and thus the map  $\pi_{(X,U,n)}^{-1}(X_{\text{sm}}) \rightarrow X_{\text{sm}}$  is an isomorphism.  $\square$

*Proof of (2) in Theorem 3.3.* Let  $p : Z \rightarrow \mathcal{X}$  be an étale cover by a separated scheme  $Z$  (note that  $Z \times_{\mathcal{X}} Z$  is also a separated scheme since  $\mathcal{X}$  has finite diagonal and thus  $Z \times_{\mathcal{X}} Z \rightarrow Z \times_k Z$  is finite). Let  $X_{\text{sing}}$  be the singular locus of  $X$  and put  $V := Z - p^{-1}(f^{-1}(X_{\text{sing}}))$ . Note that since  $X$  is a normal variety, the codimension of  $X_{\text{sing}}$  is bigger than 1. Thus the codimension of  $p^{-1}(f^{-1}(X_{\text{sing}}))$  is bigger than 1. Let  $\text{pr}_1, \text{pr}_2 : Z \times_{\mathcal{X}} Z \rightrightarrows Z$  be natural projections. By (1) in Theorem 3.3,  $f \circ p|_V : V \rightarrow X$  and  $f \circ p \circ \text{pr}_i|_{V \times_{\mathcal{X}} V} : V \times_{\mathcal{X}} V \rightarrow X$  ( $i = 1, 2$ ) are uniquely lifted to morphisms into  $\mathcal{X}_{(X,U,n)}$ . We abuse notation and denote by  $f \circ p|_V$  and  $f \circ p \circ \text{pr}_i|_{V \times_{\mathcal{X}} V}$  ( $i = 1, 2$ ) lifted morphisms into  $\mathcal{X}_{(X,U,n)}$ . Then by the purity lemma due to Abramovich and Vistoli ([1, 3.6.2] [3, Lemma 2.4.1]),  $f \circ p|_V$  and  $f \circ p \circ \text{pr}_i|_{V \times_{\mathcal{X}} V}$  ( $i = 1, 2$ ) are extended to  $Z$  and  $Z \times_{\mathcal{X}} Z$  respectively. These extensions are unique up to a unique isomorphism. Since  $f \circ p \circ \text{pr}_1 = f \circ p \circ \text{pr}_2$ , there exists a unique morphism  $\phi : \mathcal{X} \rightarrow \mathcal{X}_{(X,U,n)}$  such that  $\pi_{(X,U,n)} \circ \phi \cong f$ .  $\square$

**§3.4. Stabilizer group schemes of points on  $\mathcal{X}_{(X,U,n)}$**

In this subsection, we calculate the stabilizer group schemes (i.e. automorphism group schemes) of points on the stack  $\mathcal{X}_{(X,U,n)}$  for a good toroidal embedding  $(X, U)$  of level  $\mathbf{n} = \{n_i\}_{i \in I}$ . For the definition of points on an algebraic stack, we refer to [19, Chapter 5]. In this subsection, by a geometric point to an algebraic stack  $\mathcal{X}$  we mean a morphism  $\text{Spec } K \rightarrow \mathcal{X}$  with an algebraically closed field  $K$ .

Let us calculate the stabilizer groups of points on  $\mathcal{X}_{(X,U,n)}$ . Let  $\bar{x} : \text{Spec } K \rightarrow \mathcal{X}_{(X,U,n)}$  be a point on  $\mathcal{X}_{(X,U,n)}$  with an algebraically closed field  $K$ . Note that the point  $\bar{x}$  can be naturally regarded as the point on  $X$  via the coarse moduli map. Suppose that  $\bar{x} : \text{Spec } K \rightarrow \mathcal{X}_{(X,U,n)}$  corresponds to an admissible FR morphism  $(\pi_{(X,U,n)} \circ \bar{x}, h) : (\text{Spec } K, \mathcal{M}_K) \rightarrow (X, \mathcal{M}_X, \mathbf{n})$ . Thus  $\text{Isom}(\bar{x}, \bar{x})$  is the group scheme over  $K$ , which represents the contravariant-functor

$$G : (K\text{-schemes}) \rightarrow (\text{groups}),$$

$\{v : S \rightarrow \text{Spec } K\} \mapsto \{\text{the group of isomorphisms of the log structure } v^*\mathcal{M}_K \text{ which are compatible with } v^*h : (\pi_{(X,U,n)} \circ \bar{x} \circ v)^*\mathcal{M}_X \rightarrow v^*\mathcal{M}_K\}$ . (We

shall call  $\text{Isom}(\bar{x}, \bar{x})$  the stabilizer (or automorphism) group scheme of point  $\bar{x}$ .) Set  $P = (\pi_{(X,U,n)} \circ \bar{x})^* \mathcal{M}_X$  and  $F = \overline{\mathcal{M}}_K$ . Since  $K$  is algebraically closed, there exist isomorphisms  $(\pi_{(X,U,n)} \circ \bar{x})^* \mathcal{M}_X \cong K^* \oplus P$  and  $\mathcal{M}_K \cong K^* \oplus F$ . Therefore we have  $G(\{S \rightarrow \text{Spec } K\}) = \text{Hom}_{\text{group}}(F^{\text{gp}}/P^{\text{gp}}, \Gamma(S, \mathcal{O}_S^*))$ . (Note that by Lemma 2.11 the natural map  $F/P \rightarrow F^{\text{gp}}/P^{\text{gp}}$  is an isomorphism.) Hence we conclude that  $G$  is the Cartier dual of  $F^{\text{gp}}/P^{\text{gp}}$  over  $K$ . Thus we have the following result:

**Proposition 3.9.** *Let  $\bar{x} : \text{Spec } K \rightarrow \mathcal{X}_{(X,U,n)}$  be a point with an algebraically closed field  $K$ . Set  $\bar{y} = \pi_{(X,U,n)}(\bar{x})$ . Then the stabilizer group scheme of  $\bar{x}$  is isomorphic to the Cartier dual of  $(F)^{\text{gp}}/\overline{\mathcal{M}}_{X,\bar{y}}^{\text{gp}}$  over  $K$ , where  $\overline{\mathcal{M}}_{X,\bar{y}} \rightarrow F$  is an admissible free resolution of type  $\{n_i\}$  at  $\bar{y}$  (cf. Definition 3.2).*

*Tame algebraic stacks.* The recent paper [2] introduced the notion of tame algebraic stacks, which is a natural generalization of that of tame Deligne-Mumford stacks. We will show that the moduli stack  $\mathcal{X}_{(X,U,n)}$  is a tame algebraic stack. Let us recall the definition of tame algebraic stacks. Let  $\mathcal{X}$  be an algebraic stack over a base scheme  $S$ . Suppose that the inertia stack  $\mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}$  is finite over  $\mathcal{X}$ . Let  $\pi : \mathcal{X} \rightarrow X$  be a coarse moduli map for  $\mathcal{X}$  (Keel-Mori theorem implies the existence). Let  $\text{QCoh } \mathcal{X}$  (resp.  $\text{QCoh } X$ ) denote the abelian category of quasi-coherent sheaves on  $\mathcal{X}$  (resp.  $X$ ). The algebraic stack  $\mathcal{X}$  is said to be *tame* if the functor  $\pi_* : \text{QCoh } \mathcal{X} \rightarrow \text{QCoh } X$  is exact. By [2, Theorem 3.2],  $\mathcal{X}$  is tame if and only if for any geometric point  $\text{Spec } K \rightarrow \mathcal{X}$  with an algebraically closed field  $K$ , its stabilizer group is a *linearly reductive* group scheme over  $K$ . By Proposition 3.9, for any geometric point  $\text{Spec } K \rightarrow \mathcal{X}_{(X,U,n)}$  with an algebraically closed field  $K$ , its stabilizer group is *diagonalizable*, thus we obtain:

**Corollary 3.10.** *The algebraic stack  $\mathcal{X}_{(X,U,n)}$  is a tame algebraic stack.*

### §3.5. Log structures on $\mathcal{X}_{(X,U,n)}$

Let  $(X, U, \mathbf{n} = \{n_i\}_{i \in I})$  be a good toroidal embedding of level  $\mathbf{n}$  and  $\mathcal{X}_{(X,U,n)}$  the associated stack. Let us define a canonical (tautological) log structure on  $\mathcal{X}_{(X,U,n)}$ . Let us denote by  $\text{Lis-ét}(\mathcal{X}_{(X,U,n)})$  the *lisse-étale site* of  $\mathcal{X}_{(X,U,n)}$ . (Recall the definition of the lisse-étale site. The underlying category of  $\text{Lis-ét}(\mathcal{X}_{(X,U,n)})$  is the full subcategory of  $\mathcal{X}_{(X,U,n)}$ -schemes whose objects are smooth  $\mathcal{X}_{(X,U,n)}$ -schemes. A collection of morphisms  $\{f_i : S_i \rightarrow S\}_{i \in I}$  of smooth  $\mathcal{X}_{(X,U,n)}$ -schemes is a covering family in  $\text{Lis-ét}(\mathcal{X}_{(X,U,n)})$  if the morphism  $\sqcup_i f_i : \sqcup_i S_i \rightarrow S$  is étale surjective.) We define a log structure  $\mathcal{M}_{(X,U,n)}$

as follows. Let  $s : S \rightarrow \mathcal{X}_{(X,U,\mathbf{n})}$  be a smooth  $\mathcal{X}_{(X,U,\mathbf{n})}$ -scheme, i.e., an object in  $\text{Lis-ét}(\mathcal{X}_{(X,U,\mathbf{n})})$ . This amounts exactly to an admissible FR morphism  $(f, \phi) : (S, \mathcal{M}_S) \rightarrow (X, \mathcal{M}_X, \mathbf{n})$  such that  $f = \pi_{(X,U,\mathbf{n})} \circ s$ . By attaching to  $s : S \rightarrow \mathcal{X}_{(X,U,\mathbf{n})} \in \text{Lis-ét}(\mathcal{X}_{(X,U,\mathbf{n})})$  the log structure  $\mathcal{M}_S$ , we define a fine and saturated log structure  $\mathcal{M}_{(X,U,\mathbf{n})}$ . We shall refer this log structure as the *canonical log structure* on  $\mathcal{X}_{(X,U,\mathbf{n})}$ . (For the notion of log structures on algebraic stacks, we refer to [21, section 5].) Moreover by considering the homomorphisms  $\phi : f^* \mathcal{M}_X \rightarrow \mathcal{M}_S$ , we have a natural morphism of log stacks

$$(\pi_{(X,U,\mathbf{n})}, \Phi) : (\mathcal{X}_{(X,U,\mathbf{n})}, \mathcal{M}_{(X,U,\mathbf{n})}) \rightarrow (X, \mathcal{M}_X).$$

**Proposition 3.11.** *Let  $(X, U, \mathbf{n})$  be a good toroidal embedding with a level  $\mathbf{n} = \{n_i\}_{i \in I}$ , where  $I$  is the set of irreducible elements of  $X - U$ . Then we have the followings.*

- (1) *The subscheme  $\mathcal{D} = \mathcal{X}_{(X,U,\mathbf{n})} - U$  with reduced closed subscheme structure is a normal crossing divisor. The log structure  $\mathcal{M}_{(X,U,\mathbf{n})}$  is isomorphic to the log structure arising from the divisor  $\mathcal{D} = \mathcal{X}_{(X,U,\mathbf{n})} - U$ .*
- (2) *Suppose further that  $(X, U)$  is a tame toroidal embedding (cf. Section 1.2) and  $n_i$  is prime to the characteristic of the base field for all  $i$ . The morphism  $(\pi_{(X,U,\mathbf{n})}, \Phi) : (\mathcal{X}_{(X,U,\mathbf{n})}, \mathcal{M}_{(X,U,\mathbf{n})}) \rightarrow (X, \mathcal{M}_X)$  is a Kummer log étale morphism.*

We postpone the proof of this Proposition, and it will be given in Section 4.3 because it follows from the case when  $(X, U)$  is a toric variety.

*Remark 3.12.* One may regard  $\mathcal{X}_{(X,U,\mathbf{n})}$  as a sort of “stacky toroidal embedding” endowed with the log structure  $\mathcal{M}_{(X,U,\mathbf{n})}$ .

### §4. Toric Algebraic Stacks

We define *toric algebraic stacks*.

#### §4.1. Some combinatorics

Let  $N = \mathbb{Z}^d$  be a lattice and  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  the dual lattice. Let  $\langle \bullet, \bullet \rangle : M \times N \rightarrow \mathbb{Z}$  be the dual pairing.

• A pair  $(\Sigma, \mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)})$  is called a *simplicial fan with a level structure*  $\mathbf{n}$  if  $\Sigma$  is simplicial fan in  $N_{\mathbb{R}}$  and  $\mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)}$  is the set of positive integers indexed by the set of rays  $\Sigma(1)$ .

- A pair  $(\Sigma, \Sigma^0)$  is called a *stacky fan* if  $\Sigma$  is simplicial fan in  $N_{\mathbb{R}}$  and  $\Sigma^0$  is a subset of  $|\Sigma| \cap N$  such that for any cone  $\sigma$  in  $\Sigma$  the restriction  $\sigma \cap \Sigma^0$  is a submonoid of  $\sigma \cap N$  which has the following properties: (i)  $\sigma \cap \Sigma^0$  is isomorphic to  $\mathbb{N}^r$  where  $r = \dim \sigma$ , (ii)  $\sigma \cap \Sigma^0$  is close to  $\sigma \cap N$ . Put another way. If  $(\Sigma, \Sigma^0)$  is a stacky fan and  $\rho$  is a ray of  $\Sigma$ , then there exists the first point  $w_\rho$  of  $\rho \cap \Sigma^0$ . Since  $\sigma \in \Sigma$  is simplicial and  $\mathbb{N}^{\dim \sigma} \cong \sigma \cap \Sigma^0$  is a submonoid that is close to  $\sigma \cap N$ , thus  $\sigma \cap \Sigma^0$  is the free monoid  $\bigoplus_{\rho \in \sigma(1)} \mathbb{N} \cdot w_\rho (\subset \sigma)$ . (Each irreducible element of the monoid  $\sigma \cap \Sigma^0$  lies on a unique ray of  $\sigma$ .) Therefore the data  $\Sigma^0$  is determined by the set of points  $\{w_\rho\}_{\rho \in \Sigma(1)}$ . Let us denote by  $v_\rho$  the first lattice point on a ray  $\rho \in \Sigma(1)$ . For a ray  $\rho \in \Sigma(1)$ , if  $n_\rho \cdot v_\rho$  ( $n_\rho \in \mathbb{Z}_{\geq 0}$ ) is the first point  $w_\rho$  of  $\Sigma^0 \cap \rho$ , then we shall call  $n_\rho$  the *level of  $\Sigma^0$  on  $\rho$* .

- Let  $(\Sigma, \mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)})$  be a simplicial fan with a level structure  $\mathbf{n} = \{n_\rho\}$ . The *free-net* of  $\Sigma$  associated to level  $\mathbf{n}$  is the subset  $\Sigma_{\mathbf{n}}^0 \subset |\Sigma| \cap N$  such that for each cone  $\sigma \in \Sigma$  the set  $\sigma \cap \Sigma_{\mathbf{n}}^0$  is the free submonoid generated by  $\{n_\rho \cdot v_\rho\}_{\rho \in \sigma(1)}$ , where  $v_\rho$  denotes the first lattice point on a ray  $\rho \in \Sigma(1)$ . (This free-net may be regarded as the *geometric realization* of the level  $\mathbf{n}$ .) The *canonical free-net*  $\Sigma_{\text{can}}^0$  of  $\Sigma$  is the free-net associated to the level  $\{n_\rho = 1\}_{\rho \in \Sigma(1)}$ .

Note that for any stacky fan  $(\Sigma, \Sigma^0)$  there exists a unique level  $\mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)}$  such that  $(\Sigma, \Sigma^0) = (\Sigma, \Sigma_{\mathbf{n}}^0)$ .

- Let  $S$  be a scheme (or a ringed space). A stacky fan  $(\Sigma, \Sigma_{\mathbf{n}}^0)$  is called *tame over  $S$*  if for any cone  $\sigma \in \Sigma$  the multiplicity  $\text{mult}(\sigma)$  is invertible on  $S$ , and for any  $\rho \in \Sigma(1)$  the level  $n_\rho$  is invertible on  $S$ . For a stacky cone  $(\sigma, \sigma_{\mathbf{n}}^0)$ , we define the *multiplicity*, denoted by  $\text{mult}(\sigma, \sigma_{\mathbf{n}}^0)$ , of  $(\sigma, \sigma_{\mathbf{n}}^0)$  to be  $\text{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} n_\rho$ . A stacky fan  $(\Sigma, \Sigma_{\mathbf{n}}^0)$  is tame over  $S$  if and only if  $\text{mult}(\sigma, \sigma_{\mathbf{n}}^0)$  is invertible on  $S$  for any cone  $\sigma \in \Sigma$ .

- A stacky fan  $(\Sigma, \Sigma_{\mathbf{n}}^0)$  is called *complete* if  $\Sigma$  is a finite and complete fan, i.e.,  $\Sigma$  is a finite set and the support  $|\Sigma|$  is the whole space  $N_{\mathbb{R}}$ .

*Remark 4.1.* The notion of stacky fans was first introduced in [5].

### §4.2. Toric algebraic stacks

Fix a base scheme  $S$ .

**Definition 4.2.** Let  $(\Sigma \subset N_{\mathbb{R}}, \mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)})$  be a simplicial fan with a level structure  $\mathbf{n}$ . Let  $(\Sigma, \Sigma_{\mathbf{n}}^0)$  be the associated stacky fan. Define a fibered category

$$\mathcal{X}_{(\Sigma, \Sigma_{\mathbf{n}}^0)} \longrightarrow (S\text{-schemes})$$

as follows. The objects over a  $S$ -scheme  $X$  are triples

$$(\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$$

such that:

- (1)  $\mathcal{S}$  is an étale sheaf of sub-monoids of the constant sheaf  $M$  on  $X$  determined by  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  such that for every geometric point  $\bar{x} \rightarrow X$ ,  $\mathcal{S}_x \cong \mathcal{S}_{\bar{x}}$ . Here  $x \in X$  is the image of  $\bar{x}$  and  $\mathcal{S}_x$  (resp.  $\mathcal{S}_{\bar{x}}$ ) denotes the Zariski (resp. étale) stalk. (The condition  $\mathcal{S}_x \cong \mathcal{S}_{\bar{x}}$  for every  $\bar{x}$  means that the étale sheaf  $\mathcal{S}$  is arising from the Zariski sheaf  $\mathcal{S}|_{X_{\text{zar}}}$ .)
- (2)  $\pi : \mathcal{S} \rightarrow \mathcal{O}_X$  is a map of monoids where  $\mathcal{O}_X$  is a monoid under multiplication.
- (3) For  $s \in \mathcal{S}$ ,  $\pi(s)$  is invertible if and only if  $s$  is invertible.
- (4) For each point  $x \in X$ , there exists some (and a unique)  $\sigma \in \Sigma$  such that  $\mathcal{S}_x = \sigma^\vee \cap M$ .
- (5)  $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$  is a fine log structure on  $X$ .
- (6)  $\eta : \mathcal{S} \rightarrow \mathcal{M}$  is a homomorphism of sheaves of monoids such that  $\pi = \alpha \circ \eta$ , and for each geometric point  $\bar{x} \rightarrow X$ , the homomorphism  $\bar{\eta} : \bar{\mathcal{S}}_{\bar{x}} = (\mathcal{S}/(\text{invertible elements}))_{\bar{x}} \rightarrow \bar{\mathcal{M}}_{\bar{x}}$  is isomorphic to the composite

$$\bar{\mathcal{S}}_{\bar{x}} \xrightarrow{r} F \xrightarrow{t} F,$$

where  $r$  is the minimal free resolution of  $\bar{\mathcal{S}}_{\bar{x}}$  and  $t$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$  where  $e_\rho$  denotes the irreducible element of  $F$  corresponding to a ray  $\rho \in \Sigma(1)$  (see Lemma 4.3) and  $n_\rho$  is the level of  $\Sigma^0$  on  $\rho$ .

A set of morphisms

$$\begin{aligned} (\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M}) \\ \rightarrow (\pi' : \mathcal{S}' \rightarrow \mathcal{O}_X, \alpha' : \mathcal{M}' \rightarrow \mathcal{O}_X, \eta' : \mathcal{S}' \rightarrow \mathcal{M}') \end{aligned}$$

over  $X$  is the set of isomorphisms of log structures  $\phi : \mathcal{M} \rightarrow \mathcal{M}'$  such that  $\phi \circ \eta = \eta' : \mathcal{S} = \mathcal{S}' \rightarrow \mathcal{M}'$  if  $(\mathcal{S}, \pi) = (\mathcal{S}', \pi')$  and is the empty set if  $(\mathcal{S}, \pi) \neq (\mathcal{S}', \pi')$ . With the natural notion of pullbacks,  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a fibered category. According to [4, Theorem on page 10], for any  $S$ -scheme  $X$ , there exists an isomorphism

$$\text{Hom}_{S\text{-schemes}}(X, X_\Sigma) \cong \{ \text{all pairs } (\mathcal{S}, \pi) \text{ on } X \text{ satisfying (1), (2), (3), (4)} \},$$

which commutes with pullbacks. Here  $X_\Sigma$  is the toric variety associated to  $\Sigma$  over  $S$ . Therefore there exists a natural functor

$$\pi_{(\Sigma, \Sigma_n^0)} : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \longrightarrow X_\Sigma$$

which simply forgets the data  $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$  and  $\eta : \mathcal{S} \rightarrow \mathcal{M}$ . Moreover  $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$  and  $\eta : \mathcal{S} \rightarrow \mathcal{M}$  are morphisms of the étale sheaves and thus  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a stack with respect to the étale topology.

Objects of the form  $(\pi : M \rightarrow \mathcal{O}_X, \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X, \pi : M \rightarrow \mathcal{O}_X^*)$  determine a substack of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ , i.e., the natural inclusion

$$i_{(\Sigma, \Sigma_n^0)} : T_\Sigma = \text{Spec } \mathcal{O}_S[M] \hookrightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}.$$

We shall call  $i_{(\Sigma, \Sigma_n^0)}$  the *canonical torus embedding*. This commutes with the torus-embedding  $i_\Sigma : T_\Sigma \hookrightarrow X_\Sigma$ .

**Lemma 4.3.** *With the same notation as in Definition 4.2, let  $e$  be an irreducible element in  $F$  and let  $n$  be a positive integer such that  $n \cdot e \in r(\overline{\mathcal{S}}_{\bar{x}})$ . Let  $m \in \mathcal{S}_{\bar{x}}$  be a lifting of  $n \cdot e$ . Suppose that  $\mathcal{S}_{\bar{x}} = \sigma^\vee \cap M$ . Then there exists a unique ray  $\rho \in \sigma(1)$  such that  $\langle m, v_\rho \rangle > 0$ . Here  $v_\rho$  is the first lattice point of  $\rho$ , and  $\langle \bullet, \bullet \rangle$  is the dual pairing. It does not depend on the choice of liftings. Moreover this correspondence defines a natural injective map*

$$\{\text{Irreducible elements of } F\} \rightarrow \Sigma(1).$$

*Proof.* Since the kernel of  $\mathcal{S}_{\bar{x}} \rightarrow \overline{\mathcal{S}}_{\bar{x}}$  is  $\sigma^\perp \cap M$ , thus  $\langle m, v_\rho \rangle$  does not depend upon the choice of liftings  $m$ . Taking a splitting  $N \cong N' \oplus N''$  such that  $\sigma \cong \sigma' \oplus \{0\} \subset N'_\mathbb{R} \oplus N''_\mathbb{R}$  where  $\sigma'$  is a full-dimensional cone in  $N'_\mathbb{R}$ , we may and will assume that  $\sigma$  is a full-dimensional cone, i.e.,  $\sigma^\vee \cap M$  is sharp. By Proposition 2.9 (2), there is a natural embedding  $\sigma^\vee \cap M \hookrightarrow F \hookrightarrow \sigma^\vee$  such that each irreducible element of  $F$  lies on a unique ray of  $\sigma^\vee$ . It gives rise to a bijective map from the set of irreducible elements of  $F$  to  $\sigma^\vee(1)$ . Since  $\sigma$  and  $\sigma^\vee$  are simplicial, we have a natural bijective map  $\sigma^\vee(1) \rightarrow \sigma(1)$ ;  $\rho \mapsto \rho^*$ , where  $\rho^*$  is the unique ray which does not lie in  $\rho^\perp$ . Therefore the composite map from the set of irreducible elements of  $F$  to  $\sigma(1)$  is a bijective map. Hence it follows our claim.  $\square$

*Remark 4.4.* (1) In what follows, we refer to a homomorphism  $\pi : \mathcal{S} \rightarrow \mathcal{O}_X$  with properties (1), (2), (3), (4) in Definition 4.2 as a *skeleton*. If  $\pi : \mathcal{S} \rightarrow \mathcal{O}_X$  corresponds to  $X \rightarrow X_\Sigma$ , then  $\pi : \mathcal{S} \rightarrow \mathcal{O}_X$  is called the *skeleton for*  $X \rightarrow X_\Sigma$ .

- (2) Let  $(\Sigma, \Sigma_{\text{can}}^0)$  be a stacky fan such that  $\Sigma$  is non-singular and  $\Sigma_{\text{can}}^0$  is the canonical free net. Then  $\mathcal{X}_{(\Sigma, \Sigma_{\text{can}}^0)}$  is the toric variety  $X_\Sigma$  over  $S$ . Indeed, for any object  $(\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$  in  $\mathcal{X}_{(\Sigma, \Sigma_{\text{can}}^0)}$  and any point  $x \in X$ , the monoid  $\overline{\mathcal{S}}_x$  has the form  $\mathbb{N}^r$  for some  $r \in \mathbb{N}$ . Thus in this case, (5) and (6) in Definition 4.2 are vacant.
- (3) Toric algebraic stack can be constructed over  $\mathbb{Z}$  and pull back from there to any other scheme. Therefore, for the proof of Proposition 4.5 and Theorem 4.6 (1), (2), (3), we may assume that the base scheme is  $\text{Spec } \mathbb{Z}$ .

*Torus Action functor.* The torus action functor

$$a : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma \longrightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$$

is defined as follows. Let  $\xi = (\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$  be an object in  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ . Let  $\phi : M \rightarrow \mathcal{O}_X$  be a map of monoids from a constant sheaf  $M$  on  $X$  to  $\mathcal{O}_X$ , i.e., an  $X$ -valued point of  $T_\Sigma := \text{Spec } \mathcal{O}_S[M]$ . Here  $\mathcal{O}_X$  is viewed as a sheaf of monoids under multiplication. We define  $a(\xi, \phi)$  by  $(\phi \cdot \pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \phi \cdot \eta : \mathcal{S} \rightarrow \mathcal{M})$ , where  $\phi \cdot \pi(s) := \phi(s) \cdot \pi(s)$  and  $\phi \cdot \eta(s) := \phi(s) \cdot \eta(s)$ . (Note that  $\mathcal{S}$  is a subsheaf of the constant sheaf of  $M$ .) Let  $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$  be a morphism in  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma$  from  $(\xi_1, \phi)$  to  $(\xi_2, \phi)$ , where  $\xi_i = (\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M}_i \rightarrow \mathcal{O}_X, \eta_i : \mathcal{S} \rightarrow \mathcal{M}_i)$  for  $i = 1, 2$ , and  $\phi : M \rightarrow \mathcal{O}_X$  is an  $X$ -valued point of  $T_\Sigma$ . We define  $a(h)$  to be  $h$ . It gives rise to the functor  $a : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma \longrightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  over  $(S$ -schemes), which makes the following diagrams

$$\begin{array}{ccc}
 \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma \times T_\Sigma & \xrightarrow{\text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \times m} & \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma \\
 \downarrow a \times \text{Id}_{T_\Sigma} & & \downarrow a \\
 \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma & \xrightarrow{a} & \mathcal{X}_{(\Sigma, \Sigma_n^0)}, \\
 \\ 
 \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times T_\Sigma & \xrightarrow{a} & \mathcal{X}_{(\Sigma, \Sigma_n^0)} \\
 \text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \times e \uparrow & & \uparrow \text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \\
 \mathcal{X}_{(\Sigma, \Sigma_n^0)} & \xlongequal{\quad} & \mathcal{X}_{(\Sigma, \Sigma_n^0)}.
 \end{array}$$

commutes in the strict sense, i.e.,  $a \circ (a \times \text{Id}_{T_\Sigma}) = a \circ (\text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \times m)$  and  $a \circ (e \times \text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}}) = \text{Id}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}}$ , where  $m : T_\Sigma \times T_\Sigma \rightarrow T_\Sigma$  is the natural action and  $e : S \rightarrow T_\Sigma$  is the unit section. Thus the functor  $a$  defines an action of  $T_\Sigma$  on  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  which extends the action of  $T_\Sigma$  on itself to the whole stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$

Here the notion of group actions on stacks is taken in the sense of [23, Definition 1.3]. This action makes the coarse moduli map  $\pi_{(\Sigma, \Sigma_n^0)} : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \rightarrow X_\Sigma$  torus-equivariant.

**Proposition 4.5.** *Let  $(\Sigma, \mathbf{n} = \{n_\rho\}_{\rho \in \Sigma(1)})$  be a simplicial fan  $\Sigma$  with level  $\mathbf{n}$ . Then there exists a canonical isomorphism between the stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  and the moduli stack  $\mathcal{X}_{(X_\Sigma, T_\Sigma, \mathbf{n})}$  of admissible FR morphisms to  $X_\Sigma$  of type  $\mathbf{n}$  (cf. Definition 2.15 and Section 3) over  $X_\Sigma$ . Here  $X_\Sigma$  is the toric variety over  $S$ .*

*Proof.* We will explicitly construct a functor  $F : \mathcal{X}_{(X_\Sigma, \mathbf{n})} \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$ . To this aim, consider the skeleton  $\pi_u : \mathcal{U} \rightarrow \mathcal{O}_{X_\Sigma}$  for the identity element in  $\text{Hom}(X_\Sigma, X_\Sigma)$ , i.e., the tautological object (cf. [4, Theorem on page 10]). Observe that the log structure associated to  $\pi_u : \mathcal{U} \rightarrow \mathcal{O}_{X_\Sigma}$  is isomorphic to the canonical log structure  $\mathcal{M}_\Sigma$ . Indeed let us recall the construction of  $\pi_u : \mathcal{U} \rightarrow \mathcal{O}_{X_\Sigma}$  (cf. [4, page 11]). By [4, page 11], we have  $\mathcal{U} = \{ \text{union of subsheaves } (\sigma^\vee \cap M)_{X_\sigma} \text{ of the constant sheaf } M \text{ on } X_\Sigma \text{ of all } \sigma \in \Sigma \}$ . For a cone  $\sigma \in \Sigma$ , we have  $\mathcal{U}(X_\sigma) = \sigma^\vee \cap M$ , and  $\mathcal{U}(X_\Sigma) \rightarrow \mathcal{O}_{X_\Sigma}(X_\sigma)$  is the natural map  $\sigma^\vee \cap M \rightarrow \mathcal{O}_S[\sigma^\vee \cap M]$ . Furthermore it is easy to see that if  $V \subset X_\sigma$ , then any element  $m \in \mathcal{U}(V)$  has the form  $m_1 + m_2$  where  $m_1 \in \mathcal{U}(X_\sigma)$  and  $m_2$  is an invertible element of  $\mathcal{U}(V)$ . (If  $\sigma \succ \tau$ , then any element  $m \in \tau^\vee \cap M$  has the form  $m_1 + m_2$  where  $m_1 \in \sigma^\vee \cap M$  and  $m_2$  is an invertible element of  $\tau^\vee \cap M$ .) Therefore  $\pi_u : \mathcal{U} \rightarrow \mathcal{O}_{X_\Sigma}$  induces a homomorphism  $\zeta : \mathcal{U} \rightarrow \mathcal{M}_\Sigma$  which makes  $\mathcal{M}_\Sigma$  the log structure associated to  $\mathcal{U} \rightarrow \mathcal{M}_\Sigma$ . Let  $(f, h) : (X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma)$  be an object in  $\mathcal{X}_{(X_\Sigma, T_\Sigma, \mathbf{n})}$  over  $f : X \rightarrow X_\Sigma$ . Define  $F((f, h) : (X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma))$  to be

$$(f^{-1}\pi_u : f^{-1}\mathcal{U} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, h \circ (f^{-1}\zeta) : f^{-1}\mathcal{U} \rightarrow \mathcal{M}).$$

Then  $(f^{-1}\pi_u : f^{-1}\mathcal{U} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, h \circ (f^{-1}\zeta) : f^{-1}\mathcal{U} \rightarrow \mathcal{M})$  is an object in  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  because  $(f, h)$  is an admissible FR morphism of type  $\mathbf{n}$  and the log structure  $\mathcal{M}_\Sigma$  is arising from the prelog structure  $\pi_u : \mathcal{U} \rightarrow \mathcal{O}_{X_\Sigma}$ . Clearly, a morphism of log structure  $\phi : \mathcal{M} \rightarrow \mathcal{M}$  commutes with  $h \circ (f^{-1}\zeta) : f^{-1}\mathcal{U} \rightarrow \mathcal{M}$  if and only if it commutes with  $h : f^*\mathcal{M}_\Sigma \rightarrow \mathcal{M}$  because the log structure the  $f^*\mathcal{M}_\Sigma$  is arising from the natural morphism  $f^{-1}\mathcal{U} \xrightarrow{f^{-1}\zeta} f^{-1}\mathcal{M}_\Sigma \rightarrow f^*\mathcal{M}_\Sigma$ . Thus  $F$  is fully faithful over  $X \rightarrow X_\Sigma$ . Finally, we will show that  $F$  is essentially surjective over  $X \rightarrow X_\Sigma$ . Let  $(f^{-1}\pi_u : f^{-1}\mathcal{U} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, l : f^{-1}\mathcal{U} \rightarrow \mathcal{M})$  be an object of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  over  $f : X \rightarrow X_\Sigma$ . (Note that every object in  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is of this form.) The homomorphism  $l : f^{-1}\mathcal{U} \rightarrow \mathcal{M}$  induces the homomorphism  $l^a : f^*\mathcal{M}_\Sigma \rightarrow \mathcal{M}$ . Then  $F((f, l^a) : (X, \mathcal{M}) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma))$  is

isomorphic to  $(f^{-1}\pi_u : f^{-1}\mathcal{U} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, l : f^{-1}\mathcal{U} \rightarrow \mathcal{M})$ . Hence  $F$  is an isomorphism.  $\square$

**Theorem 4.6.** *The stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a smooth tame algebraic stack that is locally of finite type over  $S$  and has finite diagonal. Furthermore it satisfies the additional properties such that:*

- (1) *The natural functor  $\pi_{(\Sigma, \Sigma_n^0)} : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \longrightarrow X_\Sigma$  (cf. Definition 4.2) is a coarse moduli map,*
- (2) *The canonical torus embedding  $i_{(\Sigma, \Sigma_n^0)} : T_\Sigma = \text{Spec } \mathcal{O}_S[M] \hookrightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is an open immersion identifying  $T_\Sigma$  with a dense open substack of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ .*
- (3) *If  $\Sigma$  is a finite fan, then  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is of finite type over  $S$ .*

*If  $(\Sigma, \Sigma_n^0)$  is tame over the base scheme  $S$ , then  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a Deligne-Mumford stack over  $S$ . (Moreover there is a criterion for Deligne-Mumfordness. see Corollary 4.17.)*

*If  $\Sigma$  is a non-singular fan and  $\Sigma_{\text{can}}^0$  denotes the canonical free net, then  $\mathcal{X}_{(\Sigma, \Sigma_{\text{can}}^0)}$  is the toric variety  $X_\Sigma$  over  $S$ .*

*Proof.* By Proposition 4.5, Proposition 3.5, and Proposition 3.6 (see also Remark 3.7),  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a smooth algebraic stack that is locally of finite type over  $S$  and has finite diagonal. Moreover  $\pi_{(\Sigma, \Sigma_n^0)} : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \rightarrow X_\Sigma$  is a coarse moduli map by Proposition 3.6 (2), and thus by Keel-Mori theorem (cf. [17], see also [7, Theorem 1.1])  $\pi_{(\Sigma, \Sigma_n^0)}$  is proper. Hence if  $\Sigma$  is a finite fan, then  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is of finite type over  $S$ . To see that  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is tame, we may assume that the base scheme is a spectrum of a field because the tameness depends only on automorphism group schemes of geometric points (cf. [2, Theorem 3.2]). Thus the tameness follows from Proposition 4.5 and Corollary 3.10. Since the restriction  $\mathcal{M}_\Sigma|_{\text{Spec } \mathcal{O}_S[M]}$  of  $\mathcal{M}_\Sigma$  to  $T_\Sigma = \text{Spec } \mathcal{O}_S[M]$  is the trivial log structure, thus by taking account of Proposition 4.5 the morphism  $\pi_{(\Sigma, \Sigma_n^0)}^{-1}(T_\Sigma) \rightarrow T_\Sigma$  is an isomorphism. Hence  $i_{(\Sigma, \Sigma_n^0)} : T_\Sigma = \text{Spec } \mathcal{O}_S[M] \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is an open immersion. If  $(\Sigma, \Sigma_n^0)$  is tame over the base scheme  $S$ , the same argument as in *Proof of algebraicity* of Theorem 3.3 shows that  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a Deligne-Mumford stack. The last claim follows from Remark 4.4.  $\square$

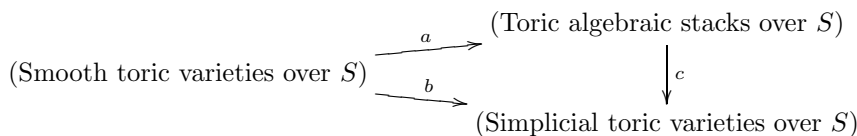
**Definition 4.7.** Let  $(\Sigma, \Sigma_n^0)$  be a stacky fan. We shall call the stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  the *toric algebraic stack* associated to  $(\Sigma, \Sigma_n^0)$  (or  $(\Sigma, \mathfrak{n})$ ).

*Remark 4.8.* (1) Let (Smooth toric varieties over  $S$ ), (resp. (Simplicial toric varieties over  $S$ )) denote the category of smooth (resp. simplicial) toric

varieties over  $S$  whose morphisms are  $S$ -morphisms. Let  $(\text{Toric algebraic stacks over } S)$  denote the 2-category whose objects are toric algebraic stacks over  $S$ , and a 1-morphism is an  $S$ -morphism between objects, and a 2-morphism is an isomorphism between 1-morphisms. Given a 1-morphism  $f : \mathcal{X}_{(\Sigma, \Sigma_n^0)} \rightarrow \mathcal{X}_{(\Delta, \Delta_n^0)}$ , by the universality of coarse moduli spaces, there exists a unique morphism  $f_0 : X_\Sigma \rightarrow X_\Delta$  such that  $\pi_{(\Delta, \Delta_n^0)} \circ f = f_0 \circ \pi_{(\Sigma, \Sigma_n^0)}$ . By attaching  $f_0$  to  $f$ , we obtain a functor

$$c : (\text{Toric algebraic stacks over } S) \rightarrow (\text{Simplicial toric varieties over } S), \mathcal{X}_{(\Sigma, \Sigma_n^0)} \mapsto X_\Sigma.$$

Therefore there is the following diagram of (2)-categories,



where  $a$  and  $b$  is fully faithful functors and  $c$  is an essentially surjective functor.

(2) *Gluing pieces together.* Let  $(\Sigma, \Sigma_n^0)$  be a stacky fan and  $\sigma$  be a cone in  $\Sigma$ . By definition there exists a natural fully faithful morphism  $\mathcal{X}_{(\sigma, \sigma_n^0)} \hookrightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  where  $(\sigma, \sigma_n^0)$  is the restriction of  $(\Sigma, \Sigma_n^0)$  to  $\sigma$ . The image of this functor is identified with the open substack  $\pi_{(\Sigma, \Sigma_n^0)}^{-1}(X_\sigma)$ , where  $X_\sigma (\subset X_\Sigma)$  is the affine toric variety associated to  $\sigma$ . That is to say, Definition 4.2 allows one to have a natural gluing construction  $\cup_{\sigma \in \Sigma} \mathcal{X}_{(\sigma, \sigma_n^0)} = \mathcal{X}_{(\Sigma, \Sigma_n^0)}$ .

**Proposition 4.9.** *Let  $(\sigma, \sigma_n^0)$  be a stacky fan such that  $\sigma$  is a simplicial cone in  $N_{\mathbb{R}}$  where  $N = \mathbb{Z}^d$ . Here  $N \cong \mathbb{Z}^d$ . Suppose that  $\dim(\sigma) = r$ . Then the toric algebraic stack  $\mathcal{X}_{(\sigma, \sigma_n^0)}$  has a finite fppf morphism*

$$p : \mathbb{A}_S^r \times \mathbb{G}_{m,S}^{d-r} \rightarrow \mathcal{X}_{(\sigma, \sigma_n^0)}$$

where  $\mathbb{A}_S^r$  is an  $r$ -dimensional affine space over  $S$ . Furthermore  $\mathcal{X}_{(\sigma, \sigma_n^0)}$  is isomorphic to the quotient stack  $[\mathbb{A}_S^r/G] \times \mathbb{G}_m^{d-r}$  where  $G$  is a finite flat group scheme over  $S$ . If  $\sigma$  is a full-dimensional cone, the quotient  $[\mathbb{A}_S^r/G]$  coincides with the quotient given in Section 3.2 (cf. Proposition 3.5).

If  $(\sigma, \sigma_n^0)$  is tame over the base scheme  $S$ , then we can choose  $p$  to be a finite étale cover and  $G$  to be a finite étale group scheme.

*Proof.* As in Remark 3.7 we choose a splitting

$$\begin{aligned} N &\cong N' \oplus N'' \\ (\sigma, \sigma_n^0) &\cong (\tau, \tau_n^0) \oplus \{0\} \\ X_\sigma &\cong X_\tau \times \mathbb{G}_{m,S}^{d-r} \end{aligned}$$

where  $\sigma \cong \tau \subset N'_\mathbb{R}$  is a full-dimensional cone. Notice that  $P := \tau^\vee \cap M'$  is a simplicially toric sharp monoid. Let  $\iota : P \rightarrow F \cong \mathbb{N}^r$  be the homomorphism of monoids defined as the composite  $P \rightarrow F \xrightarrow{n} F$  where  $P \rightarrow F$  is the minimal free resolution and  $n : F \rightarrow F$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$  for each ray  $\rho \in \sigma(1)$ . Here  $e_\rho$  is the irreducible element of  $F$  corresponding to  $\rho$  (cf. Lemma 2.14). Then by Proposition 3.5 and Proposition 4.5, the stack  $\mathcal{X}_{(\tau, \tau_n^0)}$  is isomorphic to the quotient stack  $[\text{Spec } \mathcal{O}_S[F]/G]$  where  $G$  is the Cartier dual  $(F^{\text{gp}}/\iota(P)^{\text{gp}})^D$ . The group scheme  $G$  is a finite flat over  $S$ . We remark that  $\mathcal{X}_{(\sigma, \sigma_n^0)} \cong \mathcal{X}_{(\tau, \tau_n^0)} \times \mathbb{G}_{m,S}^{d-r}$  by Proposition 4.5, and thus our claim follows. The last assertion is clear because in such case  $G$  is a finite étale group scheme.  $\square$

**Proposition 4.10.** *The toric algebraic stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is proper over  $S$  if and only if  $(\Sigma, \Sigma_n^0)$  is complete (cf. Section 4.1).*

*Proof.* If  $\Sigma$  is a finite and complete fan, then since the coarse moduli map  $\pi_{(\Sigma, \Sigma_n^0)}$  is proper, properness of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  over  $S$  follows from the fact that  $X_\Sigma$  is proper over  $S$  (cf. [10, Proposition in page 39] or [6, Chapter IV, Theorem 2.5 (viii)]). Conversely, suppose that  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is proper over  $S$ . We will show that  $\Sigma$  is a finite fan and the support  $|\Sigma|$  is the whole space  $N_\mathbb{R}$ . By [22, Lemma 2.7(ii)], we see that  $X_\Sigma$  is proper over  $S$ . By ([10, Proposition in page 39] or [6, Chapter IV, Theorem 2.5 (viii)]), it suffices to show only that  $\Sigma$  is a finite fan (this is well known, but we give the proof here because [10] assumes that all fans are finite). Let  $\Sigma_{\text{max}}$  be the subset of  $\Sigma$  consisting of the maximal elements with respect to the face relation. The set  $\{X_\sigma\}_{\sigma \in \Sigma_{\text{max}}}$  is an open covering of  $X_\Sigma$ , but  $\{X_\sigma\}_{\sigma \in \Sigma'}$  is not an open covering for any proper subset  $\Sigma' \subset \Sigma_{\text{max}}$  because for any cone  $\sigma$ , set-theoretically  $X_\sigma = \sqcup_{\sigma \succ \tau} Z_\tau$  (cf. Section 1.1). Since  $X_\Sigma$  is quasi-compact,  $\Sigma_{\text{max}}$  is finite. Hence  $\Sigma$  is a finite fan (note that  $\Sigma$  is simplicial in our case).  $\square$

*Remark 4.11.* Here we will explain the relationship between toric Deligne-Mumford stacks introduced in [5] and toric algebraic stacks introduced in this section. We assume that the base scheme is an algebraically closed field of characteristic zero (since the results of [5] only hold in characteristic zero). First, since both took different approaches, it is not clear that if one begins

with a given stacky fan, then two associated stacks in the sense of [5] and us are isomorphic to each other. But in the subsequent paper [12] we prove the *geometric characterization theorem* for toric algebraic stacks in the sense of us. One can deduce from it that they are (non-canonically) isomorphic to each other. (See [12, Section 5].) Next, toric Deligne-Mumford stacks in the sense of [5] admits a finite abelian gerbe structure, while our toric algebraic stacks do not. However, such structures can be obtained from our toric algebraic stacks by the following well-known technique. We here work over  $\mathbb{Z}$ . Let  $\mathcal{L}_D$  be an invertible sheaf on  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ , associated to a torus invariant divisor  $D$  (see the next subsection). Consider the triple

$$(u : U \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}, \phi : u^* \mathcal{L}_D \cong \mathcal{M}^{\otimes n})$$

where  $\mathcal{M}$  is an invertible sheaf on  $U$ , and  $\phi$  is an isomorphism of sheaves. Morphisms of triples are defined in the natural manner. Then it forms an algebraic stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D^{1/n})$  and there exists the natural forgetting functor  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D^{1/n}) \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$ , which is a smooth morphism. The composition of this procedure, that is,

$$\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_{D_1}^{1/n_1}) \times_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \cdots \times_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_{D_r}^{1/n_r}) \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$$

yields a gerbe that appears on toric Deligne-Mumford stacks. The stack  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D^{1/n})$  associated to  $\mathcal{L}_D$  can be viewed as the fiber product

$$\begin{array}{ccc} \mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D^{1/n}) & \longrightarrow & B\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathcal{X}_{(\Sigma, \Sigma_n^0)} & \longrightarrow & B\mathbb{G}_m, \end{array}$$

where the lower horizontal arrow is associated to  $\mathcal{L}_D$ ,  $B\mathbb{G}_m$  is the classifying stack of  $\mathbb{G}_m$ , and  $B\mathbb{G}_m \rightarrow B\mathbb{G}_m$  is associated to the homomorphism  $\mathbb{G}_m \rightarrow \mathbb{G}_m, a \mapsto a^n$ . This interpretation has the following direct generalization. Let  $N$  be a finitely generated abelian group and  $h : \mathbb{Z} \rightarrow N$  be a homomorphism of abelian groups. Then it gives rise to  $G = \text{Spec } \mathbb{Z}[N] \rightarrow \mathbb{G}_m$  and  $BG \rightarrow B\mathbb{G}_m$ . Then we can define  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D, h)$  to be the fiber product

$$\begin{array}{ccc} & & BG \\ & & \downarrow \\ \mathcal{X}_{(\Sigma, \Sigma_n^0)} & \longrightarrow & B\mathbb{G}_m. \end{array}$$

Notice that  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D, h)$  is smooth, but does not have the finite inertia stack, that is, automorphism groups of objects in  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}(\mathcal{L}_D, h)$  can be positive dimensional.

**§4.3. Torus-invariant cycles and log structures on toric algebraic stacks**

In this subsection we define torus-invariant cycles and a canonical log structure on toric algebraic stacks. It turns out that if a base scheme  $S$  is regular the complement of a torus embedding in a toric algebraic stack is a *divisor with normal crossings* relative to  $S$  (unlike simplicial toric varieties), and it fits nicely in the notions of torus-invariant divisors and canonical log structure.

*Torus-invariant cycles.* Let  $(\Sigma \subset N_{\mathbb{R}}, \Sigma_n^0)$  be a stacky fan, where  $N = \mathbb{Z}^d$ . The torus-invariant cycle  $V(\sigma) \subset X_{\Sigma}$  associated to  $\sigma$  is represented by the functor

$$F_{V(\sigma)} : (S\text{-schemes}) \rightarrow (\text{Sets}),$$

$$X \mapsto \{\text{all pairs } (\mathcal{S}, \pi : \mathcal{S} \rightarrow \mathcal{O}_X) \text{ of skeletons with property } (*)\}$$

where  $(*) = \{\text{For any point } x \in X, \text{ there exists a cone } \tau \text{ such that } \sigma \prec \tau \text{ and } \mathcal{S}_x = \tau^{\vee} \cap M, \text{ and } \pi(s) = 0 \text{ if } s \in \tau^{\vee} \cap \sigma_o^{\vee} \cap M \subset \mathcal{S}_x\}$  (see Section 1.1 for the definition of  $\sigma_o^{\vee}$ ). Indeed, to see this, we may and will suppose that  $X_{\Sigma}$  is an affine toric variety  $X_{\tau} = \text{Spec } \mathcal{O}_S[\tau^{\vee} \cap M]$  with  $\sigma \prec \tau$ . Let  $\pi : \mathcal{U} \rightarrow \mathcal{O}_{X_{\tau}}$  be the skeleton for the identity morphism  $X_{\tau} \rightarrow X_{\tau}$ . As in the proof of Proposition 4.5,  $\mathcal{U} = \{\text{union of subsheaves } (\gamma^{\vee} \cap M)_{X_{\gamma}} \text{ of the constant sheaf } M \text{ on } X_{\tau} \text{ of all } \gamma \prec \tau\}$ . Let  $\mathcal{U}' := \{\text{union of subsheaves } (\gamma^{\vee} \cap \sigma_o^{\vee} \cap M)_{X_{\gamma}} \text{ of the constant sheaf } M \text{ on } X_{\tau} \text{ of all } \gamma \prec \tau\}$ . Then  $F_{V(\sigma)}$  is represented by  $\text{Spec}(\mathcal{O}_S[\tau^{\vee} \cap M]/\pi(\mathcal{U}'))$ . Since  $\gamma^{\vee} \cap \sigma_o^{\vee} \cap M = \mathcal{U}'(X_{\gamma}) \subset \gamma^{\vee} \cap M = \mathcal{U}(X_{\gamma})$  for  $\gamma \prec \tau$ , it is easy to check that  $\text{Spec}(\mathcal{O}_S[\tau^{\vee} \cap M]/\pi(\mathcal{U}')) = \text{Spec}(\mathcal{O}_S[\tau^{\vee} \cap M]/\tau^{\vee} \cap \sigma_o^{\vee} \cap M) = V(\sigma)$ .

Now let us define the torus-invariant cycle  $\mathcal{V}(\sigma)$  associated to  $\sigma \in \Sigma$ . Consider the substack  $\mathcal{V}(\sigma)$  of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ , that consists of objects  $(\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$  such that the condition  $(**)$  holds, where  $(**) = \{\text{For any geometric point } \bar{x} \rightarrow X, \alpha(m) = 0 \text{ if } m \in \mathcal{M}_{\bar{x}} \text{ and there exists a positive integer } n \text{ such that the image of } n \cdot m \text{ in } \overline{\mathcal{M}}_{\bar{x}} \text{ lies in } \eta(\sigma_o^{\vee} \cap \mathcal{S}_{\bar{x}})\}$ . Note that if  $\Sigma$  is non-singular and  $\Sigma_n^0$  is the canonical free net, i.e.,  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} = X_{\Sigma}$ , then  $(**)$  is equal to  $(*)$  for any cone in  $\Sigma$ . Clearly, the substack  $\mathcal{V}(\sigma)$  is stable under the torus action. The following lemma shows that  $\mathcal{V}(\sigma)$  is a closed substack of  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ .

**Lemma 4.12.** *With the same notation as above, the condition  $(**)$  is represented by a closed subscheme of  $X$ . In particular,  $\mathcal{V}(\sigma) \subset \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a closed substack. Moreover if the base scheme  $S$  is reduced, then  $\mathcal{V}(\sigma)$  is reduced.*

*Proof.* Let  $\phi : X \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  be the morphism corresponding to  $(\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$ . Since our claim is smooth local on  $X$ , thus we may assume that  $\Sigma$  is a full-dimensional simplicial cone  $\tau$  and  $\sigma \prec \tau$ . Then by Proposition 4.9, we have the diagram

$$\text{Spec } \mathcal{O}_S[F] \xrightarrow{p} \mathcal{X}_{(\tau, \tau_n^0)} \xrightarrow{\pi_{(\tau, \tau_n^0)}} X_\tau = \text{Spec } \mathcal{O}_S[\tau^\vee \cap M]$$

where  $p$  is an fppf morphism and the composite is defined by  $\tau^\vee \cap M \xleftarrow{i} F \xrightarrow{n} F$ . Here  $i$  is the minimal free resolution and  $n$  is defined by  $e_\rho \mapsto n_\rho \cdot e_\rho$ , where  $e_\rho$  is the irreducible element of  $F$  corresponding to  $\rho \in \tau(1)$ . Since  $p$  is fppf, there exists étale locally a lifting  $X \rightarrow \text{Spec } \mathcal{O}_S[F]$  of  $\phi : X \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$ , and thus we can assume  $X := \text{Spec } \mathcal{O}_S[F]$  and that the log structure  $\mathcal{M}$  is induced by the natural map  $F \rightarrow \mathcal{O}_S[F]$  (cf. Proposition 2.9). What we have to prove is that the condition  $(**)$  is represented by a closed subscheme on  $X$ . Consider the following natural diagram

$$\begin{array}{ccccc} \tau^\vee \cap \sigma_0^\vee \cap M & \longrightarrow & \tau^\vee \cap M & \xrightarrow{noi} & F \\ & \searrow & \downarrow & \swarrow h & \\ & & \mathcal{O}_S[F] & & \end{array}$$

Set  $A = \{f \in F \mid \text{there is } n \in \mathbb{Z}_{\geq 1} \text{ such that } n \cdot f \in \text{Image}(\tau^\vee \cap \sigma_0^\vee \cap M)\}$ . Let  $I$  be the ideal of  $\mathcal{O}_S[F]$  that is generated by  $h(A)$ . We claim that the closed subscheme  $\text{Spec } \mathcal{O}_X/I$  represents the condition  $(**)$ . To see this, note first that  $\tau^\vee \cap \sigma_0^\vee \cap M$  is  $(\tau^\vee \cap M) \setminus (\sigma^\perp \cap \tau^\vee \cap M)$  and by embeddings  $(\tau^\vee \cap M) \subset F \subset \tau^\vee$ , each irreducible element of  $F$  lies on a unique ray of  $\tau^\vee$  (cf. Proposition 2.9 (2)). Set  $F = \mathbb{N}^r \oplus \mathbb{N}^{d-r}$  and suppose that  $\mathbb{N}^{d-r}$  generates the face  $(\sigma^\perp \cap \tau^\vee)$  of  $\tau^\vee$ . Since  $\tau^\vee \cap M$  is close to  $F$ , thus  $I$  is the ideal generated by  $\mathbb{N}^r \oplus \mathbb{N}^{d-r} \setminus \{0\} \oplus \mathbb{N}^{d-r} (\subset F)$ . Let  $e_i$  be the  $i$ -th standard irreducible element of  $F = \mathbb{N}^d$ . Let  $\bar{x} \rightarrow X = \text{Spec } \mathcal{O}_S[F]$  be a geometric point such that  $\mathcal{S}_{\bar{x}} = \gamma^\vee \cap M$  where  $\sigma \subset \gamma \subset \tau$ , and let  $\overline{\mathcal{M}}_{\bar{x}} = \langle e_{i_1}, \dots, e_{i_s} \rangle \subset F$ . In order to prove our claim, let us show that in  $\mathcal{O}_{X, \bar{x}}$ , the ideal  $I$  coincides with the ideal generated by  $L := \{m \in \overline{\mathcal{M}}_{\bar{x}} \mid \text{there is } n \in \mathbb{Z}_{\geq 1} \text{ such that } n \cdot m \in \text{Image}(\gamma^\vee \cap \sigma_0^\vee \cap M)\}$ . After reordering, we suppose  $1 \leq i_1 < \dots < i_t \leq r < i_{t+1} < \dots < i_s$  ( $t \leq s$ ). Then we have  $(\mathcal{O}_X/I)_{\bar{x}} = \mathcal{O}_{X, \bar{x}}/(e_{i_1}, \dots, e_{i_t})$ . Consider the following natural commutative diagram

$$\begin{array}{ccccccc} \gamma^\vee \cap \sigma_0^\vee \cap M & \longrightarrow & \gamma^\vee \cap M & \longrightarrow & (\gamma^\vee \cap M)^{\text{gp}} & \longrightarrow & M \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow F^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \\ & & \downarrow = & & & & \downarrow \\ \mathcal{S}_{\bar{x}} & \longrightarrow & \mathcal{M}_{\bar{x}} & \longrightarrow & \overline{\mathcal{M}}_{\bar{x}} & \longrightarrow & \overline{\mathcal{M}}_{\bar{x}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}. \end{array}$$

Since  $\tau^\vee \cap \sigma_0^\vee \cap M$  is equal to  $(\tau^\vee \cap M) \setminus (\sigma^\perp \cap \tau^\vee \cap M)$  and the image of  $\sigma^\perp (\subset M \otimes_{\mathbb{Z}} \mathbb{Q})$  in  $\overline{\mathcal{M}}_{\bar{x}}^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$  is the vector space spanned by  $e_{i_{t+1}}, \dots, e_{i_s}$ , thus the image of  $\gamma^\vee \cap \sigma_0^\vee \cap M$  in  $\overline{\mathcal{M}}_{\bar{x}}$  is contained in the set

$$L' := \mathbb{N} \cdot e_{i_1} \oplus \dots \oplus \mathbb{N} \cdot e_{i_s} \setminus \mathbb{N} \cdot e_{i_{t+1}} \oplus \dots \oplus \mathbb{N} \cdot e_{i_s}$$

in  $\overline{\mathcal{M}}_{\bar{x}}$ . Since the image of  $\gamma^\vee \cap M$  in  $\overline{\mathcal{M}}_{\bar{x}}$  is close to  $\overline{\mathcal{M}}_{\bar{x}}$ , we have  $L = L'$ . Therefore the ideal of  $\mathcal{O}_{X, \bar{x}}$  generated by  $e_{i_1}, \dots, e_{i_t}$  is the ideal of  $\mathcal{O}_{X, \bar{x}}$  generated by  $L$ . Hence the ideal  $I$  coincides with the ideal generated by  $L$  in  $\mathcal{O}_{X, \bar{x}}$  and thus they are the same in  $\mathcal{O}_X$  because  $I$  is finitely generated. Thus  $\text{Spec } \mathcal{O}_X/I$  represents the condition (\*\*).

Finally, we will show the last assertion. Suppose that  $S$  is reduced. With the same notation and assumption as above,  $\mathcal{O}_S[F]/I$  is reduced. Since  $\text{Spec } \mathcal{O}_S[F] \rightarrow \mathcal{X}_{(\tau, \tau_n^0)}$  is fppf, thus  $\mathcal{V}(\sigma)$  is reduced. □

**Definition 4.13.** We shall call the closed substack  $\mathcal{V}(\sigma)$  the *torus-invariant cycle* associated to  $\sigma$ .

**Proposition 4.14.** *Let  $\tau$  and  $\sigma$  be cones in a stacky fan  $(\Sigma, \Sigma_n^0)$  and suppose that  $\sigma \prec \tau$ . Then we have  $\mathcal{V}(\tau) \subset \mathcal{V}(\sigma)$ , i.e.,  $\mathcal{V}(\tau)$  is a closed substack of  $\mathcal{V}(\sigma)$ .*

*Proof.* By the definition and Proposition 4.12, it is enough to prove that for any cone  $\gamma$  such that  $\sigma \prec \tau \prec \gamma$ , we have  $\gamma^\vee \cap \sigma_0^\vee \cap M \subset \gamma^\vee \cap \tau_0^\vee \cap M$ . Clearly  $\gamma^\vee \cap \sigma_0^\vee \subset \gamma^\vee \cap \tau_0^\vee$ , thus our claim follows. □

*Stabilizer groups.* By the calculation of stabilizer group scheme in Section 3.4 and Proposition 4.5, we can easily calculate the stabilizer group schemes of points on a toric algebraic stack.

**Proposition 4.15.** *Let  $(\Sigma, \Sigma_n^0)$  be a stacky fan and  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  the associated stack. Let  $\bar{x} : \text{Spec } K \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  be a geometric point on  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  such that  $\bar{x}$  lies in  $\mathcal{V}(\sigma)$ , but does not lie in any torus-invariant proper substack  $\mathcal{V}(\tau)$  ( $\sigma \prec \tau$ ). Here  $K$  is an algebraically closed  $S$ -field. Let  $\mathcal{S} \rightarrow \mathcal{O}_K$  be the skeleton for  $\pi_{(\Sigma, \Sigma_n^0)} \circ \bar{x} : \text{Spec } K \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)} \rightarrow X_\Sigma$ . Then  $\mathcal{S}_{\bar{x}} = \sigma^\vee \cap M$ , and the stabilizer group scheme  $G$  at  $\bar{x}$  is the Cartier dual of  $F^{\text{gp}}/\iota((P)^{\text{gp}})$ . Here  $P := (\sigma^\vee \cap M)/(\text{invertible elements})$  and  $\iota : P \rightarrow F$  is *t o r* in Definition 4.2 (6). The rank of the stabilizer group scheme over  $\bar{x}$ , i.e.,  $\dim_K \Gamma(G, \mathcal{O}_G)$  is  $\text{mult}(\sigma, \sigma_n^0) = \text{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} n_\rho$ .*

*Proof.* By our assumption and the definition of  $\mathcal{V}(\sigma)$ , clearly,  $\mathcal{S}_{\bar{x}} = \sigma^\vee \cap M$ . Taking account of Proposition 3.9 and Proposition 4.5, the stabilizer group

scheme  $G$  at  $\bar{x}$  is the Cartier dual of  $F^{\text{gp}}/\iota((P)^{\text{gp}})$ . To see the last assertion, it suffices to show that the order of  $F^{\text{gp}}/\iota((P)^{\text{gp}}$  is equal to  $\text{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} n_\rho$ . It follows from the fact that the multiplicity of  $P$  is equal to  $\text{mult}(\sigma)$  (cf. Remark 2.7 (3)).  $\square$

*Remark 4.16.* The order of stabilizer group on a point on a Deligne-Mumford stack is an important invariant to intersection theory with rational coefficients on it (cf. [25]).

The stabilizer groups on a toric algebraic stacks are fundamental invariants because they have data arising from levels of the stacky fan and multiplicities of cones in the stacky fan. In particular, the stabilizer group scheme on a generic point on a torus-invariant divisor  $\mathcal{V}(\rho)$  ( $\rho \in \Sigma(1)$ ) is isomorphic to  $\mu_{n_\rho}$ .

**Corollary 4.17.** *Let  $(\Sigma, \Sigma_n^0)$  be a stacky fan and  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  the associated stack. Then  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is a Deligne-Mumford stack if and only if  $(\Sigma, \Sigma_n^0)$  is tame over  $S$ .*

*Proof.* First of all, the “if” direction follows from Theorem 4.6. Thus we will show the “only if” direction. Assume that  $(\Sigma, \Sigma_n^0)$  is not tame over  $S$ . Then there exists a cone  $\sigma \in \Sigma$  such that  $r := \text{mult}(\sigma, \sigma_n^0)$  is not invertible on  $S$ . Then there exists an algebraically closed  $S$ -field  $K$  such that the characteristic of  $K$  is a prime divisor of  $r$ . Consider a point  $w : \text{Spec } K \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)} \times_S K$  that corresponds to a triple  $(\mathcal{S} \rightarrow \mathcal{O}_{\text{Spec } K}, \mathcal{M} \rightarrow \mathcal{O}_{\text{Spec } K}, \mathcal{S} \rightarrow \mathcal{M})$  such that  $\mathcal{S} = \sigma^\vee \cap \mathcal{M}$ . Then by Proposition 4.15, the stabilizer group scheme of  $w$  is the Cartier dual of a finite group that has the form  $\mathbb{Z}/p_1^{c_1} \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_l^{c_l} \mathbb{Z}$  such that  $p_i$ ’s are prime numbers and  $r = \prod_{0 \leq i \leq l} p_i^{c_i}$ . After reordering, assume that  $p_1$  is the characteristic of  $K$ . Then the stabilizer group scheme of  $w$  is not étale over  $K$ . Thus by [19, Lemma 4.2],  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is not a Deligne-Mumford stack. Hence our claim follows.  $\square$

*Remark 4.18.* The first version [11] of this paper only treats the case when toric algebraic stacks are Deligne-Mumford stacks, although our method can apply to the case over arbitrary base schemes.

*Log structure on  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$ .* A canonical log structure on  $\text{Lis-ét}(\mathcal{X}_{(\Sigma, \Sigma_n^0)})$  is defined as follows. To a smooth morphism  $X \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  corresponding to a triple  $(\pi : \mathcal{S} \rightarrow \mathcal{O}_X, \alpha : \mathcal{M} \rightarrow \mathcal{O}_X, \eta : \mathcal{S} \rightarrow \mathcal{M})$ , we attach the log structure  $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$ . It gives rise to a fine and saturated log structure  $\mathcal{M}_{(\Sigma, \Sigma_n^0)}$  on  $\text{Lis-ét}(\mathcal{X}_{(\Sigma, \Sigma_n^0)})$ . We shall refer it as the *canonical log structure*. Through the identification  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} = \mathcal{X}_{(\Sigma, T_\Sigma, \mathbf{n})}$  (cf. Proposition 4.5), it is equivalent to the

canonical log structure  $\mathcal{M}_{(X_\Sigma, T_\Sigma, n)}$  on  $\mathcal{X}_{(\Sigma, T_\Sigma, n)}$ . As in Section 3.5, we have a natural morphism of log stacks

$$(\pi_{(\Sigma, \Sigma_n^0)}, \phi_{(\Sigma, \Sigma_n^0)}) : (\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)}) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma)$$

whose underlying morphism is the canonical coarse moduli map.

**Theorem 4.19.** *Assume that the base scheme  $S$  is regular. Then the complement  $\mathcal{D} = \mathcal{X}_{(\Sigma, \Sigma_n^0)} - T_\Sigma$  with reduced closed substack structure is a divisor with normal crossings relative to  $S$ , and the log structure  $\mathcal{M}_{(\Sigma, \Sigma_n^0)}$  is arising from  $\mathcal{D}$ . Furthermore  $\mathcal{D}$  coincides with the union of  $\cup_{\rho \in \Sigma(1)} \mathcal{V}(\rho)$ .*

For the proof we need the following lemma.

**Lemma 4.20** (log EGA IV 17.7.7). *Let  $(f, \phi) : (X, \mathcal{L}) \rightarrow (Y, \mathcal{M})$  and  $(g, \psi) : (Y, \mathcal{M}) \rightarrow (Z, \mathcal{N})$  be morphisms of fine log schemes respectively. Suppose that  $(g, \psi) \circ (f, \phi)$  is log smooth and  $g$  is locally of finite presentation. Assume further that  $(f, \phi)$  is a strict faithfully flat morphism that is locally of finite presentation. Then  $(g, \psi)$  is log smooth.*

*Proof.* Let

$$\begin{array}{ccc} (T_0, \mathcal{P}_0) & \xrightarrow{s} & (Y, \mathcal{M}) \\ \downarrow i & & \downarrow \\ (T, \mathcal{P}) & \xrightarrow{t} & (Z, \mathcal{N}) \end{array}$$

be a commutative diagram of fine log schemes where  $i$  is a strict closed immersion defined by a nilpotent ideal  $I \subset \mathcal{O}_T$ . It suffices to show that there exists étale locally on  $T$  a morphism  $(T, \mathcal{P}) \rightarrow (Y, \mathcal{M})$  filling in the diagram. To see this, after replacing  $T_0$  by an étale cover we may suppose that  $s$  has a lifting  $s' : (T_0, \mathcal{P}_0) \rightarrow (X, \mathcal{L})$ . Since  $(g, \psi) \circ (f, \phi)$  is log smooth, there exists étale locally on  $T$  a lifting  $(T, \mathcal{P}) \rightarrow (X, \mathcal{L})$  of  $s'$  filling in the diagram. Hence our assertion easily follows. □

*Proof of Theorem 4.19.* By Proposition 4.5 and Proposition 3.5 (together with Remark 3.7), there is an fppf strict morphism  $(X, \mathcal{L}) \rightarrow (\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)})$  from a scheme  $X$  such that the composite  $(X, \mathcal{L}) \rightarrow (S, \mathcal{O}_S^*)$  is log smooth. Here  $\mathcal{O}_S^*$  denotes the trivial log structure on  $S$ . Then by applying Lemma 4.20 we see that  $(\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)}) \rightarrow (S, \mathcal{O}_S^*)$  is log smooth. Let  $U \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  be a smooth surjective morphism from an  $S$ -scheme  $U$ . Then by [14, Theorem 4.8] there exists étale locally on  $U$  a smooth morphism  $U \rightarrow S \times_{\mathbb{Z}} \mathbb{Z}[P]$  where  $P$  is

a toric monoid ([14, Theorem 4.8] worked over a field, but it is also applicable to our case). In addition, the natural map  $P \hookrightarrow \mathcal{O}_S[P]$  gives rise to a chart for  $\mathcal{M}_{(\Sigma, \Sigma_n^0)}$ . Since  $\mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is smooth over  $S$ , thus after shrinking  $S \times_{\mathbb{Z}} \mathbb{Z}[P]$  we may suppose that  $P$  is a free monoid. Note that the support of  $\overline{\mathcal{M}}_{(\Sigma, \Sigma_n^0)}$  is the complement  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} - T_\Sigma$ . Therefore  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} - T_\Sigma$  with reduced substack structure is a divisor with normal crossings relative to  $S$ . Next we will prove that  $\cup_{\rho \in \Sigma(1)} \mathcal{V}(\rho)$  is reduced. (Set-theoretically  $\cup_{\rho \in \Sigma(1)} \mathcal{V}(\rho) = \mathcal{X}_{(\Sigma, \Sigma_n^0)} - T_\Sigma$ .) It follows from Lemma 4.12. Finally, we will show  $\mathcal{M}_{(\Sigma, \Sigma_n^0)} \cong \mathcal{M}_{\mathcal{D}}$  where  $\mathcal{M}_{\mathcal{D}}$  is the log structure associated to  $\mathcal{D}$ . Note that  $\mathcal{M}_{\mathcal{D}} = \mathcal{O}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \cap i_{(\Sigma, \Sigma_n^0)*} \mathcal{O}_{T_\Sigma}^*$  where  $i_{(\Sigma, \Sigma_n^0)} : T_\Sigma \rightarrow \mathcal{X}_{(\Sigma, \Sigma_n^0)}$  is the canonical torus embedding. Hence  $\mathcal{M}_{\mathcal{D}}$  is a subsheaf of  $\mathcal{O}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}}$ . Thus we may and will assume that  $\mathcal{X}_{(\Sigma, \Sigma_n^0)} = \mathcal{X}_{(\tau, \tau_n^0)}$  where  $\tau$  is a full-dimensional cone. Let  $p : \text{Spec } \mathcal{O}_S[F] \rightarrow \mathcal{X}_{(\tau, \tau_n^0)}$  be an fppf cover given in Proposition 4.9. By the fppf descent theory for fine log structures ([21, Corollary A.5]) together with the fact that  $\mathcal{M}_{\mathcal{D}}$  is a subsheaf  $\mathcal{O}_{\mathcal{X}_{(\tau, \tau_n^0)}}$ , it is enough to show  $p^* \mathcal{M}_{(\tau, \tau_n^0)} \cong p^* \mathcal{M}_{\mathcal{D}}$ . Since  $p^* \mathcal{M}_{(\tau, \tau_n^0)}$  is induced by the natural map  $F \rightarrow \mathcal{O}_S[F]$  (cf. Proposition 4.5 and Proposition 2.16), thus we have  $p^* \mathcal{M}_{(\tau, \tau_n^0)} \cong p^* \mathcal{M}_{\mathcal{D}}$ .  $\square$

**Proposition 4.21.** *With the same notation and assumptions as above, suppose further that  $(\Sigma, \Sigma_n^0)$  is tame over  $S$ . Then  $(\pi_{(\Sigma, \Sigma_n^0)}, \phi_{(\Sigma, \Sigma_n^0)}) : (\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)}) \rightarrow (X_\Sigma, \mathcal{M}_\Sigma)$  is Kummer log étale. In particular, there exists an isomorphism of  $\mathcal{O}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}}$ -modules*

$$\Omega^1((\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)}) / (S, \mathcal{O}_S^*)) \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_{(\Sigma, \Sigma_n^0)}} \otimes_{\mathbb{Z}} M,$$

where  $\Omega^1((\mathcal{X}_{(\Sigma, \Sigma_n^0)}, \mathcal{M}_{(\Sigma, \Sigma_n^0)}) / (S, \mathcal{O}_S^*))$  is the sheaf of log differentials (cf. [14, 5.6]).

*Proof.* It follows from the next Lemma, Proposition 4.5 and Proposition 3.5.  $\square$

**Lemma 4.22.** *With the same notation as in Proposition 2.16, if the order of  $F^{\text{gp}} / \iota^{\text{gp}}(P^{\text{gp}})$  is invertible on  $R$ , then*

$$(f, \phi) : (\text{Spec } R[F], \mathcal{M}_F) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$$

is Kummer log étale. In particular, the induced morphism (cf. Section 3.2)

$$([\text{Spec } R[F]/G], \mathcal{M}) \rightarrow (\text{Spec } R[P], \mathcal{M}_P)$$

is Kummer log étale. Here  $G$  is the Cartier dual of  $F^{\text{gp}} / \iota(P^{\text{gp}})$  and  $\mathcal{M}$  is the log structure associated to a natural chart  $F \rightarrow R[F]$ .

*Proof.* Since  $(f, \phi)$  is an admissible FR morphism, thus  $(f, \phi)$  is Kummer. By the toroidal characterization of log étaleness ([15, Theorem 3.5]),  $(f, \phi) : (\mathrm{Spec} R[F], \mathcal{M}_F) \rightarrow (\mathrm{Spec} R[P], \mathcal{M}_P)$  is log étale. Since  $\mathrm{Spec} R[F] \rightarrow [\mathrm{Spec} R[F]/G]$  is étale, thus our claim follows.  $\square$

*Proof of Proposition 3.11.* With the same notation as in Proposition 3.11, we may assume that  $X$  is an affine simplicial toric variety. Then Proposition 3.11 follows from Proposition 4.5, Theorem 4.19 and Proposition 4.21.  $\square$

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