Quenched tail estimate for the random walk in random scenery and in random layered conductance

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Joint work with J.-D. Deuschel (TU Berlin).
Slides will be available at my webpage.
Random walk in random scenery (RWRS)

- $\{z(x)\}_{x \in \mathbb{Z}^d, \mathbb{P}}$: IID random variables,
- $(S_n)_{n \in \mathbb{Z}_+}$: Random walk on $\mathbb{Z}^d$.

Random walk in random scenery:

$$W_n := \sum_{k=1}^{n} z(S_k).$$
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\]

- Scaling limit (under \( \mathbb{P} \otimes P_0 \)) yields a self-similar process.
  
  \( d = 1, z: \alpha\)-stable, \( S: \beta_{>1}\)-stable \( \Rightarrow \) index \( = 1 - \frac{1}{\beta} + \frac{1}{\alpha \beta} \).

- CLT holds in transient case.
- \( d = 2 \) too (!) but for \( \frac{1}{\sqrt{n \log n}} W_n \) (Bolthausen 1989).
RWRS: continuous time

In this talk,

- \( \{z(x)\}_{x \in \mathbb{Z}^d}, \mathbb{P} \): IID, \( \geq 0 \) with \( \mathbb{P}(z(x) \geq r) = r^{-\alpha + o(1)} \),
- \( (S_t)_{t \geq 0}, (P_x)_{x \in \mathbb{Z}^d} \): continuous time simple random walk.

Continuous time version of RWRS:

\[
A_t := \int_0^t z(S_u)du.
\]
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Naturally appears in random media:

- \( E_x[ f(X_t)e^{A_t} ] \) is a solution of \( \partial_t u = \Delta u + zu, u(0, x) = f(x) \).
- \( (S_1^t, S_2^t + A_1^t)_{t \geq 0} \): diffusion in random shear flow.
- \( (S_1^{A_t}, S_2^t)_{t \geq 0} \): random walk in layered conductance (later).
Tail estimates for RWRS

Many *annealed* results: $\mathbb{P} \otimes P_0(A_t \geq t^\rho)$.

- Natural tail assumption is $\mathbb{P}(z(x) \geq r) \approx \exp(-r^\alpha)$.
  
  "z has high exceedance" & "RW use it": both exponential.
  (To be explained more in the next slide.)

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  - Google search "Large and moderate deviations for random walks in random scenery: a review" by F. Castell.
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Not so many *quenched* results: $P_0(A_t \geq t^\rho)$ for typical $z$.

- Brownian motion in Gaussian scenery,
  - Large deviation for $\frac{1}{t \sqrt{\log t}} A_t$: Asselah-Castell (2003),
  - Moderate deviations: Castell (2004),

- Brownian motion in bounded scenery,
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No results in Borodin and Kesten-Spitzer setting.
Light tail vs Heavy tail

Let

\[ \ell_t := \int_0^t \delta_{s_u} \, du \Rightarrow A_t = \langle \ell_t, z \rangle. \]

The strategy for \( \{A_t \geq t^\rho\} \) under \( \mathbb{P} \otimes P_0 \) is

\[ \{z(\cdot) \approx a_t \psi(\cdot)\} \text{ and } \{\langle \ell_t, \psi \rangle \geq t^\rho / a_t\}. \]

Assume \( \mathbb{P}(z(x) \geq r) \approx \exp(-r^\alpha) \).

- \( \alpha < 1 \Rightarrow \text{optimal } \psi = \delta_0. \)
- \( \alpha > 1 \Rightarrow \text{optimal } \psi \text{ has a non-trivial profile.} \)
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- \( \alpha < 1 \Rightarrow \) optimal \( \psi = \delta_0 \).
- \( \alpha > 1 \Rightarrow \) optimal \( \psi \) has a non-trivial profile.

The quenched results by Asselah-Castell corresponds to the second regime. Search for a high exceedance in the \textit{physical space} instead of the probability space.
Main result 1: $\mathbb{P}(z(x) \geq r) = r^{-\alpha + o(1)}$.

Let $\rho > 0$. Then $\mathbb{P}$-almost surely,

$$P_0 (A_t \geq t^\rho) = \exp \left\{ -t^{p(\alpha,\rho)+o(1)} \right\}$$

as $t \to \infty$, where

$$p(\alpha, \rho) = \begin{cases} \frac{2\alpha \rho}{\alpha+1} - 1, & \rho \in \left( \frac{\alpha+1}{2\alpha} \lor 1, \frac{\alpha+1}{\alpha} \right], \\ \alpha(\rho - 1), & \rho > \frac{\alpha+1}{\alpha} \end{cases}$$

for $d = 1$ and

$$p(\alpha, \rho) = \begin{cases} \frac{2\alpha \rho - d}{2\alpha + d}, & \rho \in \left( \frac{d}{2\alpha} \lor 1, \frac{\alpha+d}{\alpha} \right], \\ \alpha(\rho - 1) / d, & \rho > \frac{\alpha+d}{\alpha} \end{cases}$$

for $d \geq 2$. 
Illustration: $d = 1, \alpha \leq 1 (d \geq 2, \alpha \leq \frac{d}{2}$ is similar)

For $\rho < \frac{\alpha + 1}{2\alpha}$, we in fact have a polynomial decay. The threshold $\frac{\alpha + 1}{2\alpha}$ is the self-similar index found by Borodin and Kesten-Spitzer.
Illustration: $d = 1, \alpha > 1 \ (d \geq 2, \alpha > \frac{d}{2} \text{ is similar})$

When $\alpha > 1$, $\mathbb{E}[z(x)] < \infty$ and $P_0(A_t \geq ct) \to 1$ for $c < \mathbb{E}[z(x)]$. For $c \geq \mathbb{E}[z(x)]$, this is the standard large deviation regime.
Main result II: $P(z(x) \geq r) = r^{-\alpha + o(1)}$.

Let $d = 1$ and $\alpha > 1$ or $d \geq 2$ and $\alpha > \frac{d}{2}$. Then for any $c > \mathbb{E}[z(x)]$, $P$-almost surely,

$$P_0(A_t \geq ct) = \begin{cases} 
\exp \left\{ -t^{\alpha-1 \alpha+1} + o(1) \right\}, & d = 1, \\
\exp \left\{ -t^{2\alpha-d \alpha+d} + o(1) \right\}, & d \geq 2
\end{cases}$$

as $t \to \infty$. (I.e., the extrapolation gives the correct exponent.)
Outline of the argument

Let us see how to get for \( d = 1 \) and \( \alpha > 1 \)

\[
P_0(A_t \geq ct) = \exp \left\{ -t \frac{\alpha-1}{\alpha+1} + o(1) \right\}.
\]

**Lower bound:** Let the random walk

- explore \([ -t \frac{\alpha}{\alpha+1}, t \frac{\alpha}{\alpha+1} ]\),
- there is \( z(x) \sim t \frac{1}{\alpha+1} \) almost surely,
- leave the local time \( \ell_t(x) \gtrsim t \frac{\alpha}{\alpha+1} \).

The first and the third event have probability

\[
\approx \exp\left\{ -t \frac{\alpha-1}{\alpha+1} \right\}.
\]
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$$P_0(A_t \geq ct) = \exp \left\{ -t^{\frac{\alpha-1}{\alpha+1}} + o(1) \right\}.$$

Upper bound:

- Consider level sets $\mathcal{H}_k = \left\{ |x| \leq t^{\frac{\alpha}{\alpha+1}} : t^{(k-1)\epsilon} \leq z(x) < t^{k\epsilon} \right\},$
- fine control on the “geometry” of $\mathcal{H}_k$,
- a tail estimate for additive functional by Xia Chen (2001),

$\Rightarrow$ Too difficult to get contribution from lower level sets:

$$P_0 \left( \ell_t(\mathcal{H}_k) \geq t^{1-k\epsilon} \right) \ll \exp \left\{ -t^{\frac{\alpha-1}{\alpha+1}} \right\}.$$
Xia Chen’s theorem, Stoch. Proc. & Appl. 2001

Suppose $f \geq 0$ and

$$E_x \left[ \int_0^t f(S_u) du \right] \lesssim a(t)$$

uniformly $x \in \text{supp} f$. Let $0 \ll b(t) \ll t$. Then for $\lambda > 4$,

$$P_0 \left( \int_0^t f(S_u) du \geq \lambda a \left( \frac{t}{b(t)} \right) b(t) \right) \leq \exp \{ -c(\lambda)b(t) \}.$$ 

Remark

- There is a corresponding lower bound.
- If $a(\cdot)$ varies regularly, sharper bound available.
- Simpler case dates back to Khas’minskii (1959).
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\]

Apply this to \( f = 1_{\mathcal{H}_k} \).

- \( E_x[\ell_t(\mathcal{H}_k)] = E_x[\int_0^t f(S_u)du] = \sum_y 1_{\mathcal{H}_k}(y) \int_0^t p_u(x, y)du \),
- Concentration inequality & Borel-Cantelli \( \Rightarrow \) uniform estimate.
- \( Q: E_x[\int_0^t z(S_u)du] \sim \sum_y G(x, y)z(y): \) extreme values?
Application to random layered conductance model

Let \(((X_t)_{t \geq 0}, (P^\omega_x)_{x \in \mathbb{Z}^{1+d}})\) be a continuous time Markov chain on \(\mathbb{Z}^2\) with jump rates

\[
\omega(x, x \pm e_i) = \begin{cases} 
z(x_2), & i = 1, \\ 
1, & i = 2. 
\end{cases}
\]

By using CTSRW \((S^1, S^2)\) on \(\mathbb{Z}^2\),

\[
(X^1_t, X^2_t)_{t \geq 0} = (S^1_{A^2_t}, S^2_{t})_{t \geq 0} \text{ with } A^2_t = \int_0^t z(S^2_u)du.
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Anomalous behavior expected (Andres-Deuschel-Slowik 2016) to

\[
P^\omega_0 \left( X_t = t^\delta e_1 \right) \approx P_0 \left( S^1_{A^2_t} = t^\delta e_1 \right) = E_0 \left[ p_{A^2_t}(0, t^\delta e_1) \right].
\]
Tail estimate for layered conductance model

For $\mathbb{P}$-almost every $\omega$,

$$P_0^\omega \left( X_t = t^\delta e_1 \right) = \exp \left\{ -t^{q(\alpha, \delta) + o(1)} \right\}$$

as $t \to \infty$, where

$$q(\alpha, \delta) = \begin{cases} 
0, & \delta < \frac{1}{2} \lor \frac{\alpha+1}{4\alpha}, \\
2\delta - 1, & \delta \in \left[ \frac{1}{2}, \frac{\alpha}{\alpha+1} \right), \\
\frac{4\alpha\delta-\alpha-1}{3\alpha+1}, & \delta \in \left[ \frac{\alpha}{\alpha+1} \lor \frac{\alpha+1}{4\alpha}, \frac{2\alpha+1}{2\alpha} \right], \\
\frac{\alpha(2\delta-1)}{\alpha+1}, & \delta \in \left( \frac{2\alpha+1}{2\alpha}, \frac{\alpha}{(\alpha-1)_+} \right), \\
\delta, & \delta \geq \frac{\alpha}{(\alpha-1)_+}.
\end{cases}$$

Extensions to higher dimensions & non-horizontal displacement.
Thank you!