Annealed Brownian motion in a heavy tailed Poissonian potential

Ryoki Fukushima

Tokyo Institute of Technology

Workshop on Random Polymer Models and Related Problems, National University of Singapore, May 21-25, 2012
1. Setting

- \( \left( \{ B_t \}_{t \geq 0}, P_x \right) \): \( \kappa \Delta \)-Brownian motion on \( \mathbb{R}^d \)
- \( \left( \omega = \sum_i \delta_{\omega_i}, \mathbb{P} \right) \): Poisson point process on \( \mathbb{R}^d \) with unit intensity
1. Setting

- \( \left( \{ B_t \}_{t \geq 0}, P_x \right) : \kappa \Delta \)-Brownian motion on \( \mathbb{R}^d \)
- \( \left( \omega = \sum_i \delta_{\omega_i}, \mathbb{P} \right) : \) Poisson point process on \( \mathbb{R}^d \) with unit intensity

Potential
For a non-negative and integrable function \( v \),

\[
V_\omega(x) := \sum_i v(x - \omega_i).
\]

(Typically \( v(x) = 1_{B(0,1)}(x) \) or \( |x|^{-\alpha} \land 1 \) with \( \alpha > d \).)
Annealed measure

We are interested in the behavior of Brownian motion under the measure

\[
Q_t(\cdot) = \exp\left\{-\int_0^t V_\omega(B_s)ds\right\} \mathbb{P} \otimes P_0(\cdot) = \mathbb{E} \otimes E_0 \left[ \exp\left\{-\int_0^t V_\omega(B_s)ds\right\} \right].
\]

The configuration is not fixed and hence Brownian motion and \(\omega_i\)'s tend to avoid each other.
\[ \exp\left\{ - \int_0^t V_\omega (B_s) ds \right\} : \text{large}, \quad \mathbb{P} : \text{large}, \quad P_0 : \text{small} \]
\[
\exp\{-\int_0^t V_\omega(B_s)\,ds\} : \text{large, \quad } \mathbb{P} : \text{small, \quad } P_0 : \text{large}
\]
2. Light tailed case

Donsker and Varadhan (1975)

When \( \nu(x) = o(|x|^{-d-2}) \) as \(|x| \to \infty\),

\[
\mathbb{E} \otimes E_0 \left[ \exp \left\{ - \int_0^t V_\omega(B_s) \, ds \right\} \right] \\
= \exp \left\{ -c(d, \kappa)t^{\frac{d}{d+2}}(1 + o(1)) \right\} \\
= P_0 \left( B_{[0,t]} \subset B(x, t^{\frac{1}{d+2}}R_0) \right) \mathbb{P} \left( \omega(B(x, t^{\frac{1}{d+2}}R_0)) = 0 \right),
\]

as \( t \to \infty \).

Remark

\[ c(d, \kappa) = \inf_U \{ \kappa \lambda^D(U) + |U| \}. \]
One specific strategy gives dominant contribution to the partition function.

It occurs with high probability under the annealed path measure.
Sznitman (1991, $d = 2$) and Povel (1999, $d \geq 3$)

When $\nu$ has a compact support, there exists

$$D_t(\omega) \in B(0, t^{\frac{1}{d+2}}(R_0 + o(1)))$$

such that

$$Q_t \left( B_{[0,t]} \subset B(D_t(\omega), t^{\frac{1}{d+2}}(R_0 + o(1))) \right) \xrightarrow{t \to \infty} 1.$$

Remark

Bolthausen (1994) proved the corresponding result for two-dimensional random walk model.
3. Heavy tailed case

Pastur (1977)

When \( v(x) \sim |x|^{-\alpha} \) (\( \alpha \in (d, d + 2) \)) as \( |x| \to \infty \),

\[
E \otimes E_0 \left[ \exp \left\{ - \int_0^t V_\omega(B_s) ds \right\} \right] = \exp \left\{ -a_1 t^\frac{d}{\alpha} (1 + o(1)) \right\},
\]

where \( a_1 = |B(0, 1)|\Gamma\left(\frac{\alpha-d}{\alpha}\right) \).
In fact, Pastur’s proof goes as follows:

\[ \mathbb{E} \otimes E_0 \left[ \exp \left\{ - \int_0^t V_\omega(B_s)ds \right\} \right] \]
\[ \approx \mathbb{E}[\exp\{-tV_\omega(0)\}] \]
\[ \sim \exp\{-a_1 t^{d/\alpha}\}. \]
In fact, Pastur’s proof goes as follows:

\[ \mathbb{E} \otimes E_0 \left[ \exp \left\{ - \int_0^t V_\omega(B_s) ds \right\} \right] \]

\[ \approx \mathbb{E} [\exp \{ - t V_\omega(0) \}] \]

\[ \sim \exp \{ - a_1 t \frac{d}{\alpha} \}. \]

The effort of the Brownian motion is hidden in the lower order terms.
I\ \R\ (t_{\frac{1}{\alpha}})

B_t

O(t^{\frac{1}{\alpha}})

o(t^{\frac{1}{\alpha}})
F. (2011)

When $\nu(x) = |x|^{-\alpha} \wedge 1$ ($d < \alpha < d + 2$),

$$
\mathbb{E} \otimes E_0 \left\{ \exp \left\{ -\int_0^t V_\omega(B_s) \, ds \right\} \right\}
= \exp \left\{ -a_1 t^\frac{d}{\alpha} - (a_2 + o(1)) t^\frac{\alpha + d - 2}{2\alpha} \right\},
$$

where

$$
a_2 := \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + C(d, \alpha) |x|^2 \phi(x)^2 \, dx \right\}.
$$

**Remark**

The proof is an application of the general machinery developed by Gärtner-König 2000.
Recalling the Donsker-Varadhan LDP

\[ P_0 \left( \frac{1}{t} \int_0^t \delta_{B_s} ds \sim \phi^2(x) dx \right) \approx \exp \left\{ -t \int \kappa |\nabla \phi(x)|^2 dx \right\}, \]

we expect the second term explains the behavior of the Brownian motion.
Recalling the Donsker-Varadhan LDP

\[ P_0 \left( \frac{1}{t} \int_0^t \delta_{B_s} ds \sim \phi^2(x) dx \right) \approx \exp \left\{ -t \int \kappa |\nabla \phi(x)|^2 dx \right\} , \]

we expect the second term explains the behavior of the Brownian motion.

In particular, since \( P_0 (B_{[0,t]} \subset B(x, R)) \approx \exp\{ -tR^{-2} \} \), the localization scale should be

\[ tR^{-2} = t \frac{\alpha + d - 2}{2\alpha} \iff R = t \frac{\alpha - d + 2}{4\alpha} . \]
Recalling the Donsker-Varadhan LDP

\[ P_0 \left( \frac{1}{t} \int_0^t \delta_{B_s} ds \sim \phi^2(x) dx \right) \approx \exp \left\{ -t \int \kappa |\nabla \phi(x)|^2 dx \right\}, \]

we expect the second term explains the behavior of the Brownian motion.

In particular, since \( P_0 \left( B_{[0,t]} \subset B(x, R) \right) \approx \exp\{-tR^{-2}\} \), the localization scale should be

\[ tR^{-2} = t \frac{\alpha + d - 2}{2\alpha} \Leftrightarrow R = t \frac{\alpha - d + 2}{4\alpha}. \]

In addition, the term \( \int C(d, \alpha) |x|^2 \phi(x)^2 \, dx \) says that \( V_\omega \) (locally) looks like a quadratic function.
$O(t^{\frac{1}{\alpha}})$

$B_t$

$O(t^{\frac{\alpha-d+2}{4\alpha}})$
the below is the section along this line

\[ V_\omega(x) \]

Heavy tailed case

\[ th(t) \gg t/r(t)^2 \]
the below is the section along this line

Light tailed case

\[ th(t) \ll t/r(t)^2 \]
Main Theorem (F. 2012)

$$Q_t \left( B_{[0,t]} \subset B \left( 0, t^{\frac{\alpha-d+2}{4\alpha}} \left( \log t \right)^{\frac{1}{2}+\epsilon} \right) \right) \underset{t \to \infty}{\longrightarrow} 1,$$

$$Q_t \left( V_\omega(x) - V_\omega(m_t(\omega)) \sim t^{-\frac{\alpha-d+2}{\alpha}} C(d, \alpha) |x - m_t(\omega)|^2 \right.$$  

in $$B(0, t^{\frac{\alpha-d+2}{4\alpha} + \epsilon})$$  $$\underset{t \to \infty}{\longrightarrow} 1,$$

$$\left\{ t^{-\frac{\alpha-d+2}{4\alpha}} B \frac{\alpha-d+2}{2\alpha} t^{\frac{\alpha-d+2}{2\alpha}} s \right\}_{s \geq 0} \quad \text{in law} \quad \text{OU-process with}$$

“random center”,

where $$m_t(\omega)$$ is the minimizer of $$V_\omega$$ in $$B(0, t^{\frac{\alpha-d+2}{4\alpha}} \log t).$$
Main Theorem (F. 2012)

$$Q_t \left( B_{[0,t]} \subset B \left( 0, t^{\frac{\alpha-d+2}{4\alpha}} (\log t)^{\frac{1}{2}+\varepsilon} \right) \right) \xrightarrow{t \to \infty} 1,$$

$$Q_t \left( V_\omega(x) - V_\omega(m_t(\omega)) \sim t^{-\frac{\alpha-d+2}{\alpha}} C(d, \alpha) |x - m_t(\omega)|^2 \right. \left. \quad \text{in } B(0, t^{\frac{\alpha-d+2}{4\alpha}+\varepsilon}) \right) \xrightarrow{t \to \infty} 1,$$

$$\left\{ t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha} s}} \right\}_{s \geq 0} \xrightarrow{\text{in law}} \text{OU-process with}
\text{“random center”, }$$

where $m_t(\omega)$ is the minimizer of $V_\omega$ in $B(0, t^{\frac{\alpha-d+2}{4\alpha} \log t})$.

Remark: $X_s := t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha} s}} \Rightarrow B_t = t^{\frac{\alpha-d+2}{4\alpha}} X_{t^{\frac{\alpha+d-2}{2\alpha}}}$
4. Outline of the proof (of localization)

**Observation:** The 1st statement implies the 2nd one and vice versa.
4. Outline of the proof (of localization)

Observation: The 1st statement implies the 2nd one and vice versa.

Indeed, it is easy to believe $2nd \Rightarrow 1st$. 
4. Outline of the proof (of localization)

Observation: The 1st statement implies the 2nd one and vice versa.

Indeed, it is easy to believe 2nd \( \Rightarrow \) 1st.

To see 1st \( \Rightarrow \) 2nd, let \( L_t := \frac{1}{t} \int_0^t \delta_{B_s} \, ds \) and rewrite

\[
Q_t (d\omega) = \frac{1}{Z_t} E_0 \left[ \mathbb{E} \left[ e^{-t \langle L_t, V_\omega \rangle} \right] \mathbb{P}_t^L (d\omega) \right],
\]

where \( Z_t \) is the normalizing constant and

\[
\mathbb{P}_t^L (d\omega) := \frac{\exp \{ -t \langle L_t, V_\omega \rangle \} \mathbb{P} (d\omega)}{\mathbb{E} [\exp \{ -t \langle L_t, V_\omega \rangle \}]}.
\]
Assuming the 1st statement, we may replace $L_t$ by $\delta_{m_{L_t}}$ with $m_{L_t} = \int xL_t(dx)$ in the following:

$$
\mathbb{P}_t^{L_t} \left( V_\omega(x) - V_\omega(m_t(\omega)) \sim C(d, \alpha)t^{-\frac{\alpha-d+2}{\alpha}}|x|^2 \right) \to 1?
$$
Assuming the 1st statement, we may replace $L_t$ by $\delta_{m_{L_t}}$ with $m_{L_t} = \int x L_t(\mathrm{d}x)$ in the following: Let $m_{L_t} = 0$ for simplicity.

$$P_{t}^{\delta_0} \left( V_{\omega}(x) - V_{\omega}(0) \sim C(d, \alpha) t^{-\frac{\alpha - d + 2}{\alpha}} |x|^2 \right) \to 1?$$

This can be easily proved since $(\omega, P_{t}^{\delta_0})$ is nothing but the Poisson point process with intensity $e^{-t(|x|^{-\alpha} \wedge 1)} \mathrm{d}x$. 
Assuming the 1st statement, we may replace $L_t$ by $\delta_{m_{L_t}}$ with $m_{L_t} = \int xL_t(dx)$ in the following: Let $m_{L_t} = 0$ for simplicity.

$$\mathbb{P}_t^{\delta_0} \left( V_\omega(x) - V_\omega(0) \sim C(d, \alpha) t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \right) \to 1?$$

This can be easily proved since $(\omega, \mathbb{P}_t^{\delta_0})$ is nothing but the Poisson point process with intensity $e^{-t(|x|^{-\alpha \wedge 1})} dx$.

Remark
In fact, a slightly weaker localization bound is enough to do the above replacement.
This observation is useless (circular argument) as it is. But due to the last remark, there is a chance to go as follows:

- crude control on the potential,
  \[\Rightarrow\] crude control on the trajectory,
  \[\Rightarrow\] fine control on the potential,
  \[\Rightarrow\] fine control on the trajectory.
This observation is useless (circular argument) as it is. But due to the last remark, there is a chance to go as follows:

- crude control on the potential,
  - ⇒ crude control on the trajectory,
  - ⇒ fine control on the potential,
  - ⇒ fine control on the trajectory.

Assume $V_\omega$ attains its local minimum at 0 for simplicity.
4.1 Crude control on the potential

Lemma 1

\[ Q_t \left( V_\omega(0) \in \frac{d}{\alpha} a_1 t^{\frac{\alpha - d}{\alpha}} + t^{\frac{3\alpha - 3d + 2}{4\alpha}} (-M_1, M_1) \right) \rightarrow 1. \]
4.1 Crude control on the potential

**Lemma 1**

\[ Q_t \left( V_\omega(0) \in \frac{d}{\alpha} a_1 t^{\frac{\alpha-d}{\alpha}} + t^{\frac{3\alpha-3d+2}{4\alpha}} (-M_1, M_1) \right) \to 1. \]

**Idea**

\[ Z_t \leq \mathbb{E}[\exp\{-tV_\omega(0)\}] \approx \sup_{h > 0} \left[ e^{-th} \mathbb{P}(V_\omega(0) \approx h) \right]. \]
4.1 Crude control on the potential

Lemma 1

\[ Q_t \left( V_\omega(0) \in \frac{d}{\alpha} a_1 t^{-\frac{\alpha-d}{\alpha}} + t^{-\frac{3\alpha-3d+2}{4\alpha}} (-M_1, M_1) \right) \to 1. \]

Idea

\[ Z_t \leq \mathbb{E} [\exp\{-tV_\omega(0)\}] \begin{cases} = \exp \left\{ -a_1 t \frac{d}{\alpha} \right\}, \\ \approx \sup_{h>0} \left[ e^{-th} \mathbb{P}(V_\omega(0) \approx h) \right]. \end{cases} \]

\[ \frac{d}{\alpha} a_1 t^{-\frac{\alpha-d}{\alpha}} = h(t) \text{ is the maximizer.} \]

\[ \Rightarrow \mathbb{E} \otimes E_0 \left[ \exp \left\{ - \int_0^t V_\omega(B_s) \, ds \right\} : V_\omega(0) \text{ is far from } h(t) \right] \]

\[ \leq \mathbb{E} \left[ \exp\{-tV_\omega(0)\} : V_\omega(0) \text{ is far from } h(t) \right] = o(Z_t). \]
Lemma 2

\[ Q_t \left( V_\omega(0) + V_\omega(x) \right) \geq 2h(t) + c_1 t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \]

for \( t \frac{\alpha-d+6}{8\alpha} < |x| < M_2 t^{\frac{\alpha-d+6}{8\alpha}} \) → 1.
Lemma 2

\[ Q_t \left( V_\omega(0) + V_\omega(x) \right) \geq 2h(t) + c_1 t^{-\frac{\alpha-d+2}{\alpha}} |x|^2 \]

for \( t \frac{\alpha-d+6}{8\alpha} < |x| < M_2 t \frac{\alpha-d+6}{8\alpha} \) \( \rightarrow 1 \).

Idea

By Lemma 1,

\[ \exp \left\{ -\int_0^t V_\omega(B_s) \, ds \right\} \lessapprox \exp \{ -t h(t) \} = \exp \left\{ -\frac{d}{\alpha} a_1 t^\frac{d}{\alpha} \right\} \]
Lemma 2

\[
Q_t \left( V_\omega(0) + V_\omega(x) \right) \geq 2h(t) + c_1 t^{-\frac{\alpha-d+2}{\alpha}} |x|^2
\]

for \( t \frac{\alpha-d+6}{8\alpha} < |x| < M_2 t \frac{\alpha-d+6}{8\alpha} \) \( \rightarrow 1 \).

Idea

By Lemma 1,

\[
\exp \left\{ -\int_0^t V_\omega(B_s) ds \right\} \lesssim \exp \left\{ -th(t) \right\} = \exp \left\{ -\frac{d}{\alpha} a_1 t^{\frac{d}{\alpha}} \right\}
\]

Then, use

\[
\mathbb{E} \left[ \exp \left\{ -\frac{t}{2}(V_\omega(0) + V_\omega(x)) \right\} \right] \approx \exp \left\{ -a_1 t^{\frac{d}{\alpha}} - c_2 t^{\frac{d-2}{\alpha}} |x|^2 \right\}
\]

and Chebyshev’s inequality.
4.2 Crude control on the trajectory

\[ V_\omega(x) - V_\omega(0) \]

\[ -M_2 t \frac{\alpha - d + 6}{8\alpha} \quad -t \frac{\alpha - d + 6}{8\alpha} \quad 0 \quad t \frac{\alpha - d + 6}{8\alpha} \quad M_2 t \frac{\alpha - d + 6}{8\alpha} \]
4.2 Crude control on the trajectory

\[ V_\omega(x) - V_\omega(0) \]

By “penalizing a crossing”,

\[ Q_t \left( B_{[0,t]} \subset B \left( 0, M_2 t \frac{\alpha-d+6}{8\alpha} \right) \right) \rightarrow 1. \]
4.3 Fine control on the potential

The “crude control on the trajectory” is good enough to yield

\[ Q_t \left( V_{\omega}(x) - V_{\omega}(0) \sim C(d, \alpha) t^{-\frac{\alpha - d + 2}{\alpha}} |x|^2 \right. \]

\[ \left. \text{in } B(0, t^{\frac{\alpha - d + 2}{4\alpha} + \epsilon}) \rightarrow 1. \right \]
4.4 Fine control on the trajectory

\[ V_\omega(x) - V_\omega(0) \]

By "penalizing a crossing",

\[ Q_t \left( B_{[0, t]} \right) \subset B_{\alpha - d + 2 \frac{\alpha}{4\alpha} \left( \log t \right)^{\frac{1}{2} + \epsilon}} \]

\[ \rightarrow 1 \]
4.4 Fine control on the trajectory

\[ V_\omega(x) - V_\omega(0) \]

By “penalizing a crossing”,

\[ Q_t \left( B_{[0,t]} \subset B \left( 0, t \frac{\alpha-d+2}{4\alpha} (\log t)^{\frac{1}{2}+\epsilon} \right) \right) \to 1. \]
Thank you!