Extended Affine Root System V
(Elliptic Eta-product and their Dirichlet series)

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Abstract. According to the decomposition $\prod_i (\lambda^i - 1)^{e(i)}$ of the characteristic polynomial of the Coxeter element of a marked elliptic root system (R, G), we attach the product $\eta_{(R,G)}(\tau) := \prod_i \eta(i\tau)^{e(i)}$ of Dedekind eta-function and call it an elliptic eta-product (1.2.1).

Theorem. *The Fourier coefficients at infinity of the elliptic eta-product $\eta_{(R,G)}$ are non-negative integers if and only if the elliptic eta-product is not a cusp form. This is the case when (R, G) is one of the 4 types: $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$.*

One direction of the theorem: *an eta-product is not a cusp form if all Fourier coefficients at ∞ are non-negative* is a general fact (§2 Lemma 3). The proof of the opposit direction is achieved by a study of the attached Dirichlet series. We explain this below. To state it, we use a numerical invariant $\nu_{(R,G)}$ of (R, G), called the *dual rank* ((2.2.2) or (A3.5)).

1) We show that *an elliptic eta-product $\eta_{(R,G)}$ is holomorphic (resp. cuspidal) if and only if $\nu_{(R,G)}$ is non-positive (resp. negative)* ((2.9) Lemma 4). In fact, *$\nu_{(R,G)}$ is always non-positive, and is equal to 0 if and only if the weight of the eta-product is 1 and (R, G) is simply laced* ((2.9) Lemma 5). The elliptic root systems with $\nu_{(R,G)} = 0$ are classified into types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$ (see Appendix 1 and its Example).

2) We show that *the Dirichlet series $L_{(R,G)}$ attached to an elliptic eta-product $\eta_{(R,G)}$ of weight 1 is equal to either an Artin L-function or a difference of two Artin L-functions attached to rank 2 representations of $Gal(E_{(R,G)}/\mathbf{Q})$ where $E_{(R,G)}$ is a Kummer extension $\mathbf{Q}(\zeta_{m^{red}}, x^{1/m^*})$ of $\mathbf{Q}(\zeta_{m^{red}})$ for $\zeta_{m^{red}} := \exp(2\pi\sqrt{-1}/m^{red})$ and m^{red} and m^* are numerical invariants of* (R, G) ((2.5.1), (2.7.2)). *The extension is trivial ($\Leftrightarrow m^* = 1$) if and only if $\nu_{(R,G)} = 0$* ((3.2) Theorem). The $E_{(R,G)}$ is either $\mathbf{Q}(\zeta_4)$, $\mathbf{Q}(\zeta_3)$, $\mathbf{Q}(\zeta_8)$ or $\mathbf{Q}(\zeta_{12})$ according to the 4 types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$ of (R, G) with $\nu_{(R,G)} = 0$.

3) As a corollary of 1) and 2), if $\nu_{(R,G)} = 0$ then each summand of $L_{(R,G)}$ decomposes into a product of two Dirichlet L-functions, where Euler factors for bad primes are trivial. This implies the non-negativity of the Dirichlet coefficients of $L_{(R,G)}$ (§4 (4.1)).

The theorem gives a partial affirmative answer to a conjecture ([Sa3,§13]) on non-negativity of Fourier coefficients of eta-products attached to a regular system of weights.

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§1. Elliptic eta-product

In §1, we define the elliptic eta-product (1.2.1) associated to an elliptic root system and then explain the contents of the paper. We summarize necessary facts on elliptic root systems in Appendices 1 and 3 and Tables 1 and 2 (for details, see [Sa1]).

(1.1) We fix a notion: *eta-product* studied by many authors (eg.[H-M][D-K-M][Koi][Ma][G-O]). Let h be a positive integer, and call it a *Coxeter number* in the sequel. An element $\varphi \in \mathbf{Q}(\lambda)$ is called a *cyclotomic function belonging to h*, if it has an expression:

(1.1.1)
$$\varphi(\lambda) = \prod_{i|h} (\lambda^i - 1)^{e(i)},$$

for some $e(i) \in \mathbf{Z}$ (which may be negative). Note: i) the $e(i)$ is uniquely determined from φ, ii) the h is a common multiple of the i's with $e(i) \neq 0$ but is not uniquely determined from φ. The multiplicity of zero (= − the order of pole) of $\varphi(\lambda)$ at $\lambda = 1$ is given by

(1.1.2)
$$2a_0 := \sum_{i|h} e(i) \ .$$

We call the $a_0 \in \frac{1}{2}\mathbf{Z}$ the *weight* for a reason explained below (or the *genus* for a geometric background [Sa2,3]). The *eta-product* attached to φ is defined by

(1.1.3)
$$\eta_\varphi(\tau) := \prod_{i|h} \eta(i\tau)^{e(i)},$$

where $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$ for $q = \exp(2\pi\sqrt{-1}\tau)$ and $\tau \in \mathbf{H} := \{\tau \in \mathbf{C} \mid Im(\tau) > 0\}$ is the Dedekind eta-function. Note that the eta-product does not depend on a choice of the Coxeter number h. The Coxeter number h shall play a role when we introduce the dual eta-product in the next section (2.2.2). The η_φ is an automorphic form of weight a_0, and it can be developed in a series in fractional powers of q, whose coefficients will be referred to as *Fourier coefficients* at ∞ (see §2 for details on automorphicity of $\eta_\varphi(\tau)$).

In the remaining part of §1, we apply the correspondence: $\varphi \mapsto \eta_\varphi$ to the characteristic polynomial $\varphi_{(R,G)}$ of a Coxeter element for an elliptic root system (R, G) (for elliptic root systems and their Coxeter elements, see Appendix 1. or [Sa1]). A reader, who wants a quick view in the general properties of the eta-product, may jump to §2.

Remark. Let the φ (1.1.1) be a proportion φ_+/φ_- of two characteristic polynomials of two linear transformations $c_\pm$. Then the eta product $\eta_\varphi(\tau)$ is described in (1.1.4) as a trace of the level 1 representation $S(c_+) \otimes \Lambda(-c_-)$ of $c_\pm$ in conformal field theory. Since the author could not find a reference for the description, we give a sketch of proof. In the setting of the present article, c_+ is the elliptic Coxeter element c (Appendix 1) and $c_- = \emptyset$ (i.e. $\varphi = det(\lambda \cdot id - c)$). But we shall not use the description in the present article.

Let $H_\pm$ be finite dimensional vector spaces equipped with the actions $c_\pm \in GL(H_\pm)$ such that $\varphi = \varphi_+/\varphi_-$ for $\varphi_\pm := det(\lambda \cdot id_{H_\pm} - c_\pm)$. Let $H_\pm(-1)$, $H_\pm(-2)$, $H_\pm(-3)$,$\cdots$ be an infinite sequence of copies of $H_\pm$, respectively. We consider the vector space:

$$V := S\left(H_+(-1) \oplus H_+(-2) \oplus H_+(-3) \oplus \cdots\right) \otimes \Lambda\left(H_-(-1) \oplus H_-(-2) \oplus H_-(-3) \oplus \cdots\right),$$

where $S(\cdot)$, $\Lambda(\cdot)$ expresses the symmetric tensor algebra and the Grassmann algebra, respectively. The space V is graded by counting non-trivial elements of $H_\pm(-n)$ to be of degree n for $n \in \mathbf{Z}_{>0}$. So, $V = \oplus_{n=1}^\infty V_n$ where each graded piece is of finite dimensional. Let deg be the degree operator on V (i.e. $deg(x) = n \cdot x$ for $x \in V_n$).

The copies of the actions of $\pm c_\pm$ on the $H_\pm(-n)$ induce a diagonal action on V, denoted by $S(c_+) \otimes \Lambda(-c_-)$. It preserves each graded piece V_n. Then one has the formula:

$$(1.1.4) \qquad \eta_\varphi(\tau) = q^{(rank(H_+) - rank(H_-))/24} / Tr_V\left(S(c_+) \otimes \Lambda(-c_-) \cdot q^{deg}\right).$$

This is a trivial consequence of a formula: $Tr_V\left(S(c_+) \otimes \Lambda(-c_-) \cdot q^{deg}\right) = \prod_{n=1}^\infty \phi_-/\phi_+(q^n)$, where the notation are as above except $\phi_\pm(\lambda) := det(id - \lambda \cdot c_\pm)$ for arbitrary (not necessarily quasi-unipotent) linear transformations $c_\pm$ on $H_\pm$. If $\varphi = \lambda^\mu \cdots \pm 1$ is a cyclotomic polynomial then $\eta_\varphi(\tau) = q^\mu \prod_{n=1}^\infty \phi(q^n)$ for $\phi := \pm\varphi$.

(1.2) Recall [Sa1] that a *marked elliptic root system* is a pair (R, G) where R is a generalized root system (Appendix 1) belonging to a semipositive root lattice (Q, q) of sign $(l, 2, 0)$, and G, called a marking, is a rank 1 subspace of 2-dimensional $radical(q) \otimes \mathbf{Q}$. One attaches an *elliptic Dynkin diagram* $\Gamma(R, G)$ (see Appendix 1 (A1.3) and its following statements) to (R, G), whose vertices form a basis of the root system R. A *Coxeter element* $c(R, G)$ is a product of reflexions on $Q(R)$ attached to all vertices of $\Gamma(R, G)$ in a suitable sequence. The $c(R, G)$ is of finite order $m(R, G)$ (A1.5) and its conjugacy class in $Aut(R)$ is unique. Thus, the polynomial $\varphi_{(R,G)}(\lambda) := det(\lambda I - c(R, G))$ is a cyclotomic polynomial well defined for (R, G). For explicit descriptions of $\varphi_{(R,G)}$, one is referred to (A1.7), Table 1 or (A3.2).

Definition. An *elliptic eta-product for* (R, G) is the eta-product (1.1.3) attached to the characteristic polynomial $\varphi_{(R,G)}$ of a Coxeter element of (R, G):

$$(1.2.1) \qquad \eta_{(R,G)}(\tau) := \eta_{\varphi_{(R,G)}}(\tau).$$

As for the Coxeter number of the cyclotomic function $\varphi_{(R,G)}(\lambda)$, put $h := m(R, G) =:$ the order of the Coxeter element $c(R, G)$ (see (A1.5) for an explicit formula).

The goal theorem of the paper is formulated in Abstract and is proven in §4 (4.1). It is inspired from a duality theory of regular system of weights [Sa3,§13]. Namely, we conjecture that the Fourier coefficients at ∞ of the eta-product attached to a regular weight system are non-negative if and only if it is not a cusp-form. The elliptic eta-products of types

$E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ treated in the present paper are the first non-trivial affirmative examples of the conjecture (see Appendix 2 for more details).

Remark. An eta-product in general may have negative Fourier coefficients at ∞ even if it is not a cusp form. See examples at (4.2) taken from the Conway group.

(1.3) The contents of the present paper are as follows.

In Appendix 1, we recall definitions of an elliptic root system (R, G) and its diagram $\Gamma(R, G)$ ([Sa1]). We describe the characteristic polynomial $\varphi_{(R,G)}$ (A1.7), the Coxeter number $m(R, G)$ (A1.5) and the genus a_0 (A1.8) in terms of the diagram $\Gamma(R, G)$ (summarized in **Table 1**). In Appendix 2, we explain the motivation of the present article and the relationship with the duality theory of regular system of weights [Sa2,3].

In the first half of §2, we study automorphicity of eta-products in general. Level N_φ and character ε_φ of an eta-product is calculated in (2.5.1), (2.5.2) and (2.5.3) (Lemma 1). We show that an eta-product is holomorphic (resp. cuspidal) at the cusps at $\mathbf{Z}$, if and only if the numerical invariant ν_φ, called the dual rank (2.2.2), is ≤ 0 (resp. < 0) (Lemma 2) and that if $\nu_\varphi < 0$ then all Fourier coefficients at ∞ cannot be non-negative (Lemma 3).

On the other hand, we prove that i) the elliptic eta-product is always holomorphic and it is cuspidal if and only if the dual rank $\nu_{(R,G)} < 0$ (Lemma 4), and ii) $\nu_{(R,G)} = 0$ if and only if (R, G) is 1-codimensional (A1.4) and simply laced (Lemmas 5 and 6). Here, the 1-codimensionality is equivalent to the weight a_0 of the eta-product being equal to 1 (A1.8). The classification (**Table 2** in Appendix 3) says that the elliptic root system with $\nu_{(R,G)} = 0$ are the types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$.

In §3, we study the Dirichlet series $L_{(R,G)}(s)$ (3.1.1) attached to the elliptic eta-product. (3.2) **Theorem** and its following **Table 3** describe the $L_{(R,G)}(s)$ for the 1-codimensional elliptic root system (R, G) (except for one case where a datum x (see below) is yet undetermined) as follows.

The Dirichlet series attached to a 1-codimensional elliptic root system is either equal to an Artin L-function $L(s, \rho)$ attached to a representation $\rho : Gal(E_{(R,G)}/\mathbf{Q}) \to GL_2(\mathbf{Z})$, a difference $\frac{1}{4}\left(L(s, \rho^{(+)}) - L(s, \rho^{(-)})\right)$ of two Artin L-functions attached to two representations $\rho^{(\pm)} : Gal(E_{(R,G)}/\mathbf{Q}) \to GL_2(\mathbf{Z})$, or a difference $\frac{\sqrt{-1}}{4}(L(s, \rho) - L(s, \bar{\rho}))$ of two Artin L-functions attached to a representation $\rho : Gal(E_{(R,G)}/\mathbf{Q}) \to GL_2(\mathbf{Z}[\sqrt{-1}])$ with the conductor $= N_{(R,G)}$ and the character $det(\rho) = \varepsilon_{(R,G)}$ given in Lemma 1, respectively. Here $E_{(R,G)}$ is a Kummer field $\mathbf{Q}(\zeta_{m^{red}}, x^{1/m^*})$ for $m^{red}, m^* \in \mathbf{Z}_{>0}$ in (2.5.1), (2.7.2) and some $x \in \mathbf{Z}$. The Kummer extension $E_{(R,G)}/E^{ab}_{(R,G)}$ is trivial if and only if the dual rank $\nu_{(R,G)}$ is 0. Any of $L(s, \rho)$, $L(s, \rho^{(\pm)})$ or $L(s, \bar{\rho})$, called the Artin summand, has trivial Euler factors for the primes p with $p|N_{(R,G)}$.

The proof of the theorem is achieved by inspection on Fourier coefficients of the elliptic eta-product for each type. One, first, give a guessing form of the Dirichlet series as described above. Then, one applies theorems by Hecke [36], Weil [W] and by Deligne-Serre [D-S] on the bijections between the set of normalized new forms and Eisenstein series and the set of odd complex 2-dimensional representations of Galois groups ((3.2) Assertion). Explicit descriptions of $E_{(R,G)}$, ρ, $\rho^{(\pm)}$ and $L(s, \rho)$, $L(s, \rho^{(\pm)})$ are given in **Table 3**.

If $\nu_{(R,G)} = 0$, the Galois group is an abelian group isomorphic to $(\mathbf{Z}/m^{red}\mathbf{Z})^{\times}$ and hence the representation(s) split into direct sums of 1-dimensional representations. So, the Artin summand(s) decompose into product(s) of Dirichlet L-functions ([Hecke 36, §10]). In fact, the $L_{(R,G)}(s)$ is either of the forms $L(s,1)L(s,\varepsilon)$ or $\frac{1}{4}\left(L(s,1)L(s,\varepsilon)\ -L(s,\chi)L(s,\chi\varepsilon)\right)$ for some characters ε and $\chi \in Hom((\mathbf{Z}/m^{red}\mathbf{Z})^{\times}, \{\pm 1\})$. Some elementary calculations in §4 (4.1) using the Euler product expression of the Dirichlet L-functions confirm the non-negativity of Dirichlet coefficients for $L_{(R,G)}$. This proves the goal Theorem stated at Abstract: *Fourier coefficients for the types* $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ *and* $E_8^{(1,1)}$ *are non-negative.* Explicit formulae of the Fourier-Dirichlet coefficients for 1-codimensional elliptic root systems are given in **Table 4**.

The proof seems a bit involved and non-conceptual. One should look for a conceptual understanding of the non-negativity of Fourier coefficients, which should lead to an answer to the general conjecture in [Sa3,§13] (cf. Appendix 2).

After the present paper is written, Victor Kac has pointed out to the author about a coincidence of the elliptic eta-products for the types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$ with the theta-functions for the lattices $Q_R + \mathbf{Z}\rho/h$ for finite root lattices Q_R of rank 2 (i.e. of types $A_1 \times A_1$, A_2, B_2 and G_2) together with the Weyl vector ρ (see [Ka, (3.34)]). Prof. Atkin has pointed out that these eta-products are unary-theta-functions in the sense [?], which can be expressed by Eisenstein serieses. Also D. Zagier explained the author that the proof of the goal theorem could be reduced to the classical theory of qudratic forms so far as one concerns only on the four types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. To clarify these facts may ask another work, but still their connection with the general conjecture (including the cases when the eta-products are no more holomorphic) seems still unclear to the author. Therefore, the author dicided to publish the paper in the present (old) form, since it contains some other aspects.

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Appendix 1. Marked elliptic root systems and their diagram

We recall ([Sa1]) the definition of a marked elliptic root system (R,G) and its diagram $\Gamma(R,G)$ and codimension $cod(R,G)$. We give the explicit formula of the characteristic polynomial $\varphi_{(R,G)}$ (A1.6-7), the Coxeter number $m(R,G)$ (A1.5) and the genus a_0 (A1.8).

Definition. Let us call a set R of non-isotropic elements in an even lattice (Q,q) (i.e. a pair of a free abelian group Q of finite rank and a quadratic form q on it) a *generalized root system belonging to* (Q,q) if 1) R generates Q, 2) for all α and $\beta \in R$ one has $I(\alpha^{\vee},\beta) \in \mathbf{Z}$, where $\alpha^{\vee} := \alpha/q(\alpha)$ and $I(x,y) := q(x+y) - q(x) - q(y)$, 3)the reflexion w_α w.r.t. $\alpha \in R$ (i.e. $w_\alpha(u) = u - \alpha^{\vee}I(\alpha,u)$) preserves the set R, and 4) if $R = R_1 \cup R_2$ and $R_1 \perp R_2$ w.r.t. q then either $R_1 = \phi$ or $R_2 = \phi$. The group $W(R)$ generated by reflexions w_α for all $\alpha \in R$ is called the *Weyl group* of the root system R.

One has the equivalence: $\#R < \infty \iff \#W(R) < \infty \iff q$ is definite. This is the case of finite root systems studied in classical literatures ([B]). If q is semi-definite with 1-dimensional radical $rad(q) := Q^{\perp}$, then R is an affine root system in the sense of Macdonald. Our interest in the paper is on the next case with two dimensional radicals:

Definition. A *marked elliptic root system* is a pair (R, G) where R is a generalized root system belonging to a semipositive lattice (Q, q) with $rank(rad(q)) = 2$ and G is a rank 1 subspace of $rad(q) \otimes_{\mathbf{Z}} \mathbf{Q}$. Put $l := rank(Q/rad(q)) = rank(Q) - 2$.

The image R_a of R by the projection $Q \to Q_a := Q/G \cap Q$ is an affine root system, which we assume to be reduced. Once for all, we choose and fix a set $\Gamma_{aff} \subset R$ which is projected bijective to a simple root basis of R_a. The Γ_{aff} is unique up to an isomorphism of (R, G). As usual (eg. [B,chVI,§4]), an affine Dynkin diagram structure for the root system R_a is attached to Γ_{aff}, identifying Γ_{aff} with the set of vertices of the diagram.

Let $n_\alpha \in \mathbf{Z}_{>0}$ for $\alpha \in \Gamma_{aff}$ be a system of integers such that $gcd\{n_\alpha \mid \alpha \in \Gamma_{aff}\} = 1$ and $b := \sum_{\alpha \in \Gamma_{aff}} n_\alpha \alpha$ belongs to $rad(q)$ (i.e. the projection of b in Q_a is a base of the set of null roots of R_a). Fix a generator a of $G \cap Q \simeq \mathbf{Z}$. Then, a and b form an integral basis of the radical of q (i.e. $rad(q) = \mathbf{Z}a \oplus \mathbf{Z}b$), and one has $Q = \sum \mathbf{Z}\Gamma_{aff} \oplus \mathbf{Z}a$. For any root $\alpha \in R$, put $k(\alpha) := inf\{k \in \mathbf{Z}_{>0} \mid \alpha + k \cdot a \in R\}$ and $\alpha^* := \alpha + k(\alpha) \cdot a$.

The *exponents* of the marked elliptic root system (R, G) are, by definition, 0 and

$$(A1.1) \qquad m_\alpha := \frac{q(\alpha)}{k(\alpha)} \cdot n_\alpha$$

for $\alpha \in \Gamma_{aff}$. Let $m_{max} := max\{m_\alpha \mid \alpha \in \Gamma_{aff}\}$ be the largest exponent. Put

$$(A1.2) \qquad \Gamma_{max} := \{\alpha \in \Gamma_{aff} \mid m_\alpha = m_{max}\} \quad \text{and} \quad \Gamma^*_{max} := \{\alpha^* \mid \alpha \in \Gamma_{max}\}.$$

Finally, we define the *root basis* for the marked elliptic root system (R, G) by

$$(A1.3) \qquad \Gamma(R, G) = \Gamma_{aff} \cup \Gamma^*_{max}.$$

The $\Gamma(R, G)$ is called the root basis, since it has properties: i) $Q(R) = \sum_{\alpha \in \Gamma(R,G)} \mathbf{Z}\alpha$, ii) $W(R) = < w_\alpha \mid \alpha \in \Gamma(R, G) >$ and iii) $R = \cup_{\alpha \in \Gamma(R,G)} W(R) \cdot \alpha$. To $\Gamma(R, G)$, we attach the *elliptic diagram* defined by i) vertices are in one to one correspondence with $\Gamma(R, G)$, ii) bonds among the vertices are defined according to usual convention (e.g. [B]), except for newly introduced double dotted bond o===o between vertices α and $\alpha*$ for $\alpha \in \Gamma_{max}$ (characterized by $I(\alpha^\vee, \alpha^*) = I(\alpha, \alpha^{*\vee}) = 2$ ([Sa1,I,§9], see the Example below). *The elliptic diagram is uniquely determined from the isomorphism class of* (R, G) *independent of choices of* Γ_{aff} *and* a. *Conversely, the elliptic diagram* $\Gamma(R, G)$ *determines the isomorphism class of the marked elliptic root system* (R, G) ([Sa1,I,(9.6) Theorem]). We shall identify the elliptic diagram with the root basis $\Gamma(R, G)$.

We call a marked elliptic root system (R, G) to be *simply laced* if the bonds of its diagram $\Gamma(R, G)$ are either simply laced (o—o) or doubly dotted (o= = =o). Simply laced elliptic root systems are the types $A_l^{(1,1)}$ $(l \geq 2)$, $D_l^{(1,1)}$ $(l \geq 4)$ and $E_l^{(1,1)}$ $(l = 6, 7, 8)$.

The cardinality of Γ_{max} is called the *codimension* of (R, G):

$$(A1.4) \qquad codim(R, G) := \#\Gamma_{max} = \#\Gamma^*_{max} = \#\{\circ === \circ\}.$$

One observes that i) the compliment $\Gamma_{aff} \setminus \Gamma_{max} = \Gamma(R, G) \setminus \left(\Gamma_{max} \cup \Gamma^*_{max}\right)$ is a disjoint union of A_l-type diagrams, say $\Gamma(A_{l_1}), \ldots, \Gamma(A_{l_r})$, ii) one has the equality:

$$(A1.5) \qquad m(R, G) := max\{l_1 + 1, \cdots, l_r + 1\} = lcm\{l_1 + 1, \cdots, l_r + 1\},$$

and iii) the exponents on the branch $\Gamma(A_{l_i})$ are given by an arithmetic progression:

$$(A1.6) \qquad \frac{1}{l_i + 1} \cdot m_{max}, \cdots, \frac{l_i}{l_i + 1} \cdot m_{max}$$

([Sa1,I,(8.4)iv)]). A *Coxeter element* $c(R, G)$ is defined as a product of reflexions w_α for $\alpha \in \Gamma(R, G)$ in such sequence that w_{α^*} comes next to w_α for $\alpha \in \Gamma_{max}$. Since Γ_{aff} is a tree, the conjugacy class of $c(R, G)$ in $W(R)$ does not depend on the order of the product.

Lemma A ([Sa1,I,§9(9.7)Lemma A]). *A Coxeter element $c(R, G)$ of (R, G) is of finite order $m(R, G)$. The characteristic polynomial $\varphi_{(R,G)} := det(\lambda I - c(R, G))$ is given by*

$$(A1.7) \qquad \varphi_{(R,G)}(\lambda) = (\lambda - 1) \prod_{\alpha \in \Gamma_{aff}} (\lambda - exp(2\pi\sqrt{-1} m_\alpha / m_{max})).$$

Of course by definition, $deg(\varphi_{(R,G)}) = rank(Q(R)) = l + 2$. The (A1.2), (A1.6) and (A1.7) determine $\varphi_{(R,G)}$ from the diagram $\Gamma(R, G)$. Comparing (A1.4) with the fact: $2a_0$ (1.1.2) is the multiplicity of zeros of $\varphi_{(R,G)}(\lambda) = 0$ at $\lambda = 1$, we obtain:

Corollary. *The genus a_0 of a marked elliptic root system is given by*

$$(A1.8) \qquad 2a_0 = codim(R, G) + 1.$$

This implies: $a_0 = 1 \Leftrightarrow codim(R, G) = 1$. That is: *the weight a_0 of the eta-product $\eta_{(R,G)}$ of a marked elliptic root system (R, G) is equal to 1, if and only if it is 1-codimensional.*

Example. We exhibit diagrams $\Gamma(R, G)$ together with their exponents for simply laced and 1-codimensional elliptic root systems (R, G). One has $m_{max} = m(R, G)$.

$D_4^{(1,1)}$

$E_7^{(1,1)}$

$E_6^{(1,1)}$

$E_8^{(1,1)}$

Table 1. Marked elliptic root systems and their exponents.

Type	m(R,G)	exponents	codim	$\varphi_{(R,G)}$
$A_l^{(1,1)}$ $l \geq 1)$	1	0, $m_i = 1$ $(0 \leq i \leq l)$	$l+1$	$(\lambda-1)^{l+2}$
$A_1^{(1,1)*}$	2	0, 1/2, 1	1	$(\lambda^2-1)(\lambda-1)$
$B_l^{(1,1)}$ $(l \geq 3)$	2	0, 2, 2, 2, $m_i = 4$ $(3 \leq i \leq l)$	$l-2$	$(\lambda^2-1)^3(\lambda-1)^{l-4}$
$B_l^{(1,2)}$ $(l \geq 2)$	2	0, 1, 1, $m_i = 2$ $(2 \leq i \leq l)$	$l-1$	$(\lambda^2-1)^2(\lambda-1)^{l-2}$
$B_l^{(2,1)}$ $(l \geq 2)$	2	0, 1, 1, $m_i = 2$ $(2 \leq i \leq l)$	$l-1$	$(\lambda^2-1)^2(\lambda-1)^{l-2}$
$B_l^{(2,2)}$ $(l \geq 2)$	1	0, $m_i = 1$ $(0 \leq i \leq l)$	$l+1$	$(\lambda-1)^{l+2}$
$C_l^{(1,1)}$ $(l \geq 2)$	1	0, $m_i = 1$ $(0 \leq i \leq l)$	$l+1$	$(\lambda-1)^{l+2}$
$C_l^{(1,2)}$ $(l \geq 2)$	2	0, 1, 1, $m_i = 2$ $(2 \leq i \leq l)$	$l-1$	$(\lambda^2-1)^2(\lambda-1)^{l-2}$
$C_l^{(2,1)}$ $(l \geq 2)$	2	0, 1, 1, $m_i = 2$ $(2 \leq i \leq l)$	$l-1$	$(\lambda^2-1)^2(\lambda-1)^{l-2}$
$C_l^{(2,2)}$ $(l \geq 3)$	2	0, 1, 1, 1, $m_i = 2$ $(3 \leq i \leq l)$	$l-2$	$(\lambda^2-1)^3(\lambda-1)^{l-4}$
$B_l^{(2,2)*}$ $(l \geq 2)$	2	0, 1/2, $m_i = 1$ $(1 \leq i \leq l)$	l	$(\lambda^2-1)(\lambda-1)^{l}$
$C_l^{(1,1)*}$ $(l \geq 2)$	2	0, 1, $m_i = 2$ $(1 \leq i \leq l)$	l	$(\lambda^2-1)(\lambda-1)^{l}$
$BC_l^{(2,1)}$ $(l \geq 1)$	2	0, 2, $m_i = 4$ $(1 \leq i \leq l)$	l	$(\lambda^2-1)(\lambda-1)^{l}$
$BC_l^{(2,4)}$ $(l \geq 1)$	2	0, 1, $m_i = 2$ $(1 \leq i \leq l)$	l	$(\lambda^2-1)(\lambda-1)^{l}$
$BC_l^{(2,2)}(1)$ $(l \geq 2)$	2	0, 2, 2, $m_i = 4$ $(2 \leq i \leq l)$	$l-1$	$(\lambda^2-1)^2(\lambda-1)^{l-2}$
$BC_l^{(2,2)}(2)$ $(l \geq 2)$	1	0, $m_i = 2$ $(0 \leq i \leq l)$	$l+1$	$(\lambda-1)^{l+2}$
$D_l^{(1,1)}$ $(l \geq 4)$	2	0, 1, 1, 1, 1, $m_i = 2$ $(4 \leq i \leq l)$	$l-3$	$(\lambda^2-1)^4(\lambda-1)^{l-6}$
$E_6^{(1,1)}$	3	0, 1, 1, 1, 2, 2, 2, 3	1	$(\lambda^3-1)^3(\lambda-1)^{-1}$
$E_7^{(1,1)}$	4	0, 1, 1, 2, 2, 2, 3, 3, 4	1	$(\lambda^4-1)^2(\lambda^2-1)(\lambda-1)^{-1}$
$E_8^{(1,1)}$	6	0, 1, 2, 2, 3, 3, 4, 4, 5, 6	1	$(\lambda^6-1)(\lambda^3-1)(\lambda^2-1)(\lambda-1)^{-1}$
$F_4^{(1,1)}$	3	0, 2, 2, 4, 4, 6	1	$(\lambda^3-1)^2$
$F_4^{(1,2)}$	4	0, 1, 2, 2, 3, 4	1	$(\lambda^4-1)(\lambda^2-1)$
$F_4^{(2,1)}$	4	0, 1, 2, 2, 3, 4	1	$(\lambda^4-1)(\lambda^2-1)$
$F_4^{(2,2)}$	3	0, 1, 1, 2, 2, 3	1	$(\lambda^3-1)^2$
$G_2^{(1,1)}$	2	0, 3, 3, 6	1	$(\lambda^2-1)^2$
$G_2^{(1,3)}$	3	0, 1, 2, 3	1	$(\lambda^3-1)(\lambda-1)$
$G_2^{(3,1)}$	3	0, 1, 2, 3	1	$(\lambda^3-1)(\lambda-1)$
$G_2^{(3,3)}$	2	0, 1, 1, 2	1	$(\lambda^2-1)^2$

Fact. The characteristic polynomial of the next types are non-reduced (cf. (2.7)): $B_2^{(1,2)}, B_2^{(2,1)}$, $C_2^{(1,2)}, C_2^{(2,1)}, B_4^{(1,1)}, C_4^{(2,2)}, F_4^{(1,1)}, F_4^{(1,2)}, F_4^{(2,1)}, F_4^{(2,2)}, G_2^{(1,1)}, G_2^{(3,3)}$

Appendix 2. Eta-products arising from Weight Systems

In this appendix we review the conjecture mentioned at the end of (1.2). Recall [Sa2] that a system $W = (a, b, c; h)$ of 4 integers with $0 < a, b, c < h$ is called *regular* if the rational function:

$$\chi_W(T) := T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$$

does not have a pole except at $T=0$. Then, one has a finite sum development $\chi_W(T) = \sum_{i=1}^{\mu} T^{m_i}$, where $m_i \in \mathbf{Z}$ are called *exponents*. Then $\varphi_W(\lambda) := \prod_{i=1}^{\mu}(\lambda - exp(2\pi\sqrt{-1}m_i/h))$ is a cyclotomic polynomial of Coxeter number h. We consider the associated eta-product

$$\eta_W(\tau) := \eta_{\varphi_W}(\tau).$$

The weight a_0 (which we call genus to avoid the confusion with the weights a, b and c) is integral due to the symmetry of exponents: $\chi_W(T) = T^h\chi_W(T^{-1})$. The eta-product η_W is holomorphic (resp. cuspidal) if and only if the dual-rank ν_W (2.2.2) is non-positive (resp. negative) ([Sa3, Lemma 13.4], cf. (2.9) Lemma 4). Then,

Conjecture ([Sa3,§13 Conj.13.5]). *Fourier coefficients at ∞ of an eta-product η_W attached to a regular weight system are non-negative if and only if η_W is not a cusp form.*

Ex. The following is the list of weight systems whose exponents are non-negative ([Sa2]).

Type	weights	Coxeter #	genus	rank	dual-rank	φ_W
A_l	$(1, b, l+1-b; l+1)$	$l+1$	0	l	l	$\frac{(\lambda^{l+1}-1)}{(\lambda-1)}$
D_1	$(2, l-2, l-1; 2l-2)$	$2l-2$	0	l	l	$\frac{(\lambda^{2(l-1)}-1)(\lambda^2-1)}{(\lambda^{l-1})(\lambda-1)}$
E_6	$(3,4,6;12)$	12	0	6	6	$\frac{(\lambda^{12}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^6-1)(\lambda^4-1)(\lambda-1)}$
E_7	$(4,6,9;18)$	18	0	7	7	$\frac{(\lambda^{18}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^9-1)(\lambda^6-1)(\lambda-1)}$
E_8	$(6,10,15;30)$	30	0	8	8	$\frac{(\lambda^{30}-1)(\lambda^5-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^{15}-1)(\lambda^{10}-1)(\lambda^6-1)(\lambda-1)}$
$\tilde{E}_6$	$(1,1,1;3)$	3	1	8	0	$\frac{(\lambda^3-1)^3}{(\lambda-1)}$
$\tilde{E}_7$	$(1,1,2;4)$	4	1	9	0	$\frac{(\lambda^4-1)^2(\lambda^2-1)}{(\lambda-1)}$
$\tilde{E}_8$	$(1,2,3;6)$	6	1	10	0	$\frac{(\lambda^6-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda-1)}$

The set of exponents for the weight system A_l, D_l, E_l or $\tilde{E}_l$ coincides with that of a root system of type A_l, D_l, E_l or $E_l^{(1,1)}$, respectively. So, by definition, the eta-product for a root system above coincides with that for the correspoinding weight system.

The characteristic polynomial φ_W for the types A_l, D_l and E_l are selfdual (see Remark below). Then the eta-product η_W is neither cuspidal nor holomorphic, and its Fourier coefficients are non-negative (see [Sa3, §12,§13 Assertion 13.6]).

The eta-products for $\tilde{E}_l$ (= elliptic eta-products for $E_l^{(1,1)}$) for $l = 6, 7$ and 8 are non-cuspidal holomorphic automorphic forms (see §2 Lemma 4). In [Sa3, §13 Ex.13.7], we gave a sketch of a proof of the non-negativity of their Fourier coefficients using their L-functions. Actually, the present paper shall give its complete proof, and so, gives the first non-trivial answer to the conjecture above.

Remark. Two polynomials $\varphi(\lambda) = \prod_{i|h}(\lambda^i - 1)^{e(i)}$ and $\varphi^*(\lambda) = \prod_{j|h}(\lambda^j-1)^{e^*(j)}$ of the same Coxeter # h are called dual to each other if $e(i) + e^*(h/i) = 0$ for $i \mid h$ ([Sa3]). Eg. A_l, D_l and E_l are selfdual. The Fourier coefficients of η_φ and η_{φ^*} for such dual pair are trivially non-negative. The concept of duality extends in a obvious manner to arbitrary cyclotomic functions (see (2.2)), which is a key concept to understand eta-products in §2.

§2. Automorphicity of the eta-product

In this section, we first study the automorphic properties of eta products in general. There are several literatures (e.g. [H-M][D-K-M][Ko][Ha]) on the subject when the weight a_0 is an (even) integer. We modify and sharpen them using the concept of the duality (2.2) in order to include the half integral weight case to cover all elliptic eta-products. Precisely, first, the character and the level of an eta-product are determined by a help of dual numerical invariants (Lemma 1). Then, we formulate a criterion on a eta-product to be holomorphic or cuspidal in terms of the dual-rank ν_φ (2.2.2). Namely, the eta product η_φ is holomorphic (resp. cuspidal) at the cusps in $\mathbf{Z}$ if and only if its dual rank is non-positive (resp. negative) (Lemma 2). On the other hand, one observes that the dual rank ν_φ is non-negative if all Fourier coefficients at ∞ are non-negative (Lemma 3).

In the latter half of the section and in the Appendix 3, we prove stronger results on elliptic eta-products: 1) an elliptic eta-product $\eta_{(R,G)}$ is holomorphic (resp. cuspidal) if and only if the dual rank $\nu_{(R,G)}$ is non-positive (resp. negative) (Lemma 4), and 2) the dual rank $\nu_{(R,G)}$ is, in fact, non-positive for any elliptic root system (R,G) and is equal to 0 if and only if the elliptic root system is 1-codimensional and simply laced (Lemma 5). Such root system (R,G) are classified into the types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. That is: an elliptic eta product is always holomorphic and it is not a cusp form if it is one of the above 4 types.

Notation. The argument of the value $\sqrt{z}$ for $z \in \mathbf{C}$ is chosen in the interval $(-\pi/2, \pi/2]$.

(2.1) First, we introduce numerical invariants for a cyclotomic function φ (1.1.1). Put

$$\text{(2.1.1)} \qquad \text{rank:} \qquad \mu_\varphi := deg(\varphi) = \sum_{i|h} i \cdot e(i) \in \mathbf{Z},$$

$$\text{(2.1.2)} \qquad \text{discriminant:} \qquad d_\varphi := \prod_{i|h} i^{e(i)} \in \mathbf{Q}.$$

We shall denote by $d_{sf} \in \mathbf{Z}_{>0}$ the square free part of d_φ (ie. the smallest positive integer with $d_\varphi/d_{sf} \in \left(\mathbf{Q}^\times\right)^2$). Obviously, one has $d_{sf}|h$.

(2.2) The *dual cyclotomic function* with the Coxeter number h and the *dual eta-product* are defined by

$$\text{(2.2.1)} \qquad \begin{aligned} \varphi^*(\lambda) &:= \prod_{i|h} (\lambda^{h/i} - 1)^{-e(i)}, \\ \eta^*_\varphi(\tau) &:= \eta_{\varphi^*}(\tau) = \prod_{i|h} \eta((h/i)\cdot\tau)^{-e(i)}. \end{aligned}$$

Obviously, one has $\varphi^{**} = \varphi$ and $\eta^{**}_\varphi = \eta_\varphi$. We define the dual numerical invariants.

$$\text{(2.2.2)} \qquad \begin{cases} \text{dual genus:} & 2a_0^* = -\sum_{i|h} e(i), \\ \text{dual rank:} & \nu_\varphi := \mu_{\varphi^*} = deg(\varphi^*) = -\sum_{i|h} (h/i)\cdot e(i), \\ \text{dual discriminant:} & d^*_\varphi := d_{\varphi^*} = \prod_{i|h} (h/i)^{-e(i)}. \end{cases}$$

The next numerical relations follow immediately from the definitions.

$$a_0 + a_0^* = 0. \tag{2.2.3}$$

$$d_\varphi = h^{2a_0} \cdot d_\varphi^* \tag{2.2.4}$$

This implies $d_{sf} = d_{sf}^*$ if $2a_0$ is an even integer, and $lcm(d_{sf}, d_{sf}^*)/gcd(d_{sf}, d_{sf}^*)$ is equal to the square free part of h if $2a_0$ is an odd integer. Sometimes, it is convenient to use a notion of a cyclotomic function $\hat{\varphi} := (\varphi^*)^{-1}$ instead of the dual function φ^*:

$$\hat{\varphi}(\lambda) := \prod_{i|h} (\lambda^{h/i} - 1)^{e(i)} \tag{2.2.5}$$

Numerical invariants (1.1.2), (2.1.1) and (2.1.2) attached to $\hat{\varphi}$ are $\hat{a}_0 = a_0$, $\mu_{\hat{\varphi}} = -\nu_\varphi$ and $d_{\hat{\varphi}} = (d_\varphi^*)^{-1}$ (hence $\hat{d}_{sf} = d_{sf}^*$). The attached eta-product is

$$\hat{\eta}_\varphi(\tau) := \eta_{\hat{\varphi}}(\tau) = \big(\eta_\varphi^*(\tau)\big)^{-1} = \prod_{i|h} \eta((h/i)\tau)^{e(i)}. \tag{2.2.6}$$

(2.3) The duality between φ and φ^* are equivalent to (cf [Sa3, (13.3)])

$$\begin{aligned} \eta_\varphi(-1/h\tau) \cdot \eta_\varphi^*(\tau) &= (\tau/\sqrt{-1})^{a_0}/\sqrt{d_\varphi^*}, \\ \eta_\varphi(\tau) \cdot \eta_\varphi^*(-1/h\tau) &= (\sqrt{-1}/\tau)^{a_0}/\sqrt{d_\varphi}, \end{aligned} \tag{2.3.1}$$

(use the fact: $\eta(-/\tau) = \sqrt{\tau/\sqrt{-1}}\eta(\tau)$). So, we obtain a formula for $\hat{\eta}_\varphi$ in terms of η_φ:

$$\hat{\eta}_\varphi(\tau) = \sqrt{d_\varphi^*}\ \eta_\varphi(-1/h\tau)(\tau/\sqrt{-1})^{-a_0}. \tag{2.3.2}$$

(2.4) We recall automorphic forms of half-integral weights [Sh]. We fix notation according to [Kob]. For an odd integer d and any integer c, let us define the residue symbol $\left(\frac{c}{d}\right)$ as follows. If $d > 0$ and is prime it is the Legendre symbol. It is extended for all odd $d > 0$ multiplicatively. If $d < 0$ then $\left(\frac{c}{d}\right) = \left(\frac{c}{|d|}\right)$ for $d < 0$ and $c > 0$ and $\left(\frac{c}{d}\right) = -\left(\frac{c}{|d|}\right)$ for $d < 0$ and $c < 0$. Put $\left(\frac{0}{\pm 1}\right) := 1$. For odd d, we define also

$$\epsilon_d := \begin{cases} 1 & if \quad d \equiv 1 \quad mod \quad 4; \\ \sqrt{-1} & if \quad d \equiv 3 \quad mod \quad 4. \end{cases} \tag{2.4.1}$$

So $\epsilon_d = \sqrt{\left(\frac{-1}{d}\right)}$. Recall that $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \bmod N \right\}$ for a positive integer N. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $\tau \in \mathbf{H}$, put

(2.4.2) $$j(\tilde{A},\tau) := \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{c\tau+d}.$$

In particular, $j(\tilde{A},\tau)=1$ for $A=\begin{pmatrix}-1 & b\\ 0 & -1\end{pmatrix}$ and $b\in\mathbf{Z}$. It is known ([Kob, ch.IV]) that they satisfy the cocycle condition: $j(\tilde{AB},\tau)=j(\tilde{A},\tau)j(\tilde{B},\tau)$ for $A,B\in\Gamma_0(4)$.

Let $2k, N\in\mathbf{Z}_{>0}$ and assume that $N\equiv 0$ mod 4 if $2k$ is odd. Let ε be a Dirichlet character mod N such that $\varepsilon(-1)=1$ or $(-1)^k$ according as $2k$ is odd or even. A holomorphic function $f(\tau)$ on the complex upper half plane $\mathbf{H}$ is called a *weakly holomorphic automorphic form of type* (k,ε) *and of level* N, if for all $A\in\Gamma_0(N)$ one has:

(2.4.3) $$f|_k\tilde{A}(\tau) := j(\tilde{A},\tau)^{-2k}f\left(\frac{a\tau+b}{c\tau+d}\right)=\varepsilon(d)f(\tau) \qquad \text{if} \quad 2k \quad \text{is} \quad \text{odd},$$

(2.12) $$f|_kA(\tau) := (c\tau+d)^{-k}f\left(\frac{a\tau+b}{c\tau+d}\right)=\varepsilon(d)f(\tau) \qquad \text{if} \quad 2k \quad \text{is} \quad \text{even}.$$

(2.5) We state a lemma which determine the type and level of eta-products (c.f. [H-M, theorem 1],[Ha, (3.6) Theorem] for even $2a_0$).

Lemma 1. *Let a cyclotomic function $\varphi(\lambda)$ (1.1.1) and its dual $\varphi^*(\lambda)$ (2.2.1) be given. Let $\eta_\varphi(\tau)$ (1.1.3) and $\eta^*_\varphi(\tau)$ (2.2.1) be the attached eta-product and dual eta-product. Then $\eta_\varphi(m_\varphi\tau)$ (resp. $1/\eta^*_\varphi(m^*_\varphi\tau)$) is a weakly holomorphic (i.e. holomorphic except at cusps) automorphic form of type $(a_0,\varepsilon_\varphi)$ (resp. $(a_0,\varepsilon^*_\varphi)$) on the group $\Gamma_0(N_\varphi)$. Here*

(2.5.1) $$m_\varphi := 24/gcd(24,\mu_\varphi), \quad m^*_\varphi := 24/gcd(24,\nu_\varphi),$$
$$N_\varphi := hm_\varphi m^*_\varphi,$$

and ε_φ and ε^*_φ are Dirichlet characters mod N_φ given as follows. If $2a_0$ is even,

(2.5.2) $$\varepsilon_\varphi(d)=\varepsilon^*_\varphi(d)=\begin{cases}\left(\dfrac{d_{sf}(-1)^{a_0}}{d}\right) & \textit{for } d \textit{ odd},\\ \left(\dfrac{d}{d_{sf}}\right) & \textit{for } d \textit{ even}.\end{cases}$$

If $2a_0$ is odd, then $N_\varphi\equiv 0$ mod 4. Then the characters ε_φ and ε^*_φ mod N_φ are given by

(2.5.3) $$\begin{cases}\varepsilon_\varphi(d)=\left(\dfrac{2m_\varphi d_{sf}}{d}\right) & \textit{for } d \textit{ odd},\\ \varepsilon^*_\varphi(d)=\left(\dfrac{2m^*_\varphi d^*_{sf}}{d}\right) & \textit{for } d \textit{ odd}.\end{cases}$$

Proof. We prove the rules (2.4.3) or (2.4.4) only for $f(\tau) := \eta_\varphi(m_\varphi \tau)$, since similar rule holds for $\eta^*_\varphi(m^*_\varphi \tau)$ by duality, and then one gets the rule for $\hat{\eta}_\varphi(m^*_\varphi\tau) = 1/\eta^*_\varphi(m^*_\varphi\tau)$. Note that the definitions of m_φ and m^*_φ (2.5.1) imply:

$$(2.5.4) \qquad m_\varphi \cdot \mu_\varphi \equiv 0 \mod 24, \quad m^*_\varphi \cdot \nu_\varphi \equiv 0 \mod 24.$$

Recall a transformation rule for $\eta(\tau)$ ([Ra, p.163]). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ with $c > 0$. Then one has $\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)\sqrt{\frac{c\tau+d}{\sqrt{-1}}}\eta(\tau)$ with

$$\varepsilon(a,b,c,d) = \begin{cases} \left(\dfrac{d}{c}\right) exp\left(\dfrac{\pi\sqrt{-1}(1-c)}{4}\right) exp\left(\dfrac{\pi\sqrt{-1}}{12}\big(c(a+d-bcd)+bd\big)\right) & \text{if } c \text{ odd,} \\ \left(\dfrac{c}{d}\right) exp\left(\dfrac{\pi\sqrt{-1}d}{4}\right) exp\left(\dfrac{\pi\sqrt{-1}}{12}\big(c(a-d-ad^2)+bd\big)\right) & \text{if } d \text{ odd.} \end{cases}$$

Case A. $2a_0 \equiv 0 \bmod 2$.

One has to show $f(\tau)|_{a_0} A = \varepsilon_\varphi(d) f(\tau)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_\varphi)$.

If $c = 0$, then $a = d = \pm 1$, and so $(c\tau+d)^{a_0} = (\pm 1)^{a_0} = \varepsilon_\varphi(d)$. Hence, $f(\tau)|_{a_0}A = (\pm 1)^{a_0}\prod_{i|h}\eta(mi(\tau \pm b))^{e(i)} = \varepsilon_\varphi(\pm 1) f(\tau) exp(\pm\frac{\pi\sqrt{-1}}{12} bm_\varphi \sum_{i|h} ie(i))$ where the last exponential factor reduces to 1 due to (2.5.4).

Let $c \neq 0$. Since $\varepsilon_\varphi(-1) = (-1)^{a_0}$, by replacing A by $-A$ if necessary, we may assume $c > 0$. Put $c = km_\varphi m^*_\varphi h$ for $k \in \mathbf{Z}_{>0}$. Since $(c,d) = 1$, we separate two cases.

Case A.1. $d \equiv 1 \bmod 2$

$$\begin{aligned} f|_{a_0}A(\tau) &= \prod_{i|h}\Big\{\eta\Big(\frac{am_\varphi i\tau + m_\varphi ib}{(km_\varphi m^*_\varphi h/m_\varphi i)m_\varphi i\tau + d}\Big)\sqrt{km_\varphi m^*_\varphi h\tau + d}^{\,-1}\Big\}^{e(i)} \\ &= \prod_{i|h}\Big\{\varepsilon(a, bm_\varphi i, km^*_\varphi h/i, d)\sqrt{-1}^{\,-1/2}\eta(m_\varphi i\tau)\Big\}^{e(i)} \\ &= f(\tau)\prod_{i|h}\Big(\Big(\frac{km^*_\varphi h/i}{d}\Big)exp\Big(\frac{\pi\sqrt{-1}(d-1)}{4}\Big)\Big)^{e(i)} \\ &\quad exp\Big(\frac{\pi\sqrt{-1}}{12}\Big((a-d-ad^2)\sum_{i|h}e(i)km^*_\varphi h/i + dbm_\varphi\sum_{i|h}ie(i)\Big)\Big). \end{aligned}$$

Due to (2.5.4), the last exponential factor reduces to 1. Recalling the definitions of the genus a_0, dual discriminant d^*_φ and its square free part d^*_{sf}, we get easily

$$= f(\tau)\Big(\frac{km^*_\varphi}{d}\Big)^{2a_0}.\Big(\frac{d^*_{sf}}{d}\Big)exp\Big(\frac{\pi\sqrt{-1}(d-1)2a_0}{4}\Big).$$

Noting that $2a_0$ is an even integer, we get

$$= f(\tau)(\frac{d_{sf}}{d})exp(\frac{\pi\sqrt{-1}(d-1)a_0}{2}).$$

This is the formula to be proven.

Case A.2. $c \equiv 1 \bmod 2$
By assumption, $c/m_\varphi i = km^*_\varphi h/i$ is also an odd integer. Therefore, one proceeds:

$$\begin{aligned}
f|_{a_0}A(\tau) &= \prod_{i|h}\Big\{\eta\Big(\frac{am_\varphi i\tau + m_\varphi ib}{(km_\varphi m^*_\varphi h/m_\varphi i)m_\varphi i\tau + d}\Big)\sqrt{km_\varphi m^*_\varphi h\tau + d}^{\,-1}\Big\}^{e(i)} \\
&= \prod_{i|h}\{\varepsilon(a, bm_\varphi i, km^*_\varphi h/i, d)\sqrt{-1}^{\,-1/2}\eta(m_\varphi i\tau)\}^{e(i)} \\
&= f(\tau)\prod_{i|h}\Big((\frac{d}{km^*_\varphi h/i})exp(-\frac{\pi\sqrt{-1}km^*_\varphi h/i}{4})\Big)^{e(i)} \\
&\quad exp\Big\{\frac{\pi\sqrt{-1}}{12}\Big((a+d-bcd)\sum_{i|h} e(i)km^*_\varphi h/i + dbm_\varphi \sum_{i|h} ie(i)\Big)\Big\}.
\end{aligned}$$

Due to (2.5.4), the last exponential factor reduces to 1. Recalling the definitions of the genus a_0, dual discriminant d^*_φ and its square free part d^*_{sf}, we get easily

$$= f(\tau)(\frac{d}{km^*_\varphi})^{2a_0}(\frac{d}{d^*_{sf}})exp(\frac{\pi\sqrt{-1}km^*_\varphi\nu}{4}).$$

Noting that $2a_0$ is an even integer and that $24|m^*_\varphi\nu$, we get

$$= f(\tau)(\frac{d}{d_{sf}}).$$

Case B. $2a_0 \equiv 1 \bmod 2$.
One has to show $f|_{a_0}\tilde{A}(\tau) = \varepsilon_\varphi(d)f(\tau)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_\varphi)$.

Let us prove $N_\varphi \equiv 0\ (4)$. If $h \equiv 0\ (4)$ then there is nothing to prove. If $2\not| h$, then $i \equiv h/i \equiv 1 \bmod 2$ for all i with $e(i) \neq 0$, and hence, by definitions, $\mu_\varphi \equiv \nu_\varphi \equiv 2a_0 \equiv 1 \bmod 2$. So, $m_\varphi \equiv m^*_\varphi \equiv 0\ (8)$. If $2|h$ and $4\not| h$, then $i + h/i \equiv 1$ for all i with $e(i) \neq 0$, and hence, by definitions, $\mu_\varphi + \nu_\varphi \equiv 2a_0 \equiv 1 \bmod 2$. So, either $m_\varphi \equiv 0\ (8)$ or $m^*_\varphi \equiv 0\ (8)$.

Since $4|N_\varphi|c$, c is even and hence d is odd. If $c = 0$, then $a = d = \pm 1$, and so $j(\tilde{A}, \tau) = 1 = \varepsilon_\varphi(d)$. Hence, $f(\tau)|_{a_0}\tilde{A} = \prod_{i|h}\eta(m_\varphi i(\tau \pm b))^{e(i)} = f(\tau)exp(\pm_\varphi \frac{\pi\sqrt{-1}}{12}bm\sum_{i|h} ie(i))$ where the last exponential factor reduces to 1 due to (2.5.4).

Let $c \neq 0$. Since $\varepsilon_\varphi(-1) = 1$, by replacing A by $-A$ if necessary, we may assume $c > 0$. Put $c = km_\varphi m^*_\varphi h$ for $k \in \mathbf{Z}_{>0}$.

$$\begin{aligned}
f|_{a_0}\tilde{A}(\tau) &= \prod_{i|h}\Big\{\eta\Big(\frac{am_\varphi i\tau + m_\varphi ib}{(km_\varphi m^*_\varphi h/m_\varphi i)m_\varphi i\tau + d}\Big)\Big(\frac{c}{d}\Big)^{-1}\epsilon_d\sqrt{km_\varphi m^*_\varphi h\tau + d}^{\,-1}\Big\}^{e(i)} \\
&= \prod_{i|h}\{\varepsilon(a, bm_\varphi i, km^*_\varphi h/i, d)\sqrt{-1}^{\,-1/2}\Big(\frac{c}{d}\Big)^{-1}\epsilon_d\eta(m_\varphi i\tau)\}^{e(i)} \\
&= f(\tau)\prod_{i|h}\Big((\frac{c/m_\varphi i}{d})exp(\frac{\pi\sqrt{-1}(d-1)}{4})\Big(\frac{c}{d}\Big)^{-1}\epsilon_d\Big)^{e(i)} \\
&\quad exp\Big\{\frac{\pi\sqrt{-1}}{12}\Big((a-d-ad^2)\sum_{i|h}e(i)km^*_\varphi h/i + dm_\varphi\sum_{i|h}ie(i)\Big)\Big\}.
\end{aligned}$$

Due to (2.5.4), the last exponential factor reduces to 1. A direct calculation shows that $exp\big(\frac{\pi\sqrt{-1}(d-1)}{4}\big)\cdot\epsilon_d = (-1)^{(d^2-1)/8} = \big(\frac{2}{d}\big)$. Recalling the definitions of the genus a_0, discriminant d_φ and its square free part d_{sf}, we get easily

$$= f(\tau)(\frac{m_\varphi}{d})^{2a_0}(\frac{d_{sf}}{d})(\frac{2}{d})^{2a_0}.$$

Since $2a_0$ is odd, this is the formula to be proven. Q.E.D.

(2.6) **Remark.** 1. Formula (2.3.1) is reformulated as

$$\hat{\eta}_\varphi(m^*_\varphi\tau) = \left(\frac{m_\varphi}{m^*_\varphi}\right)^{a_0/2}\frac{\sqrt{D}}{\sqrt{-1}^{a_0}}\eta_\varphi(m_\varphi\cdot)|_{a_0}\sigma(\tau), \tag{2.6.1}$$

where $D := h^{-a_0}d_\varphi = h^{a_0}d_{\varphi^*}$, $\sigma := \begin{pmatrix} 0 & 1 \\ -N_\varphi & 0\end{pmatrix}$ and $f(m^*_\varphi\cdot)$ means the function $f(m^*_\varphi\tau)$.

(2.7) If φ is *non-reduced* (see definition below), then one needs a slight careful treatment of the eta product η_φ, on which we discuss in this paragraph. The goal results are given in **Facts**, which are used in §3 and 4. For a given cyclotomic function φ (1.1.1), put

$$g := gcd\{i \mid e(i) \neq 0\} \quad \text{and} \quad g^* := h/lcm\{i \mid e(i) \neq 0\}.$$

We say φ is *reduced* if $g = 1$ and *dual-reduced* if $g^* = 1$. So, φ is reduced (resp. dual-reduced) iff φ^* is dual-reduced (resp. reduced). We say φ is *non-reduced* if $g \neq 1$.

Let us define the reduction φ^{red} of φ with the reduced Coxeter number h^{red} by

$$\varphi^{red}(\lambda) := \prod_{i|h}(\lambda^{i/g}-1)^{e(i)} \quad \text{and} \quad h^{red} := h/g. \tag{2.7.1}$$

Then $(\varphi^{red})^* = \varphi^*$ is dual-reduced with respect to the Coxeter number h^{red}. One has the relations: $\eta_\varphi(\tau) = \eta_{\varphi^{red}}(g\tau)$ and $\eta^*_\varphi(\tau) = \eta^*_{\varphi^{red}}(\tau)$. The numerical invariants are changed as: $a_0^{red} = a_0$, $\mu^{red}_\varphi = \mu_\varphi/g$, $\nu^{red}_\varphi = \nu_\varphi$. Therefore,

$$m^{red}_\varphi := 24/gcd(24,\mu^{red}) \quad \text{and} \quad m^{*red}_\varphi := 24/gcd(24,\nu^{red}) = m^*_\varphi. \tag{2.7.2}$$

satisfies: $m_\varphi^{red}/(m_\varphi^{red}, g) = m_\varphi$ and $m_\varphi^{red} h^{red} \mid m_\varphi h$. In particular, one has:

$$\eta_\varphi(m_\varphi \tau) = \eta_{\varphi^{red}}((g/gcd(m_\varphi^{red}, g))m_\varphi^{red}\tau). \tag{2.7.3}$$

Let us summarize some **Facts**, which will be used in §3 and 4 (for a proof of Fact 2, use Tables 1 and 2 and their following Facts. For a proof of (2.7.4), use (2.7.3)).

Fact 1. *The next 4 conditions are equivalent:* i) $g \mid m_\varphi^{red}$, ii) $m_\varphi^{red} h^{red} = m_\varphi h$, iii) $N_\varphi = N_{\varphi^{red}}$ and iv) $\eta_\varphi(m_\varphi\tau) = \eta_{\varphi^{red}}(m_\varphi^{red}\tau)$. We shall call such φ *tamely non-reduced.*

2. *The characteristic polynomial of an elliptic root system is either reduced or tamely non-reduced.*

3. *Let φ be reduced or tamely non-reduced. Suppose $p|m_\varphi^{red}$ for a prime p, then*

$$\eta_\varphi(m_\varphi\tau)|U_p = 0 \ , \tag{2.7.4}$$

where U_p is one of the Hecke operators acting on formal power series in q given by $(\sum c(n)q^n)|U_p := \sum c(pn)q^n$ (cf. [Kob, 5.12]). This implies that the p-th Euler factor (if it exists) for the Dirichlet series attached to $\eta_\varphi(m_\varphi\tau)$ is equal to 1.

(2.8) We analyze Fourier expansions of eta-products at a *cusp point* $a/c \in \mathbf{Q} \cup \{\infty\}$ for $a, c \in \mathbf{Z}$ with $(a, c) = 1$. Recall ([O],[Kob]) that a weakly holomorphic automorphic form f of weight k is *holomorphic* (resp. *vanishing*) at the cusp if the expansion of $f|_k A(\tau)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ in the powers of $q := exp(2\pi\sqrt{-1}\tau)$ has only non-negative (resp. positive) exponents. The f is called a cusp form if it is vanishing at all cusp points.

For a given φ (1.1.1), put $\Phi_\varphi(\xi) = \Phi(\xi) := \sum_{i|h} h\frac{(i,\xi)^2}{i\xi,} e(i)$ for $\xi \in \mathbf{Z}$. The $\xi \cdot \Phi(\xi)$ depends only on ξ *mod* h. Recall Definitions (2.1.1) and (2.2.2) so that one has $\Phi(h) = \mu_\varphi$ and $\Phi(1) = -\nu_\varphi$. The Fourier expansion of the eta-product (1.1.3) at a/c for $a, c \in \mathbf{Z}$ with $(a, c) = 1$ starts with the term $q^{c\cdot\Phi(c)}$ for $q := exp(2\pi\sqrt{-1}\tau/24mh)$ (eg. [H-M][Sa3]). Hence one obtains:

Lemma 2. *The eta-product $\eta_\varphi(\tau)$ is holomorphic (resp. vanishing) at a cusp point a/c for $a, c \in \mathbf{Z}$ with $gcd(a, c) = 1$ if and only if $c \cdot \Phi_\varphi(c) \geq 0$ (resp. > 0). In particular,*

$$\textit{the eta product } \eta_\varphi \textit{ is } \begin{cases} \textit{holomorphic} \begin{cases} \textit{at the } \infty & \textit{if and only if} \quad \mu_\varphi \geq 0 \ , \\ \textit{at cusps in } \mathbf{Z} & \textit{if and only if} \quad \nu_\varphi \leq 0 \ , \end{cases} \\ \textit{vanishing} \begin{cases} \textit{at the } \infty & \textit{if and only if} \quad \mu_\varphi > 0 \ , \\ \textit{at cusps in } \mathbf{Z} & \textit{if and only if} \quad \nu_\varphi < 0 \ . \end{cases} \end{cases}$$

The proof of the next lemma given here is due to Borcherds. This simplifies the original version of the present article, where one proved case by case for particular eta-products.

Lemma 3. *If all Fourier coefficients at ∞ of η_φ are non-negative, then $\nu_\varphi \geq 0$.*

Proof. Suppose $\nu_\varphi < 0$. Then, the Fourier expansion of η_φ at 0 implies: $\lim_{t\to 0} \eta_\varphi(\sqrt{-1}t) = 0$. On the other hand, the Fourier expansion $\sum_n c(n)q^n$ at ∞ implies: $\lim_{t\to 0} \eta_\varphi(\sqrt{-1}t) = \lim_{s\to 1} \sum_n c(n)s^n$, which cannot be 0 if all $c(n)$ are non-negative. q.e.d.

Remark. 1. One has $\Phi(\xi) = \Phi((\xi, h))$. Therefore, an eta-product is holomorphic (resp. cuspidal) if and only if $\Phi(c) \geq 0$ (resp. > 0) for $c \in \mathbf{Z}_{>0}$ with $c|h$.

2. For $k \in \mathbf{Z}_{>0}$, one has $\Phi_{\varphi_k}(\xi) = \frac{(k,\xi)}{k}\Phi_\varphi(\frac{\xi}{(k,\xi)})$ where $\varphi_k(\lambda) := \varphi(\lambda^k)$. Therefore, an eta-product η_φ is holomorphic or cuspidal if and only if $\eta_{\varphi^{red}}$ is so.

3. One has: $\Phi_\varphi(h/\xi) = \Phi_{\hat{\varphi}}(\xi) = -\Phi_{\varphi^*}(\xi)$ for $\xi|h$. This implies that η_φ is holomorphic or cuspidal if and only if $\hat{\eta}_\varphi$ is so.

4. The following sharpening of Lemma 3. gives constraint on the Fourier coefficients of $\eta_\varphi(m\tau) = \sum_{n\in\mathbf{Z}} c(n)q^n$ at ∞, but, we shall not use this fact in the present article.

$$\sum_{n\in\mathbf{Z}} c(n)e^{-VT} \sim O\Big(T^{-a_0}exp(\nu_\varphi/\,(24m_\varphi h_\varphi T))\Big) \quad (T\downarrow 0)\ . \tag{2.8.1}$$

Here the major part of the right hand side is the second exponential factor whose exponent $-\nu_\varphi/24m_\varphi h_\varphi$ is equal to $-n/N_\varphi$ where $-n$ is the leading degree of $\eta^*_\varphi(m^*_\varphi\tau)$ in $q = exp(2\pi\sqrt{-1}\tau)$ and N_φ is the level given in (2.5.1).

(2.9) Let us return to the study of elliptic eta-products. First, from Lemma 2, one obtains a numerical criterion for elliptic eta-products to be holomorphic or cuspidal as follows.

Lemma 4. *An elliptic eta-product $\eta_{(R,G)}$ is holomorphic (resp. cuspidal), if and only if the dual rank $\nu_{(R,G)} := \nu_{\varphi_{(R,G)}}$ is non-positive (resp. negative).*

Proof. One has to show that $\Phi(\xi) \geq 0$ (resp. > 0) for all $\xi \in \mathbf{Z}_{>0}$ with $\xi|m(R,G)$ if and only if $\nu_{(R,G)} \leq 0$ (resp. < 0). Since $\Phi_\varphi(1) = -\nu_\varphi$, the condition is necessary. Since $\Phi_\varphi(h) = \mu_\varphi =: \mu_{(R,G)} = l + 2 > 0$, the condition is already sufficient if the Coxeter number $h := m(R,G)$ is a prime number (recall the above Remark 1). These cover almost all types of elliptic root systems except for the types $E_7^{(1,1)}$, $E_8^{(1,1)}$, $F_4^{(1,2)}$ and $F_4^{(2,1)}$ where teh Coxeter number $m(R,G)$ is 4, 6, 4 and 4, respectively (see Table 2). In view of Remark 2 after Lemma 2, the proof for the types $F_4^{(1,2)}$ and $F_4^{(2,1)}$ are done already. For the proof of remaining cases, the following calculations are sufficient.

$\Phi_{E_7^{(1,1)}}(2) = 4\frac{2^2}{2\cdot 4}2 + 4\frac{2^2}{2\cdot 2} - 4\frac{1}{2\cdot 1} = 6 > 0,$

$\Phi_{E_8^{(1,1)}}(2) = 6\frac{2^2}{2\cdot 6} + 6\frac{1^2}{2\cdot 3} + 6\frac{2^2}{2\cdot 2} - 6\frac{1}{2\cdot 1} = 6 > 0,$

$\Phi_{E_8^{(1,1)}}(3) = 6\frac{3^2}{3\cdot 6} + 6\frac{3^2}{3\cdot 3} + 6\frac{1^2}{3\cdot 2} - 6\frac{1}{3\cdot 1} = 8 > 0.$ Q.E.D.

In fact, we shall prove the next lemma in Appendix 3.

Lemma 5. i) *For all elliptic root system (R,G), one has $\nu_{(R,G)} \leq 0$.*

ii) *The equality $\nu_{(R,G)} = 0$ holds if and only if the root system (R,G) is (a) 1-codimensional and (b) simply laced. For definitions of simply lacedness and 1-codimensionality, see Appendix 1 and its Example.*

As a result of Lemmas 4 and 5 and the classification (Table 2), one obtains:

Corollary. *All elliptic eta-products are holomorphic. An elliptic eta-product is not cuspidal if and only if the root system is one of types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$.*

Appendix 3. Rank, dual-rank and σ-rank of an elliptic root system

In this Appendix, we give a proof of Lemma 5. The proof is achieved by introducing a numerical invariant, called σ-rank, and proving its invariance by foldings and mean foldings of elliptic diagrams (Lemma 6). At the end of Appendix 3, we give Table 2 of rank, dual-rank, σ-rank, conductors and discriminants for elliptic root systems.

First, let us define the σ-rank. For a characteristic function $\varphi(\lambda)$ (1.1.1) with non-vanishing genus a_0, we introduce the σ-rank by

$$\sigma_\varphi := (\mu_\varphi - \nu_\varphi)/a_0, \qquad (A3.1)$$

where μ_φ, ν_φ and a_0 are the rank (2.1.1), dual rank (2.2.2) and genus (1.1.2) for φ. One easily checks that the σ-rank has the stability:

$$\sigma_\varphi \;=\; \sigma_{\hat{\varphi}} \;=\; \sigma_{\varphi^*}.$$

For an elliptic root system (R,G), we define the elliptic σ-rank $\sigma_{(R,G)} := \sigma_{\varphi_{(R,G)}}$. The next Lemma 6 ii) on elliptic σ-rank will be used in a proof of Lemma 5 ii) b).

Lemma 6. i) *The σ-rank for any elliptic root system is a positive integer.*

ii) *The σ-rank does not change by a folding or mean folding of an elliptic diagram, where the (mean) foldings are defined below.*

We shall give proofs of Lemmas 5 and 6 simultaneously.
First, we recall the definition of foldings and mean foldings of elliptic root systems ([Sa1,I,§12]). Assume that (R,G) is neither of the types $A_1^{(1,1)*}$, $B_l^{(2,2)*}$, $C_l^{(1,1)*}$, $BC_l^{(2,2)}(1)$ nor $BC_l^{(2,2)}(2)$. Let $\Gamma_{aff} \subset R$ be, as in Appendix 1, a lifting of a simple basis of $R_a := R/G$, which we identify with the affine diagram for R_a. An automorphism h of the affine diagram Γ_{aff} can be lifted to an automorphism of the elliptic diagram $\Gamma(R,G)$ (A1.3) and extended to an automorphism $r(h) \in Aut(R, rad(q))$ of the elliptic root system such that $r(h)|\Gamma_{aff} = h$, since h preserves the exponents (A1.1). In particular, the action on $\Gamma(R,G)$ preserves the maximal exponent part Γ_{max} (A1.2) and permutates the components of $\Gamma_{aff} \setminus \Gamma_{max}$. Let H be a subgroup of $Aut(\Gamma_{aff})$ such that there exists at least a vertex of Γ_{aff} which is fixed by H. For any $\alpha \in R$, put $Tr^H(\alpha) := \sum_{\beta \in H\alpha} \beta \in F^H$ and $Tr_H(\alpha) := (\#H\alpha)^{-1} \sum_{\beta \in H\alpha} \beta \in F^H$, where F^H is the H-fixed point subspace of $F = Q(R) \otimes \mathbf{R}$.

Lemma-Definition([Sa1,I,(12.3)]) The image sets $Tr^H(\Gamma(R,G))$ and $Tr_H(\Gamma(R,G))$ in F^H of the simple basis $\Gamma(R,G)$ form simple basis of certain marked elliptic root systems (R^H, G) and (R_H, G) belonging to F^H, respectively. We call $\Gamma(R^H, G) := Tr^H(\Gamma(R,G))$ and $\Gamma(R_H, G) := Tr_H(\Gamma(R,G))$ the *folding* and *mean folding* of $\Gamma(R,G)$, respectively.

Proof of Lemmas 5 and 6. As in Appendix 1, let $\Gamma(A_{l_1}), \ldots, \Gamma(A_{l_r})$ be the components of $\Gamma_{aff} \setminus \Gamma_{max}$. Put $c := codim(R,G)$. Using the description (A1.6) of exponents on $\Gamma(A_{l_i})$ $(1 \le i \le r)$, we obtain descriptions of the characteristic polynomial (A1.7) and its rank:

$$\varphi_{(R,G)}(\lambda) := (\lambda - 1)^{c+1} \prod_{i=1}^{r} \left((\lambda^{l_i+1} - 1)/(\lambda - 1) \right) \qquad (A3.2)$$

$$(A3.3) \qquad \mu_{(R,G)} := \mu_{\varphi_{(R,G)}} = deg(\varphi_{(R,G)}) = 1 + \#(\Gamma_{aff}) = 1 + c + \sum_{i=1}^{r} l_i \ .$$

Putting $m := m(R,G)$ (A1.5) (which should not be confused with $m_{(R,G)}$ (2.5.1)), we get the dual characteristic function and the dual rank:

$$(A3.4) \qquad \varphi^*_{(R,G)}(\lambda) := (\lambda^m - 1)^{-(c+1)} \prod_{i=1}^{r} \Big((\lambda^m - 1)/(\lambda^{m/(l_i+1)} - 1) \Big)$$

$$(A3.5) \qquad \nu_{(R,G)} := \mu_{\varphi^*_{(R,G)}} = deg(\varphi^*_{(R,G)}) = -\sum_{i=1}^{r} \left(\frac{m}{l_i + 1} - m \right) - (c+1)m$$

Let us show i) the non-positivity: $\nu_{(R,G)} \le 0$, and ii) $\nu_{(R,G)} = 0$ implies $c = 1$.

If $r = 0$ then clearly $\nu_{(R,G)} = -(c+1)m < 0$. For $r > 0$ rewrite (A3.5) as $\nu_{(R,G)} = -\sum_{i=1}^{r} m/(l_i+1) - (c+1-r)m$. The first term is always strictly negative. The second term becomes positive when $r > c+1 \ge 2$. This is the cases when $r = 3$ and $c = 1$, or when $r = 4$ and $c \le 2$. In the first case, there are 3 types of elliptic diagrams: $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. Then the equality $\nu_{(R,G)} = m(1 - \sum_{i=1}^{3} 1/(l_i+1)) = 0$ is trivial and well-known for the three. In the latter case of $r = 4$, the diagrams are of type $D_l^{(1,1)}$ ($l \ge 4$). Then one has $m = 2$ and $c = l - 3$. Thus, $\nu_{(R,G)} = 2(1-c)$, which is non-positive and is equal to 0 only if $c = 1$ when (R,G) is of type $D_4^{(1,1)}$. Thus, Lemma 5 i) and ii) a) are proven.

Before Lemma 5 ii) b), we prove Lemma 6. Using (A1.8), (A3.3) and (A3.5), one gets:

$$(A3.6) \qquad \sigma_{(R,G)} = (a_0)^{-1} \sum_{i=1}^{r} \left(l_i + \frac{m}{(l_i+1)} - m \right) + 2(m+1) \ .$$

Let us calculate the contribution of the branch $\Gamma(A_{l_i})$ in the first term of (A3.6). Since $(l_i+1)|m$ (cf. (A1.5)) and $m = 1, 2, 3, 4$ or 6, one can list all cases easily as follows.

i) Let $m = 1, 2, 3, 4$ or 6 and $l_i + 1 = m$. Then, one has: $l_i + m/(l_i+1) - m = 0$. Let us call this case a longest branch. So, a longest branch gives no-contribution to $\sigma_{(R,G)}$.

ii) Let $m = 4$ and $l_i = 1$. Then, one has $c = 1$ and $l_i + m/(l_i+1) - m = -1$.

iii) Let $m = 6$ and $l_i = 1$. Then, one has $c = 1$ and $l_i + m/(l_i+1) - m = -2$.

iv) Let $m = 6$ and $l_i = 2$. Then, one has $c = 1$ and $l_i + m/(l_i+1) - m = -2$.

Since the number of non-longest branches is at most 2, we obtain i) of Lemma 6.

Let us consider (mean) folding by a group H. By the action of H on $\Gamma(R,G)$, the longest branches are permuted, so the Coxeter number m (A1.5) is unchanged. On the other hand, the set of non-longest branches are fixed by H. Therefore, in view of the formula (A3.6) and above fact i), the σ-rank is unchanged by the folding. This proves the ii) of Lemma 6.

Let us prove ii) b) of Lemma 5: to prove that $\nu_{(R,G)} = 0$ implies $\Gamma(R,G)$ is simply laced. Suppose not. This implies that the $\Gamma(R,G)$ can be realized as a non-trivial (mean) folding of another elliptic diagram, say $\Gamma(\hat{R},\hat{G})$, where we may assume $c = codim(R,G) = codim(\hat{R},\hat{G}) = 1$ and $\mu_{(R,G)} < \mu_{(\hat{R},\hat{G})}$ ([Sa1,I, (12.5)]). Then the equality $\sigma_{(R,G)} = \sigma_{(\hat{R},\hat{G})}$ implies $\nu_{(R,G)} - \nu_{(\hat{R},\hat{G})} = \mu_{(R,G)} - \mu_{(\hat{R},\hat{G})} < 0$. This is a contradiction to $\nu_{(\hat{R},\hat{G})} \le 0$.

These complete the proofs of Lemmas 5 and 6. Q.E.D.

Table 2. rank, dual-rank, σ-rank, conductors and discriminants

Type	$\mu_{(R,G)}$	$\nu_{(R,G)}$	$\sigma_{(R,G)}$	m, m^{red}, m^*	$N_{(R,G)}$	$d_{(R,G)}$	$d^*_{(R,G)}$
$A_l^{(1,1)}$ $l \geq 1)$	$l+2$	$-l-2$	4			1	1
$A_1^{(1,1)*}$	3	-3	6	8, 8, 8	128	2	1/2
$B_l^{(1,1)}$ $(l \geq 3)$	$l+2$	$-2l+5$	6			2^3	2^{4-l}
$B_3^{(1,1)}$	5	-1	6	24, 24, 24	1152	2^3	2
$B_l^{(1,2)}$ $(l \geq 2)$	$l+2$	$-2l+2$	6			2^2	2^{2-l}
$B_2^{(1,2)}$	4	-2	6	6, 12, 12	144	2^2	1
$B_l^{(2,1)}$ $(l \geq 2)$	$l+2$	$-2l+2$	6			2^2	2^{2-l}
$B_2^{(2,1)}$	4	-2	6	6, 12, 12	144	2^2	1
$B_l^{(2,2)}$ $(l \geq 2)$	$l+2$	$-l-2$	4			1	1
$C_l^{(1,1)}$ $(l \geq 2)$	$l+2$	$-l-2$	4			1	1
$C_l^{(1,2)}$ $(l \geq 2)$	$l+2$	$-2l+2$	6			2^2	2^{2-l}
$C_2^{(1,2)}$	4	-2	6	6, 12, 12	144	2^2	l
$C_l^{(2,1)}$ $(l \geq 2)$	$l+2$	$-2l+2$	6			2^2	2^{2-l}
$C_2^{(2,1)}$	4	-2	6	6, 12, 12	144	2^2	l
$C_l^{(2,2)}$ $(l \geq 3)$	$l+2$	$-2l+5$	6			2^3	2^{4-l}
$C_3^{(2,2)}$	5	-1	6	24, 24, 24	1152	2^3	2
$B_l^{(2,2)*}$ $(l \geq 2)$	$l+2$	$-2l-1$	6			2	2^{-l}
$C_l^{(1,1)*}$ $(l \geq 2)$	$l+2$	$-2l-1$	6			2	2^{-l}
$BC_l^{(2,1)}$ $(l \geq 1)$	$l+2$	$-2l-1$	6			2	2^{-l}
$BC_1^{(2,1)}$	3	-3	6	8, 8, 8	128	2	1/2
$BC_l^{(2,4)}$ $(l \geq 1)$	$l+2$	$-2l-1$	6			2	2^{-l}
$BC_1^{(2,4)}$	3	-3	6	8, 8, 8	128	2	1/2
$BC_l^{(2,2)}(1)$ $(l \geq 2)$	$l+2$	$-2l+2$	6			2^2	2^{2-l}
$BC_2^{(2,2)}(1)$	4	-2	6	6, 12, 12	144	2^2	l
$BC_l^{(2,2)}(2)$ $(l \geq 2)$	$l+2$	$-l-2$	4			1	1
$D_l^{(1,1)}$ $(l \geq 4)$	$l+2$	$-2l+8$	6			2^4	2^{6-l}
$D_4^{(1,1)}$	6	0	6	4, 4, 1	8	2^4	2^2
$E_6^{(1,1)}$	8	0	8	3, 3, 1	9	3^3	3
$E_7^{(1,1)}$	9	0	9	8, 8, 1	32	2^5	2
$E_8^{(1,1)}$	10	0	10	12, 12, 1	72	$2^2 3^2$	1
$F_4^{(1,1)}$	6	-2	8	4, 12, 12	144	3^2	1
$F_4^{(1,2)}$	6	-3	9	4, 8, 8	128	2^3	1/2
$F_4^{(2,1)}$	6	-3	9	4, 8, 8	128	2^3	1/2
$F_4^{(2,2)}$	6	-2	8	4, 12, 12	144	3^2	1
$G_2^{(1,1)}$	4	-2	6	6, 12, 12	144	2^2	1

$G_2^{(1,3)}$	4	-4	8	6, 6, 6	108	3	1/3
$G_2^{(3,1)}$	4	-4	8	6, 6, 6	108	3	1/3
$G_2^{(3,3)}$	4	-2	6	6, 12, 12	144	2^2	1

Here, $m := m_{(R,G)} := 24/gcd(24, \mu_{(R,G)})$, $m^* := m^*_{(R,G)} := 24/gcd(24, \nu_{(R,G)})$ (2.5.1) and $m^{red} := m^{red}_{(R,G)}$ (2.7.2). By definition, one has i) if $\nu_{(R,G)} = 0 \Rightarrow m^*_{(R,G)} = 1$, and ii) $m_{(R,G)} | m^{red}_{(R,G)}$.

We state some facts which are observed from the Table 2 and are used in the sequel.

Fact 1. i) $m^*_{(R,G)} = 1 \Rightarrow \nu_{(R,G)} = 0$ and ii) $m^*_{(R,G)} | m^{red}_{(R,G)}$ for 1-codimensional (R, G). These 2 facts are more precisely formulated as:

$$(T2.1) \qquad m^*_{(R,G)} = \begin{cases} 1 & \text{if } \quad \nu_{(R,G)} = 0, \\ m^{red}_{(R,G)} & \text{else.} \end{cases}$$

Fact 2. i) For all (R, G), the discriminant $d_{(R,G)}$ is an integer and one has the equality:
$\{p \mid \text{prime number with } p | m(R, G)\} = \{p \mid \text{prime number with } p | d_{(R,G)}\}$.

ii) For all 1-codimensional (R, G), one has the equality:
$\{p \mid \text{prime number with } p | m^{red}_{(R,G)}\} = \{p \mid \text{prime number with } p | N_{(R,G)}\}$

iii) One has the inclusion relation: the set of i) $\subset$ the set of ii) $\subset \{2, 3\}$.

Remark. Due to Lemma 2., the positivity $\mu(R, G) > 0$ implies that the dual elliptic eta-product $\eta^*_{(R,G)}$ is neither a cuspidal nor a holomorphic form. So, the "dual consideration" to the goal theorem in the abstract suggests that their Fourier coefficients of $\eta^*_{(R,G)}$ are non-negative integers. In fact, this is true and is proved in a stronger form as follows.

Assertion. *All Fourier coefficients of a dual elliptic eta-product are positive integers.*

Proof. It is sufficient to show the positivity of the Taylor coefficients of $\varphi^*_{(R,G)}(\lambda)$ at $\lambda = 0$, due to an expression $\eta^*_\varphi(\tau) = q^{\nu/24} \prod_{n=1}^{\infty} \varphi^*(q^n)$. If $e^*(i) \le 0$ for all i, this is trivial. Rest cases to be checked are the types $B_3^{(1,1)}$, $C_3^{(2,2)}$, $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$, for which we have the following explicit descriptions.

$$\varphi^*_{B_3^{(1,1)}} = \varphi^*_{C_3^{(2,2)}} = \frac{(\lambda^2-1)}{(\lambda-1)^3} = \frac{1+\lambda}{(1-\lambda)^2} = 1 + 3\lambda + 5\lambda^2 + 7\lambda^3 + 9\lambda^4 + 11\lambda^5 + \cdots,$$
$$\varphi^*_{D_4^{(1,1)}} = \frac{(\lambda^2-1)^2}{(\lambda-1)^4} = 1 + \frac{4\lambda}{(1-\lambda)^2} = 1 + 4\lambda + 8\lambda^2 + 12\lambda^3 + 16\lambda^4 + 20\lambda^5 + \cdots,$$
$$\varphi^*_{E_6^{(1,1)}} = \frac{(\lambda^3-1)}{(\lambda-1)^3} = 1 + \frac{3\lambda}{(1-\lambda)^2} = 1 + 3\lambda + 6\lambda^2 + 9\lambda^3 + 12\lambda^4 + 15\lambda^5 + \cdots,$$
$$\varphi^*_{E_7^{(1,1)}} = \frac{(\lambda^4-1)}{(\lambda^2-1)(\lambda-1)^2} = 1 + \frac{2\lambda}{(1-\lambda)^2} = 1 + 2\lambda + 4\lambda^2 + 6\lambda^3 + 8\lambda^4 + 10\lambda^5 + \cdots,$$
$$\varphi^*_{E_8^{(1,1)}} = \frac{(\lambda^6-1)}{(\lambda^3-1)(\lambda^2-1)(\lambda-1)} = 1 + \frac{\lambda}{(1-\lambda)^2} = 1 + \lambda + 2\lambda^2 + 3\lambda^3 + 4\lambda^4 + 5\lambda^5 + \cdots.$$

Remark. The rank, discriminant and dual discriminant for a regular system W of weights are positive integers. It is proven [Sa3,§13] that an eta-product η_W is holomorphic (resp. cuspidal) if and only if the dual-rank ν_W is non-positive (resp. negative).

§3. Dirichlet series for 1-codimensional elliptic root systems

In this section, we formulate the main theorem of the present paper, which describes the Dirichlet series attached to the elliptic eta-products for 1-codimensional elliptic root systems by a use of Artin L-functions. The explicit results are exhibited in Table 3.

(3.1) First, for a given cyclotomic function φ (1.1.1), we define Dirichlet series:

$$L_\varphi(s) := \sum_{n=1}^{\infty} c(n) n^{-s} \tag{3.1.1}$$

$$L^*_\varphi(s) := \sum_{n=1}^{\infty} c^*(n) n^{-s} \tag{3.1.1*}$$

$$\hat{L}_\varphi(s) := \sum_{n=1}^{\infty} \hat{c}(n) n^{-s} \tag{3.$\hat{1}$.1}$$

where $c(n)$, $c^*(n)$ and $\hat{c}(n)$ are Fourier coefficients at infinity of the expansions:

$\eta_\varphi(m_\varphi \tau) = \sum_n c(n) q^n$, $\eta^*_\varphi(m^*_\varphi \tau) = \sum_n c^*(n) q^n$ and $\hat{\eta}_\varphi(m^*_\varphi \tau) = \sum_n \hat{c}(n) q^n$

of the eta-products attached to φ (cf. §2 Lemma 1). It is well known that the Dirichlet series converge on the right complex half s-plane and extend meromorphically to the whole s-plane. The duality formula (2.3.1) is reformulated as a functional equation:

$$\Lambda_\varphi(a_0 - s) = c_\varphi \Lambda^*_\varphi(s), \tag{3.1.2}$$

where $\Lambda_\varphi(s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L_\varphi(s)$, $\Lambda^*_\varphi(s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L^*_\varphi(s)$ and c_φ: a constant.

(3.2) We denote by $L_{(R,G)}$ and $\hat{L}_{(R,G)}$ the Dirichlet series (3.1.1) and (3.$\hat{1}$.1) attached to the characteristic polynomials $\varphi_{(R,G)}$ (A1.7) and $\hat{\varphi}_{(R,G)}(\lambda)$ (2.2.5) for a marked elliptic root system (R,G), respectively. Before we state the main theorem of the present article, we recall some numerical invariants: $m := 24/gcd(24, \mu_{(R,G)})$ (2.5.1), $m^{red} := 24/gcd(24, \mu^{red}_{(R,G)})$ (2.7), $m^* := 24/gcd(24, \nu_{(R,G)})$ (2.5.1), $N_{(R,G)} := mm^*m(R,G)$ (2.5.1), $\varepsilon_{(R,G)}$:= the Dirichlet character (2.5.2) attached to a marked elliptic root system $\varphi_{(R,G)}$.

Theorem. *Let (R,G) be a 1-codimensional marked elliptic root system (cf. (A1.8)). There exist a Kummer extension $E_{(R,G)} := \mathbf{Q}(\zeta_{m^{red}}, x^{1/m^*})$ of the cyclotomic field $E^{ab}_{(R,G)} = \mathbf{Q}(\zeta_{m^{red}})$ and one or two representation(s), say ρ, or $\rho^{(+)}$ and $\rho^{(-)}$: $Gal(E_{(R,G)}/\mathbf{Q}) \to GL_2(\mathbf{Z})$ or $GL_2(\mathbf{Z}[\sqrt{-1}])$ with $ker(\rho)$ or $ker(\rho^{(+)}) \cap ker(\rho^{(-)}) = \{1\}$, $det(\rho)$ or $det(\rho^{(\pm)})$ $= \varepsilon_{(R,G)}$ (through the identification $Gal(E^{(ab)}_{(R,G)}/\mathbf{Q}) \simeq \big(\mathbf{Z}/m^{red}\mathbf{Z}\big)^\times$) and the conductor $= N_{(R,G)}$ such that for the Artin L-functions $L(s,\rho)$ or $L(s,\rho^{(+)})$ and $L(s,\rho^{(-)})$ attached to them, one has the following i) - v).*

i) The Dirichlet series $L_{(R,G)}(s)$ is equal to one of the next three forms:

$$L(s,\rho), \quad \tfrac{1}{4}(L(s,\rho^{(+)}) - L(s,\rho^{(-)})) \quad \textit{or} \quad \tfrac{\sqrt{-1}}{4}(L(s,\rho^{(+)}) - L(s,\rho^{(-)})).$$

ii) $L(s,\rho), L(s,\rho^{(+)})$ and $L(s,\rho^{(-)})$ have trivial Euler factor for the prime with $p|N_{(R,G)}$.

iii) The Dirichlet series $\hat{L}_{(R,G)}(s)$ is a linear combination of $L(s,\rho)$ or of $L(s,\rho^{(+)})$ and $L(s,\rho^{(-)})$ with coefficients in Euler factors for the primes p with $p|N_{(R,G)}$.

iv) The extension $E_{(R,G)}/E^{ab}_{(R,G)}$ is trivial if and only if $\nu_{(R,G)} = 0$.

v) If $\nu_{(R,G)} = 0$, then the representation(s) decompose(s) as $\rho = 1 \oplus \varepsilon_{(R,G)}$, or $\rho^{(+)} = 1 \oplus \varepsilon_{(R,G)}$ and $\rho^{(-)} = \chi_{(R,G)} \oplus \varepsilon_{(R,G)}\chi^{-1}_{(R,G)}$ for a character $\chi_{(R,G)}$ on $(\mathbf{Z}/m^{red}\mathbf{Z})^{\times}$.

Definition. We shall call $L(s,\rho)$, $L(s,\rho^{(+)})$ and $L(s,\rho^{(-)})$ the *Artin summand* of the Dirichlet series $L_{(R,G)}$.

(3.3) The remaining of the present section gives a plan of the proof of the theorem.

The proof is achieved by explicit descriptions of $L_{(R,G)}$ and $\hat{L}_{(R,G)}$ obtained for each types separately by 4 steps explained below. Detailed results are summarized in the **Table 3**. Since the process to describe $L_{(R,G)}$ and $\hat{L}_{(R,G)}$ are similar, we explain only for $L_{(R,G)}$. According to theorems due to Hecke, Weil, Langlands, Deligne and Serre ([He, 22-36][W][D-S][Se]), the Mellin transform induces 1) a bijection between the set of suitably normalized new forms of weight 1 with level N and an odd character ε and the set of Artin L-functions of two dimensional linear irreducible representation ρ of the Galois groups over the rational numbers with conductor N and $\varepsilon = det(\rho)$, and also 2) a bijection between the set of normalized primitive Eisenstein series of weight 1 and the set of reducible 2-dimensional Galois representations (see [Se] for detailed account on the subject, and [He36] for the case of reducible representations). Therefore, if an eta-product is already a Hecke eigenform and is a new form, then the main part of the theorem is a consequence of this general theorem. But, in fact, the elliptic eta-product is not always an eigenform, and hence the following facts still need to be proven.

i) An elliptic eta-product $\eta_{(R,G)}$ is either a normalized new form (and a Hecke eigenform) or a difference of two normalized new forms so that its Mellin transform is either an Artin L-function or a difference of two Artin L-functions as described in the theorem.

ii) The Euler factor for p with $p|N_{(R,G)}$ in $L_{(R,G)}$ and in its summands is trivial due to (2.7.4) and §2 Table 2 **Fact 2.** ii) (this is not always true for all eta-products, see (4.2)).

iii) The dual $\hat{L}_{(R,G)}(s)$ is a linear combination of the summands with coefficients in Euler factors in primes p with $p|N_{(R,G)}$.

iv) If an elliptic eta-product $\eta_{(R,G)}$ is not cuspidal, then each summand in the above i) should not be a cusp form and should belong to Galois representations of the same cyclotomic field $\mathbf{Q}(\zeta_{m^{red}})$. The decomposition of ρ or $\rho^{(\pm)}$ takes some particular form as described in the theorem. Let us explain how we proceed these.

1. Let a 1-codimensional marked elliptic root system (R,G) be given. Its characteristic polynomial $\varphi_{(R,G)}$ is given in (A1.7). Using §2 Lemma 1, we know the level $N = N_{(R,G)}$ and the character $\varepsilon_{(R,G)}$ of the eta-product $\eta_{(R,G)}(m\tau)$ $\Big(= \eta^{red}_{(R,G)}(m^{red}\tau)\Big)$.

2. Calculate the Fourier coefficients $c(n)$ for $n \in \mathbf{Z}_{>0}$ of $\eta_{(R,G)}(m\tau)$ at ∞ until a degree n_0 which is larger than $\frac{N_{(R,G)}}{12} \cdot \prod_{p|N_{(R,G)}}(1+p^{-1})$.

3. Construct a Dirichlet series L, which is one of the following 3 forms:

$$L(s), \quad \tfrac{1}{4}(L^{(+)}(s) - L^{(-)}(s)) \quad \text{or} \quad \tfrac{\sqrt{-1}}{4}(L^{(+)}(s) - L^{(-)}(s))$$

where each summand $L(s)$ or $L^{(+)}(s)$ and $L^{(-)}(s)$ has an Euler product expansion of the form $\prod_{p\not| N}(1-a_p p^{-s} + \varepsilon_{(R,G)}(p)p^{-2s})^{-1}$, such that the Fourier coefficients in step 2 agree with the Dirichlet coefficients of L up to degree n_0. Here, the Euler factor for the prime p with $p|N_{(R,G)}$ does not appear because of (2.7.4) and §2 Table 2 **Fact 2.** ii).

This step is nontrivial and quite involved. It is achieved by inspection for each cases. We omit the calculations.

4. Find a Galois field $E_{(R,G)}$ and complex 2-dimensional linear representation(s) ρ or $\rho^{(\pm)}$ of the Galois group $Gal(E_{(R,G)}/\mathbf{Q})$ with the constrains $ker(\rho) = \{1\}$ or $ker(\rho^{(+)}) \cap ker(\rho^{(-)}) = \{1\}$ and the conductor$=N_{(R,G)}$ such that the Artin L-function(s) attached to the representation(s) are equal to the summand(s) that appeared in the step 3. That is:

$L(s) = L(s,\rho)$ or $L^{(+)}(s) = L(s,\rho^{(+)})$ and $L^{(-)}(s) = L(s,\rho^{(-)})$.

In fact, $E_{(R,G)}$ is obtained as a Kummer extension of the cyclotomic field $E^{ab}_{(R,G)} = \mathbf{Q}(\zeta_{m^{red}})$. The extension is determined by inspection of Euler factors in the step 3. In particular we observe that the extension is trivial if and only if $\nu_{(R,G)} = 0$.

5. **Assertion.** *Under the 1-4 above, the Dirichlet series $L_{(R,G)}$ coincides with L.*

Proof. In view of the step 4, the Mellin inverse transform f_L of L is an automorphic form of type $(1, \varepsilon_{(R,G)})$ on the group $\Gamma_0(N)$ due to the theorems of Hecke[H36] in case ρ is reducible and Weil [W] in case ρ is irreducible. The coincidence of the Fourier coefficients at infinity up to the degree n_0 (step 3.) implies $\eta_{(R,G)}(m\tau) = f_L$ (cf. [Ogg, Prop.7]). Hence the Mellin transform $L_{(R,G)}$ of $\eta_{(R,G)}(m\tau)$ should coincide with that L of f_L. Q.E.D.

Table 3. Dirichlet series for 1-codimensional elliptic root systems

Let (R,G) be a marked elliptic root system and let $\varphi^{red}_{(R,G)}$ be the attached reduced characteristic polynomial ((A1.7) and (2.7.1)). Recall the facts (2.7) that the elliptic eta-products: $\eta_{(R,G)}(m\tau)$, $\hat{\eta}_{(R,G)}(m^*\tau)$, the Dirichlet series: $L_{(R,G)}(s)$, $\hat{L}_{(R,G)}(s)$ and the numerical invariants: $m^{red}_{(R,G)}$, $m^*_{(R,G)}$, $N_{(R,G)}$ depend only on $\varphi^{red}_{(R,G)}$.

There are 8 reduced characteristic polynomials attached to 1-codimensional marked elliptic root systems. Accordingly, the table is divided into 8 groups, labeled by roman numeral I-VIII. For each P$\in$ {I,II,III,$\cdots$,VIII}, the following data 0)-vi) are exhibited.

0) The list of types of elliptic root systems (R,G) belonging to the group.

i) Characteristic functions $\varphi_P := \varphi^{red}_{(R,G)}$, $\varphi_{(R,G)}$ and $\hat{\varphi}_P := \hat{\varphi}_{(R,G)}$.

ii) Elliptic eta-products $\eta_P(\tau) := \eta_{(R,G)}(m\tau)$ and $\hat{\eta}_P(\tau) := \hat{\eta}_{(R,G)}(m^*\tau)$.

iii) The numerical invariants: $N_P := N_{(R,G)}$, $m^{red}_P := m^{red}_{(R,G)}$, $m^*_P := m^*_{(R,G)}$.

The Dirichlet character $\varepsilon_P := \varepsilon_{(R,G)}$ mod m^{red}_P for the eta-products.

iv) a) The expressions of the Dirichlet series $L_P(s) := L_{(R,G)}(s)$ and $\hat{L}_P(s) := \hat{L}_{(R,G)}(s)$

$$L(s,\rho), \quad \tfrac{1}{4}(L(s,\rho^{(+)})-L(s,\rho^{(-)})) \quad \text{or} \quad \tfrac{\sqrt{-1}}{4}(L(s,\rho^{(+)})-L(s,\rho^{(-)}))$$

b) Euler product expression of each summand: $L(s,\rho)$, $L(s,\rho^{(+)})$ and $L(s,\rho^{(-)})$.

c) Decomposition of each summand into Dirichlet L-functions (in case of $\nu_{(R,G)}=0$).

v) The Kummer extension $E_P := E_{(R,G)}$ over $E_P^{ab} = \mathbf{Q}(\zeta_{m^{red}})$.
A presentation of the Galois group $Gal(E_{(R,G)}/\mathbf{Q})$ in terms of suitably chosen Frobenius σ_p for primes p and their relations.

vi) Representation(s) ρ or $\rho^{(\pm)}$: $Gal(E_{(R,G)}/\mathbf{Q}) \to GL_2(\mathbf{Z}[\sqrt{-1}])$.
The decomposition of the representation(s) (in case of $\nu_{(R,G)}=0$):

$$\rho = 1 \oplus \varepsilon_{(R,G)}, \quad \text{or} \quad \rho^{(+)} = 1 \oplus \varepsilon_{(R,G)} \ \text{ and } \ \rho^{(-)} = \chi_{(R,G)} \oplus \varepsilon_{(R,G)}\chi_{(R,G)}^{-1}.$$

I. Types: $B_2^{(2,1)}$, $C_2^{(1,2)}$, $BC_2^{(2,2)}(1)$, $G_2^{(1,1)}$, $G_2^{(3,3)}$, $F_4^{(1,1)}$, $F_4^{(2,2)}$

i) characteristic functions $\quad \varphi_I(\lambda) = \hat{\varphi}_I(\lambda) := (\lambda-1)^2$

$$(\lambda^2-1)^2 = \varphi_{B_2^{(2,1)}}(\lambda) = \varphi_{C_2^{(1,2)}}(\lambda) = \varphi_{BC_2^{(2,2)}(1)}(\lambda) = \varphi_{G_2^{(1,1)}}(\lambda) = \varphi_{G_2^{(3,3)}}(\lambda)$$
$$(\lambda^3-1)^2 = \varphi_{F_4^{(1,1)}}(\lambda) = \varphi_{F_4^{(2,2)}}(\lambda)$$
$$(\lambda-1)^2 = \hat{\varphi}_{B_2^{(2,1)}}(\lambda) = \hat{\varphi}_{C_2^{(1,2)}}(\lambda) = \hat{\varphi}_{BC_2^{(2,2)}(1)}(\lambda) = \hat{\varphi}_{G_2^{(1,1)}}(\lambda) = \hat{\varphi}_{G_2^{(3,3)}}(\lambda) = \hat{\varphi}_{F_4^{(1,1)}}(\lambda) = \hat{\varphi}_{F_4^{(2,2)}}(\lambda)$$

ii) elliptic eta-products

$$\begin{aligned}\eta_I(\tau) := \eta(12\tau)^2 =& \eta_{B_2^{(2,1)}}(6\tau) = \eta_{C_2^{(1,2)}}(6\tau) = \eta_{BC_2^{(2,2)}(1)}(6\tau) = \eta_{G_2^{(1,1)}}(6\tau) = \eta_{G_2^{(3,3)}}(6\tau)\\ =& \eta_{F_4^{(1,1)}}(4\tau) = \eta_{F_4^{(2,2)}}(4\tau)\\ \hat{\eta}_I(\tau) := \eta(12\tau)^2 =& \hat{\eta}_{B_2^{(2,1)}}(12\tau) = \hat{\eta}_{C_2^{(1,2)}}(12\tau) = \hat{\eta}_{BC_2^{(2,2)}(1)}(12\tau) = \hat{\eta}_{G_2^{(1,1)}}(12\tau) = \hat{\eta}_{G_2^{(3,3)}}(12\tau)\\ =& \hat{\eta}_{F_4^{(1,1)}}(12\tau) = \hat{\eta}_{F_4^{(2,2)}}(12\tau)\\ =& q - 2q^{13} - q^{25} + 2q^{37} + q^{49} + 2q^{61} - 2q^{73} - 2q^{97} - 2q^{109} + q^{121} + 2q^{157} + 3q^{169}\cdots\end{aligned}$$

This eta-product was already studied in [H 22,23, pp425,426,448], [Se, pp242,243,244].

iii) Level N, m^{red}, m^* and character

$$N_I = 144, \quad m_I^{red} = 12, \quad m_I^* = 12, \quad \varepsilon_I(d) = \varepsilon_I^*(d) = \left(\frac{-1}{d}\right).$$

$$\varepsilon_I(1) = \varepsilon_I(5) = 1, \quad \varepsilon_I(7) = \varepsilon_I(11) = -1.$$

iv) Dirichlet series

$$\begin{aligned}L_I(s) \ &= \ \hat{L}_I(s) \ = \ L(s,\rho)\\ &= \prod_{\substack{p\equiv 1(12)\\ 12\equiv\exists x^4(p)}} \frac{1}{(1-p^{-s})^2} \prod_{\substack{p\equiv 1(12)\\ 12\not\equiv\forall x^4(p)}} \frac{1}{(1+p^{-s})^2} \prod_{p\equiv 5(12)} \frac{1}{1+p^{-2s}} \prod_{p\equiv 7 or 11(12)} \frac{1}{1-p^{-2s}}.\end{aligned}$$

v) Kummer extension and Galois group

$$E_I := \mathbf{Q}(\sqrt{-1}, \sqrt[4]{12}) = \mathbf{Q}(\zeta_{12}, \sqrt[4]{12}) \supset E^{ab} = \mathbf{Q}(\zeta_{12}), \quad where \quad \zeta_{12} = (\sqrt{3} + \sqrt{-1})/2.$$

$$Gal(E_I/\mathbf{Q}) =< \sigma_{1B}, \sigma_5, \sigma_7, \sigma_{11} \mid \sigma_{1B}^2 = \sigma_7^2 = \sigma_{11}^2 = (\sigma_5\sigma_7)^2 = (\sigma_5\sigma_{11})^2 = 1,$$
$$\sigma_5^2 = (\sigma_7\sigma_{11})^2 = \sigma_{1B} \in center, \quad \sigma_5\sigma_7\sigma_{11} \in < \sigma_{1B} >>,$$

where σ_{1B} stands for a Frobenius for the primes p in $B := \{p \in \mathbf{Z}_{>0} \mid$ prime number $p \equiv 1$ (12) and $\forall x^4 \not\equiv 12 \ (p)\} = \{13, 73, 97, 109, 181, 229, 241, 277, 339, 409, 421, 457, 541, \cdots\}$. One has the abelianization:

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow Gal(E_I/\mathbf{Q}) \longrightarrow (\mathbf{Z}/12\mathbf{Z})^\times \longrightarrow 1,$$

where the kernel $\mathbf{Z}/2\mathbf{Z}$ is generated by σ_{1B}.

vi) Representation

ρ is the irreducible two-dimensional linear representation of $Gal(E_I/\mathbf{Q})$.

$\rho(\sigma_{1B}) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(\sigma_5) = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \rho(\sigma_7) = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho(\sigma_{11}) = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

Here, the sign $\pm$ may be chosen independent and arbitrary.

II. Types $A_1^{(1,1)*}$, $BC_1^{(2,1)}$, $BC_1^{(2,4)}$, $F_4^{(1,2)}$, $F_4^{(2,1)}$

i) characteristic functions $\quad \varphi_{II}(\lambda) = \hat{\varphi}_{II}(\lambda) := (\lambda^2 - 1)(\lambda - 1)$

$$(\lambda^2 - 1)(\lambda - 1) = \varphi_{A_1^{(1,1)*}}(\lambda) = \varphi_{BC_1^{(2,1)}}(\lambda) = \varphi_{BC_1^{(2,4)}}(\lambda)$$
$$(\lambda^4 - 1)(\lambda^2 - 1) = \varphi_{F_4^{(1,2)}}(\lambda) = \varphi_{F_4^{(2,1)}}(\varphi)$$
$$(\lambda^2 - 1)(\lambda - 1) = \hat{\varphi}_{A_1^{(1,1)*}}(\lambda) = \hat{\varphi}_{BC_1^{(2,1)}}(\lambda) = \hat{\varphi}_{BC_1^{(2,4)}}(\lambda) = \hat{\varphi}_{F_4^{(1,2)}}(\lambda) = \hat{\varphi}_{F_4^{(2,1)}}(\lambda)$$

ii) elliptic eta-products

$$\begin{aligned}
\eta_{II}(\tau) := \eta(16\tau)\eta(8\tau) &= \eta_{A_1^{(1,1)*}}(8\tau) = \eta_{BC_1^{(2,1)}}(8\tau) = \eta_{BC_1^{(2,4)}}(8\tau) \\
&= \eta_{F_4^{(1,2)}}(4\tau) = \eta_{F_4^{(2,1)}}(4\tau) \\
\hat{\eta}_{II}(\tau) := \eta(16\tau)\eta(8\tau) &= \hat{\eta}_{A_1^{(1,1)*}}(8\tau) = \hat{\eta}_{BC_1^{(2,1)}}(8\tau) = \hat{\eta}_{BC_1^{(2,4)}}(8\tau) \\
&= \hat{\eta}_{F_4^{(1,2)}}(4\tau) = \hat{\eta}_{F_4^{(2,1)}}(4\tau) \\
&= q - q^9 - 2q^{17} + q^{25} + 2q^{41} + q^{49} - 2q^{73} + q^{81} - 2^{89} - 2q^{97} + 2q^{113} - q^{121} \cdots
\end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{II} = 128, \quad m_{II}^{red} = 8, \quad m_{II}^* = 8, \quad \varepsilon_{II}(d) = \varepsilon_{II}^*(d) = \left(\frac{-2}{d}\right).$$

$$\varepsilon_{II}(1)=\varepsilon_{II}(3)=1, \quad \varepsilon_{II}(5)=\varepsilon_{II}(7)=-1.$$

iv) Dirichlet series

$$\begin{aligned} L_{II}(s) &= \hat{L}_{II}(s) = \quad L(s,\rho) \\ &= \prod_{\substack{p\equiv 1(8)\\ -4\equiv \exists x^8(p)}} \frac{1}{(1-p^{-s})^2} \prod_{\substack{p\equiv 1(8)\\ -4\not\equiv \forall x^8(p)}} \frac{1}{(1+p^{-s})^2} \prod_{p\equiv 3(8)} \frac{1}{1+p^{-2s}} \prod_{p\equiv 5 or 7(8)} \frac{1}{1-p^{-2s}} \end{aligned}$$

v) Kummer extension and Galois group

$E_{II} := \mathbf{Q}(\sqrt{-1},\sqrt{-2},\sqrt[8]{-4}) = \mathbf{Q}(\zeta_8,\sqrt[8]{-4}) \supset E_{II}^{ab} = \mathbf{Q}(\zeta_8), \quad where \quad \zeta_8 = (\sqrt{2}+\sqrt{-2})/2.$

$$\begin{aligned} Gal(E_{II}/\mathbf{Q}) = &< \sigma_{1B},\sigma_3,\sigma_5,\sigma_7 \mid \sigma_{1B}^2=\sigma_5^2=\sigma_7^2=(\sigma_3\sigma_5)^2=(\sigma_3\sigma_7)^2=1, \\ &\sigma_3^2=(\sigma_5\sigma_7)^2=\sigma_{1B}\in center, \sigma_3\sigma_5\sigma_7 \in <\sigma_{1B}>>, \end{aligned}$$

where σ_{1B} stands for a Frobenius for the primes p in $B := \{p\in\mathbf{Z}_{>0} \mid$ prime number $p\equiv 1\ (8)$ and $\forall x^8 \not\equiv -4\ (p)\} = \{17,73,89,97,193,233,241,281,369,\cdots\}$. One has the abelianization:

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow Gal(E_{II}/\mathbf{Q}) \longrightarrow (\mathbf{Z}/8\mathbf{Z})^\times \longrightarrow 1,$$

where the kernel $\mathbf{Z}/2\mathbf{Z}$ is generated by σ_{1B}.

vi) Representation

ρ is the irreducible two-dimensional linear representation of $Gal(E_{II}/\mathbf{Q})$.

$\rho(\sigma_{1B}) = -\begin{pmatrix}1&0\\0&1\end{pmatrix}, \rho(\sigma_3)=\pm\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \rho(\sigma_5)=\pm\begin{pmatrix}0&1\\1&0\end{pmatrix}, \rho(\sigma_7)=\pm\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$

Here, the sign $\pm$ may be chosen independent and arbitrary.

III. Types $B_3^{(1,1)}$, $C_3^{(2,2)}$

i) characteristic functions

$$\varphi_{III}(\lambda) := (\lambda^2-1)^3(\lambda-1)^{-1} = \varphi_{B_3^{(1,1)}}(\lambda) = \varphi_{C_3^{(2,2)}}(\lambda)$$

$$\hat{\varphi}_{III}(\lambda) := (\lambda-1)^3(\lambda^2-1)^{-1} = \hat{\varphi}_{B_3^{(1,1)}}(\lambda) = \hat{\varphi}_{C_3^{(2,2)}}(\lambda)$$

ii) elliptic eta-products

$$\begin{aligned} &\eta_{III}(\tau) := \eta(48\tau)^3\eta(24\tau)^{-1} = \eta_{B_3^{(1,1)}}(24\tau) = \eta_{C_3^{(2,2)}}(24\tau) \\ = \ & q^5+q^{29}-q^{53}-q^{101}-2q^{125}+q^{149}-q^{173}-q^{197}+q^{245}+q^{269}-q^{293}+q^{317}+2q^{365}\cdots. \end{aligned}$$

$$\begin{aligned} &\hat{\eta}_{III}(\tau) := \eta(24\tau)^3\eta(48\tau)^{-1} = \hat{\eta}_{B_3^{(1,1)}}(24\tau) = \hat{\eta}_{C_3^{(2,2)}}(24\tau) \\ = \ & q-3q^{25}+q^{49}+2q^{73}+2q^{97}-q^{121}-4q^{145}+q^{169}-2q^{193}+2q^{241}+4q^{265}-q^{289}+2q^{313}\cdots \end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{III} = 1152, \quad m_{III}^{red} = 24, \quad m_{III}^* = 24, \quad \varepsilon_{III}(d) = \varepsilon_{III}^*(d) = \left(\frac{-2}{d}\right).$$

$$\varepsilon_{III}(1) = \varepsilon_{III}(11) = \varepsilon_{III}(17) = \varepsilon_{III}(19) = 1,$$
$$\varepsilon_{III}(5) = \varepsilon_{III}(7) = \varepsilon_{III}(13) = \varepsilon_{III}(23) = -1.$$

iv) Dirichlet series

$$\begin{aligned} L_{III}(s) &= \frac{\sqrt{-1}}{4}\left(L(s,\rho^{(+)}) - L(s,\rho^{(-)})\right) \\ \hat{L}_{III}(s) &= \frac{1}{2}\left(L(s,\rho^{(+)}) + L(s,\rho^{(-)})\right) \end{aligned}$$

where for $\chi \in \{\pm\}$

$$\begin{aligned} L(s,\rho^{(\chi)}) = & \prod_{\substack{p\equiv 1(24)\\ a\equiv\exists x^2(p)}} \frac{1}{(1-p^{-s})^2} \prod_{\substack{p\equiv 1(24)\\ a\not\equiv\forall x^2(p)}} \frac{1}{(1+p^{-s})^2} \\ & \times \prod_{\substack{p\equiv 5(24)\\ U}} \frac{1}{(1-\chi\sqrt{-1}p^{-s})^2} \prod_{\substack{p\equiv 5(24)\\ V}} \frac{1}{(1+\chi\sqrt{-1}p^{-s})^2} \\ & \times \prod_{p\equiv 7,13,23(24)} \frac{1}{1-p^{-2s}} \prod_{p\equiv 11,17 or 19(24)} \frac{1}{1+p^{-2s}}. \end{aligned}$$

v) Kummer extension and Galois group

$$E_{III} := \mathbf{Q}(\sqrt{-1},\sqrt{-2},\sqrt{-3},\sqrt[2]{a}) = \mathbf{Q}(\zeta_{24},\sqrt[2]{a}) \supset E_{III}^{ab} = \mathbf{Q}(\zeta_{24}) \ for\ \zeta_{24} = \frac{\sqrt{3}+1+\sqrt{-3}-\sqrt{-1}}{2\sqrt{2}}.$$

$$\begin{aligned} Gal(E_{III}/\mathbf{Q}) = <& \sigma_{1B},\sigma_{5U},\sigma_{5V},\sigma_7,\sigma_{11},\sigma_{13},\sigma_{17},\sigma_{19},\sigma_{23} \mid \quad \sigma_{5U} = \sigma_{5V}^{-1} \in center, \\ & \sigma_{1B}^2 = \sigma_7^2 = \sigma_{13}^2 = \sigma_{23}^2 = 1, \quad \sigma_{5U}^2 = \sigma_{5V}^2 = \sigma_{1B} \\ & \sigma_7\sigma_{5U} = \sigma_{11},\ \sigma_{13}\sigma_{5U} = \sigma_{17},\ \sigma_{23}\sigma_{5U} = \sigma_{19}, \\ & \sigma_7\sigma_{13}\sigma_{23} = \sigma_{5U}^{\pm 1} >, \end{aligned}$$

where σ_{1B} stands for a Frobenius for the primes p in $B := \{p \in \mathbf{Z}_{>0} \mid$ prime number $p \equiv 1$ (24) and $\forall x^2 \not\equiv a\ (p)\} = \{193, 337, 409, 433, 457, 601, 673, 769, 937, 1129, 1153, 1249, 1297, \cdots\}$, and σ_{5U} and σ_{5V} stand for Frobenius for the primes p in $U := \{p \in \mathbf{Z}_{>0} \mid$ prime number $p \equiv$ $5\ (24)$ and $\} = \{5, 29, 149, 269, 317, 389, 509, 677, 701, 1013, 1061, 1109, 1277, 1201, 1493, 1613, 1733, \cdots\}$ and $V := \{p \in \mathbf{Z}_{>0} \mid$ prime number $p \equiv 5\ (24)$ and $\} = \{53, 101, 173, 197, 293, 461, 557, 653, 773,$ $797, 821, 941, 1181, 1229, 1373, 1637, 1709, 1949, 1973, \cdots\}$, respectively. One has the abelianization:

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow Gal(E_{III}/\mathbf{Q}) \longrightarrow (\mathbf{Z}/24\mathbf{Z})^\times \longrightarrow 1,$$

where the kernel $\mathbf{Z}/2\mathbf{Z}$ is generated by σ_{1B}.

vi) Representations: $\rho^{(\pm)}$ are the irreducible two-dimensional linear representations of $Gal(E_{III}/\mathbf{Q})$, which are complex conjugate to each other. For $\chi \in \{\pm\}$

$$\rho^{(\chi)}(\sigma_{1B}) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \rho^{(\chi)}(\sigma_{5U}) = -\rho^{(\chi)}(\sigma_{5V}) = \chi \cdot \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \rho^{(\chi)}(\sigma_7) = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\rho^{(\chi)}(\sigma_{11}) = \pm\chi\sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \rho^{(\chi)}(\sigma_{13}) = \pm\chi\sqrt{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \rho^{(\chi)}(\sigma_{17}) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho^{(\chi)}(\sigma_{19}) = \pm\chi\sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^{(\chi)}(\sigma_{23}) = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

IV. Types: $D_4^{(1,1)}$

i) characteristic functions

$$\varphi_{IV}(\lambda) = \varphi_{D_4^{(1,1)}}(\lambda) = (\lambda^2-1)^4(\lambda-1)^{-2}$$
$$\hat{\varphi}_{IV}(\lambda) = \hat{\varphi}_{D_4^{(1,1)}}(\lambda) = (\lambda-1)^4(\lambda^2-1)^{-2}$$

ii) elliptic eta-products

$$\begin{aligned}\eta_{IV}(\tau) &:= \eta(8\tau)^4\eta(4\tau)^{-2} = \eta_{D_4^{(1,1)}}(4\tau) \\ &= q + 2q^5 + q^9 + 2q^{13} + 2q^{17} + 3q^{25} + 2q^{29} + 2q^{37} + 2q^{41} + 2q^{45} + q^{49} + 2q^{53}\cdots.\end{aligned}$$

$$\begin{aligned}\hat{\eta}_{IV}(\tau) &:= \eta(\tau)^4\eta(2\tau)^{-2} = \hat{\eta}_{D_4^{(1,1)}}(\tau) \\ &= 1 - 4\left(1 - q^2 - q^4 + 2q^5 - q^8 + q^9 - 2q^{10} + 2q^{13} - q^{16} + 2q^{17} - q^{18} - 2q^{20}\cdots\right).\end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{IV} = 8, \quad m^{red}_{IV} = 4, \quad m^*_{IV} = 1, \quad \varepsilon_{IV}(d) = \varepsilon^*_{IV}(d) = \left(\frac{-1}{d}\right).$$

$$\varepsilon_{IV}(1) = 1, \quad \varepsilon_{IV}(3) = -1.$$

iv) Dirichlet series

$$\begin{aligned}L_{IV}(s) &= L(s,1)L(s,\varepsilon_{IV}), \\ \hat{L}_{IV}(s) &= -4\frac{1-2^{1-s}}{1-2^{-s}}L(s,1)L(s,\varepsilon_{IV}),\end{aligned}$$

where for $\epsilon \in \{1, \varepsilon_{IV}\}$

$$L(s,\epsilon) = \prod_{p\equiv1(4)}\frac{1}{1-p^{-s}}\prod_{p\equiv3(4)}\frac{1}{1-\epsilon(p)p^{-s}}.$$

v) Kummer extension and Galois group

$$E_{IV} = E_{IV}^{ab} := \mathbf{Q}(\sqrt{-1}) = \mathbf{Q}(\zeta_4), \quad where \quad \zeta_4 = \sqrt{-1}.$$

$$Gal(E_{IV}/\mathbf{Q}) =< \sigma_3 \mid \sigma_3^2 = 1 >= (\mathbf{Z}/4\mathbf{Z})^\times$$

vi) Representation

$$\rho = 1 \oplus \varepsilon_{IV},$$

where 1 and ε_{IV} are the trivial and non-trivial characters on $(\mathbf{Z}/4\mathbf{Z})^\times$, respectively.

V. Types: $G_2^{(1,3)}$, $G_2^{(3,1)}$

i) characteristic functions

$$\varphi_V(\lambda) = \hat{\varphi}_V(\lambda) = \varphi_{G_2^{(1,3)}}(\lambda) = \varphi_{G_2^{(3,1)}}(\lambda) = (\lambda^3 - 1)(\lambda - 1)$$

ii) elliptic eta-products

$$\eta_V(\tau) := \eta(18\tau)\eta(6\tau) = \eta_{G_2^{(1,3)}}(6\tau) = \eta_{G_2^{(3,1)}}(6\tau)$$
$$\hat{\eta}_V(\tau) := \eta(18\tau)\eta(6\tau) = \hat{\eta}_{G_2^{(1,3)}}(6\tau) = \hat{\eta}_{G_2^{(3,1)}}(6\tau)$$
$$= q - q^7 - q^{13} - q^{19} + q^{25} + 2q^{31} - q^{37} + 2q^{43} - q^{61} - q^{67} - q^{72} - q^{79} + q^{91} \cdots.$$

iii) Level N, m^{red}, m^* and character

$$N_V = 108, \quad m_V^{red} = 6, \quad m_V^* = 6, \quad \varepsilon_V(d) = \varepsilon_V^*(d) = \left(\frac{-3}{d}\right).$$

$$\varepsilon_V(1) = 1, \quad \varepsilon_V(5) = -1.$$

iv) Dirichlet series

$$\begin{aligned} L_V(s) \quad &= \quad \hat{L}_V(s) \quad = \quad L(s,\rho) \\ &= \prod_{\substack{p\equiv 1(6) \\ 2\equiv \exists x^3(p)}} \frac{1}{(1-p^{-s})^2} \prod_{\substack{p\equiv 1(6) \\ 2\not\equiv \forall x^3(p)}} \frac{1}{1+p^{-s}+p^{-2s}} \prod_{p\equiv 5(6)} \frac{1}{1-p^{-2s}}. \end{aligned}$$

v) Kummer extension and Galois group

$$E_V := \mathbf{Q}(\sqrt{-3}, \sqrt[3]{2}) = \mathbf{Q}(\zeta_6, \sqrt[3]{2}) \supset E_V^{ab} = \mathbf{Q}(\zeta_6), \quad where \quad \zeta_6 = (1+\sqrt{-3})/2.$$

$$Gal(E_V/\mathbf{Q}) =< \sigma_{1B}, \sigma_5 \mid \sigma_{1B}^3 = \sigma_5^2 = (\sigma_{1B}\sigma_5)^2 = 1 >,$$

where σ_{1B} stands for a Frobenius for the primes p in $B = \{p \in \mathbf{Z}_{>\mathbf{0}} \mid \text{prime number } p \equiv 1$ (6) and $\forall x^3 \not\equiv 2\ (p)\} = \{7, 13, 19, 37, 61, 67, 73, 79, 97, \cdots\}$. One has the abelianization

$$1 \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow Gal(E_V/\mathbf{Q}) \longrightarrow (\mathbf{Z}/6\mathbf{Z})^{\times} \longrightarrow 1,$$

where the kernel $\mathbf{Z}/3\mathbf{Z}$ is generated by σ_{1B}.

vi) Representation

ρ is the irreducible two-dimensional linear representation of $Gal(E_V/\mathbf{Q})$.

$$\rho(\sigma_{1B}) = \begin{pmatrix} -1 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}, \quad \rho(\sigma_5) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

VI. Type $E_6^{(1,1)}$

i) characteristic functions

$$\varphi_{VI}(\lambda) = \varphi_{E_6^{(1,1)}}(\lambda) = (\lambda^3 - 1)^3 (\lambda - 1)^{-1}$$

$$\hat{\varphi}_{VI}(\lambda) = \hat{\varphi}_{E_6^{(1,1)}}(\lambda) = (\lambda^3 - 1)^{-1} (\lambda - 1)^3$$

ii) elliptic eta-products

$$\begin{aligned} \eta_{VI}(\tau) &:= \eta(9\tau)^3 \eta(3\tau)^{-1} = \eta_{E_6^{(1,1)}}(3\tau) \\ &= \quad q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + 2q^{31} + 2q^{37} + 2q^{43} + 3q^{49} + \cdots. \end{aligned}$$

$$\begin{aligned} \hat{\eta}_{VI}(\tau) &:= \eta(\tau)^3 \eta(3\tau)^{-1} = \hat{\eta}_{E_6^{(1,1)}}(\tau) \\ &= \quad 1 - 3\left(q - 2q^3 + q^4 + 2q^7 - 2q^9 - 2q^{12} + 2q^{13} + q^{16} + 2q^{19} - 4q^{22} + q^{25} \cdots\right). \end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{VI} = 9, \quad m_{VI}^{red} = 3, \quad m_{VI}^{*} = 1, \quad \varepsilon_{VI}(d) = \varepsilon_{VI}^{*}(d) = \left(\frac{-3}{d}\right).$$

$$\varepsilon_{VI}(1) = 1, \quad \varepsilon_{VI}(2) = -1.$$

iv) Dirichlet series

$$\begin{aligned} L_{VI}(s) &= L(s,1)L(s,\varepsilon_V), \\ \hat{L}_{VI}(s) &= -3\frac{1 - 3^{1-s}}{1 - 3^{-s}} L(s,1)L(s,\varepsilon_V), \end{aligned}$$

where for $\epsilon \in \{1, \varepsilon_V\}$

$$L(s,\epsilon) \quad = \quad \prod_{p \equiv 1(3)} \frac{1}{1 - p^{-s}} \prod_{p \equiv 2(3)} \frac{1}{1 - \epsilon(p) p^{-s}}.$$

v) Kummer extension and Galois group

$$E_{VI} = E_{VI}^{ab} := \mathbf{Q}(\sqrt{-3}) = \mathbf{Q}(\zeta_3), \quad where \quad \zeta_3 = (-1+\sqrt{-3})/2.$$

$$Gal(E_{VI}/\mathbf{Q}) =< \sigma_2 \mid \sigma_2^2 = 1 >= (\mathbf{Z}/3\mathbf{Z})^\times$$

vi) Representation

$$\rho = 1 \oplus \varepsilon_{VI}$$

where 1 and ε_{VI} are the trivial and non-trivial characters on $(\mathbf{Z}/3\mathbf{Z})^\times$, respectively.

VII. Type $E_7^{(1,1)}$

i) characteristic functions

$$\varphi_{VII}(\lambda) = \varphi_{E_7^{(1,1)}}(\lambda) = (\lambda^4-1)^2(\lambda^2-1)(\lambda-1)^{-1}$$

$$\hat{\varphi}_{VII}(\lambda) = \hat{\varphi}_{E_7^{(1,1)}}(\lambda) = (\lambda-1)^2(\lambda^2-1)(\lambda^4-1)^{-1}$$

ii) elliptic eta-products

$$\begin{aligned}\eta_{VII}(\tau) :=& \eta(32\tau)^2\eta(16\tau)\eta(8\tau)^{-1} = \eta_{E_7^{(1,1)}}(8\tau)\\ =& \quad q^3+q^{11}+q^{19}+2q^{27}+q^{43}+2q^{51}+q^{59}+q^{67}+q^{75}+q^{83}+^{99}+q^{107}+2q^{123}\cdots.\end{aligned}$$

$$\begin{aligned}\hat{\eta}_{VII}(\tau) :=& \eta(\tau)^2\eta(2\tau)\eta(4\tau)^{-1} = \hat{\eta}_{E_7^{(1,1)}}(\tau),\\ =& \quad 1-2\left(q+q^2-2q^3-q^4+2q^6-q^8+3q^9-2q^{11}-2q^{12}-q^{16}+2q^{17}+3q^{18}\cdots\right).\end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{VII} = 32, \quad m_{VII}^{red} = 8, \quad m_{VII}^* = 1, \quad \varepsilon_{VII}(d) = \varepsilon^*(d) = \left(\frac{-2}{d}\right).$$

$$\varepsilon_{VII}(1) = \varepsilon_{VII}(3) = 1, \quad \varepsilon_{VII}(5) = \varepsilon_{VII}(7) = -1.$$

iv) Dirichlet series

$$\begin{aligned}L_{VII}(s) &= \frac{1}{4}\left(L(s,1)L(s,\varepsilon_{VII}) - L(s,\chi_{VII})L(s,\varepsilon_{VII}\chi_{VII}^{-1})\right),\\ \hat{L}_{VII}(s) &= -2\frac{1-2^{1-2s}}{1-2^{-s}}L(s,\chi_{VII})L(s,\varepsilon_{VII}\chi_{VII}^{-1}),\end{aligned}$$

where we put $L(s,1) = L_{++}(s),\ L(s,\varepsilon_{VII}) = L_{-+}(s),\ L(s,\chi_{VII}) = L_{+-}(s)$ and $L(s,\varepsilon_{VII}\chi_{VII}^{-1}) = L_{--}(s)$, and for $\varepsilon,\ \chi \in \{\pm 1\}$

$$L_{\epsilon\chi}(s) = \prod_{p\equiv 1(8)} \frac{1}{1-p^{-s}} \prod_{p\equiv 3(8)} \frac{1}{1-\chi p^{-s}} \prod_{p\equiv 5(8)} \frac{1}{1-\epsilon p^{-s}} \prod_{p\equiv 7(8)} \frac{1}{1-\epsilon\chi p^{-s}}.$$

v) Kummer extension and Galois group

$$E_{VII} = E^{ab}_{VII} := \mathbf{Q}(\sqrt{-1}, \sqrt{-2}) = \mathbf{Q}(\zeta_8), \quad where \quad \zeta_8 = (\sqrt{2} + \sqrt{-2})/2.$$

$$Gal(E_{VII}/\mathbf{Q}) =< \sigma_3, \sigma_5, \sigma_7 \mid \sigma_i^2 = (\sigma_i\sigma_j)^2 = \sigma_3\sigma_5\sigma_7 = 1 >= (\mathbf{Z}/8\mathbf{Z})^\times$$

vi) Representations

$$\rho^{(+)} = 1 \oplus \varepsilon_{VII}, \quad \rho^{(-)} = \chi_{VII} \oplus \varepsilon_{VII} \cdot \chi^{-1}_{VII},$$

where ε_{VII}, χ_{VII} and $\varepsilon_{VII} \cdot \chi^{-1}_{VII}$ are the non-trivial characters on the Galois groups $(\mathbf{Z}/8\mathbf{Z})^\times/ < \sigma_3 >$, $(\mathbf{Z}/8\mathbf{Z})^\times/ < \sigma_5 >= (\mathbf{Z}/4\mathbf{Z})^\times$ and $(\mathbf{Z}/8\mathbf{Z})^\times/ < \sigma_7 >$ corresponding to the quadratic fields $\mathbf{Q}(\sqrt{-2})$, $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{2})$, respectively.

VIII. Type $E_8^{(1,1)}$

i) characteristic functions

$$\varphi_{VIII}(\lambda) = \varphi_{E_8^{(1,1)}}(\lambda) = (\lambda^6 - 1)(\lambda^3 - 1)(\lambda^2 - 1)(\lambda - 1)^{-1}$$

$$\hat{\varphi}_{VIII}(\lambda) = \hat{\varphi}_{E_8^{(1,1)}}(\lambda) = (\lambda - 1)(\lambda^2 - 1)(\lambda^3 - 1)(\lambda^6 - 1)^{-1}$$

ii) elliptic eta-products

$$\begin{aligned} \eta_{VIII}(\tau) :=& \eta(72\tau)\eta(36\tau)\eta(24\tau)\eta(12\tau)^{-1} = \eta_{E_8^{(1,1)}}(12\tau) \\ =& \quad q^5 + q^{17} + q^{29} + q^{41} + q^{53} + 2q^{65} + q^{89} + q^{101} + q^{113} + 2q^{125} + q^{137} + q^{149} \cdots . \end{aligned}$$

$$\begin{aligned} \hat{\eta}_{VIII}(\tau) :=& \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)^{-1} = \hat{\eta}_{E_8^{(1,1)}}(\tau) \\ =& \quad 1 - \left(q + 2q^2 - q^4 - 3q^5 + 2q^8 + 4q^9 - 2q^{10} + 2q^{13} - q^{16} - 4q^{17} - 4q^{18} + 4q^{20} \cdots\right). \end{aligned}$$

iii) Level N, m^{red}, m^* and character

$$N_{VIII} = 72, \quad m^{red}_{VIII} = 12, \quad m^*_{VIII} = 1, \quad \varepsilon_{VIII}(d) = \varepsilon^*_{VIII}(d) = \left(\frac{-1}{d}\right).$$

$$\varepsilon_{VIII}(1) = \varepsilon_{VIII}(5) = 1, \quad \varepsilon_{VIII}(7) = \varepsilon_{VIII}(11) = -1.$$

iv) Dirichlet series

$$\begin{aligned} L_{VIII}(s) \quad &= \quad \frac{1}{4}\left(L(s,1)L(s,\varepsilon_{VIII}) - L(s,\chi)L(s,\varepsilon\chi^{-1})\right), \\ \hat{L}_{VIII}(s) \quad &= \quad \frac{1}{2}\frac{1-2^{1-s}}{1-2^{-s}}\frac{1-3^{2-2s}}{1-3^{-2s}}L(s,1)L(s,\varepsilon) \\ &\quad - \frac{3}{2}\frac{1+2^{1-s}}{1+2^{-s}}L(s,\chi)L(s,\varepsilon\chi^{-1}), \end{aligned}$$

where we put $L(s,1) = L_{++}(s)$, $L(s,\varepsilon_{VIII}) = L_{-+}(s)$, $L(s,\chi_{VIII}) = L_{+-}(s)$ and $L(s,\varepsilon_{VIII}\chi_{VII}^{-1}) = L_{--}(s)$, and for $\varepsilon,\ \chi \in \{\pm 1\}$

$$L_{\epsilon\chi}(s) \quad = \quad \prod_{p\equiv 1(12)} \frac{1}{1-p^{-s}} \prod_{p\equiv 5(12)} \frac{1}{1-\chi p^{-s}} \prod_{p\equiv 7(12)} \frac{1}{1-\epsilon p^{-s}} \prod_{p\equiv 11(12)} \frac{1}{1-\epsilon\chi p^{-s}}.$$

v) Kummer extension and Galois group

$$E_{VIII} = E_{VIII}^{ab} := \mathbf{Q}(\sqrt{-1},\sqrt{-3}) = \mathbf{Q}(\zeta_{12}), \quad where \quad \zeta_{12} = (\sqrt{3}+\sqrt{-1})/2.$$

$$Gal(E_{VIII}/\mathbf{Q}) =< \sigma_5, \sigma_7, \sigma_{11} \mid \sigma_i^2 = (\sigma_i\sigma_j)^2 = \sigma_5\sigma_7\sigma_{11} = 1 >= (\mathbf{Z}/12\mathbf{Z})^\times$$

vi) Representations

$$\rho^{(+)} = 1 \oplus \varepsilon_{VIII}, \quad \rho^{(-)} = \chi_{VIII} \oplus \varepsilon_{VIII}\cdot\chi_{VIII}^{-1},$$

where ε_{VIII}, χ_{VIII} and $\varepsilon_{VIII}\cdot\chi_{VIII}^{-1}$ are the non-trivial characters on the Galois groups $(\mathbf{Z}/12\mathbf{Z})^\times/ <\sigma_5>= (\mathbf{Z}/4\mathbf{Z})^\times$, $(\mathbf{Z}/12\mathbf{Z})^\times/ <\sigma_7>= (\mathbf{Z}/3\mathbf{Z})^\times$ and $(\mathbf{Z}/12\mathbf{Z})^\times/ <\sigma_{11}>$ corresponding to the quadratic fields $\mathbf{Q}(\sqrt{-1})$, $\mathbf{Q}(\sqrt{-3})$ and $\mathbf{Q}(\sqrt{3})$, respectively.

§4. Fourier Dirichlet coefficients for an elliptic root system of codimension 1

We give the final step of the proof of the goal theorem stated at Abstract. That is: we prove that *the Fourier coefficients of the non-cuspidal elliptic eta-product are non-negative.* At the end of this §, we give Table 4 of explicit formulae for the Fourier-Dirichlet coefficients of 1-codimensional elliptic eta-products.

(4.1) Due to the results (2.9) Lemmas 4 and 5 and their Cororally, the marked elliptic root systems (R,G), whose associated eta-product is non-cuspidal, are characterized by $\nu_{(R,G)} = 0$ and are classified into types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$. Due to (3.2) main Theorem, the Dirichlet series $L_{(R,G)}(s)$ for such root system (R,G) is either of the forms:

1) $L(s,1_{(R,G)})L(s,\varepsilon_{(R,G)})$ for $D_4^{(1,1)}$ and $E_6^{(1,1)}$,
2) $\frac{1}{4}\Big(L(s,1_{(R,G)})L(s,\varepsilon_{(R,G)}) - L(s,\chi_{(R,G)})L(s,\chi_{(R,G)}^{-1}\varepsilon_{(R,G)})\Big)$ for $E_7^{(1,1)}$ and $E_8^{(1,1)}$,

where $1_{(R,G)}, \varepsilon_{(R,G)}, \chi_{(R,G)}$ and $\chi_{(R,G)}^{-1}\varepsilon_{(R,G)}$ are characters on $Gal(E_{(R,G)},\mathbf{Q}) \simeq (\mathbf{Z}/m_{(R,G)}^{red}\mathbf{Z})^\times$ taking values in $\{\pm 1\}$, and the $L(s,1_{(R,G)})$, $L(s,\varepsilon_{(R,G)})$, $\cdots$ etc are the Dirichlet L-functions attached to the characters $1_{(R,G)}$, $\varepsilon_{(R,G)}$, $\cdots$ etc with the trivial Euler factors for the primes p in $\{p \mid \text{prime with } p|m_{(R,G)}^{red}\} = \{p \mid \text{prime with } p|N_{(R,G)}\} \subset \{2,3\}$ (recall (2.7.4) and **Fact** 2. ii) after Table 2).

Let a marked elliptic root system (R,G) with $\nu_{(R,G)} = 0$ be given. We divide the set of all rational prime numbers with $p \not| \ m_{(R,G)}^{red}$ into 4 groups according to the values of characters $\varepsilon_{(R,G)}$ and $\chi_{R,G)}$ (here, we interpret $\chi_{(R,G)} \equiv 1$ in case 1)):

$P_1 := \{p \in \mathbf{Z}_{>0} \mid \text{ prime s.t. } p \not| m^{red} \text{ and } \varepsilon_{(R,G)}(p) = \chi_{(R,G)}(p) = 1\}$
$P_\varepsilon := \{p \in \mathbf{Z}_{>0} \mid \text{ prime s.t. } p \not| m^{red} \text{ and } \varepsilon_{(R,G)}(p) = -1,\ \chi_{(R,G)}(p) = 1\}$
$P_\chi := \{p \in \mathbf{Z}_{>0} \mid \text{ prime s.t. } p \not| m^{red} \text{ and } \varepsilon_{(R,G)}(p) = 1,\ \chi_{(R,G)}(p) = -1\}$
$P_{\varepsilon\chi} := \{p \in \mathbf{Z}_{>0} \mid \text{ prime s.t. } p \not| m^{red} \text{ and } \varepsilon_{(R,G)}(p) = -1,\ \chi_{(R,G)}(p) = -1\}$

In the case 1), one has

$$
\begin{aligned}
L_{(R,G)}(s) &= L(s, 1_{(R,G)})L(s, \varepsilon_{(R,G)}) \\
&= \Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1-p^{-s}}\Big)\Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1+p^{-s}}\Big) \qquad (4.1.1)\\
&= \prod_{p\in P_1} \frac{1}{(1-p^{-s})^2} \prod_{p\in P_\varepsilon} \frac{1}{1-p^{-2s}}.
\end{aligned}
$$

The Dirichlet coefficients of each of the Euler factor in the last expression are non-negative due to the expansions: $1/(1-t)^2 = \sum_{n=1}^\infty nt^{n-1}$ and $1/(1-t^2) = \sum_{n=0}^\infty t^{2n}$.

In the case 2), one has

(4.1.2)

$$
\begin{aligned}
L_{(R,G)}(s) =& \frac{1}{4}\Big\{L(s, 1_{(R,G)})L(s, \varepsilon_{(R,G)}) - L(s, \chi_{(R,G)})L(s, \chi^{-1}_{(R,G)}\varepsilon_{(R,G)})\Big\} \\
=& \frac{1}{4}\Big\{\Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1-p^{-s}} \prod_{p\in P_\chi} \frac{1}{1-p^{-s}} \prod_{p\in P_{\varepsilon\chi}} \frac{1}{1-p^{-s}}\Big) \\
&\Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1+p^{-s}} \prod_{p\in P_\chi} \frac{1}{1-p^{-s}} \prod_{p\in P_{\varepsilon\chi}} \frac{1}{1+p^{-s}}\Big) \\
&- \Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1-p^{-s}} \prod_{p\in P_\chi} \frac{1}{1+p^{-s}} \prod_{p\in P_{\varepsilon\chi}} \frac{1}{1+p^{-s}}\Big) \\
&\Big(\prod_{p\in P_1} \frac{1}{1-p^{-s}} \prod_{p\in P_\varepsilon} \frac{1}{1+p^{-s}} \prod_{p\in P_\chi} \frac{1}{1+p^{-s}} \prod_{p\in P_{\varepsilon\chi}} \frac{1}{1-p^{-s}}\Big)\Big\} \\
=& \frac{1}{4}\prod_{p\in P_1} \frac{1}{(1-p^{-s})^2} \prod_{p\in P_\varepsilon} \frac{1}{1-p^{-2s}} \prod_{p\in P_{\varepsilon\chi}} \frac{1}{1-p^{-2s}} \Big\{\prod_{p\in P_\chi} \frac{1}{(1-p^{-s})^2} - \prod_{p\in P_\chi} \frac{1}{(1+p^{-s})^2}\Big\}.
\end{aligned}
$$

As in the case 1), the Dirichlet coefficients of each of the Euler factor $\prod_{p\in P_1} \frac{1}{(1-p^{-s})^2}$, $\prod_{p\in P_\varepsilon} \frac{1}{1-p^{-2s}}$ and $\prod_{p\in P_{\varepsilon\chi}} \frac{1}{1-p^{-2s}}$ in the last expression are non-negative. The Dirichlet coefficients of the term $\prod_i p_i^{-s(k_i-1)}$ for a finite set of primes $p_i \in P_\chi$ and positive integers $k_i \in \mathbf{Z}_{>0}$ in the Euler factors $\prod_{p\in P_\chi} \frac{1}{(1-p^{-s})^2}$ and $\prod_{p\in P_\chi} \frac{1}{(1+p^{-s})^2}$ are $\prod_i k_i$ and $(-1)^{\sum_i k_i+1}\prod_i k_i$, respectively. So, the difference $(1-(-1)^{\sum_i k_i+1})\prod_i k_i$, is non-negative. This completes the proof of the non-negativity of the coefficients of $L_{(R,G)}(s)$. Q.E.D.

Thus, the proof of the goal theorem stated in the abstract is completed.

(4.2) The above proof uses the fact that an Euler factor for the bad prime $p \mid N_\varphi$ is trivial. We give 3 examples (from Conway group [Ko]) of holomorphic non-cuspidal eta-products whose Fourier coefficients at ∞ contain negative integers because of bad Euler factors.
Put $\varphi 1 := (\lambda^6-1)^6(\lambda-1)/(\lambda^3-1)^3(\lambda^2-1)^2$,
$\varphi 2 := (\lambda^2-1)(\lambda^3-1)^3(\lambda^{12}-1)^3/(\lambda-1)(\lambda^4-1)(\lambda^6-1)^3$,
$\varphi 3 := (\lambda^2-1)^3(\lambda^6-1)(\lambda^{12}-1)^2/(\lambda-1)(\lambda^3-1)(\lambda^4-1)^2$.
Then $\mu_{\varphi_i}=24$, $\nu_{\varphi_i}=0$ and hence $m_{\varphi_i}=m^*_{\varphi_i}=1$ for $i=1,2,3$. The eta-products:
$\eta_{\varphi 1}(\tau) = q-q^2+q^3+q^4-q^6+2q^7-q^8+q^9+q^{12}+2q^{13}-\cdots$,
$\eta_{\varphi 2}(\tau) = q+q^2+q^3-q^4+q^6+2q^7+q^8+q^9-q^{12}+2q^{14}-\cdots$
$\eta_{\varphi 3}(\tau) = q+q^2-q^3+q^4+2q^5-q^6+q^8+q^9+2q^{10}-q^{12}+2q^{13}-\cdots$
are holomorphic ((2.8) Lemma 2) but non-cuspidal (for $\nu_{\varphi i}=0$) forms of weight 1 and levels N=6,12,12, respectively. The attached Dirichlet series have the Euler product expansions:

$$L_{\varphi 1}(s) = \frac{1}{1+2^{-s}}\frac{1}{1-3^{-s}} \prod_{p\equiv 1 \bmod (6)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 5 \bmod (6)} \frac{1}{1-p^{-2s}},$$

$$L_{\varphi 2}(s) = \frac{1+2\cdot 2^{-s}}{1+2^{-s}}\frac{1}{1-3^{-s}} \prod_{p\equiv 1 \bmod (6)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 5 \bmod (6)} \frac{1}{1-p^{-2s}},$$

$$L_{\varphi 3}(s) = \frac{1}{1-2^{-s}}\frac{1}{1+3^{-s}} \prod_{p\equiv 1,\ 5 \bmod (12)} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 7,\ 11 \bmod (12)} \frac{1}{1-p^{-2s}},$$

whose Euler factors for the bad primes 2 or 3 create negative Dirichlet coefficients.

Table 4. Fourier Dirichlet coefficients for 1-codimensional elliptic root systems

As a consequence of Table 3, we give a table of explicit formulae of the Fourier coefficients $c(n)$ for 1-codimensional elliptic eta-product $\eta_{(R,G)}(m\tau)=\sum_n c(n)q^n$. The table is divided into 8 groups according to Table 3. For each group, we recall the type of elliptic root systems, elliptic eta-product and $m^{red} := 24/gcd(24,\mu^{red}_{(R,G)}) \in \mathbf{Z}_{>0}$ (2.7), and then, we give the formula for the coefficients $c(n)$ for $n\in\mathbf{Z}_{>0}$ using following notation.

Notation for Table 4. Let $m^{red}\in\mathbf{Z}_{>0}$ be given as above.

i) Put $P_i := \{p\in\mathbf{Z} \mid \text{a prime number with } p\equiv i \bmod m^{red}\}$ for $i\in\mathbf{Z}$ with $1\le i<m^{red}$ and $(m^{red},i)=1$. We denote by p_i an index which runs through the set P_i.
For a subset A of P_i, we denote by $p_{i,A}$ an index running through the set A.

ii) By $\sum_i$ (resp. $\prod_i$) and by $\sum_{i,A}$ (resp. $\prod_{i,A}$), we mean the summation $\sum_{p_i\in P_i}$ (resp. the product $\prod_{p_i\in P_i}$) and also $\sum_{p_{i,A}\in A}$ (resp. $\prod_{p_{i,A}\in A}$).

iii) By $\prod_i p_i^{k_i}$, we mean a finite product (i.e. almost all k_i are equal to 1) of the form $\prod_{j=1}^l p_{i,j}^{k_{i,j}}$ where $p_{i,j}$ $(1\le j\le l)$ are mutually different prime numbers in P_i. Therefore, any positive integer $n\in\mathbf{Z}_{>0}$ has the unique prime decomposition:

$$(T4.1) \qquad n = \prod_{1\le i<m^{red},(m^{red},i)=1} \left(\prod_i p_i^{k_i}\right).$$

Let us call the expression (T4.1) the *prime decomposition of* n *relative to* m^{red}.
iv) For $l \in \mathbf{Z}$, $\left(\frac{l}{2}\right)$ and $\left(\frac{l}{3}\right)$ are the Legendre symbols modulo 2 and 3, respectively:

$$\left(\tfrac{l}{2}\right) := \begin{cases} 1 & if \quad l \equiv 1 \quad mod \quad 2, \\ 0 & if \quad l \equiv 0 \quad mod \quad 2, \end{cases} \quad \text{and} \quad \left(\tfrac{l}{3}\right) := \begin{cases} 1 & if \quad l \equiv 1 \quad mod \quad 3, \\ 0 & if \quad l \equiv 0 \quad mod \quad 3, \\ -1 & if \quad l \equiv -1 \quad mod \quad 3. \end{cases}$$

I. Types: $B_2^{(2,1)}$, $C_2^{(1,2)}$, $BC_2^{(2,2)}(1)$, $G_2^{(1,1)}$, $G_2^{(3,3)}$, $F_4^{(1,1)}$, $F_4^{(2,2)}$

$\eta_I(\tau) := \eta(12\tau)^2 = \sum_n c_I(n) q^n$, $\quad m_I^{red} = 12$.

Let $n = \prod_{1,A} p_{1,A}^{k_{1,A}} \prod_{1,B} p_{1,B}^{k_{1,B}} \prod_5 p_5^{k_5} \prod_7 p_7^{k_7} \prod_{11} p_{11}^{k_{11}}$ ($k_i \in \mathbf{Z}_{\geq 0}$) be a prime decomposition of $n \in \mathbf{Z}_{>0}$ relative to $m_I^{red} = 12$ (T4.1), where $A := \{p \in \mathbf{Z}_{>0} \mid$ prime numbers, $p \equiv 1(12)$ and $\exists x^4 \equiv 12(p)\}$, and $B := \{p \in \mathbf{Z}_{>0} \mid$ prime numbers, $p \equiv 1(12)$ and $\forall x^4 \not\equiv 12(p)\}$. Then one has

$$\begin{aligned} c_I(n) \quad &= \quad \prod_{1,A} (k_{1,A}+1) \prod_{1,B} \left((k_{1,B}+1)(-1)^{k_{1,B}} \right) \\ &\quad \times \prod_{5,7,11} \left(\left(\frac{k_5+1}{2}\right) \left(\frac{k_7+1}{2}\right) \left(\frac{k_{11}+1}{2}\right) (-1)^{\frac{k_5}{2}} \right), \end{aligned}$$

II. Types $A_1^{(1,1)*}$, $BC_1^{(2,1)}$, $BC_1^{(2,4)}$, $F_4^{(1,2)}$, $F_4^{(2,1)}$

$\eta_{II}(\tau) := \eta(16\tau)\eta(8\tau) = \sum_n c_{II}(n) q^n$, $\quad m_{II}^{red} = 8$.

Let $n = \prod_{1,A} p_{1,A}^{k_{1,A}} \prod_{1,B} p_{1,B}^{k_{1,B}} \prod_3 p_3^{k_3} \prod_5 p_5^{k_5} \prod_7 p_7^{k_7}$ ($k_i \in \mathbf{Z}_{\geq 0}$) be a prime decomposition of $n \in \mathbf{Z}_{>0}$ relative to $m_{II}^{red} = 8$ (T4.1), where $A := \{p \in \mathbf{Z}_{>0} \mid$ prime numbers, $p \equiv 1(8)$ and $\exists x^8 \equiv -4(p)\}$, and $B := \{p \in \mathbf{Z}_{>0} \mid$ prime numbers, $p \equiv 1(8)$ and $\forall x^8 \not\equiv -4(p)\}$. Then one has

$$\begin{aligned} c_{II}(n) \quad &= \quad \prod_{1,A} (k_{1,A}+1) \prod_{1,B} \left((k_{1,B}+1)(-1)^{k_{1,B}} \right) \\ &\quad \times \prod_{3,5,7} \left(\left(\frac{k_3+1}{2}\right) \left(\frac{k_5+1}{2}\right) \left(\frac{k_7+1}{2}\right) (-1)^{\frac{k_3}{2}} \right). \end{aligned}$$

III. Types $B_3^{(1,1)}$, $C_3^{(2,2)}$

$\eta_{III}(\tau) := \eta(48\tau)^3 \eta(24\tau)^{-1} = \sum_n c_{III}(n) q^n$, $\quad m_{III}^{red} = 24$.

Let $n = \prod_{1,A} p_{1,A}^{k_{1,A}} \prod_{1,B} p_{1,B}^{k_{1,B}} \prod_{5,U} p_{5,U}^{k_{5,U}} \prod_{5,V} p_{5,V}^{k_{5,V}} \prod_7 p_7^{k_7} \prod_{11} p_{11}^{k_{11}} \prod_{13} p_{13}^{k_{13}} \prod_{17} p_{17}^{k_{17}}$ $\prod_{19} p_{19}^{k_{19}} \prod_{23} p_{23}^{k_{23}}$ ($k_i \in \mathbf{Z}_{\geq 0}$) be a prime decomposition of $n \in \mathbf{Z}_{>0}$ relative to $m_{III}^{red} = 24$ (T4.1), where $A := \{p \in \mathbf{Z}_{>0} \mid$ prime number, $p \equiv 1(24)$ and $\exists x^4 \equiv (p)\}$, $B := \{p \in \mathbf{Z}_{>0} \mid$ prime number, $p \equiv 1(24)$ and $\forall x^4 \not\equiv (p)\}$, $U := \{p \in \mathbf{Z}_{>0} \mid$ prime number, $p \equiv 5(24)\}$ and $V := \{p \in \mathbf{Z}_{>0} \mid$ prime number, $p \equiv 5(24)\}$. Then one has

$$c_{III}(n) \quad = \quad \frac{\sqrt{-1}}{4} \left(c^{(+)}(n) - c^{(-)}(n) \right)$$

where

$$
\begin{aligned}
c^{(\chi)}(n) &= \prod_{1,A}(k_{1,A}+1)\prod_{1,B}\left((k_{1,B}+1)(-1)^{k_{1,B}}\right)\prod_{5}(k_5+1)\left(\sqrt{-1}\right)^{\chi\left(\sum k_{5,U}-\sum k_{5,V}\right)} \\
&\quad\times\prod_{7,13,23}\left(\frac{k_7+1}{2}\right)\left(\frac{k_{13}+1}{2}\right)\left(\frac{k_{23}+1}{2}\right) \\
&\quad\times\prod_{11,17,19}\left(\left(\frac{k_{11}+1}{2}\right)\left(\frac{k_{17}+1}{2}\right)\left(\frac{k_{19}+1}{2}\right)(-1)^{\frac{k_{11}}{2}+\frac{k_{17}}{2}+\frac{k_{19}}{2}}\right)
\end{aligned}
$$

for $\chi \in \{\pm\}$. Note that $c(n)=0$ if $\sum_5 k_5$ is even. In particular, $c(n^2)=0$ for all $n \in \mathbf{Z}_{>\mathbf{0}}$.

IV. Type $D_4^{(1,1)}$

$\eta_{IV}(\tau) := \eta(2\tau)^4\eta(\tau)^{-2} = \sum_n c_{IV}(n)q^n, \quad m_{IV}^{red}=4.$

Let $n=\prod_1 p_1^{k_1}\prod_3 p_3^{k_3}$ $(k_i\in\mathbf{Z}_{\geq 0})$ be a prime decomposition of $n\in\mathbf{Z}_{>0}$ relative to $m_{IV}^{red}=4$ (T4.1). Then one has

$$c_{IV}(n) = \prod_1 (k_1+1)\prod_3\left(\frac{k_3+1}{2}\right),$$

V. Types $G_2^{(1,3)}$, $G_2^{(3,1)}$

$\eta_V(\tau) := \eta(18\tau)\eta(6\tau) = \sum_n c_V(n)q^n, \quad m_V^{red}=6.$

Let $n=\prod_{1,A}p_{1,A}^{k_{1,A}}\prod_{1,B}p_{1,B}^{k_{1,B}}\prod_5 p_5^{k_5}$ $(k_i\in\mathbf{Z}_{\geq 0})$ be a prime decomposition of $n\in\mathbf{Z}_{>0}$ relative to $m_V^{red}=6$ (T4.1), where $A := \{p\in\mathbf{Z}_{>0} \mid \text{prime numbers}, p\equiv 1(6) \text{ and } \exists x^3\equiv 2(p)\}$, and $B := \{p\in\mathbf{Z}_{>0} \mid \text{prime numbers}, p\equiv 1(6) \text{ and } \forall x^3\not\equiv 2(p)\}$. Then one has

$$c_V(n) = \prod_{1,A}(k_{1,A}+1)\prod_{1,B}\left(\frac{k_{1,B}+1}{3}\right)\prod_5\left(\frac{k_5+1}{2}\right),$$

VI. Type $E_6^{(1,1)}$

$\eta_{VI}(\tau) := \eta(9\tau)^3\eta(3\tau) = \sum_n c_{VI}(n)q^n, \quad m_{VI}^{red}=3.$

Let $n=\prod_1 p_1^{k_1}\prod_2 p_2^{k_2}$ $(k_i\in\mathbf{Z}_{\geq 0})$ be a prime decomposition of $n\in\mathbf{Z}_{>0}$ relative to $m_{VI}^{red}=3$ (T4.1). Then one has

$$c_{VI}(n) = \prod_1 (k_1+1)\prod_2\left(\frac{k_2+1}{2}\right),$$

VII. Type $E_7^{(1,1)}$

$\eta_{VII}(\tau) := \eta(32\tau)^2\eta(16\tau)\eta(8\tau)^{-1} = \sum_n c_{VII}(n)q^n, \quad m_{VII}^{red}=8$

Let $n = \prod_1 p_1^{k_1} \prod_3 p_3^{k_3} \prod_5 p_5^{k_5} \prod_7 p_7^{k_7}$ $(k_i \in \mathbf{Z}_{\geq 0})$ be a prime decomposition of $n \in \mathbf{Z}_{>0}$ relative to $m_{VII}^{red} = 8$ (T4.1). Then one has

$$c_{VII}(n) \quad = \quad \frac{1}{4}\left(c^{(+)}(n) - c^{(-)}(n)\right)$$

where

$$c^{(\chi)}(n) \quad = \quad \prod_1 (k_1+1) \prod_3 \left((k_3+1)\chi^{k_3}\right) \prod_{5,7} \left(\frac{k_5+1}{2}\right)\left(\frac{k_7+1}{2}\right),$$

for $\chi \in \{\pm\}$. Note that $c(n) = 0$ if $\sum_3 k_3$ is even. In particular $c(n^2) = 0$ for all $n \in \mathbf{Z}_{>0}$.

VIII. Type $E_8^{(1,1)}$

$\eta_{VIII}(\tau) := \eta(72\tau)\eta(36\tau)\eta(24\tau)\eta(12\tau)^{-1} = \sum_n c_{VIII}(n)q^n, \quad m_{VIII}^{red} = 12.$

Let $n = \prod_1 p_1^{k_1} \prod_5 p_5^{k_5} \prod_7 p_7^{k_7} \prod_{11} p_{11}^{k_{11}}$ $(k_i \in \mathbf{Z}_{\geq 0})$ be a prime decomposition of $n \in \mathbf{Z}_{>0}$ relative to $m_{VIII}^{red} = 12$ (T4.1). Then one has

$$c_{VIII}(n) \quad = \quad \frac{1}{4}\left(c^{(+)}(n) - c^{(-)}(n)\right)$$

where

$$c^{(\chi)}(n) \quad = \quad \prod_1 (k_1+1) \prod_5 \left((k_5+1)\chi^{k_5}\right) \prod_{7,11} \left(\frac{k_7+1}{2}\right)\left(\frac{k_{11}+1}{2}\right),$$

for $\chi \in \{\pm\}$. Note that $c(n) = 0$ if $\sum_5 k_5$ is even. In particular $c(n^2) = 0$ for all $n \in \mathbf{Z}_{>0}$.

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