PRINCIPAL Γ-CONE FOR A TREE

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ABSTRACT. For any tree Γ, we introduce Γ-cones and solve a problem of enumeration of chambers contained in two particular, called the *principal*, Γ-cones. The problem can be reformulated combinatorially: the enumeration of linear extensions of the two bipartite (called principal) orderings on Γ. We characterize the principal Γ-cones by the *strict maximality* of the number of their chambers, and give a formula for this maximal number by a *finite sum of hook length formulae*. We explain the formula through the *block decomposition* of the principal Γ-cones. The results have their origin and applications in the study of topology related to Coxeter groups.

Let $V_\Pi := \oplus_{\alpha \in \Pi} \mathbb{R} v_\alpha / \mathbb{R} \sum_{\alpha \in \Pi} v_\alpha$ be the space of all configurations in $\mathbb{R}$ labeled by a finite set Π (see §1), on which the permutation group $\mathfrak{S}(\Pi)$ acts irreducibly as a reflection group. The space V_Π decomposes into chambers cut by the system of all hyperplanes fixed by reflections on V_Π associated to the transpositions in $\mathfrak{S}(\Pi)$. Given a graph Γ, having Π as its vertex set, the system of reflection hyperplanes associated to the transpositions of vertices on the edges of Γ cuts V_Π into components. Each component, which we call a Γ-*cone*, is subdivided into chambers. If Γ is a tree, through a use of the principal decomposition of Γ (§3), we choose two particular Γ-cones, called the *principal Γ-cones*. The main results of this article are i) *a characterization of the principal cones by the strict maximality of the number of their chambers* (Theorem 3.2), ii) *an enumeration formula of the chambers in the principal cones in terms of* Γ (Theorem 4.1) and iii) *an explanation of the formula by the block decomposition of the principal cone* (Cororally to Theorem 5.1).

When Γ is a Coxeter diagram $\Gamma(W)$ of a finite Coxeter group W, the principal $\Gamma(W)$-cone was introduced in [S1] in the study of the real bifurcation set, where the origin and the applications (e.g. homotopy groups of complex configuration spaces, topological types of Morsification, see §6 and [S1,3] for more details) for the present study lie.

The principal Γ-cone for an arbitrary tree Γ, introduced in the present article, is its generalization. As we shall see, the three main results can be formulated and proven purely in terms of the tree Γ and not of the group W (e.g. §5 *Example*). Therefore, in the present article, we develop the general framework for the principal Γ-cone and their chambers in a purely combinatorial manner separated from [S1].

The contents of the present paper are as follows. In §1, we fix the basic notation related to Γ-cones for any graph Γ. In §2, we prepare two assertions to count the number of chambers in a Γ-cone. After §3, we assume that Γ is a tree. In §3, we introduce the two principal orientations on Γ and the associated two principal Γ-cones E_Γ, and prove the first main Theorem of the present paper: *the principal Γ-cones are the Γ-cones which contain strictly maximal number of chambers.* In §4, as the second main Theorem, we give the formula enumerating chambers in each of the principal Γ-cone in terms of Γ. *The formula is a sum of a finite number of terms where each term is a hook length formula for a rooted tree.* We give an alternative proof of the formula in §5, where *this finite sum formula is geometrically explained by the finite block decomposition of the principal Γ-cone.* In §6, we explain a relation of the principal Γ(W)-cone with the real bifurcation set of a Coxeter group W [S1], which motivated the present study. At the end of §6, we compare Γ-cones with a somewhat similar concept: Springer cones [Ar1][Sp], and clarify the relationship between the two.

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1. The Γ-cones and their chamber decomposition

For a finite graph Γ, we fix notation and terminology of Γ-cones and their chamber decomposition. They naturally correspond to some combinatorial structures on the graph Γ (e.g. [G-Z], [St1]).

Let Π be a finite set with $\#\Pi = l \geq 1$. A *configuration* in ℝ labeled by Π is a map $v : \Pi \to \mathbb{R}$ up to the equivalence by the translation automorphism of ℝ. The set of all configurations labeled by Π is given by the vector space of rank $l-1$:

(1) $$V_\Pi := \oplus_{\alpha \in \Pi} \mathbb{R} v_\alpha / \mathbb{R} \cdot v_\Pi$$

where $v_\alpha \ (\alpha \in \Pi)$ is the delta function on Π at α (i.e. $v_\alpha(\beta) = \delta_{\alpha\beta} \ \forall \alpha, \beta$) and $v_\Pi := \sum_{\alpha \in \Pi} v_\alpha$ is the unit function on Π (i.e. $v_\Pi(\alpha) = 1 \ \forall \alpha$). The permutation group $\mathfrak{S}(\Pi)$ acts on $\{v_\alpha\}_{\alpha \in \Pi}$ fixing v_Π, and, hence, the action extends linearly to V_Π. Let $\{\lambda_\alpha\}_{\alpha \in \Pi}$ be the dual basis of $\{v_\alpha\}_{\alpha \in \Pi}$, so that the differences $\lambda_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ for $\alpha, \beta \in \Pi$ are well-defined linear forms on V_Π (forming the root system of type A_{l-1}). The zero locus $H_{\alpha\beta}$ of $\lambda_{\alpha\beta} \ (\alpha \neq \beta)$ in V_Π is the reflection hyperplane of the reflection action induced by the transposition (α, β). The complement of the union $\cup_{\alpha,\beta \in \Pi, \alpha \neq \beta} H_{\alpha\beta}$ in V_Π decomposes into $l!$ connected components, called *chambers* (in fact, Weyl chambers of type A_{l-1}, see *Remark* 1. below). The set of all chambers is naturally bijective to the set

$Ord(\Pi) := \{$all *linear orderings* on the set $\Pi\}$ by the correspondence:

$c := \{\alpha_1 <_c \ldots <_c \alpha_l\} \in Ord(\Pi) \; \leftrightarrow \; C_c := \cap_{i=1}^{l-1} \{v \in V_\Pi \mid \lambda_{\alpha_i \alpha_{i+1}}(v) < 0\}.$

Here, the order-relation with respect to $c \in Ord(\Pi)$ is denoted by $<_c$, and the corresponding chamber is denoted by C_c. If we denote by $-c$ the reversed ordering of c, then one has $C_{-c} = -C_c$.

A *graph* Γ on Π is a one-dimensional simplicial complex whose set of vertices is Π. An edge connecting vertices α and β (if it exists) is denoted by $\overline{\alpha\beta} = \overline{\beta\alpha}$. The set of all edges of Γ is denoted by $Edge(\Gamma)$. By an abuse of notation, we shall sometimes denote the set of vertices by $|\Gamma|$, and write "a vertex $\alpha \in \Gamma$" instead of "a vertex $\alpha \in \Pi$".

Definition. A Γ-*cone* is a connected component of $V_\Pi \setminus \cup_{\overline{\alpha\beta} \in Edge(\Gamma)} H_{\alpha\beta}$. By definition, each Γ-cone is subdivided into chambers.

Remark 1. One should not confuse the chambers in the present paper, which are of type A_{l-1} living in the $(l-1)$-dimensional space V_Π, with the chambers of the Weyl group $W(\Gamma)$ of the graph Γ ([B, VI,1.5]), which live in the l-dimensional space $\oplus \mathbb{R}\Pi$ (see also §6 *Note* 2).

The Γ-cones and the chambers contained in a Γ-cone are described using *acyclic orientations* on Γ defined as follows: a collection o of orientations $\alpha <_o \beta$ for all edges $\overline{\alpha\beta} \in Edge(\Gamma)$ is called *acyclic* if the oriented graph (Γ, o) does not contain an oriented cycle (see [St2]). In the present paper, we shall consider only acyclic orientations.

An acyclic orientation o on Γ defines, by transitive closure, a partial ordering on Π, which, for an abuse of notation, we shall denote by $\leq_o$. (however, $\alpha <_o \beta$, i.e. $\alpha \leq_o \beta$ and $\alpha \neq \beta$, may not imply the existence of an edge $\overline{\alpha\beta}$). We shall denote $o_1 \leq o_2$ if $\alpha <_{o_1} \beta$ implies $\alpha <_{o_2} \beta$. Put

(2) $Or(\Gamma) := \{$all acyclic orientations on $\Gamma\}.$

The followings are immediate consequences of the definition and are well-known [G-Z]. For a convenience of the reader, we sketch the proofs.

Assertion 1.1. 1. *For an orientation* $o \in Or(\Gamma)$, *define a cone:*

(3) $E_o := \cap_{\overline{\alpha\beta} \in Edge(\Gamma) \text{ oriented as } \alpha <_o \beta} \{v \in V_\Pi \mid \lambda_{\alpha\beta}(v) < 0\}.$

Then E_o is a Γ-cone. The correspondence $o \mapsto E_o$ induces a bijection

(4) $Or(\Gamma) \simeq \{\Gamma\text{-cones}\}.$

2. *A chamber C_c for $c \in Ord(\Pi)$ is contained in the Γ-cone E_o for $o \in Or(\Gamma)$ if and only if c is a linear extension of o, i.e. $o = c|_{Edge(\Gamma)}$.*

3. *The reflection hyperplane $H_{\alpha\beta}$ for $\alpha, \beta \in \Pi$ intersects with the cone E_o if and only if α and β are disconnected in Γ and the orientation o on Γ induces an acyclic orientation on the quotient graph $\Gamma/\sim$, where $\Gamma/\sim$ is obtained from Γ by the identification of α and β.*

Proof. 1. For any $o \in Or(\Gamma)$, let us show that $E_o \neq \emptyset$, that is: there exists a map $v : \Pi \to \mathbb{R}$ such that $v(\alpha) < v(\beta)$ if $\alpha <_o \beta$. We prove this by an induction on l. Acyclicity of (Γ, o) implies an existence of a *minimal* vertex $\alpha \in \Pi$: for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$, one has $\alpha <_o \beta$. Put $\Pi' := \Pi \setminus \{\alpha\}$. Then clearly $o' := o|_{\Pi'}$ is an orientation on the graph $\Gamma' := \Gamma|_{\Pi'}$. By the induction hypothesis, there exists a map $v' : \Pi' \to \mathbb{R}$ preserving the sub-orientation o'. Then, v is an extension of v' by choosing the value $v(\alpha)$ from the non-empty set $\mathbb{R} \setminus \cup_{\beta \in \Pi', \overline{\alpha\beta} \in Edge(\Gamma)} [v'(\beta), \infty)$.

Conversely, for a given Γ-cone E, define the edge $\overline{\alpha\beta} \in Edge(\Gamma)$ to have the orientation $\alpha <_E \beta$ if $\lambda_{\alpha\beta}|_E < 0$. This defines an acyclic orientation o_E on Γ. These establish the bijection (4).

2. The inclusion $C_c \subset E_o$ is equivalent to the inclusions $C_c \subset \{\lambda_{\alpha\beta} < 0\} \Leftrightarrow \alpha <_c \beta$ for any oriented edge $\overline{\alpha\beta}$ with $\alpha <_o \beta$.

3. That $H_{\alpha\beta} \cap E_o \neq \emptyset$ is equivalent that there exists a map $v : \Pi \to \mathbb{R}$ satisfying $v(\alpha) = v(\beta)$ and the inequalities (3). Apply again the argument in 1. to this setting. $\square$

According to the previous assertion, we put

$$(5) \qquad \Sigma(o) := \{ \, c \in Ord(\Pi) \mid o = c|_{Edge(\Gamma)} \, \},$$

and identify $\Sigma(o)$ with the set of chambers contained in E_o. Let us introduce a numerical invariant for the orientation $o \in Or(\Gamma)$:

$$(6) \qquad \sigma(E_o) := \sigma(o) := \#\Sigma(o) = \#\{\text{chambers contained in } E_o\}.$$

If we denote by $-o$ the reversed orientation of o, one has $E_{-o} = -E_o$ and, therefore, $\Sigma(-o) = -\Sigma(o)$ and $\sigma(-o) = \sigma(o)$.

In order to obtain the smallest oriented graph, which gives the same cone, we introduce the *reduced oriented graph* o_{red}, or the *Hasse diagram*: an oriented edge $\alpha <_o \beta$ of the oriented graph o is called *reducible* if there is a sequence $\alpha_0 = \alpha, \alpha_1, \cdots, \alpha_{k-1}, \alpha_k = \beta \in \Pi$ for some $k \in \mathbb{Z}_{>1}$ such that $\alpha_{i-1} <_o \alpha_i$ for $i = 1, \cdots, k$. Then the *reduction* o_{red} of o, also called the *Hasse diagram* of o, is uniquely defined from o by

$o_{red} :=$ the oriented graph obtained from o by deleting
all reducible edges.

One easily observes that i) *the associated cones coincide:* $E_o = E_{o_{red}}$, ii) *there is a bijection between the edges of o_{red} and the $(l\text{–}2)$-dimensional faces of $E_o = E_{o_{red}}$.* Consequently, one has: iii) $E_o = E_{o'}$ *for oriented graphs o and o' on Π, if and only if $o_{red} = o'_{red}$.*

Remark 2. As described above, the geometry of the Γ-cones in V_Π has a natural relation with the combinatorics (partially ordered structures) on the set Π. The enumeration of $\sigma(o)$ is *the basic problem of enumeration of linear extension of a partially ordered set* (e.g., see [St1]).

On the other hand, a Γ-cone with the subdivision into chambers appears naturally in the study of finite Coxeter group W as follows, where Π stands for a *simple generator system* of W. Then, a linear ordering $\alpha_1 <_c \alpha_2 <_c \cdots <_c \alpha_l$ of Π defines a *Coxeter element* $\alpha_1 \cdots \alpha_l \in W$. Two Coxeter elements coincide if the corresponding chambers belong to the same $\Gamma(W)$-cone for the Coxeter-Dynkin diagram $\Gamma(W)$ on Π. The principal $\Gamma(W)$-cone $E_{\Gamma(W)}$, which we shall introduce in §3, has the particular geometric significance, for which we refer to [S1] (see §7).

It may be worthwhile to mention that a choice of the orientations on the Coxeter-Dynkin diagram $\Gamma(W)$ plays often an important role in the studies related to root systems (recent examples: a stability condition on a triangulated category [K-S-T], the quantized Toda equations [E]).

2. A DECOMPOSITION FORMULA

We prepare two Assertions which are to help calculating $\sigma(o)$. They are used in the proof of the Theorems in §3,4 and 5. The reader may skip this section at first reading.

For any $o \in Or(\Gamma)$, $\alpha \in \Pi$ and $r \in \mathbb{Z}_{\geq 0}$, we put

(7) $\qquad \Sigma(o, \alpha, r) := \{c \in \Sigma(o) \mid \#\{\beta \in \Pi \mid \alpha <_c \beta\} = r\},$

(8) $\qquad \sigma(o, \alpha, r) := \#\Sigma(o, \alpha, r).$

Obviously, one has the disjoint decomposition $\Sigma(o) = \coprod_{r=0}^{l-1} \Sigma(o, \alpha, r)$ for any $\alpha \in \Pi$ so that $\sigma(o) = \sum_{r=0}^{l-1} \sigma(o, \alpha, r)$.

1. Let $\Gamma_1, \cdots, \Gamma_k$ ($k \in \mathbb{Z}_{>0}$) be graphs, which contain the unique common vertex α. Let us denote by

(9) $\qquad\qquad\qquad \Gamma_1 \coprod_\alpha \cdots \coprod_\alpha \Gamma_k,$

the graph obtained by the disjoint union of the graphs Γ_i ($i = 1, \cdots, k$) up to an identification of the common vertex α.

Assertion 2.1. *Let Γ be a graph decomposing as (9). For an orientation $o \in Or(\Gamma)$, put $o_i := o|_{\Gamma_i} \in Or(\Gamma_i)$ ($1 \leq i \leq k$) and $l_i := \#\Gamma_i$ ($1 \leq i \leq k$) so that $l = \sum_i l_i - k + 1$. Then, for any $r \in \mathbb{Z}_{\geq 0}$, one has the following:*
(10)
$$\sigma(o, \alpha, r) = \sum_{\substack{r_1, \cdots, r_k \in \mathbb{Z}_{\geq 0} \\ r_1 + \cdots + r_k = r}} \sigma(o_1, \alpha, r_1) \cdots \sigma(o_k, \alpha, r_k) \binom{r_1 + \cdots + r_k}{r_1, \cdots, r_k} \binom{l - 1 - r_1 - \cdots - r_k}{l_1 - r_1 - 1, \cdots, l_k - r_k - 1},$$

where $\binom{r_1 + \cdots + r_k}{r_1, \cdots, r_k} := (r_1 + \cdots + r_k)! / r_1! \cdots r_k!$ is the multinomial coefficient. By summing the formula (10) for all $r \in \mathbb{Z}_{\geq 0}$, one obtains the following:
(11)
$$\sigma(o) = \sum_{r_1, \cdots, r_k \in \mathbb{Z}_{\geq 0}} \sigma(o_1, \alpha, r_1) \cdots \sigma(o_k, \alpha, r_k) \binom{r_1 + \cdots + r_k}{r_1, \cdots, r_k} \binom{l - 1 - r_1 - \cdots - r_k}{l_1 - r_1 - 1, \cdots, l_k - r_k - 1}.$$

Proof. It is sufficient to prove only (10).

Consider the projection $\Sigma(o) \to \Sigma(o_1) \times \cdots \times \Sigma(o_k)$, $c \mapsto (c|_{\Gamma_i})_{i=1,\cdots,k}$. The projection decomposes into projections

$$(12) \qquad \Sigma(o,\alpha,r) \to \coprod_{\substack{r_1,\cdots,r_k \in \mathbb{Z}_{\geq 0} \\ r_1+\cdots+r_k=r}} \Sigma(o_1,\alpha,r_1) \times \cdots \times \Sigma(o_k,\alpha,r_k)$$

for $r \in \mathbb{Z}_{\geq 0}$. Let us see that the cardinality of the inverse image of a point $(c_1,\cdots,c_k) \in \Sigma(o_1,\alpha,r_1) \times \cdots \times \Sigma(o_k,\alpha,r_k)$ is equal to the product of two combination number: $\binom{r_1+\cdots+r_k}{r_1,\cdots,r_k}\binom{l-1-r_1-\cdots-r_k}{l_1-r_1-1,\cdots,l_k-r_k-1}$.

Put $\Gamma_i^+ := \{\beta \in |\Gamma_i| \mid \alpha <_{c_i} \beta\}$ and $\Gamma_i^- := \{\beta \in |\Gamma_i| \mid \beta <_{c_i} \alpha\}$. Then, an ordering $c \in \Sigma(o,\alpha,r)$ is in the inverse image, if the $r = r_1+\cdots+r_k$ elements $\amalg_{i=1}^k \Gamma_i^+$ lie in the right hand side of α and the $l-r-1 = (l_1-r_1-1)+\cdots+(l_k-r_k-1)$ elements $\amalg_{i=1}^k \Gamma_i^-$ lie in left hand side of α with respect to c, and the restrictions $c|_{\Gamma_i^\pm}$ are equal to the pre-fixed linear orderings $c_i^\pm := c_i|_{\Gamma_i^\pm}$ for $i = 1,\cdots,k$. Thus the set of c in the inverse image is bijective to $\Sigma(\amalg_{i=1}^k c_i^+) \times \Sigma(\amalg_{i=1}^k c_i^-)$, where $\amalg_{i=1}^k c_i^\pm$ is the partial ordering structure on $\Gamma^\pm := \amalg_{i=1}^k \Gamma_i^\pm$. Since $\Sigma(\amalg_{i=1}^k c_i^+)$ is just the set of shuffles of k sets of cardinalities $r_1,\ldots,r_k$, its cardinality is given by the combination number, i.e. $\sigma(\amalg_{i=1}^k c_i^+) = \binom{r_1+\cdots+r_k}{r_1,\cdots,r_k}$.

Similarly, one has $\sigma(\amalg_{i=1}^k c_i^-) = \binom{l-1-r_1-\cdots-r_k}{l_1-r_1-1,\cdots,l_k-r_k-1}$. □

2.   A vertex $\alpha \in \Pi$ is called *maximal* (resp. *minimal*) with respect to $o \in Or(\Gamma)$, if $\beta <_o \alpha$ (resp. $\alpha <_o \beta$) for any edge $\overline{\alpha\beta} \in Edge(\Gamma)$ at α.

Assertion 2.2. *If α is maximal with respect to o, then one has*
$$\sigma(o,\alpha,0) \geq \sigma(o,\alpha,1) \geq \cdots \geq \sigma(o,\alpha,l-2) \geq \sigma(o,\alpha,l-1).$$
If α is minimal with respect to o, then one has
$$\sigma(o,\alpha,0) \leq \sigma(o,\alpha,1) \leq \cdots \leq \sigma(o,\alpha,l-2) \leq \sigma(o,\alpha,l-1).$$
If α is non-isolated in Γ, the smallest terms in the sequences are zero.

Proof. We show only the first case. The latter case is shown similarly.

It is sufficient to show that there is an injection map $\Sigma(o,\alpha,r) \to \Sigma(o,\alpha,r-1)$ for $r > 0$. In fact the map is constructed as follows: let $c = \{A <_c \alpha <_c \beta <_c B\} \in \Sigma(o,\alpha,r)$ where $\beta \in \Pi$ and A and B are such linear sequence of inequalities of elements of Π that the length of B is equal to $r-1$ (this is possible since $r \geq 1$). Then we set $c' := \{A <_c \beta <_c \alpha <_c B\} \in \Sigma(o,\alpha,r-1)$ where c' is well defined since α is maximal. The correspondence $c \mapsto c'$ is clearly injective.

If α is non-isolated, then the set $\Sigma(o,\alpha,l-1)$ is empty, since there exists a vertex $\beta \in \Pi$ such that $\overline{\beta\alpha} \in Edge(\Gamma)$ and $\beta <_o \alpha$ and hence for any $c \in \Sigma(o)$ one has $\beta <_c \alpha$ and $c \notin \Sigma(o,\alpha,l-1)$. □

3. PRINCIPAL Γ-CONES

A graph Γ is called a *tree* if it is connected and has no cycle. For a tree Γ, we introduce two particular Γ-cones, called the principal Γ-cones. The first main result of the present paper, formulated in Theorem (3.2), is to characterize the principal Γ-cones.

The following is a characterization of trees in terms of Γ-cones.

Assertion 3.1. *Let Γ be a graph on Π. Then, $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ forms a system of coordinate hyperplanes of V_Π if and only if Γ is a tree.*

Proof. For each edge $\overline{\alpha\beta}$ of Γ, we choose one of $\lambda_{\alpha\beta}$ or $\lambda_{\beta\alpha}$. Then, it is immediate that i) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ is linearly independent if and only if Γ does not contain a cycle, and ii) $\{\lambda_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ spans the dual space of V_Π if and only if Γ is connected. $\square$

Consequently, a Γ-cone E_o is simplicial (i.e. the cone over a simplex) if and only if o_{red} is a tree. From now on in the present paper, we shall assume that Γ is a tree on Π. Then the system of coordinate hyperplanes $\{H_{\alpha\beta}\}_{\overline{\alpha\beta}\in Edge(\Gamma)}$ cuts the vector space V_Π into 2^{l-1}-number of orthants, each of which is simplicial. A question is a characterization of the Γ-vector: $(\sigma(o))_{o\in Or(\Gamma)}$ of size 2^{l-1}. One distinguished property of the Γ-vector is that it contains a unique (up to change the sign of orientations, see *Note.* below) maximal entry, which we explain now.

Definition. Let Γ be a tree (or more generally, a connected graph).

1. A *principal decomposition of Γ* is the ordered pair $\{\Pi_1, \Pi_2\}$ of subsets of Π such that i) one has the disjoint decomposition:

$$(13) \qquad\qquad \Pi = \Pi_1 \amalg \Pi_2,$$

ii) each Π_i is totally disconnected (discrete) in Γ.

2. A *principal orientation* on Γ is an element in $Or(\Gamma)$ attached to a principal decomposition $\{\Pi_1, \Pi_2\}$ as follows:

$$(14) \qquad o_{\Pi_1,\Pi_2} := \{\ \alpha <_{o_{\Pi_1,\Pi_2}} \beta \ \text{ for } \overline{\alpha\beta} \in Edge(\Gamma) \text{ with } \alpha \in \Pi_1, \beta \in \Pi_2\}.$$

3. A *principal Γ-cone* is the Γ-cone attached to a principal orientation o_{Π_1,Π_2}. That is:

$$(15) \qquad
\begin{aligned}
E_{\Pi_1,\Pi_2} \ &:= \ E_{o_{\Pi_1,\Pi_2}} \\
&= \ \{v \in V_\Pi \mid \lambda_{\alpha\beta}(v) < 0 \text{ for } \overline{\alpha\beta} \in Edge(\Gamma) \text{ with } \alpha \in \Pi_1, \beta \in \Pi_2\}.
\end{aligned}$$

It is easy to see that there exist exactly two principal decompositions for any tree $\Gamma \neq \emptyset$, and that if one is given by $\{\Pi_1, \Pi_2\}$ then the other one is given by $\{\Pi_2, \Pi_1\}$. For simplicity, we shall confuse the expression (13) with the principal decomposition $\{\Pi_1, \Pi_2\}$.

4. Since $o_{\Pi_2,\Pi_1} = -o_{\Pi_1,\Pi_2}$ and $E_{\Pi_2,\Pi_1} = -E_{\Pi_1,\Pi_2}$, two principal Γ-cones (see *Note.* below) are isomorphic to each other as abstract cones by the multiplication of -1. The isomorphism preserves the subdivisions into chambers. The isomorphism class of the pair of the cones and its subdivision into chambers is called the *principal Γ-cone*. For an abuse of notation, it is denoted by

$$(16) \qquad\qquad E_\Gamma := E_{\Pi_1,\Pi_2} \simeq E_{\Pi_2,\Pi_1}.$$

Note. If Γ is $\Gamma(A_1) =$ one point graph, then Γ has only one orientation, say o_{A_1}. Therefore, the two principal orientations o_{Π_1,Π_2} and o_{Π_2,Π_1} coincide with o_{A_1}, i.e. $o_{\Pi_1,\Pi_2} = o_{\Pi_2,\Pi_1} = \varnothing_{A_1}$. The V_Π is the zero vector space $\{0\}$, which is equal to the only principal cone $E_{\Gamma(A_1)} = \{0\}$. The cone consists of single chamber $C := \{0\}$, i.e. $\Sigma(o_{A_1}) = \{C\}$ and $\sigma(o_{A_1}) = 1$. *Except for this case, there are always two principal Γ-cones, which are in the opposite position with respect to the origin in V_Π.*

The following is the first main theorem of the present paper, which characterizes the principal Γ-cone and justifies its naming "principal".

Theorem 3.2. *Let Γ be a tree on Π. The principal Γ-cone E_Γ is the Γ-cone which contains the strictly maximal number of chambers. That is: a Γ-cone E_o for $o \in Or(\Gamma)$ is principal, i.e. o is one of the two orientations o_{Π_1,Π_2} or o_{Π_2,Π_1}, if and only if $\sigma(o) = \sigma(\Gamma)$, where*

$$(17) \qquad\qquad \sigma(\Gamma) := \max\{\sigma(p) \mid p \in Or(\Gamma)\}.$$

Proof. With the results established in §2, the proof is straightforward: suppose $o \in Or(\Gamma)$ is not principal, that is: there exist $\alpha, \beta, \gamma \in \Pi$ with $\gamma <_o \alpha <_o \beta$. Actually, Γ decomposes as $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$, where Γ_+ (resp. Γ_-) is a full subgraph of Γ containing α and any connected component of $\Gamma \setminus \{\alpha\}$ which contains a vertex β s.t. $\alpha <_o \beta$ (resp. $\alpha >_o \beta$). By the assumption on o, one has $\Gamma_\pm \neq \emptyset$.

Put $o_+ := o|_{\Gamma_+} \in Or(\Gamma_+)$ and $o_- := o|_{\Gamma_-} \in Or(\Gamma_-)$.

Assertion 3.3. *Define a new orientation $\tilde{o} \in Or(\Gamma)$ by the following rule: $\tilde{o}$ agrees with o_+ on Γ_+ and with $-o_-$ on Γ_-. Then $\sigma(\tilde{o}) > \sigma(o)$.*

Proof. For a proof of the Assertion, we apply the formula (11) in Assertion 2.1 to the decomposition $\Gamma = \Gamma_+ \coprod_\alpha \Gamma_-$ and to $o, \tilde{o} \in Or(\Gamma)$:

$$\sigma(o) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+,\alpha,r_+)\sigma(o_-,\alpha,r_-)C_{r_+,\ r_-}C_{l_+-r_+,\ l_--r_-},$$

$$\sigma(\tilde{o}) = \sum_{r_+=0}^{l_+} \sum_{r_-=0}^{l_-} \sigma(o_+,\alpha,r_+)\sigma(o_-,\alpha,r_-)C_{r_+,\ l_--r_-}C_{l_+-r_+,\ r_-},$$

where $l_+ := \#\Gamma_+ - 1 > 0$, $l_- := \#\Gamma_- - 1 > 0$ such that $l = l_+ + l_- + 1$, and the notation $C_{r_+,\,r_-}$ means the binomial coefficients $\frac{(r_+ + r_-)!}{r_+!\,r_-!}$.

We want to calculate the difference $\sigma(\tilde{o}) - \sigma(o)$ term-by-term. Observe that the terms for $r_+ = l_+/2$ (if l_+ is even) and the terms for $r_- = l_-/2$ (if l_- is even) in the two formulae give the same value and so cancel each other in the difference. Therefore, we decompose the region $[0, l_+] \times [0, l_-]$ of the summation index (r_+, r_-) into 4 regions according to whether r_+ is larger or less than $l_+/2$ and r_- is larger or less than $l_-/2$.

For an index (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we consider 4 indices (r_+, r_-), (r_+, r_-^*), (r_+^*, r_-) and (r_+^*, r_-^*) in the 4 regions simultaneously, where $r_+^* := l_+ - r_+$ and $r_-^* := l_- - r_-$. Let us explicitly write down the difference between these 4 terms in $\sigma(\tilde{o})$ and in $\sigma(o)$:

$$
\begin{aligned}
& \sigma(r_+)\sigma(r_-)C_{r_+,r_-^*}C_{r_+^*,r_-} + \sigma(r_+)\sigma(r_-^*)C_{r_+,r_-}C_{r_+^*,r_-^*} \\
+ \; & \sigma(r_+^*)\sigma(r_-)C_{r_+^*,r_-^*}C_{r_+,r_-} + \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*,r_-}C_{r_+,r_-^*} \\
- \; & \sigma(r_+)\sigma(r_-)C_{r_+,r_-}C_{r_+^*,r_-^*} - \sigma(r_+)\sigma(r_-^*)C_{r_+,r_-^*}C_{r_+^*,r_-} \\
- \; & \sigma(r_+^*)\sigma(r_-)C_{r_+^*,r_-}C_{r_+,r_-^*} - \sigma(r_+^*)\sigma(r_-^*)C_{r_+^*,r_-^*}C_{r_+,r_-},
\end{aligned}
$$

where we used the simplified notation $\sigma(r_+) := \sigma(o_+, \alpha, r_+)$, $\sigma(r_+^*) := \sigma(o_+, \alpha, r_+^*)$, $\sigma(r_-) := \sigma(o_-, \alpha, r_-)$ and $\sigma(r_-^*) := \sigma(o_-, \alpha, r_-^*)$.

One can factorize this difference as follows:

$$
(\sigma(r_+) - \sigma(r_+^*))(\sigma(r_-^*) - \sigma(r_-))(C_{r_+,r_-}C_{r_+^*,r_-^*} - C_{r_+,r_-^*}C_{r_+^*,r_-}).
$$

Let us examine the sign of the factors and demonstrate that the product turns out to be non-negative. First, recall that the vertex α is minimal in Γ_+ and maximal in Γ_- by definition. Note also that $r_+ < l_+/2 < r_+^*$ and $r_- < l_-/2 < r_-^*$. Therefore, applying §2 Assertion 2.2, we observe that $(\sigma(r_+^*) - \sigma(r_+))(\sigma(r_-) - \sigma(r_-^*)) \geq 0$. Next, let us examine the last factor. For this purpose, we use the proportion of the two terms in the last factor:

$$
\frac{C_{r_+,r_-}C_{r_+^*,r_-^*}}{C_{r_+,r_-^*}C_{r_+^*,r_-}} = \frac{(r_+^* + r_-^*)!}{(r_+ + r_-^*)!} \cdot \frac{(r_+ + r_-)!}{(r_+^* + r_-)!}.
$$

Using the fact that $r_+ < r_+^*$, one has $r_+^* + r_-^* > r_+ + r_-^*$ and $r_+ + r_- < r_+^* + r_-$. Hence, the expression can be reduced to $\prod_{k=r_++1}^{r_+^*} \frac{r_-^* + k}{r_- + k}$, where each factor is larger than 1 since $r_- + k < r_-^* + k$ and the number of the factors is $r_+^* - r_+ > 0$ so that the result is always larger than 1. These together imply that the difference of the 4 terms is non-negative.

By summing up terms for all indices (r_+, r_-) in the region $[0, l_+/2) \times [0, l_-/2)$, we see that the difference $\sigma(\tilde{o}) - \sigma(o)$ is non-negative. To show that it is strictly positive, let us calculate the term for $(r_+, r_-) = (0, 0)$.

Then, §2, Assertion 2.2 again, one has $\sigma(r_+) = \sigma(r_-^*) = 0$. Since $l_+, l_- > 0$ (non-principality of σ), one obtains a rather big number:

$$\sigma(o_+, \alpha, l_+)\sigma(o_-, \alpha, 0)(C_{l_+, l_-} - 1) \neq 0.$$

This completes the proof of the Assertion. $\qquad\square$

The Assertion says that if an orientation o on Γ is not principal, it can not attain the maximal value $\sigma(\Gamma)$ of $\sigma(o)$ for $o \in Or(\Gamma)$. In fact, starting from any orientation $o \in Or(\Gamma)$, and by a successive application of the construction in the Assertion, one arrives at one of the principal orientations. Since $E_{\Pi_1, \Pi_2} \simeq E_{\Pi_1, \Pi_2}$, one has $\sigma(o_{\Pi_1, \Pi_2}) = \sigma(o_{\Pi_1, \Pi_2})$. This number gives the maximal value $\sigma(\Gamma)$.

This completes the proof of the Theorem. $\qquad\square$

We shall call $\sigma(\Gamma)$ the *principal number* of Γ.

Remark 3. Some particular cases of Theorem 3.2 were known already.

If Γ is a linear graph of type A_l, then the (principal) Γ-cones coincide with the (principal) Springer cone of type A_{l-1} (see [Ar],[Sp] and the latter half of §7 of the present paper). In that case, the result is shown in [Sp, Prop.3], [S-Y-Z, Theo.1.2., (2)], [N],[Br].

Remark 4. Let $\Gamma = \amalg_{i=1}^k \Gamma_i$ be the decomposition of a forest into trees. For an orientation o on Γ, put $o_i := o|_{\Gamma_i}$. Since $\sigma(o) = \Pi_i \sigma(o_i)\binom{\sum \#\Gamma_i}{\#\Gamma_1, \cdots, \#\Gamma_k}$, the maximal number of chambers in a Γ-cone is attained by the orientations o such that each o_i is a principal orientation on Γ_i.

Example. The principal cone $E_{\Gamma(D_4)}$ consists of 6 chambers forming a hexagon. The $\Gamma(D_4)$-vector is $(\sigma(o))_{o \in Or(\Gamma(D_4))} = 2(6, 2, 2, 2)$.

The principal cone $E_{\Gamma(A_4)}$ consists of 5 chambers forming a spoon graph. The $\Gamma(A_4)$-vector is $(\sigma(o))_{o \in Or(\Gamma(A_4))} = 2(5, 3, 3, 1)$.

Let Γ be a cyclic graph of 4 vertices. Even though Γ is not a tree, the Γ-vector contains the maximal entry: $(\sigma(o))_{o \in Or(\Gamma)} = 2(4, 2, 2, 1, 1, 1, 1)$, where (∗) *the maximal number is attained by the Γ-cones corresponding to principal decompositions* (13). Actually, a principal decomposition for an arbitrary Γ may not exist. However, conjecturally, (∗) holds for any connected graph Γ which admits a principal decomposition.

Remark 5. ([S2]) Assume that a connected graph Γ admits a principal decomposition. Consider the lattice L_Γ spanned by Π with the symmetric bilinear form as in the usual convention in the theory of root systems. Then the "Coxeter element" defined as the product of reflections attached to the vertex in the order of a principal order (recall *Remark* 1) is i) semi-simple of finite order, or ii) quasi-unipotent if and only if i) Γ is one of the Coxeter-Dynkin diagram for a finite Coxeter group or ii) Γ is one of the affine Coxeter-Dynkin diagram, respectively.

4. Enumeration of chambers in the principal Γ cone

As the second main result of the present paper, we give an enumeration formula (19) for the principal number $\sigma(\Gamma)$. The formula depends on a choice of a principal decomposition (13). It is formulated as i) a sum whose summation index runs over certain equivalence classes $\widetilde{Ord}(\Pi_1)$ of all linear orderings on Π_1 and ii) each summand is the quotient of $(\#\Gamma)!$ by a product of cardinalities of certain subgraphs.

We will present two proofs of the formula (19). In the present section, we give a direct proof based on the principal Γ-ordering on Π. The proof in §5 is based on a decomposition of the principal Γ-cone into blocks attached to newly introduced rooted tree structures on Π and applying the hook length formula of Knuth [K2,p70].

We start with the definition of the equivalence $\sim$ on the set $Ord(\Pi_1)$.

Let $d \in Ord(\Pi_1)$ be an ordering on Π_1. For $v \in \Pi_1$, put

$$(18) \qquad \Gamma_{d,v} \quad := \quad \text{the connected component of } \Gamma \backslash \{w \in \Pi_1 \mid w <_d v\} \\ \text{containing } v.$$

In particular, one has $\Gamma_{d,v} = \Gamma$ for the smallest element v of Π_1.

Definition. Two orderings $d, d' \in Ord(\Pi_1)$ are called *equivalent* if $\Gamma_{d,v} = \Gamma_{d',v}$ for all $v \in \Pi_1$. The equivalence class of d is denoted by $\tilde{d}$ and the set of all equivalence classes is denoted by $\widetilde{Ord}(\Pi_1)$.

Theorem 4.1. *Let Γ be a tree on Π. Choose a principal decomposition (13). Then the principal number $\sigma(\Gamma)$ defined in (17) is given by*

$$(19) \qquad\qquad \sigma(\Gamma) = (\#\Gamma)! \sum_{\tilde{d} \in \widetilde{Ord}(\Pi_1)} \frac{1}{\prod_{v \in \Pi_1} \#\Gamma_{d,v}},$$

where the terms in RHS are well-defined since $\Gamma_{d,v}$ depends only on the equivalence class $\tilde{d}$ of $d \in Ord(\Pi_1)$ and on $v \in \Pi_1$.

Proof. Before we start with the proof of the formula, we reformulate the equivalence $\sim$ in terms of the partial orderings on the set Π_1.

Assertion 4.2. *Let a linear ordering $d \in Ord(\Pi_1)$ on Π_1 be given. For any two vertices $v, v' \in \Pi_1$, one has the following:*
i) There are only three cases:

$$\Gamma_{d,v} \cap \Gamma_{d,v'} = \begin{cases} \emptyset \\ \Gamma_{d,v} \\ \Gamma_{d,v'}. \end{cases}$$

ii) If $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$, then one has $v' \leq_d v$.

iii) The next three conditions are equivalent:
    a) $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$,   *b)* $\Gamma_{d,v} \subset \Gamma_{d,v'}$,   *c)* $v \in \Gamma_{d,v'}$.

Proof. i) Since d is a linear ordering, we may assume $v' <_d v$. The fact that $\Gamma \backslash \{w \in \Pi_1 \mid w <_d v\} \subset \Gamma \backslash \{w \in \Pi_1 \mid w <_d v'\}$ implies that the component $\Gamma_{d,v}$ is either contained in the component $\Gamma_{d,v'}$ or they are disjoint. Accordingly, the intersection is either $\Gamma_{d,v}$ or an empty set.

ii) Suppose the contrary $v' \not\leq_d v$. Then the fact that d is totally ordered implies $v' >_d v$. Then, by the construction, $\Gamma_{d,v'}$ cannot contain v. This contradicts the assumption $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v}$.

iii) The implications: a) $\Rightarrow$ b) $\Rightarrow$ c) are trivial. Assume c). This implies $\Gamma_{d,v} \cap \Gamma_{d,v'} \neq \emptyset$. Suppose, further, $\Gamma_{d,v} \cap \Gamma_{d,v'} \neq \Gamma_{d,v}$. Then i) implies $\Gamma_{d,v} \cap \Gamma_{d,v'} = \Gamma_{d,v'} \neq \Gamma_{d,v}$, and, hence, $\Gamma_{d,v'} \subset \neq \Gamma_{d,v}$. This means that $\Gamma_{d,v'}$ is a connected component obtained by deleting strictly more vertices than those for $\Gamma_{d,v}$. This is possible only when $v <_d v'$. Then, $v \notin \Gamma_{d,v'}$. A contradiction to the assumption c) ! $\square$

Definition. To the equivalence class $\tilde{d}$ in $\widetilde{Ord}(\Pi_1)$ of $d \in Ord(\Pi_1)$, we attach a *partial ordering* on Π_1:   for $v, v' \in \Pi_1$, put

(20) $$v' \leq_{\tilde{d}} v$$
       $\overset{\text{def}}{\Leftrightarrow}$ the conditions a), b) and c) in Assertion 4.2 iii) hold.

In other words, *there is no order relation* $\leq_{\tilde{d}}$ *between* $v, v' \in \Pi_1$ *if* $\Gamma_{d,v} \cap \Gamma_{d,v'} = \emptyset$, *otherwise the order relation* $\leq_{\tilde{d}}$ *agrees with* $\leq_d$.

Assertion 4.3. *Let* $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ *be given. For any* $v \in \Pi_1$, *the set of predecessors* $\{w \in \Pi_1 \mid w <_{\tilde{d}} v\}$ *is totally ordered by* $\leq_{\tilde{d}}$.

Proof. Suppose $w_i <_{\tilde{d}} v$ ($i = 1, 2$). This means $v \in \Gamma_{d,w_i}$ (Assertion 4.2 iii) c)), and hence $\Gamma_{d,w_1} \cap \Gamma_{d,w_2} \neq \emptyset$. Then, Assertion 4.2 i) implies that either $w_1 \leq_{\tilde{d}} w_2$ or $w_1 \geq_{\tilde{d}} w_2$. $\square$

We obtain the following characterization of the partial ordering $<_{\tilde{d}}$.

Assertion 4.4. *For two orderings* $d, d' \in Ord(\Pi_1)$, *the following two conditions are equivalent.*
    a) One has the equality $\Gamma_{d,v} = \Gamma_{d',v}$ *for all* $v \in \Pi_1$, *i.e.* $\tilde{d} = \tilde{d'}$.
    b) The partial orderings $\leq_{\tilde{d}}$ *and* $\leq_{\tilde{d'}}$ *on the set* Π_1 *coincide.*

Proof. We have only to show that the partial ordering $<_{\tilde{d}}$ determines the set $\Gamma_{d,v}$ for $v \in \Pi_1$. First, we note that the set $\Gamma_{d,v}$ is given by

$$\Gamma_{d,v} = \left(\Gamma_{d,v} \cap \Pi_1\right) \cup \bigcup_{w \in \Gamma_{d,v} \cap \Pi_1} Nbd(w),$$

where a neighborhood of a point $w \in \Gamma$ is defined by

$$(21) \qquad Nbd(w) := \{u \in \Pi \mid^{\exists} \overline{wu} \in Edge(\Gamma)\}.$$

(*Proof.* The inclusion $\subset$ follows from the connectivity of $\Gamma_{d,v}$. The opposite inclusion $\Gamma_{d,v} \supset Nbd(w)$ for $w \in \Gamma_{d,v} \cap \Pi_1$ follows also from the connectivity of $\Gamma_{d,v}$.$\square$). Here, we note that the RHS is determined only from the tree Γ and the set $\Gamma_{d,v} \cap \Pi_1$, but no ordering is involved. On the other hand, due to Assertion 4.2 iii) c), one has

$$\Gamma_{d,v} \cap \Pi_1 := \{w \in \Pi_1 \mid v \leq_{\tilde{d}} w\}.$$

Thus, $\Gamma_{d,v}$, as a set, is determined by the partial ordering $\leq_{\tilde{d}}$. $\qquad \square$

Due to Assertion 4.4, from now on, we shall identify the equivalent class $\tilde{d}$ with the partial ordering $\leq_{\tilde{d}}$ on Π_1. Thus, *the set $\widetilde{Ord}(\Pi_1)$ is regarded as the set of certain partial orderings on Π_1* (see *Remark* 6). Finally, we give an explicit description of the graph $\Gamma_{d,v}$ from $\leq_{\tilde{d}}$.

Assertion 4.5. *Let a linear ordering for $d \in Ord(\Pi_1)$ be given. One has the equality $\Gamma_{d,v} = \Gamma_{\tilde{d},v}$ for any $v \in \Pi_1$, where*

$$(22) \qquad \Gamma_{\tilde{d},v} \quad := \quad \begin{array}{l} \textit{the connected component of } \Gamma \setminus \{w \in \Pi_1 \mid w <_{\tilde{d}} v\} \\ \textit{containing } v. \end{array}$$

Proof. Since the total ordering d is a refinement of $\tilde{d}$, one has the inclusion $\Gamma_{d,v} \subset \Gamma_{\tilde{d},v}$. To show the opposite inclusion, it is sufficient to show that if $v \not\leq_{\tilde{d}} w$, then w does not belong to $\Gamma_{\tilde{d},v}$.

We may assume $w \not\leq_{\tilde{d}} v$, otherwise $w \notin \Gamma_{\tilde{d},v}$ is trivial. Consider the totally ordered set $\{u \in \Pi_1 \mid u <_{\tilde{d}} w, \ u <_{\tilde{d}} v\}$ (c.f. Fact iv)). Since it contains at least the minimum element of d and non-empty, it contains the unique maximal element, say u_0. By definition, v and w belong to the connected component $\Gamma_{d,u_0} \subset \Gamma_{\tilde{d},u_0}$. However, since they belong to different components of $\Gamma_{d,u_0} \setminus \{u_0\}$, they also belong to different components of $\Gamma_{\tilde{d},u_0} \setminus \{u_0\}$ (since $\Gamma_{\tilde{d},u_0}$ is a tree). $\qquad \square$

Let us return to the proof of the Theorem. The formula (19) is shown by induction on $l = \#\Gamma$. We first prepare an induction formula.

Let Γ be a tree. For a given principal decomposition $\{\Pi_1, \Pi_2\}$ and an attached principal orientation o_{Π_1,Π_2}, we enumerate the set $\Sigma(o_{\Pi_1,\Pi_2})$.

By definition, for any total ordering $d \in \Sigma(o_{\Pi_1,\Pi_2})$, the smallest element belongs to Π_1. Therefore, we have a decomposition:

$$\Sigma(o_{\Pi_1,\Pi_2}) = \coprod_{v \in \Pi_1} \Sigma(o_{\Pi_1,\Pi_2}, v <)$$

where $\Sigma(o_{\Pi_1,\Pi_2}, v <) := \{d \in \Sigma(o_{\Pi_1,\Pi_2}) \mid v$ is the smallest element in $d.\}$.
Put $\sigma(o_{\Pi_1,\Pi_2}, v <) := \#\Sigma(o_{\Pi_1,\Pi_2}, v <)$ so that one has

$$\sigma(\Gamma) = \sigma(o_{\Pi_1,\Pi_2}) = \sum_{v \in \Pi_1} \sigma(o_{\Pi_1,\Pi_2}, v <).$$

For $w \in Nbd(v)$, let us denote by Γ_{vw} the connected component of $\Gamma \setminus \{v\}$ containing w. One has the decomposition $\Gamma \setminus \{v\} = \coprod_{w \in Nbd(v)} \Gamma_{vw}$.

Applying (10) in Assertion 2.1 for $\alpha = v$ and $r = l - 1 = \sum_{w \in Nbd(v)} r_w$, $r_w := \#\Gamma_{vw}$, we obtain: $\sigma(o_{\Pi_1,\Pi_2}, v <) = (l-1)! \prod_{w \in Nbd(v)} \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}$.
Summing over all vertices $v \in \Pi_1$, we obtain the induction formula:

$$(23) \qquad \frac{\sigma(\Gamma)}{(\#\Gamma)!} = \frac{1}{l} \sum_{v \in \Pi_1} \prod_{w \in Nbd(v)} \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!}.$$

By the induction hypothesis, for any $v \in \Pi_1$ and $w \in Nbd(v)$, we have already the formula for Γ_{vw}:

$$*) \qquad \frac{\sigma(\Gamma_{vw})}{(\#\Gamma_{vw})!} = \sum_{\tilde{d}_w \in \widetilde{Ord}(\Gamma_{vw} \cap \Pi_1)} \frac{1}{\prod_{u \in \Gamma_{vw} \cap \Pi_1} \#(\Gamma_{vw})_{\tilde{d}_w,u}}.$$

The substitution of $*$) into RHS of (23) gives a formula summing the terms: $\frac{1}{l} \prod_{w \in Nbd(v)} \frac{1}{\prod_{u \in \Gamma_{vw} \cap \Pi_1} \#(\Gamma_{vw})_{\tilde{d}_w,u}}$, where the summation index $v \times \{\tilde{d}_w\}_{w \in Nbd(v)}$ runs over the set $\bigcup_{v \in \Pi_1} \left(v \times \prod_{w \in Nbd(v)} (\widetilde{Ord}(\Gamma_{vw} \cap \Pi_1)) \right)$.

To the index $v \times \{\tilde{d}_w\}_{w \in Nbd(v)}$, we attach the partial ordering $\tilde{d}$ of the set Π_1 defined by the rule a) v is the smallest element, b) $\tilde{d}$ agrees with $\tilde{d}_w$ on the set $\Gamma_{vw} \cap \Pi_1$ for $w \in Nbd(v)$, and c) there is no order relation between $\Gamma_{vw} \cap \Pi_1$ and $\Gamma_{vw'} \cap \Pi_1$ for different $w, w' \in Nbd(v)$.

The correspondence $v \times \{\tilde{d}_w\}_{w \in Nbd(v)} \mapsto \tilde{d}$ gives a bijection:

$$\bigcup_{v \in \Pi_1} \left(v \times \prod_{w \in Nbd(v)} (\widetilde{Ord}(\Gamma_{vw} \cap \Pi_1)) \right) \simeq \widetilde{Ord}(\Pi_1),$$

where the opposite correspondence is given by the restriction map.

On the other hand, the term $\frac{1}{l} \prod_{w \in Nbd(v)} \frac{1}{\prod_{u \in \Gamma_{vw} \cap \Pi_1} \#(\Gamma_{vw})_{\tilde{d}_w,u}}$ for the index $v \times \{\tilde{d}_w\}_{w \in Nbd(v)}$ coincides with the term $\frac{1}{\prod_{u \in \Pi_1} \#\Gamma_{\tilde{d},u}}$ in (19) given by the corresponding partial ordering $\tilde{d}$. This means that the substitution of $*$) into RHS of (23) gives RHS of the formula (19).

This completes the proof of the Theorem 4.1. $\qquad \square$

Remark 6. In the next §5, the set $\widetilde{Ord}(\Pi_1)$ plays again quite an important role, where we regard $\widetilde{Ord}(\Pi_1)$ as the set of reduced oriented graphs (see Theorem 5.1 and its corollaries).

5. Block decomposition of the principal Γ-cone

As the third main result of the present paper, we decompose a principal Γ-cone $E_{o_{\Pi_1,\Pi_2}}$ into blocks, where each block is a simplicial cone associated to a *rooted tree* and is characterized combinatorially. The number of chambers in a block is given by the hook length formula. Thus, we obtain an alternative but intrinsic proof of the formula (19).

Definition. A reduced oriented graph (Γ, o) is called a *rooted tree* if
  i) There exists a unique minimal vertex $v_o \in \Gamma$ with respect to o.
  ii) Any vertex $(\neq v_o)$ of Γ has a unique immediate predecessor.
The smallest vertex v_o is called the *root* of (Γ, o). *The definition implies that Γ is a tree.* Conversely, a pair of a tree Γ and a vertex v_o of Γ determines a unique rooted tree structure having v_o as its root.

We return to the setting of §4: Γ is a tree on Π, $\Pi = \Pi_1 \amalg \Pi_2$ is a principal decomposition (13), and $\widetilde{Ord}(\Pi_1)$ is the set of certain partial orderings on Π_1 (recall §4 Definition (20), the equivalence of a) and b) in Assertion 4.4 and the paragraph following to Assertion 4.4). We, further, identify the partial ordering $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ with its reduced oriented graph $\tilde{d} = \tilde{d}_{red}$ on the set Π_1 (= the Hasse diagram, see the paragraph before *Remark* 1. in §1). That is: $\widetilde{Ord}(\Pi_1)$ is regarded as a set of certain reduced oriented graph structures on Π_1. The followings are reformulations of what we have shown in §4.

Fact. a) *Any element $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ is a rooted tree structure on Π_1.*
b) *For any total ordering $d \in Ord(\Pi_1)$, there exists a unique $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ such that d is a linear extension of $\tilde{d}$.*
c) *The system $\{E_{\tilde{d}}\}_{\tilde{d} \in \widetilde{Ord}(\Pi_1)}$ is a simplicial cone decomposition of V_{Π_1}.*

Proof. a) Let $\tilde{d}$ be the equivalence class of $d \in Ord(\Pi_1)$. The smallest element, say v_d, of d is also the smallest with respect to $\tilde{d}$, since $\Gamma_{v_d} = \Gamma$ contains all Π_1 (c.f. Assertion 4.2, ii) c)). Then, the uniqueness of the predecessor for an element $v \neq v_d$ follows from Assertion 4.3.

  b) This follows from the definition in §4 of $\widetilde{Ord}(\Pi_1)$, where any total ordering d belongs to the unique equivalence class $\tilde{d}$.

  c) This follows from a), b) and Assertion 3.1.   □

In the following Theorem 5.1 and its corollary, we lift (in a suitable sense) the above Facts to the principal Γ-cone E_{Π_1,Π_2}.

  First, let us define the lifting $\widetilde{od}$ of $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ by

$$(24) \qquad\qquad \tilde{d} \mapsto \widetilde{od} := (o_{\Pi_1,\Pi_2} \cup \tilde{d})_{red},$$

where $(o_{\Pi_1,\Pi_2} \cup \tilde{d})_{red}$ is the reduction, i.e. Hasse diagram (recall §1), of the oriented graphs $o_{\Pi_1,\Pi_2} \cup \tilde{d}$ obtained by the union of oriented edges of o_{Π_1,Π_2} and $\tilde{d}$ (here, the union is acyclic, since i) any element of Π_2 cannot be a part of an oriented cycle since it is maximal with respect to $o_{\Pi_1,\Pi_2} \cup \tilde{d}$, and ii) the part $\tilde{d}$ on Π_1 is a tree without a cycle, Fat a)).

The following theorem gives a characterization of the elements $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ and their liftings $\tilde{od}$. Then the corollary shows that the cones $E_{\tilde{od}}$ for the liftings give arise a simplicial decomposition of the principal cone E_{Π_1,Π_2}, which we shall call the *block decomposition* of $E_{\Pi_1\Pi_2}$.

Theorem 5.1. *Let Γ be a tree on Π, and let $\Pi = \Pi_1 \amalg \Pi_2$ (13) be one of its principal decompositions.*

    1. *The set $\widetilde{Ord}(\Pi_1)$ of rooted trees on Π_1 is characterized as follows:*

$$\widetilde{Ord}(\Pi_1) = \{\ \tilde{d}\ |\ \text{i)}\ \tilde{d}\ \text{is a rooted tree structure on}\ \Pi_1,$$
$$\text{ii)}\ \tilde{d}|_{Nbd(\beta)}\ \text{is totally ordered for any}\ \beta \in \Pi_2,$$
$$\text{iii)}\ \tilde{d}\ \text{is minimal with respect to i) and ii)},$$
$$\text{i.e. if}\ \tilde{f}\ \text{satisfies i), ii) and}\ \tilde{f} \le \tilde{d}\ \text{then}\ \tilde{f} = \tilde{d}.\}$$

Here, we recall (21) for a definition of a neighborhood $Nbd(\beta)$ of β.

    2. *Let us introduce a set of rooted tree structures on Π:*

$$\widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi) := \{\ \tilde{e}\ |\ \text{i)}\ \tilde{e}\ \text{is a rooted tree structure on}\ \Pi,$$
$$\text{ii)}\ o_{\Pi_1,\Pi_2} \le \tilde{e},\ \text{i.e. if}\ x <_{o_{\Pi_1,\Pi_2}} y\ \text{then}\ x <_{\tilde{e}} y,$$
$$\text{iii) the set}\ \Pi_2\ \text{is totally disordered with repect to}\ \tilde{e},$$
$$\text{iv)}\ \tilde{e}\ \text{is minimal with respect to i), ii) and iii)},$$
$$\text{i.e. if}\ \tilde{f}\ \text{satisfies i), ii), iii) and}\ \tilde{f} \le \tilde{e}\ \text{then}\ \tilde{f} = \tilde{e}.\}$$

Then, the lifting $\tilde{d} \mapsto \tilde{od}$ (24) induces the bijection

$$(25) \qquad\qquad \widetilde{Ord}(\Pi_1)\ \simeq\ \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi).$$

    3. *The set $\Sigma(o_{\Pi_1,\Pi_2})$ of chambers in E_{Π_1,Π_2} decomposes into a union:*

$$(26) \qquad\qquad \Sigma(o_{\Pi_1,\Pi_2})\ =\ \amalg_{\tilde{e} \in \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi)} \Sigma(\tilde{e}).$$

Definition. We call the simplicial cone $E_{\tilde{e}}$ associated to a rooted tree $\tilde{e} \in \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi)$ a *block*.

The decomposition (26) can be paraphrased in terms of the block decomposition of the principal Γ-cone as follows.

Corollary. *(Block decomposition of the principal cone)*
    The principal cone E_{Π_1,Π_2} decomposes into a union of the blocks:

$$(27) \qquad\qquad \overline{E}_{\Pi_1,\Pi_2}\ =\ \coprod_{\tilde{e} \in \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi)} \overline{E}_{\tilde{e}}\ .$$

Proof. We prove only **2**, and then **1.** is its byproduct. The proof is slightly involved and is divided into steps a.i)-ii),b.i)-ii),c),d) and e).

a.i) *The lifting $\tilde{od}$ for $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ satisfies i), ii) and iii) of* **2**.

a.ii) ∗) *For any $\beta \in \Pi_2$ and for any $\tilde{d} \in \widetilde{Ord}(\Pi_1)$, the set $Nbd(\beta) \subset \Pi_1$ is totally ordered with respect to the partial ordering $\tilde{d}$.*

∗∗) *The restriction $\tilde{od}|_{\Pi_1}$ of the oriented graph $\tilde{od}$ to Π_1 is equal to $\tilde{d}$. In particular, this implies that the correspondence* (24) *is injective.*

Proof. a.i) Those ii) and iii) for $\tilde{od}$ follow immediately from the definition. The i) for $\tilde{od}$ is a consequence of a.ii) and b.ii).

a.ii)∗) For $\beta \in \Pi_2$, consider $\alpha_1, \alpha_2 \in Nbd(\beta) \subset \Pi_1$ with $\alpha_1 \neq \alpha_2$. Since $\Gamma_{\tilde{d}, \alpha_i}$ (recall (22)) contains $Nbd(\alpha_i)$, one has $\beta \in \Gamma_{\tilde{d}, \alpha_i}$ for $i = 1, 2$. That is $\Gamma_{\tilde{d}, \alpha_1} \cap \Gamma_{\tilde{d}, \alpha_2} \neq \emptyset$. Then, due to Assertion 4.2, i) and ii), either $\Gamma_{\tilde{d}, \alpha_1} \supset \Gamma_{\tilde{d}, \alpha_2}$ or $\Gamma_{\tilde{d}, \alpha_1} \subset \Gamma_{\tilde{d}, \alpha_2}$ occurs, and, hence by the definition (20), one has either $\alpha_1 <_{\tilde{d}} \alpha_2$ or $\alpha_1 >_{\tilde{d}} \alpha_2$. Thus, ∗) is shown.

a.ii)∗∗) By definition, $\tilde{od}|_{\Pi_1} \leq \tilde{d}$. If an edge $\overline{\alpha \alpha'}$ with $\alpha <_{\tilde{d}} \alpha'$ in $\tilde{d}$ is removable in $\tilde{od}$, there exists a path $\alpha_0 = \alpha <_{\tilde{od}} \alpha_1 <_{\tilde{od}} \cdots <_{\tilde{od}} \alpha_n = \alpha'$ in $\tilde{od}$ for $n \geq 2$. Since $\tilde{d}$ is reduced, there exits $1 \leq i < n$ such that $\alpha_i \in \Pi_2$. By taking largest such i, we find an edge $\alpha_i <_{\tilde{od}} \alpha_{i+1}$ with $\alpha_i \in \Pi_2$ and $\alpha_{i+1} \in \Pi_1$. This is impossible since $\tilde{od} \leq (o_{\Pi_1 \Pi_2} \cup \tilde{d})$. Thus, ∗∗) is shown. □

b.i) *Let $\tilde{e}$ be an oriented graph on Π satisfying ii) and iii) of* **2**. *Then, there does not exists a pair $\alpha \in \Pi_1$ and $\beta \in \Pi_2$ such that $\beta <_{\tilde{e}} \alpha$.*

b.ii) *Under the same assumption, $\tilde{e}$ is a rooted tree (i.e. $\tilde{e}$ satisfies i)), if and only if one has* ∗) *$\tilde{e}|Nbd(\beta)$ is totally ordered for all $\beta \in \Pi_2$ and* ∗∗) *the restriction $\tilde{e}|_{\Pi_1}$ of the oriented graph $\tilde{e}$ is a rooted tree on Π_1. In this case, the restriction to Π_1 of the ordering $\leq_{\tilde{e}}$ on Π coincides with the ordering on Π_1 generated by the oriented graph $\tilde{e}|_{\Pi_1}$ on Π_1.*

Proof. b.i) For any $\alpha \in \Pi_1$, there exists at least one $\gamma \in \Pi_2$ such that $\alpha <_{o_{\Pi_1, \Pi_2}} \gamma$. If the contrary $\beta <_{\tilde{e}} \alpha$ holds, using ii) and the transitive closure, one obtains $\beta <_{\tilde{e}} \gamma$, which contradicts to iii).

b.ii)∗) If there were no order relation among $\alpha_1, \alpha_2 \in Nbd(\beta)$, there exists some $\gamma \in \Pi$ such that $\alpha_1, \alpha_2 <_{\tilde{e}} \gamma \leq_{\tilde{e}} \beta$ and γ has at least two immediate predecessor. A contradiction! ∗∗) By b.i), the set of predecessors for any $\alpha \in \Pi_1$ in Π is equal to that in Π_1. In particular, the imediate predecessor α' is in Π_1 so that the edge $\overline{\alpha' \alpha}$ belongs to $\tilde{e}|_{\Pi_1}$.

Conversely, if $Nbd(\beta)$ is totally ordered, then $\beta \in \Pi_2$ is connected with only the maximal element of $Nbd(\beta) \subset \Pi_1$. This, together with ∗∗), implies that $\tilde{e}$ is a rooted tree. □

c) *The lifting $\tilde{od}$ for $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ satisfies iv) of Theorem 5.2.*

Proof. Assume the contrary: $\exists \tilde{f} \leq \tilde{od}$ and $\tilde{f} \neq \tilde{od}$.

We, first, show that there exists a pair $\alpha, \beta \in \Pi_1$ such that $\alpha <_{\tilde{od}} \beta$ and $\alpha \not<_{\tilde{f}} \beta$, $\alpha \not>_{\tilde{f}} \beta$. (*Proof.* By the assumption, there exist at least a pair $\alpha, \beta \in \Pi$ with $\alpha <_{\tilde{od}} \beta$ and $\alpha \not<_{\tilde{f}} \beta$. Due to iii) and b.i) for $\tilde{od}$, one has $\alpha \notin \Pi_2$. If $\alpha \in \Pi_1$ and $\beta \in \Pi_2$ then by construction of $\tilde{od}$ there exists $\gamma \in \Pi_1$ such that $\alpha <_{\tilde{d}} \gamma$ and $\gamma <_{o_{\Pi_1,\Pi_2}} \beta$. Then, one necessary has $\alpha \not<_{\tilde{f}} \gamma$, since otherwise one has $\alpha <_{\tilde{f}} \gamma <_{o_{\Pi_1,\Pi_2}} \beta$, and by ii) on $\tilde{f}$, one has the contradiction $\alpha <_{\tilde{f}} \beta$. Then, replace β by γ.)

We may assume that α is the immediate predecessor of β with respect to $\tilde{d}$, since one of the edges of the path from α to β satisfies the condition. We may assume further that the pair $\alpha, \beta \in \Pi_1$ is maximal in the sense that for any $\beta' \in \Gamma_{\tilde{d},\beta} \cap \Pi_1$ (i.e. $\beta \leq_{\tilde{d}} \beta'$) one has $\beta \leq_{\tilde{f}} \beta'$.

Since α is the immediate predecessor of β with respect to $\tilde{d}$, α is connected with a single point, say $\gamma \in \Pi_2$, of $\Gamma_{\tilde{d},\beta}$. Since $\Gamma_{\tilde{d},\beta}$ is a connected tree containing β, there exists a path in $\Gamma_{\tilde{d},\beta}$ connecting γ and β. Let $\beta' \in \Pi_1$ be the element next to γ. Since $\beta' \in \Gamma_{\tilde{d},\beta} \cap \Pi_1$, by the assumption of the maximality of the pair α, β, we have $\beta \leq_{\tilde{f}} \beta'$. On the other hand, due to ii) for $\tilde{f}$, we have also $\alpha \leq_{\tilde{f}} \gamma$ and $\beta' \leq_{\tilde{f}} \gamma$. Thus we obtain two monotonically increasing pathes attached to the two sequences $v_{\tilde{f}} \leq_{\tilde{f}} \alpha \leq_{\tilde{f}} \gamma$ and $v_{\tilde{f}} \leq_{\tilde{f}} \beta \leq_{\tilde{f}} \beta' \leq_{\tilde{f}} \gamma$. They are different, since there is no order relations between α and β with respect to $\tilde{f}$. A contradiction to that $\tilde{f}$ is a tree! Thus, iv) for $\tilde{od}$ is shown. $\square$

d) *Any element* $\tilde{e} \in \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi)$ *is a lifting* $\tilde{od}$ *for some* $\tilde{d} \in \widetilde{Ord}(\Pi_1)$.

Proof. Consider any linear extension $d \in Ord(\Pi_1)$ of $\tilde{e}|_{\Pi_1}$ (i.e. $e|_{\Pi_1} \leq d$) and its equivalence class $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ (§4 Definition). If we have shown $\tilde{d} \leq \tilde{e}|_{\Pi_1}$ (shown in the following e)), then we have done the proof since we have $\tilde{od} = (o_{\Pi_1,\Pi_2} \cup \tilde{d})_{red} \leq (o_{\Pi_1,\Pi_2} \cup \tilde{e}|_{\Pi_1})_{red} \leq \tilde{e}$ and the minimality iv) for $\tilde{e}$ implies $\tilde{od} = \tilde{e}$. Thus, the surjectivity of (25) is shown. $\square$

e) *For any* $d \in Ord(\Pi_1)$ *with* $\tilde{e}|_{\Pi_1} \leq d$, *one has* $\tilde{d} \leq \tilde{e}|_{\Pi_1}$.

Proof. For any edge $\overline{\alpha\beta}$ in $\tilde{d}$ on Π_1 with $\alpha <_{\tilde{d}} \beta$, we have to show the order relation $\alpha <_{\tilde{e}} \beta$. We show this by the "inverse induction" in the sense that we prove it by assuming the innequality $\alpha' <_{\tilde{e}} \beta'$ for all edges $\overline{\alpha'\beta'}$ in $\tilde{d}$ with the inequality: $\beta \leq_{\tilde{d}} \alpha' <_{\tilde{d}} \beta'$.

Suppose the contrally $\alpha \not<_{\tilde{e}} \beta$. Since $\alpha >_{\tilde{e}} \beta$ is not possible (else, we have $\alpha >_d \beta$, contradicting to $\alpha <_{\tilde{d}} \beta$), there should be no order relation between α and β with respect to $\tilde{e}$. The fact $\alpha <_{\tilde{d}} \beta$ implies that $\beta \in \Gamma_{d,\alpha}$ (recall (20)). The fact that α is the immediate predecessor of β means, precisely, that there exist elements $\beta' \in \Gamma_{d,\beta} \cap \Pi_1$ and $\gamma \in \Pi_2$ such that $\alpha, \beta' <_{o_{\Pi_1,\Pi_2}} \gamma$. By assumption ii) on $\tilde{e}$, we have $\alpha, \beta' <_{\tilde{e}} \gamma$.

On the other hand, by definition $\beta' \in \Gamma_{d,\beta}$ and hence $\beta \leq_{\tilde{d}} \beta'$. Then, by the inverse induction hypothesis, one obtains $\beta \leq_{\tilde{e}} \beta'$. Thus, we obtain two monotonically increasing pathes in $\tilde{e}$ connecting the bottom element $v_{\tilde{e}}$ and γ as follows: $v_{\tilde{e}} \leq_{\tilde{e}} \alpha \leq_{\tilde{e}} \gamma$ and $v_{\tilde{e}} \leq_{\tilde{e}} \beta \leq_{\tilde{e}} \beta' \leq_{\tilde{e}} \gamma$. Since, by the contraly assumption, there is no order relation between α and β with respect to $\tilde{e}$, whereas each path passes through either α or β, the two pathes are different. This is a contradiction to the fact that $\tilde{e}$ is a tree. This proves $\alpha <_{\tilde{e}} \beta$ and, hence, e) is proven. $\qquad\square$

3. For $\tilde{e} \in \widetilde{Ord}_{\Pi_1,\Pi_2}(\Pi)$, ii) implies $\Sigma(o_{\Pi_1,\Pi_2}) \supset \Sigma(\tilde{e})$. The decomposition (26) follows, since for any $d \in \Sigma(o_{\Pi_1,\Pi_2})$, there exists a unique $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ such that $(o_{\Pi_1,\Pi_2} \cup \tilde{d}) \leq d$ (**Fact** b) and c)).

Corollary is a rewriting of (26) (c.f. 2. of Assertion 1.1).

These complete the proof of Theorem 5.1 and its Corollaries. $\qquad\square$

Remark 7. The block decomposition (27) of the principal Γ-cone E_Γ depends on a choice of the principal orientation o_{Π_1,Π_2} (see Example).

As an application of the block decomposition (26) ((27)) of the principal cone, let us give an alternative proof of the formula (19). This is achieved by two steps. The first step is to recall the well known hook length formula of Knuth enumerating the chambers in a Γ-cone for a rooted tree (it is also an immediate consequence of the decomposition formula (10) with $r = \#\Gamma - 1$ and $\alpha =$ the root of the tree).

Lemma 5.2. (Knuth [K2,p70]) *Let* (Γ, o) *be a rooted tree. Then*

$$(28) \qquad\qquad \sigma(o) = \frac{(\#\Gamma)!}{\prod_{v \in \Pi} \#\Gamma_{o,v}}$$

where

$$(29) \qquad \Gamma_{o,v} := \text{ the connected component of } \Gamma_o \setminus \{w \in \Pi \mid w <_o v\} \text{ containing } v.$$

Note. The underlying oriented graph structure in $\Gamma_{\tilde{d},v}$ (22) is the principal orientation o_{Π_1,Π_2} and that for $\Gamma_{o,v}$ (29) is the rooted tree o. They are, in a sense, the most contrasting orientations. However, we show in the following a "numerical coincidence" of them.

The second step of the alternative proof of (19) is as follows. Apply (28) to $\sigma(\tilde{od})$ to count the number of chambers in $E_{\tilde{od}}$. Comparing (19) and (25), let us show the equality:

$$(30) \qquad\qquad \frac{(\#\Gamma)!}{\prod_{v \in \Pi} \#\Gamma_{\tilde{od},v}} = \frac{(\#\Gamma)!}{\prod_{v \in \Pi_1} \#\Gamma_{\tilde{d},v}}$$

for $\tilde{d} \in \widetilde{Ord}(\Pi_1)$. We note that the region of the running index v in LHS of (30) can be shrunken from Π to $\Pi_1 = \Pi \setminus \Pi_2$, since for $v \in \Pi_2$

one has $\#\Gamma_{\tilde{od},v} = 1$ because of the fact that v is maximal with respect to $o_{\Pi_1,\Pi_2} \cup \tilde{d}$. Therefore, we have only to show the formula

$$(31) \qquad\qquad\qquad \#\Gamma_{\tilde{d},v} = \#\Gamma_{\tilde{od},v}$$

for $v \in \Pi_1$. We show that the vertex sets $|\Gamma_{\tilde{d},v}|$ and $|\Gamma_{\tilde{od},v}|$ coincide (even though the graph structures are quite different, see Example).

Note the inclusion relation: $o_{\Pi_1,\Pi_2} \subset (o_{\Pi_1,\Pi_2} \cup \tilde{d}) \supset \tilde{od}$ among oriented graphs and the equality among the vertex sets: $A := \{w \in \Pi_1 \mid w <_{\tilde{d}} v\} = \{w \in \Pi \mid w <_{o_{\Pi_1,\Pi_2} \cup \tilde{d}} v\} = \{w \in \Pi \mid w <_{\tilde{od}} v\}$. Thus, one has the relation: $\Gamma_{\tilde{d},v} \subset \Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v} \supset \Gamma_{\tilde{od},v}$ among the connected components of the complements of A containing v. The sets $|\Gamma_{\tilde{d},v}|$ and $|\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}|$ coincide, since, if $v <_{\tilde{d}} w$ for $w \in \Pi_1$ then $w \in \Gamma_{\tilde{d},v}$. The sets $|\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}|$ and $|\Gamma_{\tilde{od},v}|$ coincide, since, if an element $w \in \Pi_2$ is connected with v in $\Gamma_{o_{\Pi_1,\Pi_2} \cup \tilde{d},v}$ then w is connected with v by $\tilde{od}$.

This completes the proof of the equality (31), and hence the alternative proof for (19) is completed.

Example. We illustrate the two block decompositions of type A_7 and the formula (19) for the two principal orientations on $\Gamma(A_7)$.

I. Let the principal decomposition (13) of $\Gamma(A_7)$ and the associated principal orientation $o_{A_7} := o_{\Pi_1,\Pi_2}$ be given by

There are 5 partial orderings $\tilde{d} \in \widetilde{Ord}(\Pi_1)$ and, accordingly, 5 associated rooted trees $\tilde{od}$. Three of them are illustrated as follows. Here, the black colored vertex is the root of the tree:

Two more rooted trees are obtained from $\tilde{od}_1$ and $\tilde{od}_2$ by the action of the left-right involutive diagram automorphism of $\Gamma(A_7)$. Therefore, the formula (19) of the principal number of type A_7 turns out to be

$$\sigma(A_7) = 2\sigma(\tilde{od}_1) + 2\sigma(\tilde{od}_2) + \sigma(\tilde{od}_3) = 2\frac{7!}{7 \cdot 5 \cdot 3} + 2\frac{7!}{7 \cdot 5 \cdot 3} + \frac{7!}{7 \cdot 3 \cdot 3} = 272.$$

II.   The principal decomposition of $\Gamma(A_7)$ opposite to (13) and the opposite principal orientation $-o_{A_7} := o_{\Pi_2,\Pi_1}$ are given by

$$-o_{A_7} : \begin{array}{l} \Pi_1 : \\ \Pi_2 : \end{array} \quad \text{[diagram]}$$

There are 14 partial orderings $\tilde{d} \in \widetilde{Ord}(\Pi_2)$ and, accordingly, 14 associated rooted trees $\tilde{od}$. Seven of them are illustrated as follows:

$-o_{A_7} \cup \tilde{d}_1 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_1 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_2 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_2 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_3 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_3 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_4 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_4 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_5 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_5 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_6 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_6 :$ [diagram]

$-o_{A_7} \cup \tilde{d}_7 :$ [diagram] $\xrightarrow{red}$ $\tilde{od}_7 :$ [diagram]

The remaining seven rooted trees are obtained from the above seven trees by the action of the left-right involutive diagram automorphism of $\Gamma(A_7)$. Therefore, the formula (19) of the principal number of type A_7 turns out to be

$$\sigma(A_7) = 2\big(\sigma(\tilde{od}_1) + \sigma(\tilde{od}_2) + \sigma(\tilde{od}_3) + \sigma(\tilde{od}_4) + \sigma(\tilde{od}_5) + \sigma(\tilde{od}_6) + \sigma(\tilde{od}_7)\big)$$

$$= 2\Big(\frac{7!}{7 \cdot 6 \cdot 4 \cdot 2} + \frac{7!}{7 \cdot 6 \cdot 4 \cdot 3} + \frac{7!}{7 \cdot 6 \cdot 3 \cdot 2} + \frac{7!}{7 \cdot 6 \cdot 5 \cdot 3} + \frac{7!}{7 \cdot 6 \cdot 5 \cdot 3} + \frac{7!}{7 \cdot 4 \cdot 2 \cdot 2} + \frac{7!}{7 \cdot 4 \cdot 3 \cdot 2}\Big)$$

$$= 272.$$

Note. Even though the starting diagram $\Gamma(A_7)$ is linear the blocks correspond to non-linear diagrams.

6. Geometric background

We recall briefly a theorem [S1§3], which combines the principal Γ-cones with some geometry of real bifurcation set in case of Γ being a Coxeter graph of finite type. For details, one is refereed to [S1].

Let W be a finite reflection group acting irreducibly on an $\mathbb{R}$-vector space V of rank l. Due to Chevalley Theorem [B,ch.v,5.3], the invariants $S(V^*)^W$ is freely generated by some homogeneous elements $P_1, \cdots, P_l$. Thus, the quotient variety $S_W := V/\!\!/W = Spec(S(V^*)^W)$ is a smooth affine scheme over $\mathbb{R}$ of coordinates $P_1, \cdots, P_l$. It contains the discriminant divisor D_W defined by a polynomial $\Delta_W \in S(V^*)^W$, which is the square of a basic anti-invariant [B,ch.v, 5.4]. *The Δ_W is a monic polynomial of degree l with respect to the coordinate P_l of S_W of the largest degree.* The integration $\exp(D)$ of the lowest degree vector field $D := \frac{\partial}{\partial P_l}$ on S_W (unique up to a constant factor and is called the *primitive vector field*) defines a unique (up to a scaling factor) additive group $\mathbb{G}_a$-action on S_W. The quotient $T_W := S_W /\!\!/ \mathbb{G}_a$ is an $(l-1)$-dimensional affine variety (forgetting the coordinate P_l). The restriction to D_W of the projection map $S_W \to T_W$ is an l-fold flat covering, whose ramification divisor in T_W is denoted by B_W and called the *bifurcation divisor*. The B_W decomposes into the ordinary part $B_{W,2}$ and the higher part $B_{W,\geq 3}$ according as the ramification index of the covering is equal or larger than 2.

Depending on $\varepsilon \in \{\pm 1\}$, there are real forms $T_{W,\mathbb{R}}^\varepsilon, B_{W,2,\mathbb{R}}^\varepsilon$ and $B_{W,\geq 3,\mathbb{R}}^\varepsilon$ of these schemes. On the other hand, arising from a study of eigenspaces of Coxeter elements, there is a distinguished real half axis $AO^\varepsilon \simeq \mathbb{R}_{>0}$ embedded in $T_{W,\mathbb{R}}^\varepsilon \setminus B_{W,\geq 3,\mathbb{R}}^\varepsilon$ (see [S1] for details). The connected component of $T_{W,\mathbb{R}}^\varepsilon \setminus B_{W,\geq 3,\mathbb{R}}^\varepsilon$ containing AO^ε, denoted by E_W^ε, is called the *central region*, which we determine in the following theorem.

Using P_l, consider the l-valued algebroid function $T_W \leftarrow D_W \overset{P_l|_{D_W}}{\to} \mathbb{A}$ (=affine line), defined by the polynomial equation $\Delta_W = 0$ in P_l. It is ramifying along $B_{W,\geq 3}$, but regular along $B_{W,2}$. We can indexify l branches of the function on $T_{W,\mathbb{R}}^\varepsilon$ at the base point AO^ε by the set Π of a simple generator system of W, and denote them by $\{\varphi_{\alpha,\varepsilon}\}_{\alpha \in \Pi}$ [S1].

Theorem 6.1. *Let $\Gamma(W)$ be the Coxeter graph on Π, and $E_{\Gamma(W)}$ be its principal cone. Then the algebraic correspondence $b_{W,\varepsilon} := \sum_{\alpha \in \Pi} \varphi_{\alpha,\varepsilon} \cdot v_\alpha$ from $T_{W,\mathbb{R}}^\varepsilon$ to V_Π induces a semi-algebraic homeomorphism:*

$$(32) \qquad b_{W,\varepsilon} \; : \; \overline{E}_W^\varepsilon \; \simeq \; \overline{E}_{\Gamma(W)}$$

from the closure of the central region of W to the closure of the principal cone for the Coxeter graph $\Gamma(W)$ of W on Π, and a homeomorphism:

$$(33) \qquad b_{W,\varepsilon} \; : \; E_W^\varepsilon \cap B_{W,2,\mathbb{R}}^\varepsilon \; \simeq \; E_{\Gamma(W)} \cap \left(\cup_{\alpha\beta \in \Pi} H_{\alpha\beta} \right).$$

That is: the central region E_W^ε is a simplicial cone homeomorphic to the principal $\Gamma(W)$-cone $E_{\Gamma(W)}$. Connected components of $E_W^\varepsilon \setminus B_{W,2,\mathbb{R}}^\varepsilon$ are in one to one correspondence with the set of chambers contained in the principal $\Gamma(W)$-cone $E_{\Gamma(W)}$. In particular, the number of connected components of $E_W^\varepsilon \setminus B_{W,2,\mathbb{R}}^\varepsilon$ is given by the principal number $\sigma(\Gamma(W))$.

The theorem (in a detailed form) has several important implications in the study of the topology of the configuration space S_W ([S1,3]).

Note. 1. *The correspondence b_W is independent of the choice of P_l (up to the scaling constant on the primitive vector field D).*

Proof. Since the largest exponent of W is unique, any other largest degree coordinate $\tilde{P}_l$ of S_W with $D\tilde{P}_l = 1$ is of the form $P_l + Q$ for a polynomial Q of lower degree coordinates. Then, $\tilde{\varphi}_\alpha = \varphi_\alpha + Q$ $(\alpha \in \Pi)$, whose second term is independent of α. So, the ambiguity of translation by Q is absorbed in the equivalence in the definition (1) of labeled configuration space V_Π. That is: one has $\tilde{b}_W = b_W + Qv_\Pi \equiv b_W$ in V_Π. $\square$

2. The principal $\Gamma(W)$-cone in RHS of (31) depends only on the underlying graph structure of the diagram $\Gamma(W)$ and not on the labels on the edges. The graphs $\Gamma(W)$ (forgetting the labels) of types A_l, B_l, C_l, F_4, G_2, H_3, H_4 and $I_2(p)$ are linear. Hence, the central regions E_W for them are homeomorphic to the principal cones of type A_l.

Finally in the present paper, we compare the concept of Γ-cones with a somewhat similar concept, the *Springer cones*, which we explain below.
Definition ([Ar1]). Let V_W be a real vector space with an irreducible action of a finite reflection group W. The reflection hyperplanes of W divide V_W into chambers. Let $\{H_\alpha\}_{\alpha \in \Pi}$ be the system of the walls of a chamber. A connected component of $V_W \setminus \cup_{\alpha \in \Pi} H_\alpha$ is called a *Springer cone*. A Springer cone containing the maximal number of chambers (unique up to sign [Sp1]) is called a *principal Springer cone*. This maximal number is called the *Springer number*. The Springer number has been calculated by the authors Solomon, Springer and Arnold ([So],[Sp1], [Ar1]).

There are some formal similarities between the (principal) Springer cones in V_W and the (principal) Γ-cones in V_Π (see Table below). A result similar to Theorem 3.2 is proven for Springer cones [Sp1, Prop.3].

	Springer cone	Γ-cone
The ambient vector space	V_W with W-chambers (depending on the group W)	V_Π with A_{l-1}-chambers (depending on the set Π)
The cutting hyperplanes	$\{H_\alpha\}_{\alpha \in \Pi}$ (indexed by the vertices of $\Gamma(W)$)	$\{H_{\alpha\beta}\}_{\overline{\alpha\beta} \in Edge(\Gamma)}$ (indexed by the edges of the tree Γ)

Roughly speaking, the principal Springer cones deal with the *generators* of the Artin groups, whereas the principal Γ-cones deal with the *(non-commutative) braid relations* of the Artin groups.

The only cases when a Γ-cone decomposition is simultaneously a Springer cone decomposition are listed by the following assertion.

Assertion 6.2. *For a forest Γ, the following i)–iii) are equivalent.*

i) There exists a finite Coxeter group W and a linear isomorphism: $V_\Pi \simeq V_W$ which maps chambers to chambers and the Γ-cones to the Springer-cones.

ii) The smallest number of chambers contained in a Γ-cone is equal to 1, i.e. $\inf\{\sigma(o) \mid o \in Or(\Gamma)\} = 1$.

iii) Γ is a linear graph of type A_l, and $W = W(A_{l-1})$ for $l > 1$.

Proof. i) $\Rightarrow$ ii): This follows from the definition of the Springer cone.

ii) $\Rightarrow$ iii): if a Γ-cone consists of a single chamber $\overline{C} := \{\lambda_{\alpha_1} \leq \cdots \leq \lambda_{\alpha_l}\}$, then Γ is a linear graph α_1-α_2-$\cdots$-α_l (of type A_l) on Π.

iii) $\Rightarrow$ i): If Γ is a linear graph α_1-α_2-$\cdots$-α_l, then the orientation $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ on Γ corresponds to the Γ-cone consisting only of a single chamber $\overline{C} := \{\lambda_{\alpha_1} \leq \cdots \leq \lambda_{\alpha_l}\}$ of type A_{l-1} in $V_\Pi = V_{A_{l-1}}$. $\square$

Remark 8. Assertion 6.2 is not true if Γ is not a forest (see §3 Example), since the argument ii) $\Rightarrow$ iii) fails in general.

Since Assertion 6.2, $\sigma(A_l) := \sigma(\Gamma(A_l))$ is equal to the Springer number a_{l-1} of type A_{l-1}. Since the Springer number a_n of type A_n is given by the generating function: $1 + \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n = \frac{1}{1-\sin(x)}$ ([Sp1,3.]), one has

$$(34) \qquad 1 + \sum_{n=1}^{\infty} \frac{\sigma(A_n)}{n!} x^n = 1 + \int_0^x \frac{1}{1-\sin(x)} dx = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

This formula was found by several authors repeatedly (e.g. [St, Exercise 43(c)]). Including this case, Y. Sano [Sa] gave the following explicit formula of the principal numbers for the three infinite series of Coxeter graphs of types A_n, D_n and E_n:

$$\begin{array}{rcl}
\sum_{n=1}^{\infty} \frac{\sigma(A_n)}{n!} x^n & = & \tan(\frac{x}{2} + \frac{\pi}{4}) - 1, \\
\sum_{n=3}^{\infty} \frac{\sigma(D_n)}{n!} x^n & = & (2x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) + 2 - 2x^2, \\
\sum_{n=4}^{\infty} \frac{\sigma(E_n)}{n!} x^n & = & (\frac{1}{2}x^2 - 2x + 3)\tan(\frac{x}{2} + \frac{\pi}{4}) - 3x^2 - x - 3.
\end{array}$$

Question. By an analogy to §3 Theorem, consider any system of l-reflection hyperplanes in V_W forming coordinate hyperplanes and ask a question: is there a unique (up to a sign) orthant of V_W, cut out by the hyperplanes, which contains the maximal number of chambers. The answer is apparently positive for the type A_l and $I_2(p)$ for odd $p \in 2\mathbb{Z}_{>0}$, and negative for the types B_l, C_l and $I_2(p)$ for even $p \in 2\mathbb{Z}_{>0}$.

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