Extended Affine Root Systems III
(Elliptic Weyl Groups)

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Abstract

We give a representation of an elliptic Weyl group $W(R)$ (= the Weyl group for an elliptic root system*) R) in terms of the elliptic Dynkin diagram $\Gamma(R,G)$ for the elliptic root system. The representation is a generalization of a Coxeter system: the generators are in one to one correspondence with the vertices of the diagram and the relations consist of two groups: i) elliptic Coxeter relations attached to the diagram, and ii) a finiteness condition on the Coxeter transformation attached to the diagram. The group defined only by the elliptic Coxeter relations is isomorphic to the central extension $\tilde{W}(R,G)$ of $W(R)$ by an infinite cyclic group, called the hyperbolic extension of $W(R)$.

*) an elliptic root system = a 2-extended affine root system (see the introduction and the remark at its end).

Contents

§0 Introduction

For a non-negative integer k, a k-extended affine root system is, by definition, a generalized root system belonging to a semi-positive quadratic form with k dimensional radical [Sa2, I]. It turns out that a 0-extended affine root system is a finite and hence classical root system (see, for instance [B], [H]), and that 1-extended affine root system is an affine root system in the sense of [M]. The 2-extended affine root systems are of particular interest from a view point of algebraic geometry ([Sa1]). They are classified by elliptic Dynkin diagrams and their Coxeter transformations are studied in [Sa2, I], and then, their flat invariants are introduced in [Sa2, II]. We will call 2-extended affine root systems *elliptic root systems* (see the remark at the end of this introduction). We will call the group generated by reflexions for all root vectors of an elliptic root system an *elliptic Weyl group.*

The Weyl group for a finite, affine or hyperbolic root system is well known to be a Coxeter group. That is : the group is represented by generators and relations in terms of Coxeter systems, where the relations are called the Coxeter relations (for a finite or affine root system, see [H], [B], and for a hyperbolic root system, see [Sa3]). On the other hand, for root systems of Witt index ≥ 2, there was no such known description of their Weyl groups. Then, the purpose of the present paper is to give a representation of the elliptic Weyl group, which is a generalization of the Coxeter system. Let us state a consequence of our main result.

Corollary of Theorem 1. *Let R be an elliptic root system and let $\Gamma(R, G)$ be its attached elliptic Dynkin diagram with respect to a marking G (see (1.3)). Then the elliptic Weyl group $W(R)$ is generated by the element a_α of order 2 attached to each vertex α of the diagram $\Gamma(R, G)$ and is defined by two types of relations :*
i) *a system of elliptic Coxeter relations attached to the elliptic diagram $\Gamma(R, G)$ (see (2.1)),*
ii) *a relation of the form $\tilde{c}(\Gamma(R, G))^m = 1$, where $\tilde{c}(\Gamma(R, G))$ is the hyperbolic Coxeter element (see (2.2.2)) and $m = m(R, G)$ is an integer defined from $\Gamma(R, G)$ (see (1.3.2)).*

The description of the elliptic Weyl groups for the types $A_l^{(1,1)}$, $B_l^{(1,1)}$, $C_l^{(1,1)}$ and $D_l^{(1,1)}$ was studied in [T1,2]. In the present paper, we give a proof of the description for all elliptic root systems, independent of the classification of elliptic root systems. In fact, the above description of the elliptic Weyl group is a consequence of the main theorems of the present paper:

Theorem 1. *The group $\tilde{W}(\Gamma(R, G))$ defined only by the generalized Coxeter relations is naturally isomorphic to the central extension $\tilde{W}(R, G)$ of $W(R)$ by an infinite cyclic group generated by $\tilde{c}(R, G)^m$ (see (1.6.2)).*

Theorem 2. *There exist an affine root system $(R, G)_a$ and an abelian normal subgroup $N(R, G)$ of $\tilde{W}(\Gamma(R, G))$ isomorphic to the affine root lattice of $(R, G)_a$ such that $\tilde{W}(\Gamma(R, G))/N(R, G)$ is isomorphic to the affine Weyl group of $(R, G)_a$ whose adjoint action on $N(R, G)$ is identified with the affine Weyl group action on the affine root lattice.*

Here the above central extension in Theorem 1. is known as the hyperbolic extension $\tilde{W}(R, G)$ of the Weyl group $W(R)$ (see (1.6.2)), playing a central role in the flat invariant theory for the elliptic Weyl group [Sa2] (cf. *Remark* and *Problem* at the end of the introduction).

The construction of this paper is as follows. In §1, we recall elliptic root systems and related notion such as elliptic Dynkin diagram, Coxeter element and hyperbolic extension of elliptic Weyl groups. In (2.1) of §2, the generalized Coxeter relations are introduced. The main Theorems 1 and 2 are formulated in (2.2) and (2.3), respectively. The proofs of the theorems are given in §3.

Remark. There are some reasons (which are essentially the same) why we name the 2-extended affine root system an elliptic root system.

1. The root lattice of an elliptic root system describes the lattice of vanishing cycles for a simple elliptic singularity ([Sa 1]), where the two dimensional radical of the quadratic form of the elliptic root system corresponds to the lattice of an elliptic curve.

2. The hyperbolic extension $\tilde{W}(R, G)$ acts properly discontinuously on a complex half space $\tilde{\mathbb{E}}$ of complex dimension equal to rank(R), where the orbit space of the action carries naturally the flat structure and is identified with the base space of universal unfolding of a simply elliptic singular point ([Sa2, II]).

3. Flat invariant theory for elliptic Weyl groups reveals deep connection between elliptic root systems and elliptic modular functions (see Satake [Sat1-2]. Compare also Yahiro, Yamada [Y], Pollman [P], [AABGP]).

Problem. The Dynkin diagram of a finite root system describes not only the associated root system and its Weyl group but also the associated Lie algebra [C][Se], Hecke algebra [I-M] and the Artin group (= the fundamental group of the regular orbit space of the Weyl group action on the complexified Cartan subalgebra [Br], [B-S]). Both descriptions are achieved through generators and relations, where i) the generators are attached to the vertices of the diagram, and ii) the parts of the diagram to give the (binary) relations are exactly same parts to give the Coxeter relations for the Weyl group. Therefore, the description of the elliptic Weyl group by the elliptic Coxeter relations in (2.1) (which are no more binary) seems to suggest the existence of descriptions of elliptic Lie algebra and elliptic Artin group (i.e. the fundamental group of the regular orbit space of the action of $W(R, G)$ on $\tilde{\mathbb{E}}$) associated to the elliptic diagram by generators and relations using the same part of elliptic diagram, where the power of the Coxeter element should play a role again. We ask for such descriptions of the elliptic Lie algebra and elliptic Artin group as open problems.

§1 Elliptic Root Systems

We recall a definition of *elliptic root systems* and related notion such as elliptic Dynkin diagram, Coxeter element and hyperbolic extension of the elliptic Weyl group from [Sa2, I] (in the sequel, we shall refer [Sa2, I] as [ibid]). Then we introduce a new terminology, *boundary side,* in order to describe a generalized Coxeter relation in §2. Some basic properties on elliptic root systems are summarized in Facts 0–5.

(1.1) Marked elliptic root system (R, G)

Let F be a real vector space with a symmetric bilinear form $I : F \times F \to \mathbb{R}$ of finite rank. For a non isotropic element $\alpha \in F$, put $\alpha^\vee := 2\alpha/I(\alpha,\alpha)$ and define a reflexion w_α by $w_\alpha(u) := u - I(u,\alpha^\vee)$ for $u \in F$, so that $\alpha^{\vee\vee} = \alpha$ and $w_\alpha^2 = 1$.

A set R of non isotropic vectors in F is called an *elliptic root system* if

i) I is semi-positive with $rank_{\mathbb{R}}(rad(I)) = 2$, where $rad(I) := F^\perp$.

ii) R satisfies axioms for generalized root systems belonging to I: A.1. the root lattice $Q(R)$ (:= the additive subgroup of F generated by R) satisfies $Q(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong F$, A.2. $w_\alpha(R) = R$ for all $\alpha \in R$, A.3. $I(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \mathbb{Z}$, A.4. irreducibility ([ibid, (1.2)], [Sa3]).

A subspace G of $rad(I)$ of rank 1 defined over $\mathbb{Q}$ is called a *marking.* The pair (R, G) is called a *marked elliptic root system.* The image in $F/rad(I)$ (resp. F/G) of the set R by the natural projections, denoted by R_f and R_a, respectively, forms a finite (resp. affine) root system. In the present paper as in [ibid], we consider only the case when R_a is reduced.

(1.2) Exponents

Once for all the rest of the present paper, we fix a generator a of the lattice $G \cap\ rad(I)$:

(1.2.1) $$GLlcmG \cap rad(I) = \mathbb{Z}a.$$

The generator a is unique up to a choice of sign. For any $\alpha \in R$, put

(1.2.2) $$k(\alpha) := inf\{k \in \mathbf{N} \mid \alpha + k \cdot a \in R\},$$

called the counting of α. Put

(1.2.3) $$\alpha^* := \alpha + k(\alpha) \cdot a.$$

Once for all the rest of the paper, we fix a set

(1.2.4) $$\Gamma = \{\alpha_0, \cdots, \alpha_l\}$$

of roots in R whose projection in F/G is a basis of the affine root system R_a. It is known that the Γ is unique up to an automorphism of (R, G) ([ibid (3.4)]). The Γ carries a structure of the Dynkin diagram for the affine root system R_a. Let n_α $(\alpha \in \Gamma)$ be a system of positive integers with $\gcd\{n_\alpha, \alpha \in \Gamma\} = 1$ such that the image of

(1.2.5) $$b := \sum_{\alpha \in \Gamma} n_\alpha \alpha$$

in F/G is a null root of R_a (i.e. $b \in rad(I)$). Since there exists always an element of Γ, say α_0, such that $n_{\alpha_0} = 1$ (cf. [M]), the root lattice $Q(R)$ in F has an expression:

(1.2.6) $$Q(R) = \sum_{\alpha \in \Gamma} \mathbb{Z}\alpha \oplus \mathbb{Z}a = \sum_{i=1}^{l} \mathbb{Z}\alpha_i \oplus \mathbb{Z}a \oplus \mathbb{Z}b.$$

The set of *exponents* of (R, G) is defined by the union of 0 and

$$m_\alpha := \frac{I_R(\alpha,\alpha)}{2\cdot k(\alpha)}\cdot n_\alpha \tag{1.2.7}$$

for $\alpha \in \Gamma$, where I_R is a constant multiple of I normalized such as $inf\{I_R(\alpha,\alpha) \mid \alpha \in R\}$ is equal to 2. Consider the subset of the affine diagram Γ

$$\Gamma_{\max} := \{\alpha \in \Gamma \mid m_\alpha = m_{\max}\}, \tag{1.2.8}$$

where $m_{\max} := \max\{m_\alpha \mid \alpha \in \Gamma\}$. Put

$$\Gamma^*_{\max} := \{\alpha^* \mid \alpha \in \Gamma_{\max}\}. \tag{1.2.9}$$

(1.3) Elliptic Dynkin diagram $\Gamma(R, G)$ for (R, G)

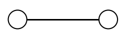
An elliptic Dynkin diagram (or, elliptic diagram) $\Gamma(R, G)$ for a marked elliptic root system (R, G) is a finite graph given by the following data:

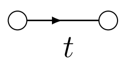
1. the vertex set of $\Gamma(R, G)$ is the union of Γ ((1.2.4)) and $\Gamma^*_{\max}$ ((1.2.9)),

$$\Gamma(R, G) = \Gamma \cup \Gamma^*_{\max}. \tag{1.3.1}$$

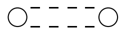
2. the bond between vertices α and β of $\Gamma(R, G)$ is given by the convention:

if $I(\alpha, \beta) = I(\beta, \alpha) = 0$,

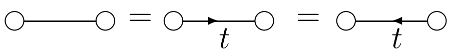

if $I(\alpha, \beta^\vee) = I(\alpha^\vee, \beta) = -1$,

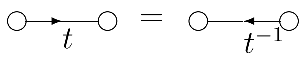


if $I(\alpha, \beta^\vee) = -t$, $\quad I(\alpha^\vee, \beta) = -1$ for $t = 2, 3$,

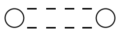

if $I(\alpha^\vee, \beta) = I(\alpha, \beta^\vee) = 2$.

We shall use a convention:



for $t = 1$, and


for $t = 2^{\pm 1}, 3^{\pm 1}$.

The bond , which we call a *double bond*, appears only between vertices $\alpha \in \Gamma_{\max}$ and $\alpha^* \in \Gamma^*_{\max}$.

Fact 0. ([ibid, (9.6) Theorem]). The diagram $\Gamma(R, G)$ is uniquely determined by the isomorphism class of (R, G) (independent of choices of the sign of a and the basis Γ). Conversely, the diagram $\Gamma(R, G)$ determines the isomorphism class of (R, G) (see Fact 4.).

Fact 1. ([ibid, (8.4)]). i) The complement $\Gamma(R, G) \setminus (\Gamma_{\max} \cup \Gamma^*_{\max}) = \Gamma \setminus \Gamma_{\max}$ is a disjoint union of A-type diagrams, say $\Gamma(A_{l_1}), \ldots \Gamma(A_{l_r})$. We have the equality:

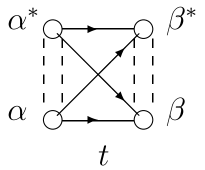
$$m(R, G) := \max\{l_1 + 1, \cdots, l_r + 1\}\{l_1 + 1, \cdots, l_r + 1\}. \tag{1.3.2}$$

ii) The exponents attached to the vertices of the component $\Gamma(A_{l_j})$ are given by the arithmetic progression : $\frac{i}{l_j+1} \cdot m_{\max}$ $(i = 1, \cdots, l_j)$.

We recall the table of elliptic diagrams for marked elliptic root systems at the end of the present paper. The number $m(R,G)$ plays a role of Coxeter number for the elliptic root system as we shall see in Fact 3.

(1.4) Boundary side

In order to define a generalized Coxeter relation in §2, we introduce a new terminology, boundary side. Consider two roots α, β in an elliptic root system R and associated roots α^* and β^* as defined in (1.2.3). Let the intersection diagram attached to them be

(1.4.1) 

for some $t = 2^{\pm 1}, 3^{\pm 1}$. Then the proportion $K(\alpha : \beta) := k(\alpha)/k(\beta) = (\alpha^* - \alpha) : (\beta^* - \beta)$ is shown to be either 1 or t ([ibid, (6.1.3)]). Obviously, $K(\alpha : \beta)K(\beta : \alpha) = 1$.

Definition. We call α the *boundary side* (or *b-side* for short) in the bond $\alpha\ \bigcirc\!\!\xrightarrow[t]{}\!\!\bigcirc\ \beta$, if $K(\alpha : \beta) = \inf\{1, t\}$.

By definition, either α or β is a b-side. The following facts show that one can determine the b-side of a bond $\alpha\ \bigcirc\!\!\xrightarrow[t]{}\!\!\bigcirc\ \beta$, in an elliptic diagram $\Gamma(R,G)$ only by the above diagram without knowing the value $K(\alpha : \beta)$.

Fact 2. i) Let α *and* $\beta \in \Gamma_{\max}$ *be connected by a bond* $\alpha\ \bigcirc\!\!\xrightarrow{t}\!\!\bigcirc\ \beta$ *for* $t = 2^{\pm 1}$. *Then α is the b-side if there are no vertices other than* α^*, β *and* β^* *which are adjascent to α in the elliptic diagram.*

ii) Let $\alpha \in \Gamma_{\max}$ *and let* $\beta \in \Gamma \setminus \Gamma_{\max}$ *be connected by a bond* $\alpha\ \bigcirc\!\!\xrightarrow{t}\!\!\bigcirc\ \beta$ *for* $t = 2^{\pm 1}$ *or* $3^{\pm 1}$, *then α is the b-side of the bond.*

For short, the above facts are paraphrased that a b-side always lies on the "boundary" of $\Gamma_{\max}$. These facts are verified immediately from the tables for $k(\alpha)$ in [ibid, §6]. They are also explained from a view point of folding of elliptic diagrams (see [ibid, §12]).

(1.5) Coxeter element $c(R,G)$

The elliptic Weyl group $W(R)$ is the subgroup of the linear isometry group $O(F,I) := \{g\imath(F) | I \circ g = I\}$ generated by the reflexion w_α for all $\alpha \in R$. It is shown that the group

$W(R)$ is generated by w_α for $\alpha \in \Gamma(R,G)$ ([ibid, §9], cf. Fact 4 below). The Coxeter element $c(R,G)$ for (R,G) is defined by the product:

$$c(R,G) := \prod_{\alpha \in \Gamma(R,G)} w_\alpha, \tag{1.5.1}$$

where the order of the product of reflexions is chosen as w_{α^*} comes next to w_α for all $\alpha \in \Gamma_{\max}$. The conjugacy class of $c(R,G)$ in $W(R)$ does not depend on the order of the product under the above condition, since the diagram obtained by collapsing each double bond in an elliptic diagram is a tree (cf. [Bo, Ch.V, §6 1. lemma 1]). ker

Fact 3. ([ibid, (9.7) Lemma A iii)]). i) The Coxeter element is of finite order $m(R,G)$. ii) The eigenvalues of the Coxeter element are 1 and $exp(2\pi\sqrt{-1}m_\alpha/m_{\max})$ for $\alpha \in \Gamma$.

We describe in Fact 4 the construction of the marked elliptic root system (R,G) from the elliptic diagram $\Gamma(R,G)$.

Fact 4. ([ibid, (9.6)]). For a given elliptic diagram $\Gamma(R,G)$, consider:
$\hat{F} :=$ *the vector space spanned by vertices of* $\Gamma(R,G)$ *over* $\mathbb{R}$,
$\hat{I} :=$ *the symmetric bilinear form on $\hat{F}$ defined (up to a positive constant factor) by the convention (1.3),*
$\hat{w}_\alpha :=$ *the reflexion w.r.t.* $\alpha \in \Gamma(R,G)$ *on* $(\hat{F},\hat{I})$,
$\hat{c} := \prod_{\alpha \in \Gamma(R,G)} \hat{w}_\alpha$, *where $\hat{w}_{\alpha^*}$ comes next to $\hat{w}_\alpha$.*
$\hat{R} := \bigcup_{\alpha \in \Gamma(R,G)} \hat{W}\cdot\alpha$, *where* $\hat{W} := <\hat{w}_\alpha, \forall \alpha \in \Gamma(R,G)>$,
$\hat{G} :=$ *the linear space spanned by $\alpha^* - \alpha$ for all $\alpha \in \Gamma_{\max}$.*
Then the space F is identified with $\hat{F}/(\hat{c}^{m(R,G)} - 1_{\hat{F}})\hat{F}$, and one has canonical isomorphisms ([ibid, (9.6) Theorem]):

$$\begin{aligned} (R,G) &\cong \text{ the image of } (\hat{R},\hat{G}) \text{ in } \hat{F}/(\hat{c}^{m(R,G)} - 1_{\hat{F}})\hat{F}, \\ W(R) &\cong \text{ the image of } \hat{W} \text{ in } GL(\hat{F}/(\hat{c}^{m(R,G)} - 1_{\hat{F}})\hat{F}). \end{aligned}$$

(1.6) Hyperbolic extension $\tilde{W}(R,G)$ of $W(R)$

We recall a concept of hyperbolic extension for a marked elliptic root system (R,G) in (F,I) [ibid, §11]. Consider the pair $(\tilde{F},\tilde{I})$ of a vector space $\tilde{F}$ over $\mathbb{R}$ and a symmetric bilinear form $\tilde{I}$ on $\tilde{F}$ such that F is a 1-codimensional subspace of $\tilde{F}$ and $\tilde{I} \mid F = I$ and $rad(\tilde{I}) = G$. Such $(\tilde{F},\tilde{I})$ exists uniquely up to an isomorphism. Let $\tilde{w}_\alpha \in O(\tilde{F},\tilde{I})$ be the reflexion w.r.t. $\alpha \in R$ considered as an element in $\tilde{F}$, and we denote by $\tilde{W}(R,G)$ the group generated by $\tilde{w}_\alpha$

for all $\alpha \in R$ and call it the *hyperbolic extension* of $W(R)$. The *hyperbolic Coxeter element* $\tilde{c}(R, G)$ is defined by the product:

$$\tilde{c}(R, G) := \prod_{\alpha \in \Gamma(R,G)} \tilde{w}_\alpha \tag{1.6.1}$$

where the order of the product of reflexions is the same as the definition of a Coxeter element $c(R, G)$ ((1.5.1)). The conjugacy class of $\tilde{c}(R, G)$ in $\tilde{W}(R, G)$ does not depend on the order of the product for the same reason in the case of $c(R, G)$.

Fact 5. ([ibid, (11.3) Lemma C ii)]). The natural map $\tilde{W}(R, G) \to W(R)$ induces a central extension :

$$1 \to K \to \tilde{W}(R, G) \to W(R) \to 1 \tag{1.6.2}$$

where the kernel K is an infinite cyclic group generated by $\tilde{c}(R, G)^m$ for $m = m(R, G)$. In particular,

$$H(R, G) := (\tilde{W}(R, G) \to W(R_f)) \tag{1.6.3}$$

is a Heisenberg group with the center generated by $\tilde{c}(R, G)^m$.

§2 The main theorems

Elliptic Coxeter relations attached to an elliptic diagram $\Gamma(R, G)$ are introduced in (2.1). The main results of the present paper, formulated in theorems 1 and 2 in (2.2) and (2.3), respectively, describe the structure of the group $\tilde{W}(\Gamma(R, G))$ defined by the generalized Coxeter relations. Their proofs are given in §3.

(2.1) Elliptic Coxeter relations

Attached to the elliptic diagram $\Gamma(R, G)$ of a marked elliptic root system (R, G), we introduce the elliptic Coxeter relations.

Generators: for each $\alpha \in \Gamma(R, G)$, we attach a generator a_α. For simplicity, we shall denote $a, a^*, b, b^*, c, c^* \cdots$ instead of $a_\alpha, a_{\alpha^*}, a_\beta, a_{\beta^*}, a_\gamma, a_{\gamma^*} \cdots$ so far as there may be no confusion.

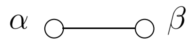
Relations: for any subdiagram of $\Gamma(R, G)$ isomorphic to one of the following list, we give a relation attached to the diagram in the following table.

0. α ○ $\qquad$ $a^2 = 1$

I.0

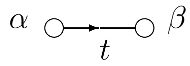
$(ab)^2 = 1$

I.1



$(ab)^3 = 1$

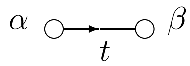
I.2



$t = 2^{\pm 1}$

$(ab)^4 = 1$

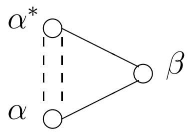
I.3



$t = 3^{\pm 1}$

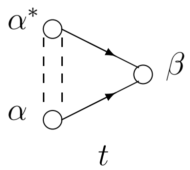
$(ab)^6 = 1$

II.1



$(aba^*b)^3 = 1$

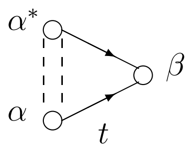
II.2



$t = 2^{\pm 1}$

$(aba^*b)^2 = 1$

II.3

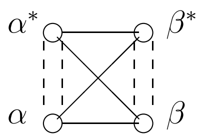


$t = 3^{\pm 1}$

$(aba^*b)^3 = 1$ and

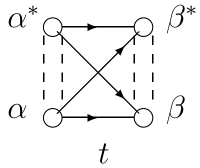
$(aba^*bab)^2 = 1$

III.1



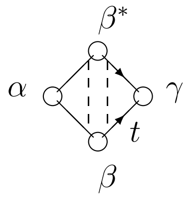
$ab^*a = a^*ba^*$

$\Leftrightarrow ba^*b = b^*ab^*$ under 0 and I.2

III.t 

$t = 2^{\pm 1}, 3^{\pm 1}$ $\qquad$ $ab^*a = a^*ba^*$,

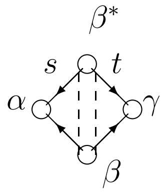
where α is the b-side of α  β in the sense of (1.4) Fact 2. i).

IV.t 

$t = 1,\ 2^{\pm 1},\ 3^{\pm 1}$ $\qquad$ $(abab^*cb^*)^2 = 1$ and $(ab^*abcb)^2 = 1$,

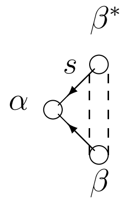
where the two relations are equivalent in case $t = 1$.

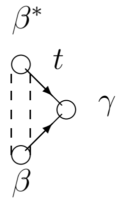
Here the relations 0 and I are well known as Coxeter relations, and the relations II, III and IV are newly introduced relations due to the double bonds in the diagram. Let us call them altogether *generalized Coxeter relations*, or *elliptic Coxeter relations*.

Remark 1. Attached to the diagram 

$t,\ s = 2^{\pm 1}$, consider the relation :

$$(ab^*abcb)^3 = 1,\ (abab^*cb^*)^3 = 1,\ (b^*ab^*cbc)^3 = 1,\ (babcb^*c)^3 = 1.$$

They can be obtained from the relations attached to its subdiagrams; 

and 

.

Remark 2. The relations obtained from elliptic Coxeter relations by substituting $a^*, b^*, \cdots$ by $a, b, \cdots$ are reduced to Coxeter relations.

(2.2)

Definition. We denote by $\tilde{W}(\Gamma(R,G))$ the group defined by the elliptic Coxeter relations given in (2.1) attached to the elliptic diagram $\Gamma(R,G)$.

Theorem 1. *The correspondence* $a_\alpha \mapsto \tilde{w}_\alpha$ *for* $\alpha \in \Gamma(R,G)$ *induces an isomorphism:*

$$\tilde{W}(\Gamma(R,G)) \quad \cong \quad \tilde{W}(R,G). \tag{2.2.1}$$

Here $\tilde{W}(R,G)$ *is the hyperbolic extension of the elliptic Weyl group* $W(R)$ *(c.f. (1.6)).*

As a consequence of the theorem, we shall get a description (2.2.3) of the elliptic Weyl group $W(R)$. Let us introduce a *hyperbolic Coxeter element* in $\tilde{W}(\Gamma(R,G))$ by

$$\tilde{c}(\Gamma(R,G)) \;=\; \prod_{\alpha\in\Gamma(R,G)} a_\alpha \tag{2.2.2}$$

where the order of product is the same as $c(R,G)$ ((1.5.1)). The conjugacy class of $\tilde{c}(\Gamma(R,G))$ in $\tilde{W}(\Gamma(R,G))$ does not depend on the choice of the order. Then, the following corollary is proven by Theorem 1 and (1.6) Fact 5, or corollaries of *Theorem* 2 stated in (2.3) (cf. *Remark* at the end of (2.3)).

Corollary. *The cyclic group generated by* $\tilde{c}(\Gamma(R,G))^m$ *is in the center of* $\tilde{W}(\Gamma(R,G))$ *for* $m = m(R,G)$. *The correspondence* $a_\alpha \mapsto w_\alpha$ *for* $\alpha \in \Gamma(R,G)$ *induces an isomorphism:*

$$\tilde{W}(\Gamma(R,G))/(\tilde{c}(\Gamma(R,G))^m) \quad \cong \quad W(R). \tag{2.2.3}$$

Remark. As a consequence of the theorem, we have the following description of the group $\hat{W} := \langle \hat{w}_\alpha, {}^\forall\alpha \in \Gamma(R,G)\rangle$ introduced in (1.5) *Fact* 4:

$$\hat{W} \cong \begin{cases} W(R) & \text{if } cod(R,G) = 1 \\ \tilde{W}(R,G) & \text{if } cod(R,G) > 1 \end{cases},$$

where $cod(R,G) := \sharp(\Gamma_{\max})$. This fact will not be used in the sequel.

Proof of Remark. Due to (3.1) Lemma 1, the generators $\hat{w}_\alpha$ of $\hat{W}$ satisfy the elliptic Coxeter relations. So we have a surjective homomorphism: $\tilde{W}(\Gamma(R,G)) \to \hat{W}$. Since $\hat{W}$ projects onto $W(R)$ (*Fact* 4), $Ker(\tilde{W}(\Gamma(R,G)) \to \hat{W})$ is contained in the center $\langle \tilde{c}(\Gamma(R,G))^{m(R,G)}\rangle$. On the other hand, we know that $\hat{c}^{m(R,G)}$ is unipotent and that its (maximal) Jordan block is of size $cod(R,G)$ (see [ibid, §11]). This implies that $\hat{c}^{m(R,G)}$ is either trivial or of infinite order according as $cod(R,G)$ is equal to 1 or > 1. □

(2.3)

Let us prepare some more notation in order to state Theorem 2. Put

$$T_\alpha := a_\alpha a_{\alpha^*} \tag{2.3.1}$$

for $\alpha \in \Gamma_{\max}$, and put

(2.3.2) $N(R,G) :=$ the smallest normal subgroup of $\tilde{W}(\Gamma(R,G))$ containing T_α for $\forall\, \alpha \in \Gamma_{\max}$.

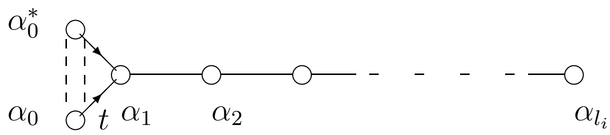
Then one has a natural isomorphism:

$$\tilde{W}(\Gamma(R,G))/N(R,G) \;\cong\; W(R_a). \tag{2.3.3}$$

(*Proof.* The L.H.S. is a group obtained from $\tilde{W}(\Gamma(R,G))$ by substituting $a^*, b^*, \cdots$,etc. by $a, b, \cdots$,etc. Therefore, in view of the *Remark* 2 at the end of (2.1), it is isomorphic to the Coxeter group associated to the affine diagram $\Gamma = \Gamma(R,G) \setminus \Gamma^*_{\max}$. The affine Weyl group $W(R_a)$ admits such description ([H]).)

Let us introduce $T_\alpha \in \tilde{W}(\Gamma(R,G))$ for all $\alpha \in \Gamma$ as follows. If $\alpha \in \Gamma_{\max}$, then T_α is defined by (2.3.1). If α belongs to a component $\Gamma(A_{l_i})$ of $\Gamma\backslash\Gamma_{\max}$ (c.f. Fact 1 in (1.3)) of the figure

(2.3.4)



for $t = 1, 2^{\pm 1}, 3^{\pm 1}$,

then we define

$$T_{\alpha_{j+1}} := a_{\alpha_{j+1}} \cdot T_{\alpha_j} \cdot a_{\alpha_{j+1}} \cdot T_{\alpha_j}^{-1}. \tag{2.3.5}$$

by induction on $0 \le j < l_i$, where T_{α_0} is already given by (2.3.1). In fact, one sees $T_\alpha \in N(R,G)$ for all $\alpha \in \Gamma$ by induction on j.

Theorem 2. *Let $N(R,G)$ be as given in (2.3.2). Then one has:*

1. *$N(R,G)$ is a free abelian group generated by T_α for all $\alpha \in \Gamma$. More precisely, one has a natural isomorphism:*

$$N(R,G) \quad\cong\quad Q((R,G)_a), \tag{2.3.6}$$

by the correspondence

$$T_\alpha \mapsto k(\alpha)\alpha^\vee \quad (\alpha \in \Gamma),$$

where $Q((R,G)_a)$ is a root lattice of an affine root system $(R,G)_a$ given in theorem -added.

2. *The adjoint action of $\tilde{W}(\Gamma(R,G))$ on $N(R,G)$, factored by $N(R,G)$, induces an equivariant isomorphism:*

$$\tilde{W}(\Gamma(R,G))/N(R,G) \simeq W((R,G)_a), \tag{2.3.7}$$

with respect to the identification (2.3.6).

3. *The power $\tilde{c}(\Gamma(R,G))^{m(R,G)}$ of the hyperbolic Coxeter element belongs to $N(R,G)$. It is expressed as*

$$\tilde{c}(\Gamma(R,G))^{m(R,G)} \;=\; \prod_{\alpha\in\Gamma((R,G)_a)} T_\alpha^{n_\alpha} \tag{2.3.8}$$

where $n_\alpha \in N$ are the coefficients of the null root of the affine root system $(R,G)_a$.

Assuming the identification (2.2.1), let us state immediate consequences of Theorem 2.

Corollary 1. *The* $N(R,G)$ *is a maximal abelian subgroup of the Heisenberg group* $H(R,G)$ *(cf. (1.6.3)).*

Corollary 2. *The center of* $\tilde{W}(\Gamma(R,G))$ *is the cyclic group generated by the null root in* $N(R,G)$.

In order to state Theorem-added, let us recall that the isomorphism classes of marked elliptic root systems are devided into four groups I $\sim$ IV in [ibid, (12.5)] from a view point of folding of elliptic diagrams.

I. $A_l^{(1,1)}$ $(l \geq 1)$, $D_l^{(1,1)}$ $(l \geq 4)$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$,

II. $B_l^{(1,2)}$ $(l \geq 3)$, $B_l^{(2,2)}$ $(l \geq 2)$, $C_l^{(1,2)}$ $(l \geq 2)$, $C_l^{(2,2)}$ $(l \geq 3)$, $BC_l^{(2,4)}$ $(l \geq 1)$, $F_4^{(1,2)}$, $F_4^{(2,2)}$, $G_2^{(1,3)}$, $G_2^{(3,3)}$,

III. $B_l^{(1,1)}$ $(l \geq 3)$, $B_l^{(2,1)}$ $(l \geq 2)$, $C_l^{(1,1)}$ $(l \geq 2)$, $C_l^{(2,1)}$ $(l \geq 3)$, $BC_l^{(2,1)}$ $(l \geq 1)$, $F_4^{(1,1)}$, $F_4^{(2,1)}$, $G_2^{(1,1)}$, $G_2^{(3,1)}$,

IV. $A_l^{(1,1)*}$, $B_l^{(2,2)*}$ $(l \geq 2)$, $C_l^{(1,1)*}$ $(l \geq 2)$, $BC_l^{(2,2)}(1)$ $(l \geq 2)$, $BC_l^{(2,2)}(2)$ $(l \geq 1)$.

Theorem-added. *The affine root system* $(R,G)_a$ *is given as follows.*
If (R,G) *belongs to the group I,II or III, then*

$$(R,G)_a := \begin{cases} R_a = R_a^\vee & \text{if } (R,G) \text{ belongs to the group I,} \\ R_a & \text{if } (R,G) \text{ belongs to the group II,} \\ R_a^\vee & \text{if } (R,G) \text{ belongs to the group III.} \end{cases}$$

If (R,G) *belongs to the group IV, then*

$$(A_l^{(1,1)*})_a :=, \qquad (B_l^{(2,2)*})_a := BC_l^2 \ (l \geq 2), \ (C_l^{(1,1)*})_a := BC_l^2 \ (l \geq 2),$$
$$(BC_l^{(2,2)}(1))_a := C_l^1 \ (l \geq 2), \ (BC_l^{(2,2)}(2))_a := B_l^2 \ (l \geq 1).$$

Remark. Since the identification (2.2.1) induces that of the center of $\tilde{W}(\Gamma(R,G))$ with the center of $\tilde{W}(R,G)$, it follows from (2.3.6), (2.3.7) and §1 Fact 5 in §1, that the cyclic group generated by L.H.S. of (2.3.8) coincides with that by R.H.S. of (2.3.8). The (2.3.8) is a strengthening of this fact, whose proof is given in (3.4) using only the elliptic Coxeter relations independent of Fact 5.

§3 The proofs of Theorems

We prepare three lemmas in (3.1), (3.2) and (3.3), which are relatively independent each other. Using them, the proofs of Theorems 1 and 2 are given in (3.4).

(3.1) Verification of elliptic Coxeter relations

We verify that the elliptic Coxeter relations listed in (2.1) are satisfied by the reflexions. Precisely, we show the following lemma.

Lemma 1. *Let H be a vector space over $\mathbb{R}$ with a symmetric bilinear form J on it, and let Δ be a diagram of I $\sim$ IV in (2.1). Suppose that there are non isotropic vectors $\alpha, \alpha^*, \beta, \beta^*, \cdots$, etc. in H which satisfy the following three conditions:*
i) *the intersection diagram among them according to the convention in (1.3) is equal to Δ,*
ii) *the differences $\alpha - \alpha^*$ and $\beta - \beta^*$ (if they exist) belong to the radical of J, and*
iii) *the α is a b-side in the sense of the definition in (1.4) if Δ is III.t for $t = 2^{\pm 1}, 3^{\pm 1}$.*
Let us denote by $a, a^*, b, b^*, \cdots$ etc. the reflexions in $O(H,J)$ w.r.t. the vertices $\alpha, \alpha^*, \beta, \beta^*, \cdots$ etc. Then they satisfy the relations attached to Δ.

Proof. We consider only the relations II, III and IV, since the result for the Coxeter relations O and I are well known. In the cases II.t ($t = 1, 2, 3$), the inner products of vertices are given by :

$$J(\alpha, \beta^\vee) = -t, \quad J(\alpha^\vee, \beta) = -1, \quad J(\alpha^*, \beta^\vee) = -t, \quad J(\alpha^{*\vee}, \beta) = -1.$$

Then we have the following formulas :

(i) $ba^*ba\ (u) = u - J(u, \alpha^\vee)\alpha - \{J(u, \alpha^\vee + \beta^\vee) + (t-2)\ J(u, \alpha^\vee)\}\ (\alpha^* + t\beta),$

(ii) $aba^*b\ (u) = u - \{J(u, \alpha^\vee) + (t-2)J(u, \alpha^\vee + \beta^\vee)\}\alpha - J(u, \alpha^\vee + \beta^\vee)\ (\alpha^* + t\beta),$

(iii) $baba^*bab\ (u) = u - \{J(u, \alpha^\vee) + (t-2)\ J(u, \alpha^\vee + \beta^\vee)\}\ \{\alpha^* + (t-2)(\alpha + t\beta)\},$

(iv) $a^*baba^*\ (u) = u - \{(t-2)\ J(u, \alpha^\vee) + J(u, \alpha^\vee + \beta^\vee)\}\ \{(t-2)\alpha^* + (\alpha + t\beta)\},$

(v) $aba^*ba\ (u) = u - \{(t-2)\ J(u, \alpha^\vee) + J(u, \alpha^\vee + \beta^\vee)\}\ \{(t-2)\alpha + \alpha^* + t\beta\}.$

In the above, if $t = 1$, then (iii) = (iv), if $t = 2$, then (i) = (ii), and if $t = 3$, then (iii) = (iv), (iii) = (v), so the relations are verified.

In the cases II.t ($t = 2^{-1}, 3^{-1}$), the inner products of vertices are given by

$$J(\alpha, \beta^\vee) = -1, \quad J(\alpha^\vee, \beta) = -s, \quad J(\alpha^*, \beta^\vee) = -1, \quad J(\alpha^{*\vee}, \beta) = -s,$$

where $s = 2, 3$ corresponding to $t = 2^{-1}, 3^{-1}$, respectively. Then we have :

(i) $ba^*ba\ (u) = u - J(u, \alpha^\vee)\alpha - \{J(u, \alpha^\vee + s\beta^\vee) + (s-2)\ J(u, \alpha^\vee)\}\ (\alpha^* + \beta)$,

(ii) $aba^*b\ (u) = u - J(u, \alpha^\vee)\alpha - J(u, \alpha^\vee + s\beta^\vee)\ (\alpha^* + \beta + (s-2)\alpha)$,

(iii) $baba^*bab\ (u) = u - \{J(u, \alpha^\vee) + (s-2)\ J(u, \alpha^\vee + s\beta^\vee)\}\ \{\alpha^* + (s-2)(\alpha + \beta)\}$,

(iv) $a^*baba^*\ (u) = u - \{(s-2)\ J(u, \alpha^\vee) + J(u, \alpha^\vee + s\beta^\vee)\}\ \{(s-2)\alpha^* + \alpha + \beta\}$,

(v) $aba^*ba\ (u) = u - \{(s-2)\ J(u, \alpha^\vee) + J(u, \alpha^\vee + s\beta^\vee)\}\ \{\alpha^* + \beta + (s-2)\alpha\}$,

from the above we see that if $s = 2$ then (i) = (ii), and if $s = 3$ then (iii) = (iv), (iii) = (v).

In the cases III.t ($t = 1, 2, 3$), similarly we obtain :

$$ab^*a\ (u) = u - J(u, t\alpha^\vee + \beta^\vee)\ (\alpha + \beta^*),$$

$$a^*ba^*\ (u) = u - J(u, t\alpha^\vee + \beta^\vee)\ (\alpha^* + \beta),$$

$$ba^*b\ (u) = u - J(u, \alpha^\vee + \beta^\vee)\ (\alpha^* + t\beta),$$

$$b^*ab^*\ (u) = u - J(u, \alpha^\vee + \beta^\vee)\ (\alpha + t\beta^*).$$

If $t = 1$, then $ab^*a = a^*ba^*$, $ba^*b = b^*ab^*$, and if $t = 2,\ 3$ and $\alpha^* + \beta = \alpha + \beta^*$ (in other words α is the $b - side$), then $ab^*a = a^*ba^*$.

In the cases III.t ($t = 2^{-1}, 3^{-1}$), we have :

$$ab^*a\ (u) = u - J(u, \alpha^\vee + \beta^\vee)\ (s\alpha + \beta^*),$$

$$a^*ba^*\ (u) = u - J(u, \alpha^\vee + \beta^\vee)\ (s\alpha^* + \beta),$$

and further using the relation $s(\alpha^* - \alpha) = \beta^* - \beta$, we get $ab^*a = a^*ba^*$.

In the cases IV.t ($t = 1, 2, 3$),

$$abab^*cb^*\ (u) = u - J(u, \alpha^\vee + \beta^\vee)(\alpha + \beta) - J(u, \gamma^\vee + t\beta^\vee)(\gamma + \beta^*) = b^*cb^*aba\ (u),$$

$$ab^*abcb\ (u) = u - J(u, \alpha^\vee + \beta^\vee)(\alpha + \beta^*) - J(u, \gamma^\vee + t\beta^\vee)(\gamma + \beta) = bcbab^*a\ (u),$$

these mean $(abab^*cb^*)^2 = 1$, and $(ab^*abcb)^2 = 1$ for any t, respectively.

In the case IV.t ($t = 2^{-1}, 3^{-1}$),

$$abab^*cb^*\ (u) = u - J(u, \alpha^\vee + \beta^\vee)(\alpha + \beta) - J(u, \gamma^\vee + \beta^\vee)(\gamma + s\beta^*) = b^*cb^*aba\ (u),$$

$$ab^*abcb\ (u) = u - J(u, \alpha^\vee + \beta^\vee)(\alpha + \beta^*) - J(u, \gamma^\vee + \beta^\vee)(\gamma + s\beta) = bcbab^*a\ (u),$$

these mean $(abab^*cb^*)^2 = 1$, and $(ab^*abcb)^2 = 1$, respectively. $\square$

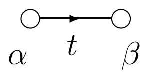
(3.2) Adjoint action of $\tilde{W}(\Gamma(R,G))$ on $N(R,G)$

Lemma 2. i) *$N(R,G)$ is an abelian group generated by T_α for $\alpha \in \Gamma$.*
ii) *Let us denote by $Ad_g(n)$ the adjoint action gng^{-1} of $g \in \tilde{W}(\Gamma(R,G))/N(R,G)$ on $n \in N(R,G)$. Then one has the formula for $\alpha, \beta \in \Gamma$.*

(0)

$$Ad_{a_\alpha}(T_\beta) = T_\beta$$

$(I)_t$

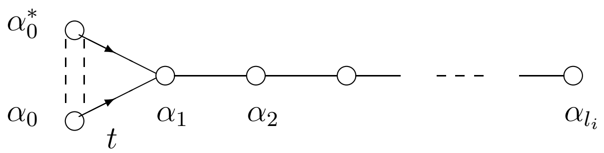


$$Ad_{a_\alpha}(T_\beta) = T_\alpha^t T_\beta \quad t = 1, 2, 3$$

$$Ad_{a_\beta}(T_\alpha) = T_\alpha T_\beta$$

where we assume α is the b-side in the diagram $(I)_t$ if $t \neq 1$.

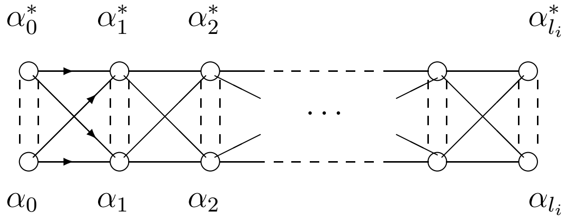
Proof. i) Let $\Gamma(A_{l_i})$ be a component of $\Gamma \backslash \Gamma_{\max}$ and consider the following diagram:



$$t = 1, 2^{\pm 1}, 3^{\pm 1},$$

where α_0 is the vertex in Γ_{max} which is connected to $\Gamma(A_{l_i})$.

Adding new vertices $\alpha_1^*, \alpha_2^*, \cdots, \alpha_{l_i}^*$ to the above diagram, we consider the following diagram Γ_i:



$$t = 1, 2^{\pm 1}, 3^{\pm 1}.$$

To the new vertices, let us attach elements $a_1^*, a_2^*, \cdots, a_{l_i}^* \in \tilde{W}(\Gamma(R,G))$, defined by the relations:

(3.2.1) $$a_0a_1^*a_0 \;=\; a_0^*a_1a_0^* \quad \text{and} \quad a_ja_{j+1}^*a_j = a_j^*a_{j+1}a_j^* \quad (1 \le j < l_i).$$

Assertion i). *The system $a_0, a_0^*, a_1, a_1^*, \cdots, a_{l_i}^*$ satisfies the elliptic Coxeter relations attached to the diagram Γ_i .*

Proof. We check the relations I.t, II.t, III.t and the relations IV.t separately

(I.t, II.t and III.t)

In the case of $t=1$, we show that the elliptic Coxeter relations among a_0, a_0^*, a_1 and a_1^* hold, which are :
(i) $(a_0^*a_1^*)^3 = 1$, (ii) $(a_0a_1^*)^3 = 1$, (iii) $a_1^*a_0a_1^* = a_1a_0^*a_1$,
(iv) $(a_0a_1^*a_0^*a_1^*)^3 = 1$, (v) $(a_1a_0a_1^*a_0)^3 = 1$.
By using (3.2.1) $a_1^* = a_0a_0^*a_1a_0^*a_0$,

$$\begin{aligned}
\text{L.H.S. of (i)} &= (a_0^*a_0a_0^*a_1a_0^*a_0)^2\\
&= (a_0^*a_0a_1a_0^*a_1a_0)^3 \quad (\text{ by I.2 })\\
&= (a_1a_0)^3 \quad (\text{ by II.1 })\\
&= 1
\end{aligned}$$

so (i) is obtained. (ii) and (iii) are similarly shown, further (iv) and (v) are obtained from (i), (ii), (iii) and (II.1). The elliptic Coxeter relations involving α_j^* $(j \ge 2)$ can be checked in a similar way by induction on j. In the cases of $t = 2^{\pm 1}, 3^{\pm 1}$, the elliptic Coxeter relations are checked due to the fact that α_0 is $b-side$ (see (1.4) Fact 2 ii)). Here we show the relations among a_0, a_0^*, a_1 and a_1^*, because of the same reason as the case $t=1$.

(Case $t = 2^{\pm 1}$)

(i) $(a_0^*a_1^*)^4 = 1$, (ii) $(a_0a_1^*)^4 = 1$, (iii) $(a_0a_1^*a_0^*a_1^*)^2 = 1$.

$$\begin{aligned}
\text{L.H.S. of (i)} &= (a_0^*a_0a_0^*a_1a_0^*a_0)^4\\
&= (a_0^*a_0a_1a_0^*a_1a_0^*a_1a_0)^4 \quad (\text{ by I.3 })\\
&= (a_0^*a_1a_0^*a_1a_0a_0^*a_1a_0)^4 \quad (\text{ by II.2 })\\
&= (a_1a_0^*a_1a_0^*a_0a_0^*a_1a_0)^4 \quad (\text{ by I.3 })\\
&= (a_1a_0^*a_1a_0a_0^*a_0a_0^*a_1)^4 \quad (\text{ by II.2 })\\
&= (a_1a_0a_0^*a_1a_0a_1a_0a_1a_0^*a_0)^2 \quad (\text{ by I.3 })\\
&= (a_1a_0a_1a_0a_1a_0^*a_0a_1a_0^*a_0)^2 \quad (\text{ by II.2 })
\end{aligned}$$

$$
\begin{aligned}
&= (a_1 a_0 a_0^* a_0 a_1 a_0^*)^2 \quad (\text{ by I.3 }) \\
&= (a_0^* a_1 a_0^* a_1)^2 \quad (\text{ by II.2 }) \\
&= 1.
\end{aligned}
$$

(Case $t = 3^{\pm 1}$)

(i) $(a_0^* a_1^*)^6 = 1$, (ii) $(a_0 a_1^*)^6 = 1$, (iii) $(a_0 a_1^* a_0^* a_1^*)^3 = 1$, (iv) $(a_0 a_1^* a_0^* a_1^* a_0 a_1^*)^2 = 1$.

For the proof, we use the relation: $aba^*ba = a^*baba^*$ $(*)$, which can be obtained from the elliptic Coxeter relations II.3 $(aba^*b)^3 = 1$ and $(aba^*bab)^2 = 1$. We set $a := a_0$, $b := a_1$, then

$$
\begin{aligned}
\text{L.H.S. of (i)} &= (a^*aa^*ba^*a)^6 \\
&= (a^*aba^*ba^*ba^*ba^*ba)^6 \quad (\text{ by I.3 }) \\
&= (a^*ba^*baba^*baa^*ba^*ba^*ba)^6 \quad (\text{ by II.3 }) \\
&= (ababaa^*ba^*)^6 \quad (\text{ by } (*)) \\
&= (bababa^*ba^*)^6 \quad (\text{ by I.3 }) \\
&= (babaa^*baba^*baa^*)^6 \quad (\text{ by II.3 }) \\
&= (baa^*a)^6 \quad (\text{ by } (*)) \\
&= (bababababa^*)^6 \quad (\text{ by I.3 }) \\
&= (abaa^*baba^*)^6 \quad (\text{ by II.3 }) \\
&= (a^*b)^6 \quad (\text{ by } (*)) \\
&= 1.
\end{aligned}
$$

The other relations (ii), (iii) and (iv) in the cases $t = 2^{\pm 1}, 3^{\pm 1}$ are cheked easily.

(IV.t)

The relations IV.t among $a_0, a_0^*, a_1, a_1^*, a_2$ and a_2^* are:

(i) $(a_2 a_1 a_2 a_1^* a_0 a_1^*)^2 = 1$, (ii) $(a_2 a_1^* a_2 a_1 a_0 a_1)^2 = 1$,

(iii) $(a_2^* a_1 a_2^* a_1^* a_0 a_1^*)^2 = 1$, (iv) $(a_2^* a_1^* a_2^* a_1 a_0 a_1)^2 = 1$,

and the relations substituting a_0 by a_0^* in the above. The relations of type IV.t involving a_j, a_j^* $(j \geq 3)$ can be checked similar way. In the previous (I.t, II.t, III.t), we have already shown the relations $a_2a_1^*a_2 = a_1a_2^*a_1$ and $a_2^*a_1a_2^* = a_1^*a_2a_1^*$. By using them (ii) and (iii) are trivial, so we show (i) and (iii).

$$\begin{aligned}
\text{(i)} \quad &\Longleftrightarrow \quad (a_2a_1a_2a_1^*a_2a_0a_2a_1^*)^2 = 1 \\
&\Longleftrightarrow \quad (a_1a_1a_2^*a_1a_0a_1a_2^*a_1)^2 = 1 \\
&\Longleftrightarrow \quad (a_2^*a_1a_2^*a_1a_0a_1)^2 = 1 \\
&\Longleftrightarrow \quad (a_1a_2^*a_1a_1a_0a_1)^2 = 1 \\
&\Longleftrightarrow \quad (a_2^*a_0)^2 = 1.
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad &\Longleftrightarrow \quad (a_2^*a_1^*a_2^*a_1a_2^*a_0a_2^*a_1)^2 = 1 \\
&\Longleftrightarrow \quad (a_1^*a_2^*a_1a_2^*a_0a_2^*a_1a_2^*)^2 = 1 \\
&\Longleftrightarrow \quad (a_1^*a_1^*a_2a_1^*a_0a_2a_1^*a_2)^2 = 1 \\
&\Longleftrightarrow \quad (a_0a_2)^2 = 1.
\end{aligned}$$

The relations substituting a_0 by a_0^* of (IV.t) can be similarly shown.

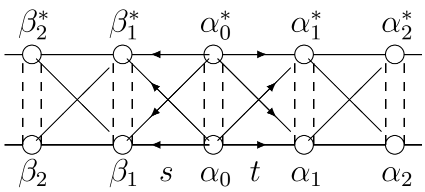
So the proof of Assertion i) is completed. □

Further we consider the diagram $\tilde{\Gamma}(R,G) := \Gamma(R,G) \cup \bigcup_{i=1}^{r} \Gamma_i$.

Assertion ii). *The elliptic Coxeter relations attached to the new diagram $\tilde{\Gamma}(R,G)$ are satisfied by the systems $\{a_\alpha, a_\alpha^* \mid \alpha \in \Gamma\}$ of $\tilde{W}(\Gamma(R,G))$.*

Proof. We have to prove the elliptic Coxeter relation for the two cases when the vertices are either on a union $\Gamma_i \cup \Gamma_{i'}$ with $\Gamma_i \cap \Gamma_{i'} \neq \phi$ for $1 \leq i < i' \leq r$, or on the union $\Gamma_i \cup \Gamma_{\max} \cup \Gamma_{\max}^*$ for $1 \leq i \leq r$.

We consider only the case of the diagram $\Gamma_i \cup \Gamma_{i'}$ with $\Gamma_i \cup \Gamma_{i'} \neq \phi$, since the other case is proven similarly.



$t = 1, 2^{\pm 1}, 3^{\pm 1}$
$s = 1, 2^{\pm 1}$.

(Case $s = 1$)

In the case of $t = 1$, we prove the elliptic Coxeter relations among $b_1^*, b_1, a_0, a_0^*, a_1$ and a_1^*, which are :

(i) $(b_1^*a_1)^2 = 1$, (ii) $(b_1^*a_1^*)^2 = 1$, (iii) $(b_1a_1^*)^2 = 1$,
(iv) $(b_1a_0b_1a_0^*a_1^*a_0^*)^2 = 1$, (v) $(b_1^*a_0b_1^*a_0^*a_1a_0^*)^2 = 1$, (vi) $(b_1^*a_0b_1^*a_0^*a_1^*a_0^*)^2 = 1$.

By using the expressions : $a_1^* := a_0a_0^*a_1a_0^*a_0$, $b_1^* := a_0a_0^*b_1a_0^*a_0$,

$$\text{L.H.S. of (i)} = (a_0a_0^*b_1a_0^*a_0a_1)^2 = 1 \quad (\text{ by IV.1 }),$$

$$\begin{aligned}\text{L.H.S. of (iv)} &= (b_1a_0b_1a_0^*a_0a_0^*a_1a_0^*a_0a_0^*)^2\\ &= (b_1a_0b_1a_0^*a_0b_1)^2 \quad (\text{ by II.1 })\\ &= (b_1a_0^*)^2 = 1.\end{aligned}$$

(ii), (iii), (v) and (vi) are similarly shown. The remaining relations and the cases of $t = 2^{\pm1}, 3^{\pm1}$ are checked in a similar way to the cases of Assertion i).

(Case $s = 2^{\pm1}$)

In this case, $t \neq 3^{\pm1}$. Due to the *Remark* 1 at the end of (2.1), the elliptic Coxeter relations among b_1^*, b_1, a_1, and a_1^* are :

(i) $(b_1^*a_1)^2 = 1$, (ii) $(b_1^*a_1^*)^2 = 1$, (iii) $(b_1a_1^*)^2 = 1$.

These are checked in a similar way to the case ($s = 1$).
Therefore the assertion ii) is completed. □

By use of the previous elements, from the definition (2.3.3) and the induction on j, one has the expression :

(3.2.2) $$T_{\alpha_{j+1}} = a_{j+1}a_{j+1}^* \quad (0 \leq j < l_i).$$

Recall that $T_\alpha \in N(R, G)$ for $\alpha \in \Gamma$.

Assertion iii). *All* T_α, $\alpha \in \Gamma$ *commute each other.*

Proof. We have only to prove the following relations, because the others can be proven by induction on the distance of α from $\Gamma_{\max}$.

(3.2.3) $$a_0a_0^*a_1a_1^* = a_1a_1^*a_0a_0^* \quad \text{in } \Gamma_i,$$

(3.2.4) $$a_1a_1^*b_1b_1^* = b_1b_1^*a_1a_1^* \quad \text{in } \tilde{\Gamma}$$

Proof of (3.2.3). For simplicity, we set $a := a_0$, $b := a_1$.

(Case $t = 1$)

$$aa^*bb^* = b^*aa^*b^* \qquad (\text{ by (3.2.1) })$$

$$
\begin{aligned}
&= b^*ab^*a^*b^*a^* && (\text{ by I.1 }) \\
&= ba^*ba^*b^*a^* && (\text{ by (3.2.1) }) \\
&= bb^*ab^*b^*a^* && (\text{ by III.1 }) \\
&= bb^*aa^*.
\end{aligned}
$$

(Case $t = 2^{\pm 1}$)

$$
\begin{aligned}
aa^*bb^* &= aba^*ba^*ba^*b^* && (\text{ by I.3 }) \\
&= ba^*baa^*ba^*b^* && (\text{ by II.2 }) \\
&= ba^*bb^*ab^* && (\text{ by (3.2.1) }) \\
&= ba^*ba^*b^*ab^*a^* && (\text{ by II.2 }) \\
&= bab^*ab^*ab^*a^* && (\text{ by (3.2.1) }) \\
&= bb^*aa^* && (\text{ by I.3 }) .
\end{aligned}
$$

(Case $t = 3^{\pm 1}$)

For the proof, we use the relation ;

$$
(*) \qquad aba^*ba = a^*baba^*
$$

obtained by $(aba^*b)^2 = 1$ and $(aba^*bab)^2 = 1$, then

$$
\begin{aligned}
aa^*bb^* &= aa^*baa^*ba^*a && (\text{ by (3.2.1) }) \\
&= abaa^*baa^*a && (\text{ by } (*)) \\
&= babababa^*baa^*a && (\text{ by I.4 }) \\
&= babaa^*baba^*a && (\text{ by II.3 }) \\
&= baa^*b && (\text{ by } (*)) \\
&= bb^*aa^* && (\text{ by (3.2.1) }) .
\end{aligned}
$$

Proof of (3.2.4) is trivial. To complete the proof of Lemma 2 i), we have to show that the subgroup generated by T_α ($\alpha \in \Gamma$) is closed under the adjoint action Ad_{a_α} for $\alpha \in \Gamma$. This is achieved in the next ii) explicitly.

ii) The relations (0) and $(\mathrm{I})_t$ are proven by direct calculations as follows.

$$
\begin{aligned}
(0) \quad Ad_{a_\alpha}(T_\beta) &= abb^*a = bb^* = T_\beta. \\
(\mathrm{I})_1 \quad Ad_{a_\alpha}(T_\beta) &= abb^*a = babab^*a = bb^*abab^*abaa = bb^*abab^*ab \\
&= bb^*aba^*ba^*b = bb^*abba^* = bb^*aa^* = T_\alpha T_\beta.
\end{aligned}
$$

$$
\begin{array}{llll}
(\mathrm{I})_2 & Ad_{a_\alpha}(T_\beta) & = & abb^*a = aa^*aaa^*bb^*a = aa^*ab^*aa^*b^*a = aa^*ab^*ab^*a^*b^*a^*b^*a^*a \\
& & = & aa^*aa^*b^*ab^*b^*a^*b^*a^*a = aa^*aa^*b^*aa^*b^*a^*a = aa^*aa^*a^*bb^*a^*a = aa^*aa^*bb^* \\
& & = & T_\alpha^2 T_\beta. \\
& Ad_{a_\beta}(T_\alpha) & = & baa^*b = bb^*aa^* = T_\alpha T_\beta. \\
(\mathrm{I})_3 & Ad_{a_\alpha}(T_\beta) & = & abb^*a = abaab^*a = abaa^*ba^* = abaa^*baaa^* \\
& & = & aa^*baa^*baa^* = aa^*baa^*ba^*aaa^*aa^* = aa^*bb^*aa^*aa^* \\
& & = & T_\alpha^3 T_\beta. \\
& Ad_{a_\beta}(T_\alpha) & = & baa^*b = bb^*aa^* = T_\alpha T_\beta.
\end{array}
$$

Thus the proof of Lemma 2 is completed. □

(3.3) Abelian normal subgroup N of $\tilde{W}(R,G)$

Lemma 3. *There is an abelian normal subgroup N of $\tilde{W}(R,G)$ such that* i) *one has an exact sequence*

$$
(3.3.1) \qquad 1 \quad \to \quad N \quad \to \quad \tilde{W}(R,G) \quad \to \quad W(R_a) \quad \to \quad 1,
$$

and ii) *N is isomorphic to the lattice $Q((R,G)_a)$ in F/G of rank $l+1$ generated by*

$$
k(\alpha)\cdot\alpha^\vee
$$

for $\alpha \in \Gamma$ through the Eichler-Siegel map E_G.

Proof. Recall the Eichler-Siegel maps

$$
E_G : F\otimes F/G \to End(\tilde{F}) \quad \text{and} \quad E : F/G\otimes F/rad(I) \to End(F/G)
$$

given by $E_G(\sum p\otimes q)(u) := u - \sum p\cdot \tilde{I}(q,u)$ and $E(\sum p\otimes q)(u) := u - \sum p\cdot I(q,u)$, respectively ([ibid, (1.14) definition, (1.17.2)]). Both are injective maps so that one obtains isomorphisms: $\tilde{W}(R,G) \cong E_G^{-1}(\tilde{W}(R,G))$ and $W(R_a)\cong E^{-1}(W(R_a))$. $E_G^{-1}(\tilde{W}(R,G))$ (resp. $E^{-1}(W(R_a))$) has the group low $\circ$ and is generated by $\alpha\otimes\alpha^\vee$ for $\alpha\in\Gamma(R,G)$ (resp. $\alpha\in\Gamma$). Then the set $E_G^{-1}(\tilde{W}(R,G))$ is contained in a lattice of $F\otimes F/G$ given by

$$
(3.3.2) \qquad \mathscr{L} := \bigoplus_{\alpha\in\Gamma}(Q_\alpha\otimes\mathbb{Z}\alpha^\vee)
$$

with $Q_\alpha := \sum_{\beta\in\Gamma}\frac{k(\alpha)}{k(\beta)}(\mathbb{Z}\beta+\mathbb{Z}\beta^*)$, since $\mathscr{L}$ contains $\alpha\otimes\alpha^\vee$ for all $\alpha\in\Gamma(R,G)$ and $\mathscr{L}\circ\mathscr{L}\subset\mathscr{L}$ (note $I(\alpha_1^\vee, Q_{\alpha_2})Q_{\alpha_1}\subset Q_{\alpha_2}$ [ibid, (6.1.2)]).

On the other hand, the exact sequence :

$$
0\to G\otimes F/G\oplus F\otimes rad(I)/G \to F\otimes F/G \to F/G\otimes F/rad(I)\to 0
$$

induces an exact sequence $0 \to N \to \tilde{W}(R,G) \to W(R_a) \to 1$, where $N := E_G^{-1}(\tilde{W}(R,G)) \cap (G \otimes F/G \oplus F \otimes rad(I)/G)$. Let us show $N \subset G \otimes F/G$. Recall that [ibid, (1.20.2)]

$$E_G^{-1}(g) = \xi(g) + p(g) + q(g) - E_0(\xi(g))^t p(g) + \frac{1}{2} I(p(g),^t p(g)) + r(g),$$

and

$$E^{-1}(\bar{g}) = \xi(g) + p(g),$$

for $g \in \tilde{W}(R,G)$ (and its image $\bar{g}$ in $W(R_a)$), where $\xi(g) \in L \otimes L$, $p(g) \in H \otimes L$, $q(g) \in G \otimes L$ and $r(g) \in A(H) \oplus G \otimes H$ for a decomposition $F = L \oplus H \oplus G$ with $rad(I) = H \oplus G$ and $A(H) :=$ anti-symmetric tensor product of $H = 0$. Thus for a given $g \in \tilde{W}(R,G)$, g belongs to N if and only if $\xi(g) = p(g) = 0$, and then $E_G^{-1}(g) = q(g) + r(g) \in G \otimes F/G$. So one has the inclusion relation $N \subset (G \otimes F/G) \cap \mathscr{L} = \bigoplus_{\alpha \in \Gamma} \mathbb{Z} k(\alpha) \cdot a \otimes \alpha^\vee$, where the group low $\circ$ coincides with the addition in the lattice. To prove the opposite inclusion relation, we consider

$$E_G^{-1}(w_\alpha w_{\alpha^*}) = (\alpha \otimes \alpha^\vee) \circ (\alpha^* \otimes \alpha^{*\vee}) = \alpha \otimes \alpha^\vee + \alpha^* \otimes \alpha^{*\vee} - \alpha \otimes \alpha^{*\vee} \cdot \tilde{I}(\alpha^\vee \otimes \alpha^*).$$

Recalling $\alpha^* := \alpha + k(\alpha) \cdot a$ (1.2.3), we obtain:

$$E_G^{-1}(w_\alpha w_{\alpha^*}) = k(\alpha) \cdot a \otimes \alpha^\vee. \tag{3.3.3}$$

By an identification $G \otimes F/G \cong F/G, a \otimes x \mapsto x$, N is identified with the lattice generated by $k(\alpha)\alpha^\vee$ $(\alpha \in \Gamma)$, which is the root lattice of $(R,G)_a$. $\square$

(3.4) Proofs of Theorems 1 and 2.

Lemma 1 implies that if the set of vertices of an elliptic diagram $\Gamma(R,G)$ is embedded in any vector space H endowed with a symmetric bilinear form J in such a way that i) the conventions in (1.3) are satisfied ii) $\alpha - \alpha^* \in rad(J)$ for any $\alpha \in \Gamma_{\max}$, and iii) any b-side $\alpha \in \Gamma_{\max}$ satisfies its condition of the definition in (1.4), then there is a natural surjective homomorphism from the group $\tilde{W}(\Gamma(R,G))$ to the subgroup of $O(H,J)$ generated by the reflexions s_α of the vertices α given by the correspondence $a_\alpha \longmapsto s_\alpha$. In particular, one has a surjective homomorphism from $\tilde{W}(\Gamma(R,G))$ onto $\tilde{W}(R,G)$. This homomorphism commutes with the projection to the affine Weyl group $W(R_a)$ given by (2.3.5) and (3.3.1). So the homomorphism induces the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & N(R,G) & \to & \tilde{W}(\Gamma(R,G)) & \to & W(R_a) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & \tilde{W}(R,G) & \to & W(R_a) & \to & 1. \end{array}$$

Since the middle arrow is surjective, the first arrow is also surjective. Then the facts that N is a free abelian group of rank $l+1$ (lemma 3) and $N(R,G)$ is an abelian group generated by $l+1$ elements (lemma 2), imply that the first arrow is bijective and hence the middle arrow is also bijective. So Theorem 1 is proven. In view of (3.3.3), T_α is identified with $k(\alpha)\alpha^\vee$ for $\alpha \in \Gamma$. This implies (2.3.6) in Theorem 2. Then Lemma 2 implies (2.3.7) in Theorem 2.

Finally, it remains to prove (2.3.8) in Theorem 2. Let us express the hyperbolic Coxeter element (2.2.2) as follows:

$$\tilde{c}((\Gamma(R,G)) = B \cdot C,$$

where

$$B := \prod_{\alpha \in \Gamma_{\max}} a_\alpha a_{\alpha^*} = \prod_{\alpha \in \Gamma_{\max}} T_\alpha$$

$$C := \Big(\prod_{i=1}^{r} c_i\Big) \quad \text{with} \quad c_i := \prod_{\alpha \in \Gamma(A_{l_i})} a_\alpha$$

and $\overset{r}{\underset{i=1}{\cup}} \Gamma(A_{l_i})$ is the disjoint decomposition of $\Gamma \setminus \Gamma_{\max}$. We know that

i) c_i $(i = 1, \cdots, r)$ commute mutually and $c_i{}^{l_i+1} = 1$ and ii) $B \in N(R,G)$.

Then

$$\begin{aligned} \tilde{c}(\Gamma(R,G))^m &= B \cdot C \cdot B \cdot C \cdots B \cdot C \\ &= B \cdot Ad_c(B) \cdot Ad_c^2(B) \cdots Ad_c^{m-1}(B) \cdot C^m \end{aligned}$$

Since $C^m = 1$ for $m = m(R,G) = lcm\{l_i + 1 \mid 1 \le i \le r\}$ (recall (1.3.2)) and each factor $Ad_c^j(B)$ belongs to $N(R,G)$ owing to the above ii), $\tilde{c}(\Gamma(R,G))^m$ belongs to $N(R,G)$. Let us determine $Ad_c^j(B)$ as an element of $N(R,G)$. For each i with $1 \le i \le r$ and j with $0 \le j \le l_i$, one has the formula: $Ad_{c_i}^j(T_{\alpha_0}) = \prod_{p=0}^{j} T_{\alpha_p} = (\prod_{p=1}^{j} T_{\alpha_p}) T_{\alpha_0}$, where we have numbered the vertices of $\Gamma(A_{l_i})$ as in (2.3.2). For $j \in \mathbf{N}$, let j_i be the integer such that $0 \le j_i \le l_i$ and $j \equiv j_i \bmod l_i + 1$. Then due to the fact i), we have $C^j = \prod_{i=1}^{r} c_i^{j_i}$, so ,

$$Ad_c^j(B) = \Big(\prod_{i=1}^{r}\Big(\prod_{p=1}^{j_i} T_{\alpha_p}\Big)\Big) \cdot B$$

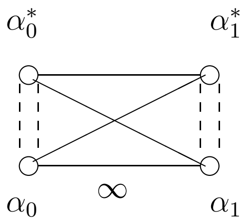
Therefore

$$\tilde{c}(\Gamma(R,G))^{m(\Gamma,R)} = \prod_{\alpha \in \Gamma} T_{\alpha_a}^{m_\alpha \cdot \frac{m(R,G)}{m_{\max}}}.$$

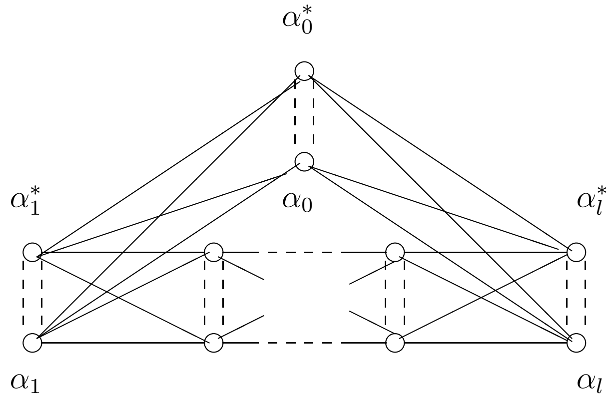
The R.H.S. is exactly the multiplicative expression of the null root of the affine root system $(R,G)_a$. This completes the proofs of Theorems 1 and 2. $\square$

Table of elliptic diagrams

$A_l^{(1,1)}$ $(l \geq 1)$

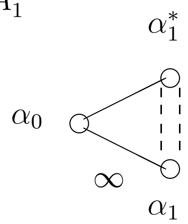


$(l = 1)$

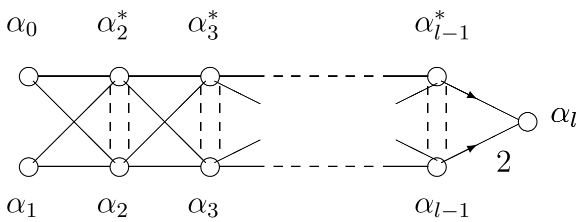


$(l \geq 2)$

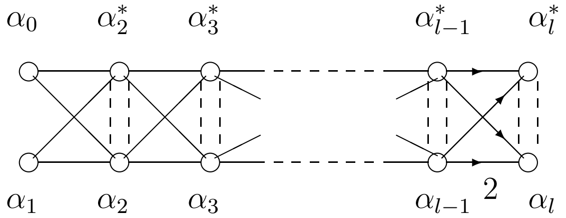
$A_1^{(1,1)*}$



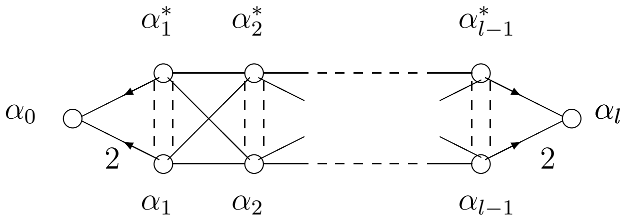
$B_l^{(1,1)}$ $(l \geq 3)$



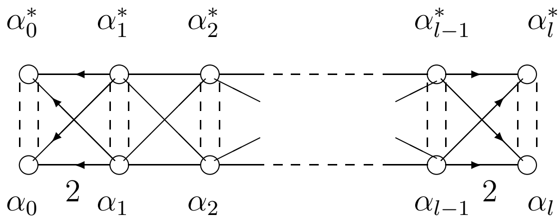
$B_l^{(1,2)}$ $(l \geq 2)$



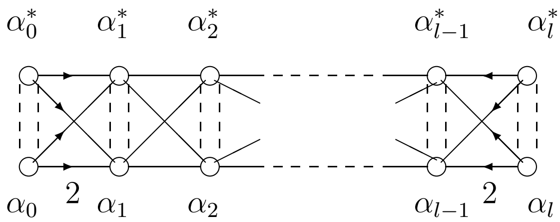
$B_l^{(2,1)}$ $(l \geq 2)$



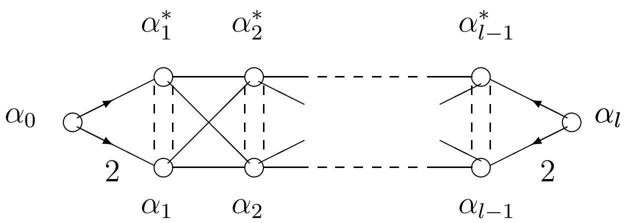
$B_l^{(2,2)}$ $(l \geq 2)$



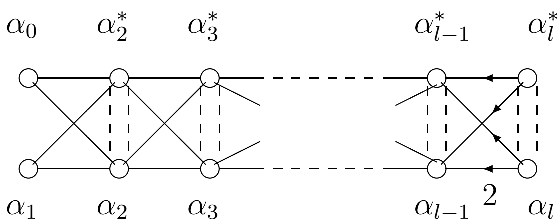
$C_l^{(1,1)}$ $(l \geq 2)$



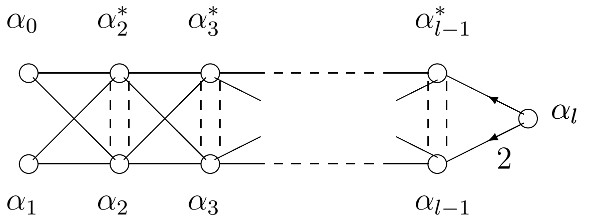
$C_l^{(1,2)}$ $(l \geq 2)$



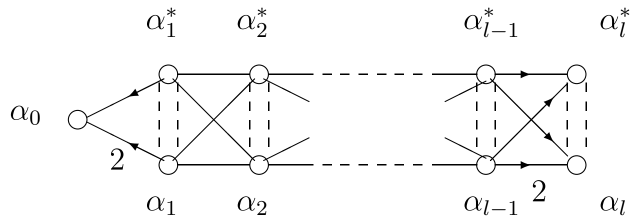
$C_l^{(2,1)}$ $(l \geq 2)$



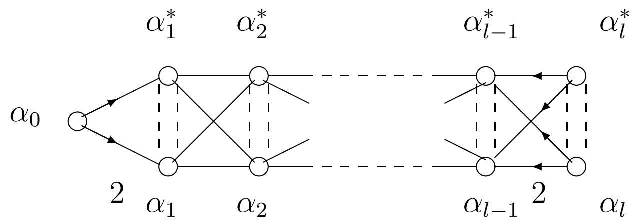
$C_l^{(2,2)}$ $(l \geq 3)$



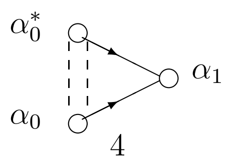
$B_l^{(2,2)*}$ $(l \geq 2)$



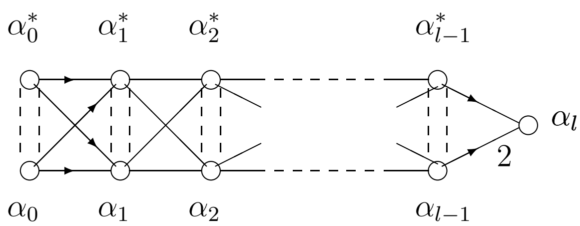
$C_l^{(1,1)*}$ $(l \geq 2)$



$BC_l^{(2,1)}$ $(l \geq 1)$

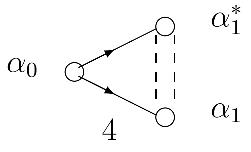


$(l = 1)$

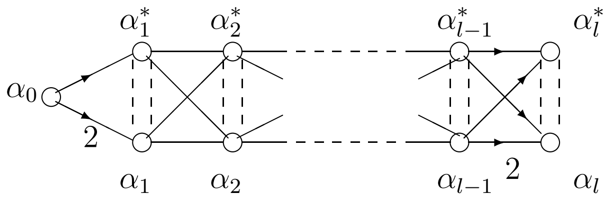


$(l \geq 2)$

$BC_l^{(2,4)}$ $(l \geq 1)$

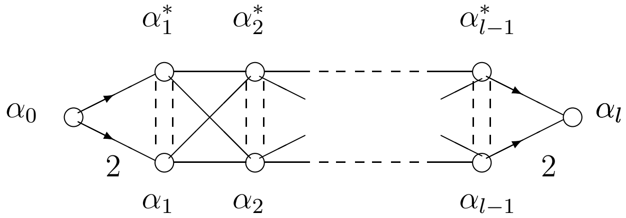


$(l = 1)$

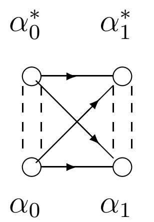


$(l \geq 2)$

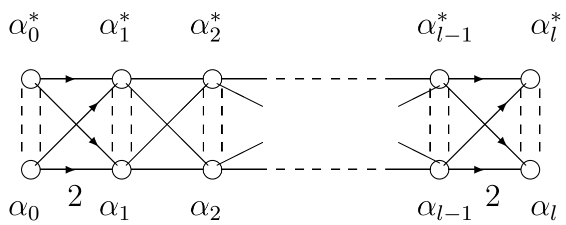
$BC_l^{(2,2)}(1)$ $(l \geq 2)$



$BC_l^{(2,2)}(2)$ $(l \geq 1)$

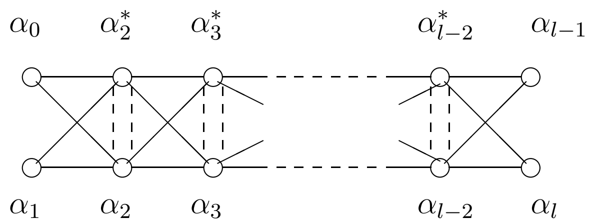


$(l = 1)$

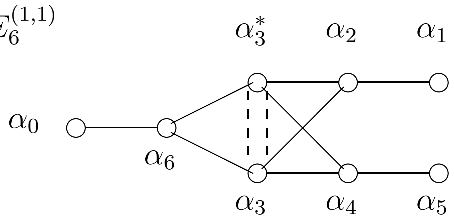


$(l \geq 2)$

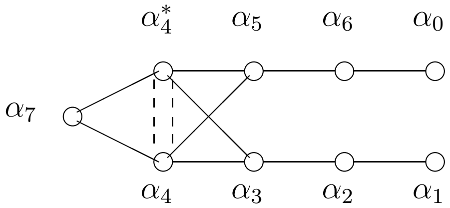
$D_l^{(1,1)}$ $(l \geq 4)$



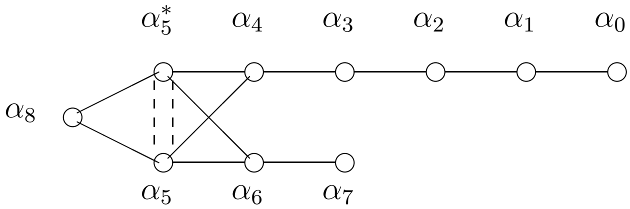
$E_6^{(1,1)}$



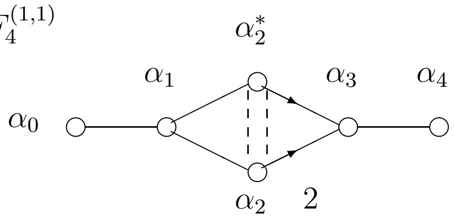
$E_7^{(1,1)}$



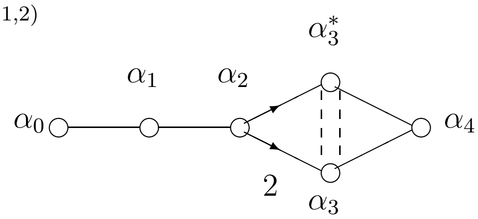
$E_8^{(1,1)}$



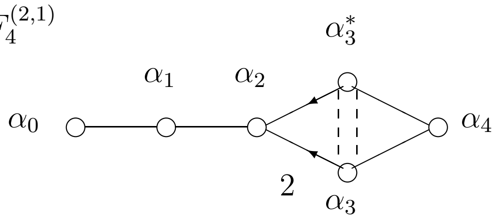
$F_4^{(1,1)}$



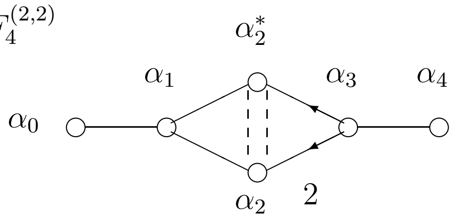
$F_4^{(1,2)}$



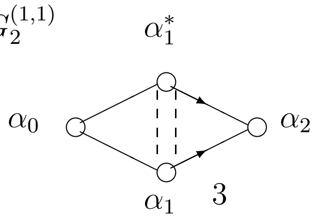
$F_4^{(2,1)}$



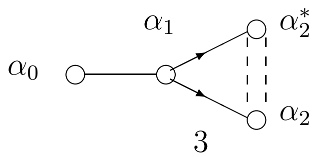
$F_4^{(2,2)}$



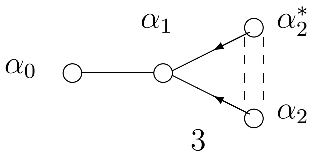
$G_2^{(1,1)}$



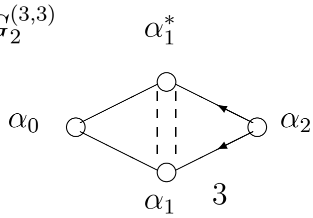
$G_2^{(1,3)}$



$G_2^{(3,1)}$



$G_2^{(3,3)}$



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