A polynomial invariant of a homology 3-sphere defined by Reidemeister torsion

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Contents

Reidemeister torsion

- 1930's Reidemeister, Franz, de Rham: Classification of lens space
- 1961-2 Milnor:
  - Hauptvermutung for polyhedrons
  - Alexander polynomial=Reidemeister torsion
- 1980's asymptotic behavior of quantum invariants
- 1990's twisted Alexander polynomial=Reidemeister torsion
- 1996 Meng-Taubes: SW=Milnor torsion

Casson invariant and Reidemeister torsion

- 1985 Casson, an integral lift of Rochlin invariant
- Late 1980's D. Johnson, *A geometric form of Casson's invariant and its connection to Reidemeister torsion*, unpublished lecture notes (hand written).

A generalization of the determinant

Roughly speaking,

$$\textit{Reidemeister torsion} = \text{"det"}(\textit{an acyclic based chain complex}).$$

- acyclic: a linear isomorphism
- based: a matrix is given under a basis

determinant for a linear isomorphism

- V: l-dimensional vector space over a field $\mathbb{F}(=\mathbb{C},\mathbb{R})$.
- $f : V \to V$: a linear isomorphism

How to define $\det(f)$?

- $\mathbf{e}_1,\ldots,\mathbf{e}_l$: a basis of V
- $F \in M(l;\mathbb{F})$: the matrix defined by f under $\mathbf{e}_1,\ldots,\mathbf{e}_l$

For $\mathbf{e}_1,\ldots,\mathbf{e}_l$, we take their image $f(\mathbf{e}_1),\ldots,f(\mathbf{e}_l)$.

$$
\begin{aligned}
\det(f,(\mathbf{e}_1,\ldots,\mathbf{e}_l)) &= \det(F) \\
&= \det\left(f(\mathbf{e}_1),\ldots,f(\mathbf{e}_l)/\mathbf{e}_1,\ldots,\mathbf{e}_l\right).
\end{aligned}
$$

Under fixing a basis of V, $\mathbf{e}_1, \ldots, \mathbf{e}_l$, we have an identification

$$\wedge^l V \cong \mathbb{F}.$$

$$f(\mathbf{e}_1)\wedge\cdots\wedge f(\mathbf{e}_l) = \det\left(f(\mathbf{e}_1), \ldots, f(\mathbf{e}_l)/\mathbf{e}_1, \ldots, \mathbf{e}_l\right)\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_l$$

determinant for a short exact sequence

- $0 \to V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \to 0$: an exact sequence of finite dimensional vector spaces over $\mathbb{F}$.
- $\mathbf{b}_i \subset V_i$: a basis of V_i for $i = 1, 2, 3$
- $f_2^{-1}(\mathbf{b}_3) \subset V_2$: a lift of $\mathbf{b}_2$
- $(f_1(\mathbf{b}_1), f_2^{-1}(\mathbf{b}_3))$: another basis of V_2

$$\tau = \det\left(f_1(\mathbf{b}_1), f_2^{-1}(\mathbf{b}_3)/\mathbf{b}_2\right)$$

Remark

τ does not depend on a choice of a lift $f_2^{-1}(\mathbf{b}_3)$.

Reidemeister torsion for a chain complex

- $\mathbb{F}$: a field
- a chain complex of finite dimensional vector spaces over $\mathbb{F}$ C_*:

$$0 \xrightarrow{\partial_{m+1}} C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

- $\mathbf{c_i}$: a basis of C_i

Definition

A chain complex C_* is acyclic if

$$Im\partial_{i+1} = Ker\partial_i \ (Z_i(C_*) = B_i(C_*), H_i(C_*) = \{0\})$$

for any $i = 0, \dots, m$.

Assumption: C_* is acyclic.

- Fix $\mathbf{b_i}$ for $B_i(C_*) = Z_i(C_*)$.

Exact sequence

$$0 \longrightarrow Z_i(C_*) \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1}(C_*) \longrightarrow 0$$

- $\widetilde{\mathbf{b}}_{i-1}$: a lift of $\mathbf{b_{i-1}}$ in C_i
- $(\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}})$: another basis of C_i
- $\left(\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}}/\mathbf{c_i}\right)$: the transformation matrix from $\mathbf{c_i}$ to $(\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}})$
- $[\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}}/\mathbf{c_i}] = \det\left(\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}}/\mathbf{c_i}\right)$:

Torsion of a based chain complex

Definition

The torsion $\tau(C_*)$ of a based chain complex C_* with $\mathbf{c}_*$

$$\tau(C_*) = \frac{\prod_{i:odd}[\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}}/\mathbf{c_i}]}{\prod_{i:even}[\mathbf{b_i}, \widetilde{\mathbf{b}}_{\mathbf{i-1}}/\mathbf{c_i}]} \in \mathbb{F} \setminus \{0\}.$$

Lemma

$\tau(C_*)$ does not depend on choices of lifts $\{\mathbf{b}_i\}$.

Lemma

$\tau(C_*)$ does not depend on choices of $\{\mathbf{b}_i\}$.

Proof.

Assume $\mathbf{b'_q}$ is another basis of B_q.
In the definition of $\tau(C_*)$, the difference between $\mathbf{b_q}$ and $\mathbf{b'_q}$ is related to the followings only two parts:

$$\left[\mathbf{b'_q}, \mathbf{b_{q-1}}/\mathbf{c_q}\right] = \left[\mathbf{b_q}, \mathbf{b_{q-1}}/\mathbf{c_q}\right]\left[\mathbf{b'_q}/\mathbf{b_q}\right]$$

$$\left[\mathbf{b_{q+1}}, \mathbf{b'}_\mathbf{q}/\mathbf{c_{q+1}}\right] = \left[\mathbf{b_{q+1}}, \mathbf{b_q}/\mathbf{c_{q+1}}\right]\left[\mathbf{b'_q}/\mathbf{b_q}\right]$$

Since $\left[\mathbf{b'_q}/\mathbf{b_q}\right]$ appears in the both of the denominator and the numerator of the definition, they are cancelled. □

Reidemeister torsion for a CW-complexe

- X:a finite(=compact) CW-complex
- $\tilde{X} \to X$:universal covering
- $\pi_1(X)$ acts on $\tilde{X}$ (from the right), cellularly.
- $C_i(\tilde{X};\mathbb{Z})$:spanned by lifts of i-cells of X over $\mathbb{Z}$.
- $C_*(\tilde{X};\mathbb{Z})$: a chain complex of right $\mathbb{Z}[\pi_1(X)]$-modules.

local coefficients

- $\rho : \pi_1(X) \to GL(V)$: a finite dimensional linear representation over $\mathbb{F}$
- V:l-dimensional vector space over $\mathbb{F}$ $(= \mathbb{C}^2, sl(2;\mathbb{C}), su(2))$
- $C_*(X; V_\rho) = C_*(\tilde{X}; \mathbb{Z}) \otimes_{\pi_1(X)} V_\rho$

Assume: $C_*(X; V_\varphi)$ is acyclic.

Definition

$$\tau_\rho(X) = \tau(C_*(X; V_\rho)) \in \mathbb{F} \setminus \{0\}.$$

well-definedness

- Choices of lifts of cells
- Orders of cells.
- Choice of a basis for V

Proposition

- $\tau_\rho(X) \in \mathbb{F}/\pm\det(\rho(\pi_1(X)))$ *does not depend on choices of lifts and orders of cells.*
- *It depends on a choice of basis* $e_1, \ldots, e_l$. *Then we fix* $e_1 \wedge \cdots \wedge e_l \in \bigwedge^l V$.

Under same assumption, we have the following.

Theorem (Kirby-Siebenmann, Chapman)

If X is a closed manifold, then $\tau_\rho(M) \in \mathbb{F}/\pm\det(\rho(\pi_1(X)))$ is a topological invariant of M.

Remark

For any acyclic even-dimensional unimodular representation $\rho : \pi_1(X) \to SL(2l; \mathbb{F})$, $\tau_\rho(X) \in \mathbb{F} \setminus \{0\}$ is well defined.

Remark

- *In the case of* $V = \mathbb{C}^2$, *we take*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- *In the case of* $V = su(2)$, *under a fixed Killing form, we take an orthonormal basis.*
- *If* ρ *is not acyclic, then we define* $\tau_\rho(X) = 0$.

Definition

We call ρ an acyclic representation if $C_*(X; V_\varphi)$ is acyclic.

Condition for acyclicity

If $C_*(X; V_\varphi)$ is acyclic, then Euler number of $H_*(X; V_\varphi)$ is

$$\begin{aligned} 0 &= \sum_i (-1)^i \dim H_i(X; V_\rho) \\ &= \sum_i (-1)^i \dim C_i(X; V_\rho) \\ &= \sum_i (-1)^i \dim C_i(X; \mathbb{Z}) \cdot \dim V \\ &= l\chi(X). \end{aligned}$$

Here $\chi(X)$ is Euler characteristic of X. Hence we have

$$\chi(X) = 0.$$

There does not exist an acyclic representation for X with $\chi(X) \neq 0$.

Johnson Theory

- M : homology 3-sphere
- $\lambda(M) \in \mathbb{Z}$: Casson invariant
- $\mathcal{R}(X) = \{\rho : \pi_1 X \to SU(2); \textit{irreducible representation}\}$
- $\hat{\mathcal{R}}(X) = \mathcal{R}(X)/SU(2)$
- $M = A \cup_{\Sigma_g} B$: Heegard decomposition
- $\hat{\mathcal{R}}(M) = \hat{\mathcal{R}}(A) \cap \hat{\mathcal{R}}(B) \subset \hat{\mathcal{R}}(\Sigma_g)$

Assume:

- $\hat{\mathcal{R}}(M)$ is a finite set.
- Any point of $\hat{\mathcal{R}}(M)$ is non degenerate.

For any $\rho \in \mathcal{R}(M)$, compose it with

$$Adj : SU(2) \to Aut(su(2)),$$

we have

$$Adj \circ \rho : \pi_1(M) \to Aut(su(2)).$$

- $dim_{\mathbb{R}} H^1(\Sigma_g; su(2)_{Adj \circ \rho}) = 6g - 6$
- $dim_{\mathbb{R}} H^1(A; su(2)_{Adj \circ \rho}) = 3g - 3$
- $dim_{\mathbb{R}} H^1(B; su(2)_{Adj \circ \rho}) = 3g - 3$
- $H^1(M; su(2)_{Adj \circ \rho}) = \{0\}$

volume forms

Johnson constructed volume forms

- vol_{Σ_g} on $H^1(\Sigma_g; su(2)_{Adj\circ\rho})$,
- vol_A on $H^1(A; su(2)_{Adj\circ\rho})$,
- vol_B on $H^1(B; su(2)_{Adj\circ\rho})$.

By the assumption, he defined $t_{[\rho]} \in \mathbb{R} \setminus \{0\}$ by

$$vol_{\Sigma_g} = t_{[\rho]}(vol_A \wedge vol_B) \in \bigwedge^{6g-6} H^1(\Sigma_g; su(2)_{Adj\circ\rho}).$$

Geometric form of Casson invariant

Theorem (Johnson)

- $$\lambda(M) = \sum_{[\rho]\in\hat{\mathcal{R}}(M)} sgn(t_{[\rho]})$$
- $t_{[\rho]} = \tau_{Adj\circ\rho}(M)$

Remark

- $Adj \circ \rho : \pi_1 M \to SU(2) \to Aut(su(2))$
- $\tau_{Adj\circ\rho}(M) = \tau(C_*(M; su(2)_{Adj\circ\rho}))$
- *A* *geometric form* *of Casson invariant:*

$$\sum_{[\rho]\in\hat{\mathcal{R}}(M)} t_{[\rho]}$$

A polynomial invariant defined by Reidemeister torsion

Johnson proposed to study

- a polynomial whose zeros are $\{t_{[\rho]}\}$:

$$\prod_{[\rho]} (t - t_{[\rho]}),$$

- a polynomial for Reidemeister torsion $\tau_\rho(M)$ for an $SL(2;\mathbb{C})$-representation.

In particular he studied $\prod_{[\rho]} (t - \tau_\rho)$ for $1/n$-surgeried manifold along the trefoil knot.

A polynomial defined by Reidemeister torsion

- M : homology 3-sphere
- $\tau_\rho(M) \in \mathbb{C}$: Reidemeister torsion for M with an irreducible $SL(2;\mathbb{C})$-representation ρ
- $\tau'_\rho(M) = 1/\tau_\rho(M)$
- Assume : $\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(M)$ is a finite set.

By assumption, $\{\tau_\rho(M) \mid \tau_\rho(M) \neq 0\}$ is a finite set.

Definition

$\sigma_M(t) \in \mathbb{C}[t]$ is a polynomial whose roots are given by the set $\left\{\tau'_\rho(M) = \frac{1}{\tau_\rho(M)} \mid \tau_\rho(M) \neq 0\right\}$.

Assumption : $\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(M)$ is an algebraic variety of dimension zero.
By using resultants, it is given as a set of algebraic numbers.

- Any representation ρ in $\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(M)$ can be realized over an algebraic extension field $\mathbb{F}$ over $\mathbb{Q}$.
- $\tau_\rho(M)$ and $\tau'_\rho(M)$ are also algebraic numbers in $\mathbb{F}$.
- Any Galois conjugate of ρ (under the action of $Gal(\mathbb{F}/\mathbb{Q})$) is a representation.

If the action of $Gal(\mathbb{F}/\mathbb{Q})$ gives all Galois conjugate of τ'_ρ by $Gal(\mathbb{Q}(\{\tau'_\rho\})/Q)$, then

- $\sigma_M(t) \in \mathbb{Q}[t]$.
- It is the minimal polynomial of $\tau'_\rho \in \mathbb{Q}(\tau'_\rho)$.

minimal polynomial

- $F \supset \mathbb{Q}$ is an extension of $\mathbb{Q}$.
- $\theta \in \mathbb{F}$

Definition

The minimal polynomial of θ is the monic polynomial of least degree among all polynomials in $\mathbb{Q}[x]$ having θ as a root.

Remark

The minimal polynomial of θ exists when θ is algebraic over $\mathbb{Q}$, that is, $f(\theta) = 0$ for some non-zero polynomial $f(x) \in \mathbb{Q}[x]$.

Brieskorn homology 3-sphere

- $\Sigma(p, q, r)$: Brieskorn homology 3-sphere

$$\{z_1^p + z_2^q + z_3^r = 0\} \cap S^5 \subset \mathbb{C}^3 = \mathbb{R}^6$$

- $p, q, r \in \mathbb{Z}$: pairwise coprime positive integers
- We may assume q and r are odd numbers.
- $\Sigma(p, q, r)$ can be given by Dehn surgery along (p, q)-torus knot

$\Sigma(p, q, |pqn + 1|)$

Now consider the case of $r = |pqn + 1|$. Then $\Sigma(p, q, |pqn + 1|)$ can be obtained by $1/n$-surgery along $T(p, q)$.
In this case any conjugacy class of irreducible representations can be represented by

$$\rho_{(a,b,k)} : \pi_1(\Sigma(p, q, |pqn + 1|)) \to SL(2; \mathbb{C})$$

with

1. $0 < a < p, 0 < b < q, a \equiv b \bmod 2,$
2. $0 < k < |pqn + 1|, k \equiv na \bmod 2,$
3. $\mathrm{tr}(\rho_{(a,b,k)}(x)) = 2 \cos \frac{a\pi}{p},$
4. $\mathrm{tr}(\rho_{(a,b,k)}(y)) = 2 \cos \frac{b\pi}{q},$
5. $\mathrm{tr}(\rho_{(a,b,k)}(\mu)) = 2 \cos \frac{k\pi}{|pqn+1|}.$

$\tau_\rho(\Sigma(p, q, |pqn + 1|))$

Johnson computed Reidemeister torsion $\tau_{\rho_{(a,b,k)}}(\Sigma(p, q, |pqn + 1|))$ as follows.

Theorem (Johnson)

1. *A representation $\rho_{(a,b,k)}$ is acyclic if and only if $a \equiv b \equiv 1, k \equiv n \mod 2$.*
2. *For an acyclic representation $\rho_{(a,b,k)}$,*

$$\begin{aligned}&\tau_{\rho_{(a,b,k)}}(\Sigma(p, q, |pqn + 1|)) \\ =&\frac{1}{2\left(1 - \cos\frac{a\pi}{p}\right)\left(1 - \cos\frac{b\pi}{q}\right)\left(1 + \cos\frac{pqk\pi}{|pqn+1|}\right)}.\end{aligned}$$

In general case, we have the following.

Proposition (Johnosn, Kitano-Tran)

$$\tau_{\rho_{(a,b,c)}}(\Sigma(p,q,r)) = \frac{1}{2\left(1-\cos\frac{a\pi}{p}\right)\left(1-\cos\frac{b\pi}{q}\right)\left(1-\cos\frac{c\pi}{r}\right)}$$

$$= \frac{1}{16\sin^2\left(\frac{\pi a}{2p}\right)\sin^2\left(\frac{\pi b}{2q}\right)\sin^2\left(\frac{\pi c}{2r}\right)}$$

where $(0,0,0) < (a,b,c) < (p,q,r)$, $a \equiv b \equiv c \equiv 1$ *mod* 2.

$\sigma_{(p,q,r)}(t)$

From here $\sigma_{\Sigma(p,q,r)}(t)$ is simply written as $\sigma_{(p,q,r)}(t)$.

$$\sigma_{(p,q,r)}(t) = C \prod_{(a,b,c)} \left(t - 16 \sin^2\left(\frac{\pi a}{2p}\right) \sin^2\left(\frac{\pi b}{2q}\right) \sin^2\left(\frac{\pi c}{2r}\right) \right)$$

where the above product can be taken over the same condition $(0,0,0) < (a,b,c) < (p,q,r)$, $a \equiv b \equiv c \equiv 1 \bmod 2$.
Now put

$$C_{(p,q,a,b)} = 2 \sin\left(\frac{\pi a}{2p}\right) \sin\left(\frac{\pi b}{2q}\right).$$

Normalized Chebyshev polynomial of first kind

Normalized Chebyshev polynomial of the first kind:

$$T_n(2\cos\theta) = 2\cos n\theta.$$

Recursive definition:

- $T_0(x) = 2,\ T_1(x) = x.$
- $T_{n+1}(x) = xT_n(x) - T_{n-1}(x).$

$$T_0(x) = 2,\, T_1(x) = x,\, T_2(x) = x^2 - 2,\, T_3(x) = x^3 - 3x.$$

Normalized Chebyshev polynomial of second kind

Normalized Chebyshev polynomial of the second kind:

$$S_n(2\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Recursive definition:

- $S_0(x) = 1,\ S_1(x) = x.$
- $S_{n+1}(x) = xS_n(x) - S_{n-1}(x).$

$$S_0(x) = 1, S_1(x) = x, S_2(x) = x^2 - 1, S_3(x) = x^3 - 2x.$$

A Relation:

$$T_n(x) = S_n(x) - S_{n-2}(x)$$

Polynomial of $\Sigma(p, q, r)$

Theorem (Kitano-Tran)

$$\begin{aligned}
\sigma_{(p,q,r)}(t) &= \prod_{(a,c)} S_{q-1}\left(\frac{\sqrt{t}}{C_{(p,r,a,c)}}\right) \\
&= \prod_{(a,b)} S_{r-1}\left(\frac{\sqrt{t}}{C_{(p,q,a,b)}}\right) \\
&= \begin{cases} \displaystyle\prod_{(b,c)} S_{p-1}\left(\frac{\sqrt{t}}{C_{(q,r,b,c)}}\right) & (p : odd) \\ \displaystyle\prod_{(b,c)} \bar{S}_{2p-1}\left(\frac{\sqrt{t}}{C_{(q,r,b,c)}}\right) & (p : even) \end{cases} .
\end{aligned}$$

Here $\bar{S}_{2p-1}(x) = S_{2p-1}(x)/x$.

Examples

$$
\begin{aligned}
\sigma_{(2,3,5)}(t) =& 4t^2 - 6t + 1 \\
\sigma_{(2,3,7)}(t) =& 8t^3 - 20t^2 + 12t - 1 \\
\sigma_{(2,3,11)}(t) =& 32t^5 - 144t^4 + 224t^3 - 140t^2 + 30t - 1 \\
\sigma_{(2,3,13)}(t) =& 64t^6 - 352t^5 + 720t^4 - 672t^3 + 280t^2 - 42t + 1 \\
\sigma_{(2,3,17)}(t) =& 256t^8 - 1920t^7 + 5824t^6 - 9152t^5 + 7920t^4 \\
& - 3693t^3 + 840t^2 - 72t + 1 \\
\sigma_{(2,3,19)}(t) =& 512t^9 - 4352t^8 + 15360t^7 - 29120t^6 + 32032t^5 \\
& - 20592t^4 + 7392t^3 - 1320t^2 + 90t - 1
\end{aligned}
$$

They are irreducible over $\mathbb{Q}$.

Remark

I add the following example by suggestions of T. Kitayama after the talk.

Example (reducible example)

$$\sigma_{(2,5,9)}(T) = 256t^8 - 2688t^7 + 9856t^6 - 15840t^5 + 12192t^4 - 4608t^3 + 820t^2 - 60t + 1$$

This is not irreducible as

$$\begin{aligned} & \sigma_{(2,5,9)}(T) \\ =& (4t^2 - 6t + 1) \\ & \times (64t^6 - 576t^5 + 1584t^4 - 1440t^3 + 492t^2 - 54t + 1) \\ =& \sigma_{(2,3,5)}(T) \\ & \times (64t^6 - 576t^5 + 1584t^4 - 1440t^3 + 492t^2 - 54t + 1) \end{aligned}$$

$SL(2;\mathbb{C})$-Casson invariant

For $M = \Sigma(2,3,6n+1)(n > 0)$,

$$\begin{aligned}\sharp\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(\Sigma(2,3,6n+1)) =& \lambda_{SL(2;\mathbb{C})}(\Sigma(2,3,6n+1)) \\ =& 3n \ (Boden - Curtis)\end{aligned}$$

On the other hand,

$$\lambda_{SU(2)}(\Sigma(2,3,6n+1)) = n.$$

$$\lambda_{SL(2;\mathbb{C})}(\Sigma(2,3,6n+1)) - 2\lambda_{SU(2)}(\Sigma(2,3,6n+1)) = n$$

What is the difference ?

$$\sharp\hat{\mathcal{R}}_{SL(2;\mathbb{R})}(\Sigma(2,3,6n+1)) = n$$

Proposition (K.-Yamaguchi)

$$\lambda_{SL(2;\mathbb{C})}(\Sigma(p,q,r)) = 2\lambda_{SU(2)}(\Sigma(p,q,r)) + \sharp\hat{\mathcal{R}}_{SL(2;\mathbb{R})}(\Sigma(p,q,r))$$

- $\rho : \pi_1(\Sigma(p,q,r)) \to SL(2;\mathbb{C})$ induces

$$\bar{\rho} : \pi_1^{orb} S^2(p,q,r) \to PSL(2;\mathbb{C}).$$

- $\pi_1^{orb} S^2(p,q,r) = \langle\ x, y, z \mid x^p = y^q = z^r = xyz = 1\ \rangle$
- Jenkins-Neumann:
  Classification of $PSL(2;\mathbb{R})$-representations of triangle groups
- Any $SU(2)$-representation that is conjugate to $SL(2;\mathbb{R})$-representation is an abelian representation.

" $SL(2;\mathbb{R})$-Casson invariant"

Problem

Can we define an $SL(2;\mathbb{R})$-Casson invariant ?

Dehn surgery along the figure-eight knot

- $K = 4_1$
- $\pi_1(S^3 \setminus K) = \langle a, b \mid w^{-1}a = bw^{-1} \rangle$ where $w = ba^{-1}b^{-1}a$.
- $M_n(K)$: $1/n$-Dehn surgery along K.
- $\rho : \pi_1(M_n(K)) \to SL(2, \mathbb{C})$: irreducible representation.
- $\mu = a \in \pi_1(M_n(K))$: meridian of K
- $x = tr(\rho(\mu))$

Reidemeister torsion for $M_n(K)$

Reidemeister torsion of $M_n(K)$ for ρ is given as follows.

Theorem (K.)

$$\tau_\rho(M_n(K)) = -\frac{2(x-1)}{x^4 - 9x^2 + 4}$$

Here $x = tr(\rho(\mu))$.

Remark

The explicit value x is determined by the surgery condition. Remark that any complex value x can not be realized.

Let $\rho : \pi_1(S^3 \setminus K) \to SL(2;\mathbb{C})$ be an irreducible representaion. By taking a conjugation, we may assume

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix}, \quad \rho(b) = \begin{bmatrix} s & 0 \\ -u & s^{-1} \end{bmatrix}.$$

Here $(s, u) \in (\mathbb{C}^*)^2$ is a solution of

$$\phi_K(s, u) = u^2 - (u+1)(s^2 + s^{-2} - 3).$$

Compute the image of longitude
$\lambda = (ab^{-1}a^{-1}b)^{-1})(ba^{-1}b^{-1}a^{-1})^{-1}$,

$$\rho(\lambda) = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

where

$$\lambda_{11} = (-s^{-2} - 1)u^3 + (s^2 - s^{-2})u + (s^{-4} + s^2 - s^{-2} - 1)u^2 + 1,$$
$$\lambda_{12} = u(s + s^{-1})(s^2 + s^{-2} - 1 - u),$$
$$\lambda_{21} = -u^2(s + s^{-1})\phi_K(s, u),$$
$$\lambda_{22} = (-s^2 - 1)u^3 + (s^4 - s^2 + s^{-2} - 1)u^2 + (s^{-2} - s^2)u + 1.$$

The condition $\phi_K(s,u)=0$ implies

$$\begin{aligned} tr\rho(\lambda) &= s^4+s^{-4}-s^2-s^{-2}-2 \\ &= x^4-5x^2+2 \end{aligned}$$

where $x=s+s^{-1}$.
Here this representation $\pi_1(S^3\setminus K)\to SL(2;\mathbb{C})$ can be extend as $\rho:\pi_1(M_n(K))\to SL(2;\mathbb{C})$ if and only if

$$\rho(\lambda^n)=\rho(\mu^{-1}).$$

By using $L = tr\rho(\lambda) = x^4 - 5x^2 + 2$,

$$\rho(\lambda^n) = \left[\begin{array}{cc} \lambda_{11}^n & \lambda_{12} S_{n-1}(L) \\ 0 & \lambda_{22}^n \end{array} \right].$$

Further the condition $\rho(\lambda^n) = \rho(\mu^{-1})$ is equivalent to the following three conditions of

1. $\lambda_{11}^n = s^{-1}$,
2. $\lambda_{22}^n = s$,
3. $\lambda_{12} S_{n-1}(L) = -1$.

By direct computation, we have

1. $u^2 - (u+1)(s^2 + s^{-2} - 3) = 0$,
2. $\lambda_{11}^n = s^{-1}$,
3. $L \neq -2$ (i.e. $x^2 \neq 4$).

Lemma

The set of s satisfying the following

1. $u^2 - (u+1)(s^2 + s^{-2} - 3) = 0$,
2. $\lambda_{11}^n = s^{-1}$

is coincided wth the set of solutions for

$$s + s^{-1} = T_n(s^4 + s^{-4} - s^2 - s^{-2} - 2).$$

By the formula of Reideimeister torsion

$$\tau_\rho = \frac{2x-2}{x^2(x^2-5)},$$

we consider the resultant

$$P_n(t) = \mathrm{Res}_x \left(T_n(x^4 - 5x^2 + 2) - x, tx^2(x^2-5) - (2x-2) \right).$$

A resultant

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

Definition (the resultant of $f(x), g(x)$)

$$Res(f(x), g(x)) = \begin{vmatrix} a_n & a_{n-1} & \cdots & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_1 & a_0 & & \\ & & a_n & \cdots & & & & \\ & & & \vdots & & & & \\ & & & a_n & \cdots & & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & \cdots & b_0 & & & \\ & b_m & b_{m-1} & \cdots & b_1 & b_0 & & \\ & & b_m & \cdots & & & & \\ & & & \vdots & & & & \\ & & & b_m & \cdots & & b_1 & b_0 \end{vmatrix}$$

$f(x,y), g(x,y)$:polynomials of 2-variable x, y By considering x is a variable and y is a constant, we can define the resultant $Res_x(f,g)$.

Proposition

There exist polynomials $F(x,y), G(x,y)$ such that

$$f(x,y)G(x,y) + g(x,y)F(x,y) = Res_x(f,g).$$

Therefore a solution (x,y) of $f(x,y) = g(x,y) = 0$ satisfies $Res_(f,g) = 0$.

Polynomial for surgeried manifold along 4_1

$M_n(K)$ is a homology 3-sphere obtained by $1/n$-surgery along $K = 4_1$.

Theorem (Kitano-Tran)

$$\sigma_{M_n(K)}(t) = \begin{cases} P_n(t)/(2t-3) & n \text{ is odd,} \\ P_n(t)/(2t+1) & n \text{ is even.} \end{cases}$$

Here

$$P_n(t) = Res_x\left(T_n(x^4 - 5x^2 + 2) - x, tx^2(x^2 - 5) - (2x - 2)\right)$$

Examples

$$
\begin{aligned}
\sigma_1(t) =& t^3 - 12t^2 + 20t - 8 \\
\sigma_2(t) =& t^7 - 56t^6 + 660t^5 - 3384t^4 + 8720t^3 - 11008t^2 \\
& + 5376t - 128 \\
\sigma_3(t) =& t^{11} - 124t^{10} + 3036t^9 - 31696t^8 + 161024t^7 \\
& - 364128t^6 + 152640t^5 + 426752t^4 - 262144t^3 \\
& - 142336t^2 + 55296t - 2048
\end{aligned}
$$

$$
\begin{aligned}
\sigma_4(t) = & t^{15} - 224t^{14} + 10320t^{13} - 211776t^{12} + 2296400t^{11} \\
& - 135709000t^{10} + 411722000t^{9} - 496721000t^{8} - 355295000t^{7} \\
& + 1563510000t^{6} - 1136530000t^{5} - 589578000t^{4} + 1159330000t^{3} \\
& - 500040000t^{2} + 5898240t - 32768
\end{aligned}
$$

Open Problems

Problem

How strong is this polynomial $\sigma_M(t)$?

- *How about for Brieskorn homology sphere ?*

$\sigma_M(t)$ is a minimal polynomial?

Problem

How is it related with minimal poly of τ'_ρ over $\mathbb{Q}$?

Problem

If there exists an epimorphism $\varphi : \pi_1(M) \to \pi_1(M')$, then $\sigma_M(t)$ can be divided by $\sigma_{M'}(t)$?

Problem

- *Does $\sigma_M(t)$ know $\sharp\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(M)$ and $SL(2;\mathbb{C})$-Casson invariant?*
- *Does τ'_ρ with the action of a Galois group know $\sharp\hat{\mathcal{R}}_{SL(2;\mathbb{C})}(M)$ and $SL(2;\mathbb{C})$-Casson invariant?*

General case

Problem

How can we treat it in the case of $\sharp\hat{\mathcal{R}}(M) = \infty$*?*

- *Only consider the componets of dimension* 0 *?*
- *How to take a perturbation ?*
- *How to treat splicing manifolds ?*

In the original definition of the Casson invariant, we need to take a perturbation to get a transverse intersection. The problem is that a representation after a perturbation does not give a representation of $\pi_1(M)$.

Problem

Can we construct $\lambda_{SL(2;\mathbb{R})}(M)$ *for (some class of) homology 3-spheres satisfying*

$$\lambda_{SL(2;\mathbb{C})}(\Sigma(p,q,r)) = 2\lambda(\Sigma(p,q,r)) + \lambda_{SL(2;\mathbb{R})}(\Sigma(p,q,r))?$$

Proposition

There exists a hyperbolic homology 3-sphere without an irreducible $SL(2;\mathbb{R})$*-representation.*

$SL(2;\mathbb{R})$-representations

$M_n(K)$: $1/n$-surgeried along the figure-eight knot

Proposition

$$\lambda_{SL(2;\mathbb{C})}(M_n(K)) = 4n - 1, \lambda(M_n(K)) = -n$$

$$\lambda_{SL(2;\mathbb{C})}(M_n(K)) - 2|\lambda(M_n(K))| = 2n - 1$$

Checked by Mathematica:

Example (n=2,...,6,8)

$$\lambda_{SL(2;\mathbb{C})}(M_n(K)) - 2|\lambda(M_n(K))| > \sharp\hat{\mathcal{R}}_{SL(2;\mathbb{R})}(M_n(K)) > 0$$

Example ($n = 7, 9$)

$$\sharp\hat{\mathcal{R}}_{SL(2;\mathbb{R})}(M_n(K)) = 0.$$

Problem

How can we construct TQFT for Johnson theory?

Witten's theory

Description from the view point of topological quantum field theory: E. Witten, *Topology-changing amplitudes in* $(2+1)$*-dimensional gravity*, Nuclear Phys. B **323** (1989).