Casson invariant and structure of the mapping class group

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January 25, 2017

Contents

Homology 3-spheres and the Torelli group (1)

$\mathfrak{M}(3) = \{$closed oriented 3-manifold$\}/$ori. pres. diffeo.

$\cup$

$\mathfrak{H}(3) = \{$closed oriented homology 3-sphere$\}/$ori. pres. diffeo.

Heegaard decomposition:

$\mathfrak{M}(3) \ni^{\forall} [M],\ M = H_g \cup_\varphi -H_g \quad (H_g$: handlebody, $\varphi \in \mathcal{M}_g)$

$\mathcal{M}_g$: mapping class group

$S^3 = H_g \cup_{\iota_g} -H_g \quad (\iota_g \in \mathcal{M}_g : 90^\circ$-rotation on each handle)

$\Rightarrow\ \mathcal{M}_g \ni [\varphi] \longmapsto [M_\varphi = H_g \cup_\varphi -H_g] \in \mathfrak{M}(3)$

Homology 3-spheres and the Torelli group (2)

Theorem (Reidemeister-Singer)

$$\left(\coprod_g \mathcal{M}_g\right) / \textit{R.S. stabilization} = \mathfrak{M}(3)$$

alternative description:

fix a Heegaard embedding $\Sigma_g \subset S^3 = H_g \cup_{\Sigma_g} -H_g$

$$\mathfrak{N}_g = \{\varphi \in \mathcal{M}_g; \varphi \text{ extends to diffeo. of } H_g\} \quad (\subset \mathcal{M}_g)$$
$$\mathfrak{N}'_g = \{\varphi \in \mathcal{M}_g; \varphi \text{ extends to diffeo. of } -H_g\} \quad (\subset \mathcal{M}_g)$$

$$\mathfrak{M}(3) = \lim_{g\to\infty} \mathfrak{N}'_g \backslash \mathcal{M}_g / \mathfrak{N}_g$$

Homology 3-spheres and the Torelli group (3)

restriction to the Torelli group :

$$\mathcal{I}_g = \mathrm{Ker}(\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z}))$$

Proposition

$$\lim_{g\to\infty} \mathcal{I}_g / \sim \; = \mathfrak{H}(3) \quad (\mathcal{I}_g \ni \varphi \longmapsto W_\varphi = H_g \cup_{\iota_g \varphi} -H_g \in \mathfrak{H}(3))$$

where $\varphi \sim \psi \Leftrightarrow \iota_g\varphi = \iota_g\psi \in \mathfrak{N}'_g \backslash \mathcal{M}_g / \mathfrak{N}_g$

two filtrations of $\mathcal{I}_g$:

$\mathcal{I}_g = \mathcal{M}_g(1) \supset \mathcal{M}_g(2) \supset \cdots$ (Johnson filtration)

$\mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) = [\mathcal{I}_g(1), \mathcal{I}_g(1)] \supset \cdots$ (lower central series)

$\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ for any k

$$\mathcal{M}_g(2) = \mathrm{Ker}(\tau_1 : \mathcal{I}_g \overset{\text{first Johnson hom.}}{\longrightarrow} \wedge^3 H/H) \ (H = H_1(\Sigma_g; \mathbb{Z}))$$

$$\mathcal{M}_g(k+1) = \mathrm{Ker}(\tau_k : \mathcal{M}_g(k) \overset{\text{Johnson hom.}}{\longrightarrow} \mathfrak{h}_g(k))$$

Theorem (Johnson)

$$\mathcal{I}_g(2) \overset{\text{finite index}}{\subset} \mathcal{M}_g(2) = \mathcal{K}_g = \langle \textit{Dehn twists on BSCC} \rangle$$

$$\mathcal{I}_g(1)/\mathcal{I}_g(2) = H_1(\mathcal{I}_g) \cong \wedge^3 H/H \oplus 2\text{-\textit{torsion}}$$

BCSS=bounding simple closed curve

Homology 3-spheres and the Torelli group (5)

Two filtrations of $\mathcal{I}_g$ induces those of $\mathfrak{H}(3)$ and $\mathbb{Q}\mathfrak{H}(3)$:

$$\mathfrak{H}(3) = \mathfrak{H}(3)^1 \supset \mathfrak{H}(3)^2 \supset \mathfrak{H}(3)^3 \supset \cdots$$
$$\mathfrak{H}(3) = \mathfrak{H}(3)_1 \supset \mathfrak{H}(3)_2 \supset \mathfrak{H}(3)_3 \supset \cdots$$

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)^1 \supset \mathbb{Q}\mathfrak{H}(3)^2 \supset \mathbb{Q}\mathfrak{H}(3)^3 \supset \cdots$$
$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)_1 \supset \mathbb{Q}\mathfrak{H}(3)_2 \supset \mathbb{Q}\mathfrak{H}(3)_3 \supset \cdots$$

where

$$\mathfrak{H}(3)^k = \lim_{g\to\infty} \mathcal{M}_g(k)/\sim \supset \ \mathfrak{H}(3)_k = \lim_{g\to\infty} \mathcal{I}_g(k)/\sim$$

Casson invariant (1985):

$\lambda : \mathfrak{H}(3) \to \mathbb{Z}$

(i) $\lambda \equiv$ Rohlin homomorphism : $\mathfrak{H}(3) \to \mathbb{Z}/2$ (mod 2)

(ii) $\lambda = \dfrac{1}{2}$ "alg. number" of $\{$irred. rep. : $\pi_1 W \to \mathrm{SU}(2)\}$/conj.

(iii) $\lambda(-W) = -\lambda(W)$, additive w.r.t. connected sum

(iv) $W \supset K$ (knot) $\Rightarrow \lambda(W_{1/n}(K)) = \lambda(W) + n\dfrac{1}{2}\bar{\Delta}''_K(1)$

Extensions by Walker (to rational homology 3-spheres) and

Lescop (to all 3-manifolds)

Casson invariant and the first MMM class (2)

Consider the mapping

$$\lambda^* : \mathcal{I}_g \to \mathbb{Z} \quad \text{defined by } \lambda^*(\varphi) = \lambda(W_\varphi)$$

NOT a homomorphism, but its restriction to $\mathcal{K}_g$

$$\lambda^* : \mathcal{K}_g \to \mathbb{Z}$$

can be shown to be a homomorphism!

What is it?

Answer: secondary class associated to the fact: the first

MMM-class vanishes in the Torelli group $\quad e_1 = 0 \in H^2(\mathcal{I}_g; \mathbb{Q})$

Casson invariant and the first MMM class (3)

$e_1 \in H^2(\mathcal{M}_g; \mathbb{Z})$

geometric meaning: signature of surface bundles over surfaces

$\Rightarrow \ e_1 = 0 \in H^2(\mathcal{I}_g; \mathbb{Q})$ because signature of any fiber bundle

$F \to E \to B$ vanishes if $\pi_1 B$ acts on $H_*(F; \mathbb{Q})$ trivially

(Chern-Hirzebruch-Serre)

There are two canonical cocycles representing e_1:

pull back of $\quad -3c_1 \in Z^2(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q})$

image of $\quad \wedge^2 (\wedge^3 H_\mathbb{Q}/H_\mathbb{Q})^{\mathrm{Sp}} \cong \mathbb{Q} \to \wedge^2 \tilde{k} \in Z^2(\mathcal{M}_g; \mathbb{Q})$

under $\quad (\tilde{k}, \rho_0) : \mathcal{M}_g \to \wedge^3 H_\mathbb{Q}/H_\mathbb{Q} \rtimes \mathrm{Sp}(2g, \mathbb{Z})$

Casson invariant and the first MMM class (4)

$$\Rightarrow 3c_1 + \wedge^2 \tilde{k} = d^{\exists} u_1, \quad u_1 \in C^1(\mathcal{M}_g; \mathbb{Q})$$

$$\begin{cases} c_1|_{\mathcal{I}_g} & = 0 \\ \wedge^2 \tilde{k}|_{\mathcal{K}_g} & = 0 \end{cases} \Rightarrow d(u_1|_{\mathcal{K}_g}) = 0$$

Theorem (M.)

$H^1(\mathcal{K}_g; \mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q}$ $(g \geq 2)$ *generated by* $d_1 := [u_1|_{\mathcal{K}_g}]$

Furthermore $d_1($*BSCC map of type* $(h, g-h)) = ct.\ h(g-h)$

Casson invariant and the first MMM class (5)

Theorem (M.)

$$\lambda^* = \frac{1}{24} d_1 + \bar{\tau}_2 : \ \mathcal{K}_g \to \mathbb{Q}$$

where

$$\bar{\tau}_2 = \mathcal{K}_g \overset{\tau_2}{\to} \mathfrak{h}_g(2) \overset{\text{certain quotient}}{\longrightarrow} \mathbb{Q}$$

Furthermore

$$\lambda^* = \frac{1}{24} d_1 \quad \text{on } \mathcal{M}_g(3)$$

so that we may say that d_1 is the core of the Casson invariant

Difference between two filtrations of the Torelli group (1)

(recall) two filtrations of $\mathcal{I}_g$:

$$\mathcal{I}_g = \mathcal{M}_g(1) \supset \mathcal{M}_g(2) = \mathcal{K}_g \supset \mathcal{M}_g(3) \cdots \text{ (Johnson filtration)}$$
$$\mathcal{I}_g = \mathcal{I}_g(1) \supset \mathcal{I}_g(2) = [\mathcal{I}_g(1), \mathcal{I}_g(1)] \supset \mathcal{I}_g(3) \cdots \text{ (lower central series)}$$

$\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ for any k

Johnson showed $\mathcal{M}_g(2)/\mathcal{I}_g(2) \otimes \mathbb{Q} = 0$ and he asked

$$[\mathcal{M}_g(k) : \mathcal{I}_g(k)] < \infty \ ?$$

Theorem (M. 1988)

The index of $\mathcal{I}_g(3) = [[\mathcal{I}_g, \mathcal{I}_g], \mathcal{I}_g]$ in $\mathcal{M}_g(3)$ is infinite

This was proved by showing that

$d_1 \neq 0$ on $\mathcal{M}_g(3)$ whereas $d_1 = 0$ on $\mathcal{I}_g(3)$

Difference between two filtrations of the Torelli group (2)

alternatively

$$\tau_1 : \mathcal{I}_g \to \wedge^3 H/H \overset{\text{over } \mathbb{Q}}{\cong} (\mathcal{I}_g/\mathcal{I}_g(2)) \otimes \mathbb{Q}$$

$$\tau_1^* : H^2(\wedge^3 H_\mathbb{Q}/H_\mathbb{Q})^{\mathrm{Sp}} \cong \mathbb{Q} \to H^2(\mathcal{I}_g; \mathbb{Q})$$

is trivial because the image is e_1 which vanishes on the Torelli group. Then by considering the Sullivan 1-minimal model of $\mathcal{I}_g$ and by taking its dual, we can conclude $(\mathcal{I}_g(2)/\mathcal{I}_g(3))^{\mathrm{Sp}} \neq 0$ whereas we know that $(\mathcal{M}_g(2)/\mathcal{M}_g(3))^{\mathrm{Sp}} = 0$

Hain determined τ_1^* completely and obtained

Theorem (Hain 1997)

$\bigoplus_{k=1}^{\infty}(\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q} \cong$ *Free Lie* $\langle \wedge^3 H/H \rangle /$*quad. relation*

Difference between two filtrations of the Torelli group (3)

Theorem (Hain)

$\bigoplus_{k=1}^{\infty}(\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q} \to \bigoplus_{k=1}^{\infty}(\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$ *is surjective so that the latter is generated by the degree* 1 *part*

It is a mystery whether an analogue of the above difference would appear for $\mathrm{Aut}\, F_n$ or not, namely comparison of

Andreadakis filtration vs l.c.s. filtration of IA_n

Andreadakis Conjecture (over $\mathbb{Q}$)

works of Satoh, recently Bartholdi, Massuyeau-Sakasai

Finite type invariants and the Torelli group (1)

Ohtsuki's (finite type) invariants:

$$\lambda_k : \mathfrak{H}(3) \to \mathbb{Q} \quad (k = 1, 2, \ldots)$$

and Ohtsuki filtration based on the LMO-invariant

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)_{(3)} \supset \mathbb{Q}\mathfrak{H}(3)_{(6)} \supset \mathbb{Q}\mathfrak{H}(3)_{(9)} \supset \cdots$$

Finite type invariants and the Torelli group (2)

$\mathcal{A}(\emptyset) =$ commutative algebra generated by vertex oriented
connected trivalent graphs/AS+IHX
degree $=$ half the number of vertices

Theorem (Garoufalidis-Ohtsuki+Le-Murakami-Ohtsuki)

There exists an isomorphism

$$\mathrm{Gr}_m\mathcal{A}(\emptyset) \cong \mathbb{Q}\mathfrak{H}(3)_{(3m)}/\mathbb{Q}\mathfrak{H}(3)_{(3m+1)}$$

Theorem (Garoufalidis-Levine)

There exists a mapping

$$\mathcal{I}_g(2m)/\mathcal{I}_g(2m+1) \otimes \mathbb{Q} \to \mathrm{Gr}_m\mathcal{A}^{\mathrm{conn}}(\emptyset)$$

which is surjective for $g \geq 5m+1$

In particular

$$(\mathcal{I}_g(2m)/\mathcal{I}_g(2m+1))^{\mathrm{Sp}} \otimes \mathbb{Q}$$

gives rise to invariants for

$$\mathbb{Q}\mathfrak{H}(3)_{(3m)}/\mathbb{Q}\mathfrak{H}(3)_{(3m+1)}$$

The case $m = 1$:

$$\mathrm{Gr}_1\mathcal{A}(\emptyset) \cong \mathbb{Q}\mathfrak{H}(3)_{(3)}/\mathbb{Q}\mathfrak{H}(3)_{(4)} \cong \mathbb{Q}$$

$$(\mathcal{I}_g(2)/\mathcal{I}_g(3))^{\mathrm{Sp}} \otimes \mathbb{Q} \cong \mathrm{Gr}_1\mathcal{A}^{\mathrm{conn}}(\emptyset) \cong \mathbb{Q}$$

given by

theta graph, Casson invariant λ and $d_1 : \mathcal{K}_g \to \mathbb{Q}$

Open problems (1)

Problem (Hain)

Is the following sequence exact?

$$\mathbb{Q} \to \bigoplus_{k=1}^{\infty} (\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q} \twoheadrightarrow \bigoplus_{k=1}^{\infty} (\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$$

In other words, the difference (over $\mathbb{Q}$) between Johnson and

l.c.s filtrations of $\mathcal{I}_g$ is only the Casson invariant?

How about Ohtsuki invariants?

Open problems (2)

Problem

Compare the following three filtrations of $\mathbb{Q}\mathfrak{H}(3)$

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)^1 \supset \mathbb{Q}\mathfrak{H}(3)^2 \supset \mathbb{Q}\mathfrak{H}(3)^3 \supset \cdots \quad (\mathcal{M}_g(k)/\sim)$$

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)_1 \supset \mathbb{Q}\mathfrak{H}(3)_2 \supset \mathbb{Q}\mathfrak{H}(3)_3 \supset \cdots \quad (\mathcal{I}_g(k)/\sim)$$

$$\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)_{(3)} \supset \mathbb{Q}\mathfrak{H}(3)_{(6)} \supset \mathbb{Q}\mathfrak{H}(3)_{(9)} \supset \cdots \quad (\textit{Ohtsuki})$$

M.: $\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)^2$

any homology 3-sphere can be obtained by pasting $\mathcal{K}_g$

Pitsch: $\mathbb{Q}\mathfrak{H}(3) = \mathbb{Q}\mathfrak{H}(3)^3$

any homology 3-sphere can be obtained by pasting $\mathcal{M}_g(3)$

Open problems (3)

Casson invariant appears in

$\mathbb{Q}\mathfrak{H}(3)_2/\mathbb{Q}\mathfrak{H}(3)_3$ and $\mathbb{Q}\mathfrak{H}(3)_{(3)}/\mathbb{Q}\mathfrak{H}(3)_{(6)}$

The following two problems are modified ones from the original after comments by M. Sato and S. Tsuji

Problem

Determine whether the Ohtsuki invariant

$$\lambda_k : \mathfrak{H}(3) \cong \mathcal{I}_g/\sim (= \mathcal{K}_g/\sim = \mathcal{M}_g(3)/\sim) \to \mathbb{Q}$$

can be described explicitly in terms of d_1 and Johnson homomorphisms. If so, give the formula. In particular determine whether its restriction to some deep subgroup $\mathcal{M}_g(m_k)$ (depending on k) becomes a homomorphism or not

Open problems (4)

$k = 1$ (Casson invariant):

λ_1 is a homomorphism on $\mathcal{K}_g$ but NOT on $\mathcal{M}_g(1) = \mathcal{I}_g$

Problem (special case of Hain's problem)

By Garoufalidis-Levine, the Ohtsuki invariant λ_k restricted to $\mathcal{I}_g(2k)$ gives a homomorphism

$$\lambda_k : \mathcal{I}_g(2k)/\mathcal{I}_g(2k+1) \to \mathbb{Q}$$

For each $k \geq 2$, determine whether it is described as a quotient of the higher Johnson homomorphism

$$\mathcal{I}_g(2k)/\mathcal{I}_g(2k+1) \overset{\otimes\mathbb{Q}}{\twoheadrightarrow} \mathcal{M}_g(2k)/\mathcal{M}_g(2k+1) \overset{\tau_g(2k)}{\subset} \mathfrak{h}_g(2k)$$

or not

Extending the above picture to a wider context (1)

$$H^2(\mathcal{M}_g;\mathbb{Q}) \ni e_1 \mapsto 0 \in H^2(\mathcal{I}_g;\mathbb{Q}) \quad \Rightarrow \quad \lambda : \mathfrak{H}(3) \to \mathbb{Z}$$

More precisely

$$\mathcal{K}_g \to \mathcal{M}_g \to \mathcal{M}_g/\mathcal{K}_g \overset{\mathbb{Q}}{\cong} U_Q \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \quad (U = \wedge^3 H/H)$$

$$H^1(\mathcal{M}_g;\mathbb{Q}) = 0 \to H^1(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q} \to$$
$$H^2(U_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) \cong \mathbb{Q}^2 \to H^2(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q}$$

the difference of two natural cocycles for e_1 $\Rightarrow$

$$H^1(\mathcal{K}_g;\mathbb{Q})^{\mathcal{M}_g} \cong \mathbb{Q} \quad \Rightarrow \quad \text{Casson invariant}$$

Extending the above picture to a wider context (2)

extending $\mathcal{M}_g \Rightarrow \mathcal{H}_{g,1}$ and $e_1 \Rightarrow \tilde{t}_{2k+1}$, ultimate goal:

$$H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \ni \tilde{t}_{2k+1} \mapsto 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q}) \ \Rightarrow \ \nu_k : \boldsymbol{\Theta}^3 \to \mathbb{Q}$$

Garoufalidis-Levine (based on Goussarov and Habiro)

$\mathcal{H}_{g,1}^{\mathrm{smooth}} = \{$homology cylinder over $\Sigma_{g,1}\}/$smooth H-cobordism

$$\mathcal{H}_{0,1}^{\mathrm{smooth}} = \boldsymbol{\Theta}^3 = \mathfrak{H}(3)/\text{smooth H-cobordism} \overset{\text{central}}{\subset} \mathcal{H}_{g,1}^{\mathrm{smooth}}$$

$$\boldsymbol{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} = \mathcal{H}_{g,1}^{\mathrm{smooth}}/\boldsymbol{\Theta}^3 \quad \text{(central extension)}$$

Extending the above picture to a wider context (3)

exact sequence:

$$0 \to H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q}) \to H^1(\mathbf{\Theta}^3;\mathbb{Q})$$
$$\cong \mathrm{Hom}(\mathbf{\Theta}^3,\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}} \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

Theorem (Furuta, Fintushel-Stern)

$\mathbf{\Theta}^3$ *has infinite rank*

$\Rightarrow \mathbf{\Theta}^3/\text{torsion} \subset \mathbb{Q}^\infty$ (because $\mathbf{\Theta}^3$ is countable)

$\Rightarrow \mathrm{Hom}(\mathbf{\Theta}^3,\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ (direct product of countably many $\mathbb{Q}$)

so there exist (uncountably) many homomorphisms

$$\mathbf{\Theta}^3 \to \mathbb{Q}$$

but explicitly known one(s): Frϕyshov and Ozsváth-Szabó

Extending the above picture to a wider context (4)

Problem

How is the huge group $H^1(\mathbf{\Theta}^3;\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}}$ *divided into*

$$\mathrm{Coker}\left(H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right) \quad \textit{and}$$

$$\mathrm{Ker}\left(H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})\right) ?$$

Coker is non-trivial $\Leftrightarrow$

$^\exists$ homomorphism $\mathbf{\Theta}^3 \to \mathbb{Q}\ (\neq 0)$ which extends to $\mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathbb{Q}$

many works on $\mathcal{H}_{g,1}$ by

Sakasai, Habiro, Massuyeau, Cha-Friedl-Kim,...

Extending the above picture to a wider context (5)

Mal'cev completion of $\pi_1\Sigma_{g,1}$: $\cdots \to N_d \to \cdots \to N_1 = H_{\mathbb{Q}}$

Theorem (Garoufalidis-Levine)

$\exists\ \tilde{\rho}_\infty : \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \varprojlim_{d\to\infty} \mathrm{Aut}_0\, N_d$ (*symplectic auto. groups*)

each factor $\tilde{\rho}_d : \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \mathrm{Aut}_0\, N_d$ *is surjective over* $\mathbb{Z}$

candidates for Ker: constructed a homomorphism

$$\tilde{\rho} : \overline{\mathcal{H}}_{g,1} \to \left(\wedge^3 H_{\mathbb{Q}} \oplus \prod_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right) \rtimes \mathrm{Sp}(2g, \mathbb{Z})$$

and defined

$$(\wedge^2 S^{2k+1} H_{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} \ni 1 \mapsto \tilde{\mathfrak{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1}; \mathbb{Q})$$

Extending the above picture to a wider context (6)

replacing $\overline{\mathcal{H}}_{g,1}$ with more geometric object

(2008, after a comment by Orr):

$\mathcal{H}_{g,1}^{\mathrm{top}} = \{$homology cylinder over $\Sigma_{g,1}\}/$topological H-cobordism

Theorem (Freedman)

Any homology 3-sphere bounds a contractible topological 4-mfd

It follows that $\mathcal{H}_{g,1}^{\mathrm{smooth}} \rightarrow \mathcal{H}_{g,1}^{\mathrm{top}}$ factors through $\overline{\mathcal{H}}_{g,1}$

$$\mathbf{\Theta}^3 \rightarrow \mathcal{H}_{g,1}^{\mathrm{smooth}} \rightarrow \overline{\mathcal{H}}_{g,1} \rightarrow \mathcal{H}_{g,1}^{\mathrm{top}}$$

and the homomorphisms $\tilde{\rho}_\infty, \tilde{\rho}$ are actually defined on $\mathcal{H}_{g,1}^{\mathrm{top}}$

$$\Rightarrow \ \tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}}; \mathbb{Q})$$

how about Coker ?

$\mathfrak{h}_{g,1} =$ symplectic derivation Lie algebra of $\mathcal{L}(H_{\mathbb{Q}})$

extremely rich and mysterious structure

Theorem (Massuyeau-Sakasai)

(i) $\mathcal{H}_{g,1} \overset{\textit{homo.}}{\to} \hat{H}_1(\mathfrak{h}^+_{g,1}) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ *with dense image*

(ii) $H_1(\mathcal{H}_{g,1};\mathbb{Q}) \supset \mathbb{Q}$ (*sharp contrast: $\mathcal{M}_g$ is perfect ($g \geq 3$)*)

$\Rightarrow \hat{H}^1_c(\hat{\mathfrak{h}}_{g,1}) \subset H^1(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q})$ but this part comes from $H^1(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q})$ so that it vanishes in the Coker

At present, there is no information about

$$\mathrm{Coker}\left(H^1(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \to H^1(\mathcal{H}^{\mathrm{smooth}}_{g,1};\mathbb{Q})\right)$$

back to Ker

Theorem (Sakasai-Suzuki-M.)

$$\exists\ \tilde{\rho}^*_\infty : H^*_c(\hat{\mathfrak{h}}^+_{\infty,1})^{\mathrm{Sp}} \otimes H^*(\mathrm{Sp}(2\infty,\mathbb{Z})) \to H^*(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q})$$

$$\Rightarrow\ H^2_c(\hat{\mathfrak{h}}_{\infty,1}) \to H^2(\mathcal{H}^{\mathrm{top}}_{g,1};\mathbb{Q}) \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

$H^2_c(\hat{\mathfrak{h}}_{\infty,1}) \ni \mathfrak{t}_{2k+1}$ (Lie algebra version) $\mapsto \tilde{\mathfrak{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$

Conant-Kassabov-Vogtmann defined more classes on the LHS,

but, at present, only $\mathfrak{t}_3, \mathfrak{t}_5, \mathfrak{t}_7$ are known to be non-trivial...

Prospect (1)

only known homomorphism(s) (Frϕyshov and Ozsváth-Szabó)

$$\mathbf{\Theta}^3 \to \mathbb{Z}$$

candidate: Neumann-Siebenmann, Fukumoto-Furuta-Ue, Saveliev

$$\nu := \sum_{i=0}^{7} (-1)^{\frac{i(i+1)}{2}} \mathrm{rank}\, HF^i \text{ (instanton Floer homology)}$$

recall:

Theorem (Taubes)

$\sum_{i=0}^{7} (-1)^i \mathrm{rank}\, HF^i = 2\lambda$ *(Casson invariant)*

Theorem (Manolescu)

The Rohlin homomorphism $\mathbf{\Theta}^3 \to \mathbb{Z}/2$ does not split

Prospect (2)

geometric meaning of the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q})$:

Intersection numbers of higher and higher Massey products

(using works of Kitano, Garoufalidis-Levine)

Conjecture

The homomorphism $H^1(\mathbf{\Theta}^3;\mathbb{Q}) \cong \mathbb{Q}^{\mathbb{N}} \to H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$ *induced by*

$$0 \to \mathbf{\Theta}^3 \to \mathcal{H}_{g,1}^{\mathrm{smooth}} \to \overline{\mathcal{H}}_{g,1} \to 1$$

is highly non-trivial (possibly injective) and its image contains the classes $\tilde{\mathbf{t}}_{2k+1} \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \;\Rightarrow$

$$\tilde{\mathbf{t}}_{2k+1} \neq 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{top}};\mathbb{Q}) \;\; \textit{and}$$
$$\tilde{\mathbf{t}}_{2k+1} = 0 \in H^2(\mathcal{H}_{g,1}^{\mathrm{smooth}};\mathbb{Q})$$

Prospect (3)

If Conjecture is true $\Rightarrow$ obtain homomorphisms

$$\nu_k : \mathbf{\Theta}^3 \to \mathbb{Q} \quad (k = 1, 2, \ldots)$$

homology cobordism invariants