GOAL • Chern-Simons perturbation theory を $b_1 > 0$ について構成したい.

• Apply to 3-mfd topology (FTI, + more)

1. Garoufalidis-Levine filtration

$\mathbb{Z}\pi$-homology equivalence

• $f : \underset{\text{closed oritd.}}{M^3} \longrightarrow \underset{\text{fixed}}{N^3}$, conti. $\quad (\pi = \pi_1(N))$

is $\mathbb{Z}\pi$-homology equiv

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot \ \deg f = 1 \\ \cdot \ \tilde{f}_* : H_*(\tilde{M}) \xrightarrow{\cong} H_*(\tilde{N}) \end{cases}$$

Here

$$\begin{array}{ccc} \tilde{M} = f^*\tilde{N} & \longrightarrow & \tilde{N} \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{\quad f \quad} & N \end{array}$$

(簡単の為の) assumption $\pi_2 N = 0$.

• $f \simeq f' \overset{\text{def}}{\Longleftrightarrow}$
$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\exists g"', "\cong"] & N \\ M' \arrow[ur, "f'"'] & \end{tikzcd}$$
, htpy comm.

diffeo.

• $\mathcal{N}(N) := \{\mathbb{Z}\pi\text{-h.e. } M \to N\}/\cong$, a set.

represented by a null-htpic framed link in N. ★

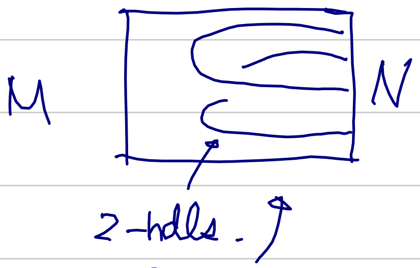
$\in$ ⊘ $\quad \Phi \big\downarrow$ "Surgery obstruction"

$B(\pi)$

$\in$ ⊘

$\mathbb{Z}\pi$ - linking matrix

idea of ★



1) $\exists f_M : \boxed{} \to N$ s.t. $f_M|_N = \mathrm{id}$.
(∵ "null-htpic")

2) f_M is unique up to htpy
(∵ $\pi_2 N = 0$)

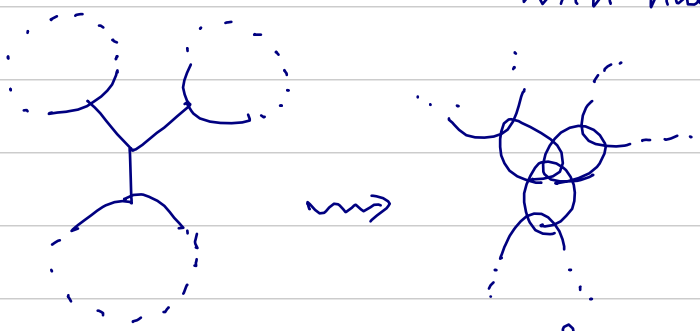
• $\mathcal{N}^{AS}(N) := \operatorname{Ker} \Phi$, represented by π-alg. split framed link in N

$\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$ linking matrix.

Garoufalidis - Levin's filtration

• $G = \{G_1, \dots, G_n\}$, Y-link in $M = (M, f: M \to N)$
with null htpic leaves.

G_i



$\rightsquigarrow$ Surgery

Y-graph　　framed link　　$(+\ \partial\pi\text{-h.e.})$

• $[M, G] = \sum_{G' \subset G} (-1)^{|G'|} M^{G'} \in \underbrace{\mathrm{Span}_{\mathbb{Q}} \mathcal{M}^{AS}(N)}_{= \mathcal{F}(N)}$

• $\mathcal{F}_n^Y(N) := \mathrm{Span}_{\mathbb{Q}} \{ [M, G] \mid M \in \mathcal{M}^{AS}(N),\ |G| = n \}$

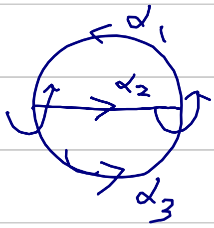
$\rightsquigarrow \mathcal{F}(N) = \mathcal{F}_0^Y(N) \supset \mathcal{F}_1^Y(N) \supset \cdots$

(Analog of Ohtsuki, Goussarov - Habiro.)

Theorem (G.L.)

$\psi_n : \underbrace{\mathcal{A}_n(\pi)}_{\text{space of } \pi\text{-decorated graphs.}} \xrightarrow{\text{Surgery}} \mathcal{F}_n^Y(N) / \mathcal{F}_{n+1}^Y(N)$ is surjective

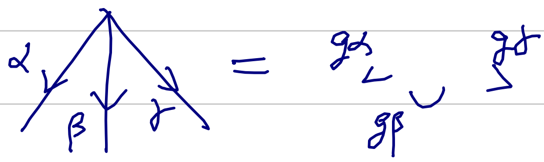
• π-decorated graph.



• trivalent
• oriented (vertices, edges)
• $\alpha_i \in \mathbb{Q}\pi$

• $\mathcal{A}_n(\pi) := \mathrm{Span}_{\mathbb{Q}} \{\pi\text{-decorated graphs, } 2n \text{ vertices}\} / \underline{\underline{\text{rel.}}}$

(AS, IHX, $\underline{\underline{\text{Holonomy}}}$, etc.)



$\alpha, \beta, \gamma \in \mathbb{Q}\pi$
$g \in \pi$

Main Theorem (Watanabe)

| $N = T^3$
| $\Rightarrow \psi_n$: isom. $(n \geq 1)$

(Rmk $N = S^3 \Rightarrow \psi_n$: isom. $(n \geq 1)$
T. Le '97, by using LMO.)

idea of proof : construct the 'inverse map'.

2. perturbative invariant

$\left(N = \mathbb{T}^3, \quad f : M \to N : \mathbb{Z}\pi\text{-h.e.} \right)$

Morse cpx

- $\Lambda = \mathbb{Q}\pi = \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$, $\quad \hat{\Lambda} = \mathbb{Q}(\Lambda)$
  quotient field

- $\Sigma_1 = 1 \times S^1 \times S^1, \quad \Sigma_2 = S^1 \times 1 \times S^1, \quad \Sigma_3 = S^1 \times S^1 \times 1$
  $\subset N$

  $\gamma : [a, b] \to N$, piecewise smooth "generic"

  $\mathrm{Hol}(\gamma) = t_1^{n_1} t_2^{n_2} t_3^{n_3}$

  $n_i := \gamma \cdot \Sigma_i$ (# of intersections)

- $h : M \to \mathbb{R}$, Morse $\quad \xi = \nabla h$.

- $C_i(\xi : \hat{\Lambda}) := \mathrm{Span}_{\hat{\Lambda}} \{ \text{crit pt. of } h \text{ ind} = i \}$

  $\partial : C_i(\xi : \hat{\Lambda}) \to C_{i-1}(\xi : \hat{\Lambda})$

  $$\partial p := \sum_q \underbrace{\Big(\# \; \overset{p}{\underset{q}{\Big\downarrow}} {\scriptstyle -\xi} \Big)}_{\text{count with hol} \in \Lambda \text{ in } N} \cdot q$$

Fact $(C_i(\xi, \hat{\lambda}), \partial)$; acyclic complex.

$\exists g : C_* \to C_{*+1}$, $\partial g + g \partial = 1$

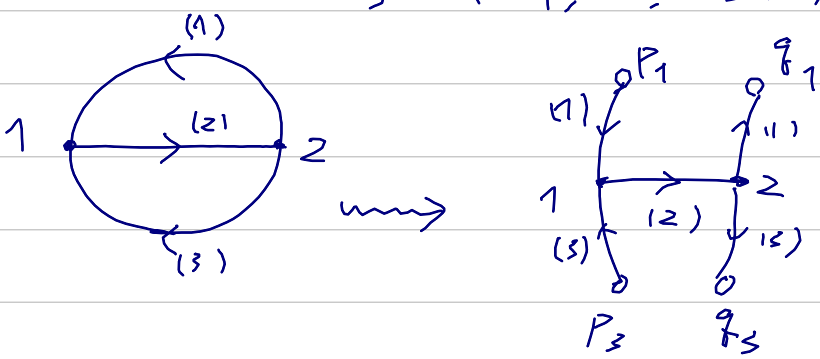
combinatorial propagator.

perturbative inv. (based on K. Fukaya's const)

- $h_1, \dots, h_{3n} : M \to \mathbb{R}$

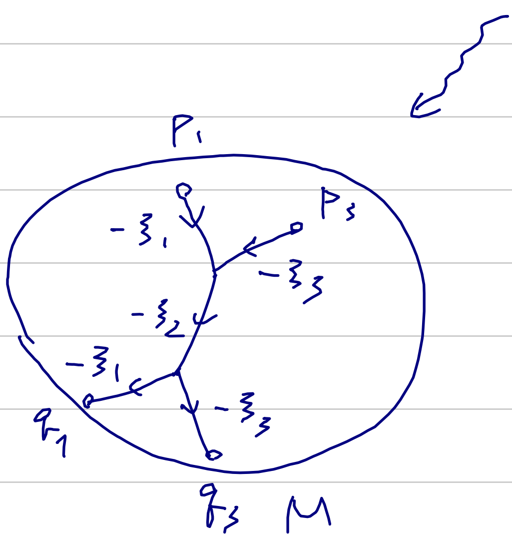
(corresponding to # edges)

$\vec{\xi} = (\xi_1, \dots, \xi_{3n})$, gradients



$\leftarrow$ crit pts. of h_1

$\leftarrow$ crit pts. of h_3



"flow-graph"

$\mathcal{M}_\Gamma(\vec{\xi}) := \{\text{flow graphs } \Gamma \to M\}$

• $\vec{\xi}$; generic, $\mathrm{ind}\, p_i - \mathrm{ind}\, q_i = 1$ $(\forall i)$

$\Rightarrow$ $\mathcal{M}_\Gamma(\vec{\xi})$: a finite set.

• $\vec{g} = (g_1, \cdots, g_{3n})$, combinatorial propagators for $C_*(\xi_i; \hat{\Lambda})$

def

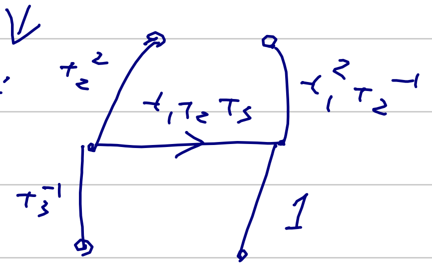
$$\hat{Z}_{2n} := \frac{1}{2^{3n}(2n)!(3n)!} \mathrm{Tr}_{\vec{g}} \Big(\sum_{\Gamma} \underline{\# \mathcal{M}_\Gamma(\vec{\xi})} \cdot \Gamma \Big)$$

count with hol in N

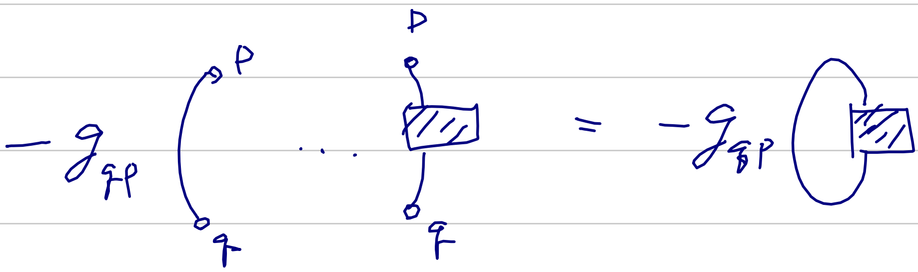
+ anomaly correction.

Sum over $2n$ vertex, tri-valent graphs possibly with $\overset{\circ\ \circ}{p_i\ q_i}$ edges.

e.g.



$\mathrm{Tr}_{\vec{g}}$; contraction



Theorem (Watanabe)

$\hat{Z}_{2n}$ is inv. of $Z\pi$-h.e..

i.e. $\hat{Z}_{2n} : \mathcal{Y}(N) \to \mathcal{A}_{2n}(\hat{\Lambda})$.

3. Idea of proof of Main theorem

Prop. $n \geq 1$,

(1) $\hat{Z}_{2n}(\tilde{\mathcal{F}}^{Y}_{2n+1}) = 0$

i.e, $\hat{Z}_{2n} : \mathcal{F}_n(N) \to \mathcal{F}^{Y}_{2n+1} \to \mathcal{A}_{2n}(\hat{\Lambda})$

(2) $\hat{Z}_{2n}(\psi_{2n}(\underbrace{[\Gamma(\alpha)]}_{\pi\text{-decorated graph}})) = [\Gamma(\alpha)]$

2n-vertices.

Then,

$$\begin{tikzcd} \mathcal{A}_{2n}(\Lambda) \arrow[r, two heads, "\psi_{2n}"] & \tilde{\mathcal{F}}^{Y}_{2n+1}/\tilde{\mathcal{F}}^{Y}_{2n} \arrow[l, bend left, dashed, "\hat{Z}_{2n}"] \end{tikzcd}$$

4. Extended Casson inv.

$N = T^3$, $f : M \longrightarrow N$, $\mathbb{Z}\pi$-h.e.

$$\lambda_\pi(M) := \hat{Z}_2(\underbrace{N}_{(N,\, id)}) - \hat{Z}_2(\underbrace{M}_{(M,\, f)}) \in \mathcal{A}_2(\hat{\Lambda})$$

inv. of $\mathbb{Z}\pi$-h.e.

$$\begin{tikzcd}
M \in \mathcal{J}^{AS}(M) \Rightarrow N - M \in \mathcal{F}_1^Y \arrow[r] & \mathcal{F}_1^Y/\mathcal{F}_3^Y \arrow[d, phantom, "\shortparallel"] \\
& \mathcal{F}_2^Y/\mathcal{F}_3^Y \arrow[d, "\hat{Z}_2"] \\
& \mathcal{A}_2(\hat{\Lambda})
\end{tikzcd}$$

clasper
calculus
(G.L.)

<u>Prop.</u> S : $\mathbb{Z}HS^3$, λ : Casson inv.

$$\lambda_\pi(N \# S^3) = \frac{\lambda(S)}{2}\,[\quad]$$