On the Casson-Walker invariant and a quantum representation of the mapping class group through the LMO invariant for genus one open books

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Backgrounds

Casson-Walker inv $\lambda(M) \in \mathbb{Q}$

- for a $\mathbb{Z}$HS M (Casson, 1980s)

$$H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$$

- for a $\mathbb{Q}$HS M (Walker, 1992)

$$H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$$

Construction in terms of mapping class groups
with regard to Heegaard splittings

- Morita (1980s)
- Cheptea, Habiro, Massuyeau (2008) etc.

$\rightsquigarrow$ How about the case of open book decompositions

Motivation A

$g = 1$ Heegaard splitting

lens spaces $L(p, q)$

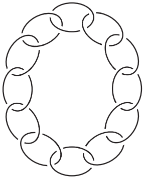
$$\lambda(L(p, q)) = -\frac{1}{2} s(q, p)$$

$(s(q, p) :$ Dedekind sum$)$

$g = 1$ open book decomposition

$g = 1$ open books M_φ

$(\varphi :$ homeo of $\Sigma_{1,1})$



$\lambda(M_\varphi) =$?

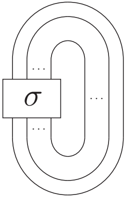
Aim A

Calculate the Casson-Walker invariant for 3-manifolds admitting a genus one open book decomposition.

quantum inv. of links

quantum rep. of a braid group

$\psi_n : B_n \to \mathrm{End}(V^{\otimes n})$



$\sigma \in B_n$, $L = \hat{\sigma}$

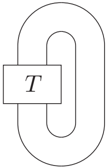
Jones poly.

$\mathrm{trace}(h^{\otimes n} \cdot \psi_n(\sigma))$

quantum inv. of 3-manifolds

quantum rep. of a mapping class group

$\rho : \mathfrak{M}_{1,1} \to \mathrm{End}(\hat{\mathcal{A}}($  $))$



$T \in \mathfrak{M}_{1,1}$, $M = S^3_{\hat{T}}$

Casson inv.

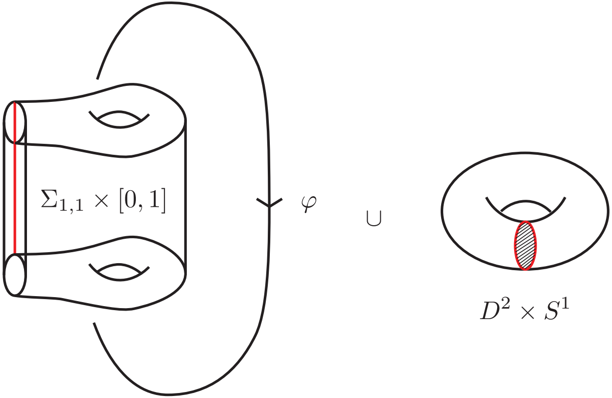
?

Aim B

Construct the representation of the mapping class group $\mathrm{MCG}(\Sigma_{1,1})$ and present the Casson-Walker invariant of $g = 1$ open books

Open book decompositions

An open book decompositions (genus 1, 1 boundary comp) of a 3-manifold M

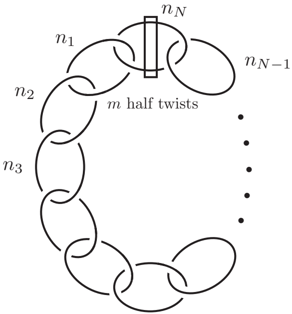
$M =$ 

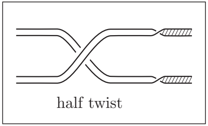
where φ : a homeo of $\Sigma_{1,1}$ s.t. $\varphi|_{\partial \Sigma_{1,1}} = \mathrm{id}_{\partial \Sigma_{1,1}}$

Surgery presentation

Fact : A 3-mfd with genus one open book decomp has the following surgery presentation.

$L_{n_1,\cdots,n_N;m}$





$$S^3 \xrightarrow{\text{surgery along } L_{n_1,\cdots,n_N;m}} M_{n_1,\cdots,n_N;m}$$

Result A

$M_{n_1,\cdots,n_N;m}$: 3-mfd with genus one open book decomp
obtained by surgery along a link $L_{n_1,\cdots,n_N;m}$
Suppose that $M_{n_1,\cdots,n_N;m}$ is a $\mathbb{Q}HS$,

Theorem A [M.]

$$\lambda(M_{n_1,\cdots,n_N;m}) = -\frac{1}{24}\left(\sum_i n_i - 3\sigma\right) - \frac{(-1)^{m+\sigma_+}}{24|H_1|}\left(2\sum_i n_i + 6N - 12m\right)$$

where σ : signature of linking matrix of $L_{n_1,\cdots,n_N;m}$
σ_+: $\sharp$ posi. eigenvalues of linking mat.
$|H_1|$: order of $H_1(M_{n_1,\cdots,n_N;m};\mathbb{Z})$

Result B

M_φ : 3-mfd with genus one open book decomp
which monodromy is φ

Suppose that M_φ is a $\mathbb{Q}HS$

$\tilde{\varphi} \in \widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/KI'$, KII, KIII

$\rho : \widetilde{\mathfrak{M}_{1,1}} \to \mathrm{End}(\hat{\mathcal{A}}($  $))$

Theorem B [M.]

The Casson-Walker invariant of M_φ can be calculated as follows.

$$\lambda(M_\varphi) = \frac{2}{\mathrm{tr}_0(\rho(\tilde{\varphi}))}\mathrm{tr}_1(\rho(\tilde{\varphi})) + \frac{1}{8}\sigma(\tilde{\varphi})$$

where $\sigma(\tilde{\varphi})$: signature of linking matrix of $L_{\tilde{\varphi}}$

$\mathrm{tr}_i(\rho(\tilde{\varphi}))$ = degree i part of “quantum trace” of $\rho(\tilde{\varphi})$

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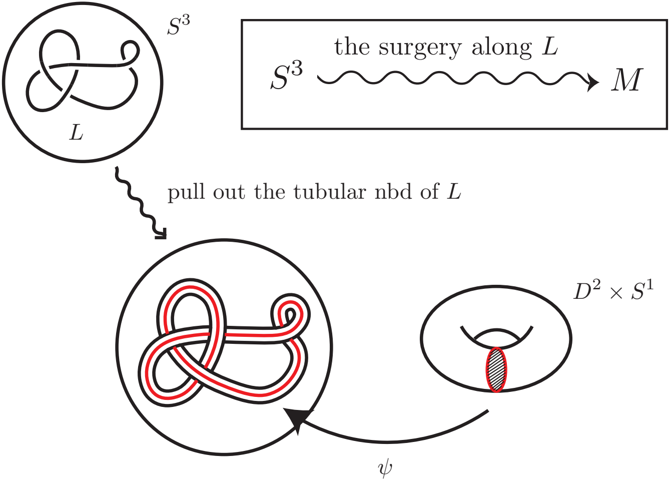
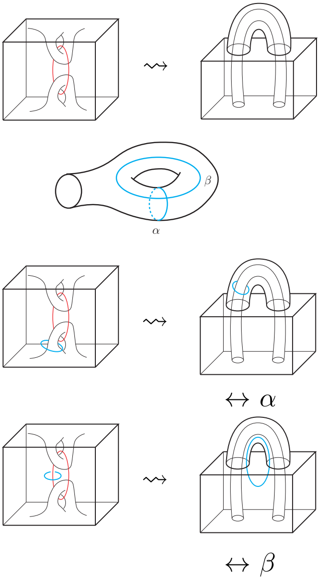
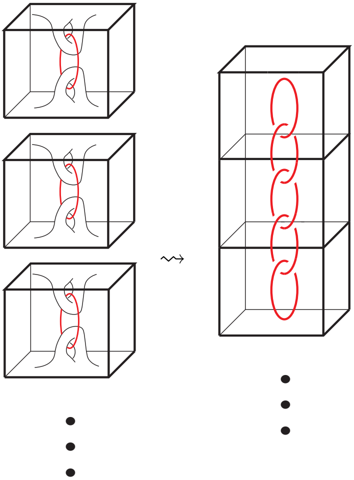
Surgery



$$M = (S^3 \setminus \text{the nbd of } L) \cup_\psi (\sqcup^\ell D^2 \times S^1)$$

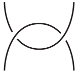
Surgery presentations and monodromies





Idea of Proof of Thm A

$\lambda(M) =$ deg=1 LMO inv $Z_1^{\mathrm{LMO}}(M)$ $\overset{\iota}{\rightsquigarrow}$ Kontsevich inv of L $Z(L)$

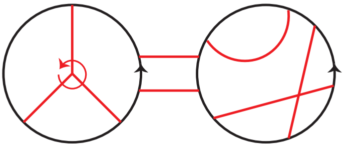
Calulculate $Z\Bigg($  $\Bigg)$

glue them

$Z_1^{\mathrm{LMO}}(M_{n_1,\cdots,n_N;m})$ $\overset{\iota}{\rightsquigarrow}$ $Z(L_{n_1,\cdots,n_N;m})$

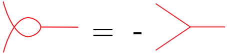
Jacobi diagrams

A Jacobi diag on a 1-manifold X

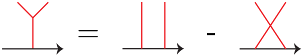


deg of a Jacobi diag $= \frac{1}{2}$ # vertices

$$\mathcal{A}(X) = \mathrm{span}_{\mathbb{C}}\ \{\text{Jacobi diags on } X\}/\text{AS, IHX, STU rel}$$

AS : 

IHX : 

STU : 

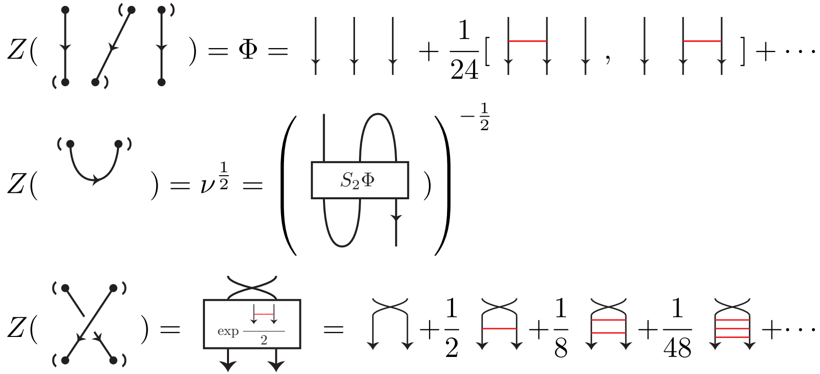
Kontsevich invariant

The Kontsevich inv is an inv of a link L

$$Z(L) = Z(T_1) \circ Z(T_2) \circ \cdots \circ Z(T_k) \in \mathcal{A}(\sqcup^{\ell} S^1)$$

(T_i : elementary q-tangle)

For example,



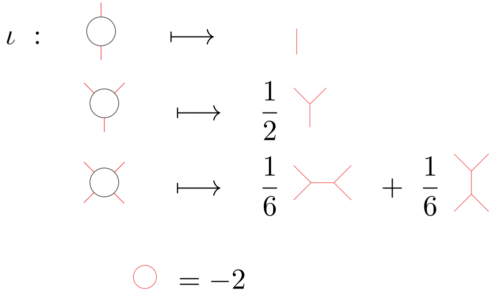
deg=1 part of LMO invariant

The deg=1 part of LMO inv

$$Z_1^{LMO}(M) = \frac{\iota(Z(L))}{\iota(Z(\infty))^{\sigma_+}\iota(Z(\infty))^{\sigma_-}} \in \mathrm{span}_{\mathbb{C}}\{\emptyset, \ominus\}$$

$$\iota : \mathcal{A}(\sqcup^{\ell} S^1) \to \mathrm{span}_{\mathbb{C}}\{\emptyset, \ominus\}$$

where



The value of the deg=1 LMO invariant of $g = 1$ open books

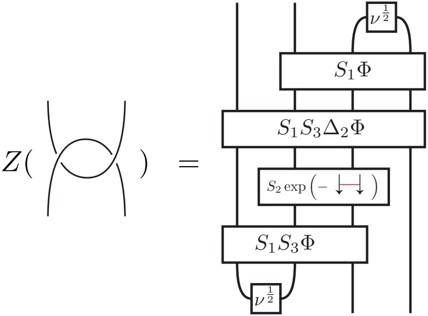
$M_{n_1,\cdots,n_N;m}$: 3-mfd with genus one open book decomp
obtained by surgery along a link $L_{n_1,\cdots,n_N;m}$
A_N : linking matrix of $L_{n_1,\cdots,n_N;m}$
Suppose $\det A_N \neq 0$

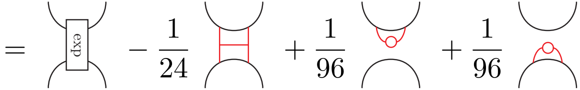
$$Z_1^{\mathrm{LMO}}(M) = c_0(M) + c_1(M)\theta$$

Prop

$$\begin{aligned} c_1\left(M_{n_1,\cdots,n_N;m}\right) = &-\frac{1}{48}(-1)^{N+\sigma_+}\det A_N(\mathrm{tr}A_N - 3\sigma) \\ &-\frac{1}{48}(-1)^{m+N+\sigma_+}\left(2\mathrm{tr}A_N + 6N - 12m\right) \end{aligned}$$

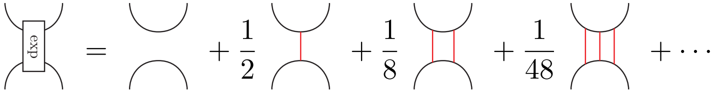
The value of a clasp





(Ohtsuki, 2007)

where



Result A

$M_{n_1,\cdots,n_N;m}$: 3-mfd with genus one open book decomp
obtained by surgery along a link $L_{n_1,\cdots,n_N;m}$

Suppose that $M_{n_1,\cdots,n_N;m}$ is a $\mathbb{Q}HS$,

Theorem A [M.]

$$\lambda(M_{n_1,\cdots,n_N;m}) = -\frac{1}{24}\left(\sum_i n_i - 3\sigma\right) - \frac{(-1)^{m+\sigma_+}}{24|H_1|}\left(2\sum_i n_i + 6N - 12m\right)$$

where σ : signature of linking matrix of $L_{n_1,\cdots,n_N;m}$

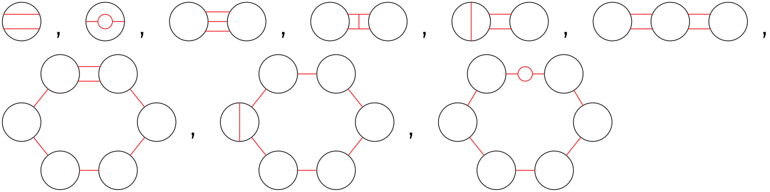
σ_+: $\sharp$ posi. eigenvalues of linking mat.

$|H_1|$: order of $H_1(M_{n_1,\cdots,n_N;m};\mathbb{Z})$

Proof of Thm A

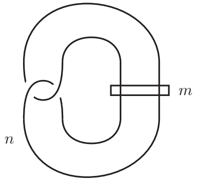
Fact : $\lambda(M) = \dfrac{2c_1(M)}{|H_1|}$ when $b_1(M) = 0$

Jacobi diags which become  through ι



As for non circular case, it can be calculated as LMO inv of lens spaces.
As for circular case, we have only to claculate above three diags. $\square$

The value of twist knots 1



$K_{n;m}$

$$M_{n;m} = S^3_{K_{n;m}}, \quad |H_1(M_{n;m})| = |n + (-1)^m 2|$$

$$c_1(S^3_{K_{n,m}}) = \begin{cases} (-1)^{\sigma_+}\left(\frac{1}{48}(n+2)(n-3\sigma) + \frac{1}{48}(2n+6+12m)\right) & m : \text{even} \\ (-1)^{\sigma_+}\left(\frac{1}{48}(n-2)(n-3\sigma) - \frac{1}{48}(2n+6+12m)\right) & m : \text{odd} \end{cases}$$

$$\lambda(M_{n;m}) = \begin{cases} -\frac{1}{24}\frac{((n+2)-1)((n+2)-2)}{n+2} - \frac{m}{2(n+2)} & (m : \text{even}, n+2 > 0) \\ -\frac{1}{24}\frac{((n-2)-1)((n-2)-2)}{n-2} - \frac{m+1}{2(n-2)} & (m : \text{odd}, n-2 > 0) \end{cases}$$

The value of twist knots 2

$$\lambda(M_{n;m}) = \begin{cases} -\frac{1}{24}\frac{((n+2)-1)((n+2)-2)}{n+2} - \frac{m}{2(n+2)} & (m : \text{even}, n+2>0) \\ -\frac{1}{24}\frac{((n-2)-1)((n-2)-2)}{n-2} - \frac{m+1}{2(n-2)} & (m : \text{odd}, n-2>0) \end{cases}$$

$$\begin{aligned} \lambda(S^3_K) =& \lambda(S^3) + \frac{1}{2p}\Delta''_K(1) - s(1,p) \\ =& -s(1,p) + \frac{1}{2p}\Delta''_K(1) \end{aligned}$$

$$s(1,p) = \frac{(p-1)(p-2)}{12p}$$

$$\Delta''_{K_{n,m}}(1) = \begin{cases} -m & m : \text{even} \\ m+1 & m : \text{odd} \end{cases}$$

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Introduction

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- Def (Open book decomp) and Prop (Surgery presentation)
- Results

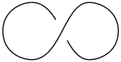
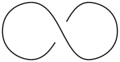
Part A

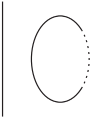
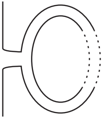
- Def (Surgery) and Explanation (Surgery presentation)
- Idea of Proof of Thm A
- Definitions (Jacobi diag, Kontsevich inv, LMO inv)
- Proof of Thm A

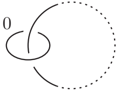
Part B

- Definitions (Kirby moves, Space of Jacobi diags, Product)
- Construction of Representation
- Proof of Thm B

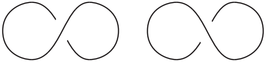
Kirby moves

the KI move :  $\longleftrightarrow \emptyset \longleftrightarrow$ 

the KII move :  $\longrightarrow$ 

the KIII move : 

$\longrightarrow \emptyset$

the KI' move :  $\longleftrightarrow \emptyset$

The space of the Jacobi diagrams on 2-tangles

$\hat{\mathcal{A}}($  $) = \{$Jacobi diagrams on 2-tangles up to AS, IHX, STU$\}/P_2,\ O_1,\ I_{>2}$

P_2 :  $+$  $+$  ~ 0

O_1 : $\bigcirc \sim -2$

$I_{>2}$: the Jacobi diagram whose $\sharp$ trivalent vertices $> 2 \sim 0$

We set 10 elements as the basis of $\mathcal{A}($  $)$.

$\mu_{00} =$  , $\mu_1 =$  , $\mu_{10} =$  , $\mu_{01} =$  , $\mu_{11} =$  ,

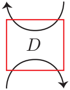
$\theta\mu_{00} = \ominus \sqcup$  , $\theta\mu_1 = \ominus \sqcup$  , $\theta\mu_{10} = \ominus \sqcup$  ,

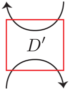
$\theta\mu_{01} = \ominus \sqcup$  , $\theta\mu_{11} = \ominus \sqcup$ 

The product on the space $\mathcal{A}($  $)$

$$\bullet : \hat{\mathcal{A}}(\ \ \ ) \otimes \hat{\mathcal{A}}(\ \ \ ) \overset{\circ}{\to} \hat{\mathcal{A}}(S^1 \sqcup \ \ \ ) \overset{\hat{\iota}}{\to} \hat{\mathcal{A}}(\ \ \ )$$

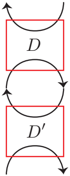
   

For diagrams $\eta =$ 

, $\eta' =$ 

$\in \mathcal{A}($  $)$,

$$\eta \bullet \eta' = \hat{\iota}\left(\quad \right) \in \hat{\mathcal{A}}(\ \ \ ),$$



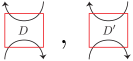
Construction of Representation

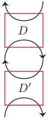
$$\rho : \widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/\mathsf{KI}', \mathsf{KII}, \mathsf{KIII} \stackrel{\hat{\iota}\check{Z}}{\to} \hat{\mathcal{A}}(\quad) \stackrel{\pi}{\to} \mathrm{End}(\hat{\mathcal{A}}(\quad))$$

$$\hat{\iota} := \sqrt{-1}^{\ell}\iota : \hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \quad) \to \hat{\mathcal{A}}(\quad)$$

$\hat{\iota}\check{Z}$ is invariat under the KI′, KII, KIII moves.

$$\bullet : \hat{\mathcal{A}}(\quad) \otimes \hat{\mathcal{A}}(\quad) \ni \Big(\quad , \quad \Big) \mapsto \hat{\iota} \Big(\quad \Big) \in \hat{\mathcal{A}}(\quad)$$





$$\pi : \hat{\mathcal{A}}(\quad) \to \mathrm{End}(\hat{\mathcal{A}}(\quad))$$

$$\pi\left(\hat{\iota}\check{Z}(R)\right) : \hat{\mathcal{A}}(\quad) \to \hat{\mathcal{A}}(\quad)$$

$$\hat{\iota}\check{Z}(T) \mapsto \hat{\iota}\check{Z}(R \circ T) = \hat{\iota}\check{Z}(R) \bullet \hat{\iota}\check{Z}(R)$$

Result B

M_φ : 3-mfd with genus one open book decomp
which monodromy is φ

Suppose that M_φ is a $\mathbb{Q}HS$

$\tilde{\varphi} \in \widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/$KI′, KII, KIII

$\rho : \widetilde{\mathfrak{M}_{1,1}} \to \mathrm{End}(\hat{A}($  $))$

Theorem B [M.]

The Casson-Walker invariant of M_φ can be calculated as follows.

$$\lambda(M_\varphi) = \frac{2}{{}^t v_0 \rho(\tilde{\varphi}) w} {}^t v_1 \rho(\tilde{\varphi}) w + \frac{1}{8}\sigma(\tilde{\varphi})$$

where $\sigma(\tilde{\varphi})$: signature of linking matrix of $L_{\tilde{\varphi}}$

$v_i =$ degree i part of ${}^t\Big(\mathrm{tr}$  $, \mathrm{tr}$  $, \mathrm{tr}$  $, \mathrm{tr}$  $, \mathrm{tr}$  $, \cdots \Big)$

$w = \hat{\imath}\check{Z}($  $) \in \hat{A}($  $)$

Proof of Thm B

It is known that

$$Z_1^{\mathrm{LMO}}(M) = c_0(M) + c_1(M)\theta,\ \ \lambda(M) = \frac{2}{c_0(M)} c_1(M),$$

$$\begin{aligned} Z_1^{\mathrm{LMO}}(M) &= \left(-1 + \frac{1}{16}\theta\right)^{-\sigma_+} \left(1 + \frac{1}{16}\theta\right)^{-\sigma_-} \iota\check{Z}(L) \\ &= (-1)^{\sigma_+} \left(1 + \frac{1}{16}\sigma\theta\right)(b_0 + b_1\theta) \\ &= (-1)^{\sigma_+} b_0 + (-1)^{\sigma_+}\left(\frac{\sigma}{16} b_0 + b_1\right)\theta. \end{aligned}$$

On the other hand, we have that

$$(\mathrm{tr}\mu_{00}, \mathrm{tr}\mu_1, \mathrm{tr}\mu_{10}, \cdots)\, \rho(\tilde{\varphi})\hat{\iota}\check{Z}($$

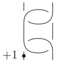
$$) = \hat{\iota}\check{Z}(L_{\tilde{\varphi}}).$$

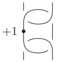
$\square$

Concrete Calculations

When the monodromy φ is periodic in $\mathfrak{M}_{1,1}$, we can set

$$\tilde{\varphi} = h^m \alpha^{n_1} \beta \cdots \alpha^{n_N} \beta,\ \forall i, n_i \leq 0,\ \exists j, n_j \neq 0,$$

where $h =$  $\circ$  , $\alpha =$ 

, $\beta =$ 

,

and $\sigma(\tilde{\varphi}) = 0$.

$\rho(h)$, $\rho(\alpha)$, $\rho(\beta)$ can be presented as matrices in $GL_{10}(\mathbb{C})$.

$$v = {}^t\!\left(0,\ -2,\ -2,\ -2,\ \frac{1}{6}\theta,\ 0,\ -2\theta,\ -2\theta,\ -2\theta,\ 0\right)$$

$$w = {}^t\!\left(0,\ -1,\ \frac{1}{2},\ \frac{1}{2},\ 0,\ \frac{1}{24},\ 0,\ 0,\ 0,\ 0\right)$$

Thus, the Casson-Walker invariant of M_φ is

$$\lambda(M_\varphi) = \frac{2}{{}^t v_0 \rho(\tilde{\varphi}) w} {}^t v_1 \rho(\tilde{\varphi}) w$$

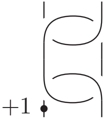
Presentation of the generators

$$\alpha = \frac{1}{24}\theta\mu_{00} + \left(-1 + \frac{1}{24}\right)\mu_1 + \frac{1}{2}\mu_{10} + \frac{1}{2}\mu_{01} + \frac{1}{4}\left(1 - \frac{1}{3}\theta\right)\mu_{11}$$

$$\beta = \left(-1 + \frac{1}{16}\theta\right)\mu_{00} - \mu_1 + \frac{1}{2}(1 - \frac{1}{48}\theta)\mu_{10} + \frac{1}{2}(1 - \frac{1}{48}\theta)\mu_{01} - \frac{1}{96}\theta\mu_{1}$$

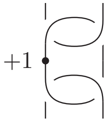
$$\begin{aligned} V &= V_1 \oplus V_2 \oplus V_3 \\ &= \mathbb{C}\langle \mu_{00}, \mu_{10}, \theta\mu_{00}, \theta\mu_{10}\rangle \oplus \mathbb{C}\langle \mu_{01}, \mu_{11}, \theta\mu_{01}, \theta\mu_{11}\rangle \oplus \mathbb{C}\langle \mu_1, \theta\mu_1\rangle \end{aligned}$$

Concrete matrix presentaion of α

$A := \rho($  $)$ is the following matrix.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{24} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{96} & 0 & \frac{1}{24} & 0 & 0 & \frac{1}{2} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{96} & -\frac{1}{24} & 0 & 0 & 0 & \frac{1}{2} & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{1}{96} & -\frac{1}{24} & \frac{1}{2} & 1
\end{pmatrix}
\oplus
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{1}{96} & -\frac{1}{24} & \frac{1}{2} & 1
\end{pmatrix}
\oplus
\begin{pmatrix}
1 & 0 \\
-\frac{1}{24} & 1
\end{pmatrix}
$$

$B := \rho($  $)$ is the following matrix.

$$
\begin{pmatrix}
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{48} & 0 & \frac{1}{24} & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\
0 & -\frac{1}{24} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{48} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{48} & \frac{1}{24} & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & -\frac{1}{48} & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{1}{48} & \frac{1}{24} & 1 & -2 \\
0 & -\frac{1}{48} & 0 & 1
\end{pmatrix} \oplus \begin{pmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{1}{48} & \frac{1}{24} & 1 & -2 \\
0 & -\frac{1}{48} & 0 & 1
\end{pmatrix} \oplus \begin{pmatrix}
1 & 0 \\
-\frac{1}{24} & 1
\end{pmatrix}
$$

Summary and Future directions

Summary

- We calculated the Casson-Walker invariant of $g = 1$ open books through the calculation of the deg=1 part of the LMO invariant.
- We constructed the representaion of $\mathfrak{M}_{1,1}$ on the space of the Jacobi diagrams on  , and we gave a formula for the calculation of the Casson-Walker invariant of $g = 1$ open books.

Future directions

- relation to rep. theory of MCG
- relation to contact topology
- general cases