G : cpt Lie grp

$\mathbb{M}$: quaternionic rep. i.e. $G \times \underbrace{Sp(1)}_{\cong Spin(3)} \curvearrowright \mathbb{M}$

Consider the SW type equation on X^3 associated with $(G, \mathbb{M})$

eg. $G = SU(2)$, $M = 0$ ⇝ Casson i.e. $F_A = 0$

$G = U(1)$, $M = \mathbb{H}$ ⇝ Seiberg-Witten equation

Choose a spin structure on X

P : G-bdle on X

A : connection on P

Φ : spinor with value in $P \times_G \mathbb{M}$

$$(\star) \left\{ \begin{array}{l} \not{D}_A \Phi = 0 \\ * \underbrace{F_A}_{\text{curvature}} = \mu(\Phi) \end{array} \right.$$

$$\mu : \mathbb{M} \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*, \quad \mathbb{R}^3 \cong \mathrm{Im}\,\mathbb{H} \cong sp(1)$$

$$q \mapsto \frac{1}{2}(I q, 3 q)$$

inv. inn. prod

$Spin(3) \curvearrowright sp(1)$ ⇝ 1-form

⇝ Hope : define invariants of X by sign-counting solutions of $(\star)$

eg Casson $= \#$ of flat $SO(3)$ connections / gauge

$F_A = 0$

Difficult to define "counting" in general

(a la Atiyah-Segal)

<u>Working hypothesis</u>

We have a 3d TQFT $Z = Z_{G,M}$.
associated with (G, M)

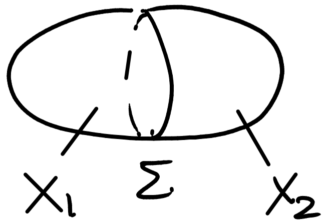
X: closed oriented 3-mfd (+ additional str. Spin^c str.)
$\rightsquigarrow Z(X)$: "number" — # of solutions

Σ^2: closed 2-mfd (+ add. str.)
$\rightsquigarrow Z(\Sigma)$: "vector space" — cohomology of moduli sp of ☆ on Σ

X^3: 3-mfd with bdry $\rightsquigarrow Z(X) \in Z(\partial X)$

with certain axioms

e.g.
- $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$
- $Z(-\Sigma) = Z(\Sigma)^*$
- gluing



$Z(X) = \langle Z(X_1), Z(X_2) \rangle$

e.g. $\varphi : \Sigma \to \Sigma$ diffeo, $X_\varphi = \Sigma \times [0,1] / (x,0) \sim (\varphi(x), 1)$

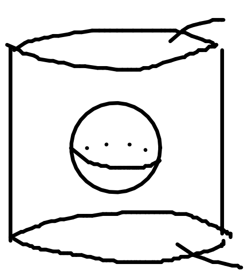
$$Z(X_\varphi) = \mathrm{trace}(Z(\varphi) : Z(\Sigma) \circlearrowleft)$$

In particular $Z(\Sigma \times S^1) = \dim Z(\Sigma)$

"<u>Cor</u>" (of axioms) $Z(S^2)$ is a commutative ring
$Z(\Sigma)$ is a its module

proof) $X = \Sigma \times [0.1] \setminus D^3$
$\partial X = \Sigma \times \{0\} \sqcup -\Sigma \times \{1\} \sqcup S^2$

$\rightsquigarrow Z(X) : Z(\Sigma) \otimes Z(S^2) \longrightarrow Z(\Sigma)$



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Example Casson–Walker–Lescop invariant of $S^2 \times S^1$
$= -1/12$

$$\therefore \quad \dim \mathcal{Z}(S^2) = -\frac{1}{12} = \zeta(-1) = 1+2+3+\cdots$$

$\Longrightarrow$ $\mathcal{Z}(S^2)$ is an infinite dimensional graded commutative ring

Idea: Use algebraic geometry to study $\mathcal{Z}(S^2)$

i.e., spectrum $\mathcal{M}_C = \mathrm{Spec}\, \mathcal{Z}(S^2)$

affine algebraic variety (scheme) $\mathbb{C}[\mathcal{M}_C] = \mathcal{Z}(S^2)$
more precisely

(grading $\rightsquigarrow$ $\mathbb{C}^\times$-action
$\mathbb{C}[\mathcal{M}_C] = \bigoplus_{d \in \mathbb{Z}} \{ f : t f = t^d f \}$)

$\mathcal{Z}(\Sigma) \rightsquigarrow$ coherent sheaf on $\mathcal{M}_C$
the space of holo. sections of holomorphic vector bdle

Toy model (corresponding to $G = U(1)$, $M = \mathbb{H}$)

$$-\frac{1}{12} = \underbrace{1}_{\mathrm{gr}\,0} + \underbrace{2}_{\mathrm{gr}\,1} + \underbrace{3}_{\mathrm{gr}\,2} + \cdots = \dim \mathbb{C}[x,y]$$

$\mathrm{gr}\, x = \mathrm{gr}\, y = 1$

$$\mathrm{Spec}\, \mathbb{C}[x,y] = \mathbb{C}^2 \circlearrowleft \mathbb{C}^\times \quad \begin{array}{l} t \cdot x = t x \\ t \cdot y = t y \end{array}$$

Rem geometric intuition is useful only when $\mathcal{M}_C$ is a "nice" space (eg. nonsingular)

Opposite extreme : $\mathrm{Spec}\, A$
$\sim$ pts, but with nontrivial scheme structure
A finite dim comm. ring

<u>Witten</u> $Z_{(G,M)}$ = TQFT associated with Rozansky-Witten theory with target $= \mathcal{M}_C$ hK or holo. sympl.

$\uparrow$ Coulomb branch of (G, M)

$(G=SU(2), M=0) \Rightarrow \mathcal{M}_C =$ Atiyah-Hitchin space
$x^2 = y^2 z + y$ in $\mathbb{C}^3$

$(G = U(1), M = \mathbb{H}) \Rightarrow \mathcal{M}_C =$ Taub-NUT space $\cong \mathbb{C}^2$

<u>Rem</u> ① RW theory is usually considered for <u>compact</u> target spaces.

$$Z(S^2) = H^*(\mathcal{M}_C, \mathcal{O}) \quad \text{finite dim'l}$$

affine variety $\Rightarrow H^{>0} = 0$

but $\dim H^0 = \infty$ unless $\mathcal{M}_C = pt$

② In general $\mathcal{M}_C$ is <u>singular</u>, hence RW theory must be used with care

In fact, $\mathcal{M}_C$ is nonsingular $\overset{?}{\Longleftrightarrow} M /\!/\!/ G = \{0\}$
$\Updownarrow$
a priori estimate for spinor Φ of SW type equation

Construction by Braverman–Finkelberg–N.

Suppose $\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$

Put a cpx str. on Σ,

Consider moduli stack of pairs

$$\begin{cases} \mathcal{P} : \text{holo. } G_{\mathbb{C}}\text{-bdle over } \Sigma, \\ \phi \in H^0((\mathcal{P} \times_{G_{\mathbb{C}}} \mathbb{N}) \otimes K_\Sigma^{1/2}) \end{cases}$$

Let $Z(\Sigma) = H_c^*(\text{moduli stack})^*$

<u>ex</u> ① $G = U(1)$, $\mathbb{M} = 0$ $\rightsquigarrow$ invariant should be 0

But $Z(S^2)$: nontrivial

<u>Claim</u> $\operatorname{Spec} Z(S^2) = \underset{\substack{\circlearrowleft \\ \mathbb{C}^\times}}{\mathbb{C}} \times \mathbb{C}^\times$

$$\text{moduli stack} = \coprod_{d \in \mathbb{Z}} \mathrm{pt}/\underset{\uparrow \text{auto}}{\mathbb{C}^\times}$$

$$\therefore \quad H_c^*(\quad)^* = \bigoplus_{d \in \mathbb{Z}} H^*_{\mathbb{C}^\times}(\mathrm{pt})^* = \mathbb{C}[w, \underbrace{z, z^{-1}}_{\text{"dim"} = 0}]$$

where $H^*_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[w]$

$z^{\pm}$ = fund. class of $d = \pm 1$

higher genus $\quad$ moduli stack $= \coprod_{d \in \mathbb{Z}} \operatorname{Jac}\Sigma / \underset{\text{trivial}}{\mathbb{C}^\times}$

$$Z(\Sigma_g) = H_c^*(\quad)^* = \underset{\substack{\| \\ \Lambda^* H^1(\Sigma)}}{H^*(\operatorname{Jac}\Sigma_g)} \otimes Z(S^2)$$

<u>Rem</u> $Z(\Sigma_g) = H^0(\mathcal{M}_C, \Lambda^* T_{\mathcal{M}_C}^{\oplus g})$

② $G = U(1)$, $M = \mathbb{H}$, $N = \mathbb{C}$

$$\underset{S^2}{\text{moduli stack}} = \coprod_{d \in \mathbb{Z}} \underbrace{H^0(\mathcal{O}(d-1))}_{= \mathbb{C}^{\max(d,0)}} / \mathbb{C}^\times$$

$$\begin{array}{cccccc} & & \{0\} & \{0\} & \mathbb{C} & \mathbb{C}^2 \\ d & \times & \times & \times & \times & \times \\ & & -1 & 0 & 1 & 2 \end{array} \quad \rightsquigarrow \quad \operatorname{Spec} \mathcal{Z}(S^2) = \mathbb{C}^2$$

$$H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[w]$$

x = fund. class for $d = 1$

y = 〃 $d = -1$

$\Rightarrow$ $w = xy$ $\therefore$ $\mathcal{Z}(S^2) = \mathbb{C}[x, y]$